Chapter 3
Lower eigenvalue estimates on closed manifolds

In this chapter we assume that M has empty boundary.

3.1 Friedrich's inequality

The most general sharp lower bound for the Dirac spectrum has been proved by T. Friedrich in [85] and is now known under the name "Friedrich's inequality". For the concept of Killing spinor we refer to Section A.1.

Theorem 3.1.1 (T. Friedrich [85]) *Any eigenvalue λ of D on an $n(\geq 2)$-dimensional closed Riemannian spin manifold (M^n, g) satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M(S), \tag{3.1}$$

where S is the scalar curvature of M.

Moreover (3.1) *is an equality for some eigenvalue λ if and only if there exists a non-zero real Killing spinor on (M^n, g).*

Proof: It follows from the Schrödinger-Lichnerowicz formula (1.15) that, for any $\varphi \in \Gamma(\Sigma M)$,

$$\int_M \langle D^2\varphi, \varphi\rangle v_g = \int_M \langle \nabla^*\nabla\varphi, \varphi\rangle v_g + \int_M \frac{S}{4}|\varphi|^2 v_g.$$

By definition of $\nabla^*\nabla$ and since D is formally self-adjoint we can write

$$\int_M |D\varphi|^2 v_g = \int_M |\nabla\varphi|^2 v_g + \int_M \frac{S}{4}|\varphi|^2 v_g. \tag{3.2}$$

Decompose now w.r.t. any local orthonormal basis $\{e_j\}_{1\leq j\leq n}$ of TM

N. Ginoux, *The Dirac Spectrum*, Lecture Notes in Mathematics 1976,
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$$\nabla\varphi = \underbrace{\nabla\varphi + \frac{1}{n}\sum_{j=1}^{n} e_j^* \otimes e_j \cdot D\varphi}_{=P\varphi\in\mathrm{Ker}(\mu)} \underbrace{- \frac{1}{n}\sum_{j=1}^{n} e_j^* \otimes e_j \cdot D\varphi}_{\in\mathrm{Ker}(\mu)^\perp},$$

where P is the so-called Penrose operator of (M^n, g), see also Appendix A. We deduce that $|\nabla\varphi|^2 = |P\varphi|^2 + \frac{1}{n}|D\varphi|^2$ (identity (A.11) in Appendix A). Replacing $|\nabla\varphi|^2$ in (3.2) one obtains

$$\int_M |D\varphi|^2 v_g = \frac{1}{n}\int_M |D\varphi|^2 v_g + \int_M |P\varphi|^2 v_g + \int_M \frac{S}{4}|\varphi|^2 v_g,$$

that is,

$$\int_M \left(|D\varphi|^2 - \frac{n}{4(n-1)}S|\varphi|^2\right) v_g = \frac{n}{n-1}\int_M |P\varphi|^2 v_g. \tag{3.3}$$

Choose φ to be a non-zero eigenvector for D associated to the eigenvalue λ. From $|P\varphi|^2 \geq 0$ one straightforward obtains the inequality (3.1).

If (3.1) is an equality for some eigenvalue λ then (3.3) implies $P\varphi = 0$ for any non-zero eigenvector φ for D associated to λ, hence any such φ is a (necessarily real) Killing spinor on (M^n, g). Conversely, if (M^n, g) carries a non-zero α-Killing spinor φ, then since M is compact α must be real. Moreover, on the one hand φ is an eigenvector for D associated to the eigenvalue $-n\alpha$, on the other hand we know from Proposition A.4.1 that the scalar curvature of (M^n, g) must be $S = 4n(n-1)\alpha^2$, in particular it must be non-negative. Therefore such a φ must be an eigenvector for D associated to the eigenvalue $\sqrt{\frac{nS}{4(n-1)}}$ or $-\sqrt{\frac{nS}{4(n-1)}}$. This shows the equivalence in the limiting-case and concludes the proof. □

Another method for the proof of (3.1), which is actually T. Friedrich's in [85], relies on the *modified connection*

$$\widetilde{\nabla}_X\psi := \nabla_X\psi + \frac{\lambda}{n}X\cdot\psi$$

for every $X \in TM$, where $D\psi = \lambda\psi$: Compute $|\widetilde{\nabla}\psi|^2$ (which plays the role of $|P\psi|^2$ above), integrate and apply the Schrödinger-Lichnerowicz formula. Alternatively but still along the same idea, it can be directly deduced from the Cauchy-Schwarz inequality that $|D\varphi| \leq \sqrt{n}|\nabla\varphi|$ for every section φ, from which (3.1) follows.

As a consequence of Theorem 3.1.1, if the scalar curvature S of (M^n, g) is positive then its Dirac operator has trivial kernel - whatever the spin structure is. This had been already noticed by A. Lichnerowicz in [175] where he had obtained as a straightforward application of (3.2) the following estimate:

$$\lambda^2 \geq \frac{1}{4}\inf_M(S). \tag{3.4}$$

It follows from (3.4) combined with the Atiyah-Singer index theorem [28] (see Theorem 1.3.9) that a Riemannian manifold with positive scalar curvature must have vanishing topological index. In particular, if the manifold has non-vanishing $\widehat{A}$-genus, then it cannot carry any Riemannian manifold with positive scalar curvature. In other words, there exists a topological obstruction to the existence of metrics with positive scalar curvature on closed spin manifolds, at least in even dimensions. The reader interested in further results in that topic - such as Gromov-Lawson's work - should refer to [173] or to [124]. The existence of Riemannian metrics for which the Dirac kernel is non-zero is discussed in Section 6.2. Moreover, we mention another closely related application of the Atiyah-Singer index theorem to geometry via the Schrödinger-Lichnerowicz formula, namely to the so-called scalar curvature rigidity issue which asks for the possibility of increasing the scalar curvature without shrinking the distances of a given metric on a fixed background manifold. For example this is not possible on the round sphere (M. Llarull [180, Thm. B]) nor on any connected closed Kähler manifold with non-negative Ricci curvature (S. Goette and U. Semmelmann [109, Thm. 0.1]), we refer to [110] for the case of symmetric spaces and references.

Although it requires the non-negativity of S to be non-trivial, Friedrich's inequality (3.1) provides fine information of geometrical nature on the Dirac spectrum. Indeed S stands for a very weak curvature invariant of a given Riemannian manifold. This shows for example a difference of behaviour with other differential operators such as the scalar Laplacian Δ: by a result of A. Lichnerowicz [174], any non-zero eigenvalue λ of Δ satisfies

$$\lambda \geq \frac{n}{n-1} \inf_M(\mathrm{Ric}),$$

where Ric is the Ricci curvature tensor of (M^n, g), which is a stronger curvature invariant. In case $\inf_M(S) \leq 0$ Friedrich's inequality (3.1) can be improved in different ways using various techniques, see Sections 3.3 to 3.7.

Besides, (3.1) is sharp since e.g. $M := \mathbb{S}^n$ ($n \geq 2$) admits non-zero Killing spinors, see Example A.1.3.2. For the classification of Riemannian spin manifolds carrying non-zero real Killing spinors we refer to Theorems A.4.2 and A.4.3 in Appendix A.

3.2 Improving Friedrich's inequality in presence of a parallel form

O. Hijazi [128] and A. Lichnerowicz [176, 177] noticed that equality in (3.1) cannot hold on those M admitting a non-zero *parallel k-form* for some $k \in \{1, \ldots, n-1\}$. This suggests (3.1) could be enhanced under this assumption.

The idea of proof for Theorems 3.2.1, 3.2.4 and 3.2.6 can be summarised as follows (see [64]): given any eigenvector φ of D to the eigenvalue λ, decompose $|\nabla\varphi|^2$ in a sharper way than for the proof of Friedrich's inequality, using the splitting of ΣM induced by the Clifford action of the parallel form.

Theorem 3.2.1 (B. Alexandrov, G. Grantcharov and S. Ivanov [5])

Any eigenvalue λ of D on an $n(\geq 3)$-dimensional closed Riemannian spin manifold (M^n, g) admitting a non-zero parallel 1-form *satisfies*

$$\lambda^2 \geq \frac{n-1}{4(n-2)} \inf_M(S), \tag{3.5}$$

where S is the scalar curvature of M.

Moreover if (3.5) *is an equality for some eigenvalue λ, then the universal cover of M is a Riemannian product of the form $\mathbb{R} \times N$, where N admits a real Killing spinor.*

We shall prove a more general result:

Theorem 3.2.2 (A. Moroianu and L. Ornea [202]) *Inequality* (3.5) *holds as soon as M^n ($n \geq 3$) admits a non-zero* harmonic 1-form of constant length. *Furthermore, if it is an equality for some eigenvalue λ, then this form is parallel.*

Proof: Let ξ be the dual vector field to the harmonic 1-form of constant length. We may assume that $g(\xi, \xi) = 1$ on M. Define the following Penrose-like operator

$$T_X\varphi := \nabla_X\varphi + \frac{1}{n-1}(X - g(X,\xi)\xi)\cdot D\varphi - \frac{1}{n-1}(ng(X,\xi) + X\cdot\xi\cdot)\nabla_\xi\varphi$$

for all $X \in TM$ and $\varphi \in \Gamma(\Sigma M)$. In case ξ is parallel this operator can be described as the sum of the orthogonal projections of $\nabla\varphi$ onto the kernels of the Clifford multiplications $T^*M \otimes \Sigma_\pm M \xrightarrow{\mu_\pm} \Sigma M$, where $\Sigma_\pm M := \mathrm{Ker}(i\xi\cdot \mp \mathrm{Id}_{\Sigma M})$, see [5, eq. (4)] for another equivalent expression (note however that it does not exactly coincide with the T defined in [202]). Nevertheless ξ need not be parallel in order for T to play its role for the estimate as we shall see in the proof.

Fix a local orthonormal frame $\{e_j\}_{1\leq j\leq n}$ of TM. For any $\varphi \in \Gamma(\Sigma M)$, we have

$$\begin{aligned}
|T\varphi|^2 &= \sum_{j=1}^n |T_{e_j}\varphi|^2 \\
&= |\nabla\varphi|^2 + \frac{1}{n-1}|D\varphi|^2 + \frac{n}{n-1}|\nabla_\xi\varphi|^2
\end{aligned}$$

$$
\begin{aligned}
&-\frac{2}{n-1}(|D\varphi|^2 + \Re e\,(\langle \xi \cdot D\varphi, \nabla_\xi \varphi\rangle)) \\
&-\frac{2}{n-1}(n|\nabla_\xi \varphi|^2 - \Re e\,(\langle \xi \cdot \nabla_\xi \varphi, D\varphi\rangle)) \\
&-\frac{2}{n-1}\Re e\,(\langle \xi \cdot \nabla_\xi \varphi, D\varphi\rangle)) \\
= {}& |\nabla\varphi|^2 - \frac{1}{n-1}|D\varphi|^2 - \frac{n}{n-1}|\nabla_\xi\varphi|^2 + \frac{2}{n-1}\Re e\,(\langle \xi \cdot \nabla_\xi \varphi, D\varphi\rangle)).
\end{aligned}
$$

Now we can express the last term on the r.h.s. through the other ones, a trick due to the authors of [202]: namely, since ξ is assumed to be harmonic, i.e., closed and co-closed, the identity (1.12) reads $D(\xi \cdot \varphi) = -\xi \cdot D\varphi - 2\nabla_\xi \varphi$ and hence

$$
\Re e\,(\langle \xi \cdot \nabla_\xi \varphi, D\varphi\rangle) = |\nabla_\xi \varphi|^2 + \frac{1}{4}(|D\varphi|^2 - |D(\xi \cdot \varphi)|^2),
$$

from which we obtain

$$
|T\varphi|^2 = |\nabla\varphi|^2 - \frac{1}{n-1}|D\varphi|^2 - \frac{n-2}{n-1}|\nabla_\xi\varphi|^2 + \frac{1}{2(n-1)}(|D\varphi|^2 - |D(\xi \cdot \varphi)|^2).
$$

Integrating this identity over M and applying the Schrödinger-Lichnerowicz formula (1.15) we have

$$
\begin{aligned}
\int_M |T\varphi|^2 v_g = \frac{n-2}{n-1}\int_M |D\varphi|^2 v_g - \frac{1}{4}\int_M S|\varphi|^2 v_g - \frac{n-2}{n-1}\int_M |\nabla_\xi\varphi|^2 v_g \\
+\frac{1}{2(n-1)}\int_M |D\varphi|^2 - |D(\xi\cdot\varphi)|^2 v_g.
\end{aligned}
$$

But choosing φ to be eigen for D for the smallest (in absolute value) eigenvalue λ, the min-max principle (see Lemma 5.0.2) implies

$$
\begin{aligned}
\int_M |D(\xi\cdot\varphi)|^2 v_g &\geq \lambda^2 \int_M |\xi \cdot \varphi|^2 v_g \\
&= \lambda^2 \int_M |\varphi|^2 v_g \\
&= \int_M |D\varphi|^2 v_g,
\end{aligned}
$$

hence

$$
\left(\frac{n-2}{n-1}\lambda^2 - \frac{1}{4}\inf_M(S)\right)\int_M |\varphi|^2 v_g \geq \int_M |T\varphi|^2 v_g + \frac{n-2}{n-1}\int_M |\nabla_\xi\varphi|^2 v_g \tag{3.6}
$$

and the inequality (3.5) follows.

If (3.5) is an equality for some eigenvalue λ, then (3.6) implies $T\varphi = 0$ and $\nabla_\xi\varphi = 0$, that is,

$$\nabla_X\varphi = -\frac{\lambda}{n-1}(X - g(X,\xi)\xi)\cdot\varphi \tag{3.7}$$

for any eigenvector φ associated to λ and any $X \in TM$. In particular its length must be constant on M. As in [202] we next show that ξ must be parallel. For this purpose we compute the curvature tensor on such a φ: let $X, Y \in TM$, then

$$\begin{aligned}
R_{X,Y}\varphi &= \nabla_{[X,Y]}\varphi - [\nabla_X, \nabla_Y]\varphi \\
&= \frac{\lambda}{n-1}\left((g(X,\nabla_Y\xi) - g(Y,\nabla_X\xi))\xi + g(X,\xi)\nabla_Y\xi - g(Y,\xi)\nabla_X\xi\right)\cdot\varphi \\
&\quad + \frac{\lambda^2}{(n-1)^2}\Big((X - g(X,\xi)\xi)\cdot(Y - g(Y,\xi)\xi) \\
&\qquad - (Y - g(Y,\xi)\xi)\cdot(X - g(X,\xi)\xi)\Big)\cdot\varphi,
\end{aligned}$$

hence using (1.9) and the fact that $g(\xi,\xi) = 1$ on M we obtain

$$\begin{aligned}
\frac{1}{2}\mathrm{Ric}(\xi)\cdot\varphi &= \sum_{j=1}^n e_j\cdot R^{\nabla}_{\xi,e_j}\varphi \\
&= \frac{\lambda}{n-1}\sum_{j=1}^n\Big((g(\xi,\nabla_{e_j}\xi) - g(e_j,\nabla_\xi\xi))e_j\cdot\xi \\
&\qquad + (g(\xi,\xi)\nabla_{e_j}\xi - g(e_j,\xi)\nabla_\xi\xi)\cdot e_j\cdot\varphi\Big) \\
&= \frac{\lambda}{n-1}(-2\nabla_\xi\xi\cdot\xi + \sum_{j=1}^n\nabla_{e_j}\xi\cdot e_j)\cdot\varphi.
\end{aligned}$$

The last sum vanishes because of (1.2) together with ξ being closed and co-closed. On the other hand $d\xi = 0$ means that $g(\nabla_X\xi, Y) - g(\nabla_Y\xi, X) = 0$ for all $X, Y \in TM$, hence for $X = \xi$ one obtains - using once again $g(\xi,\xi) = 1$ - that $g(\nabla_\xi\xi, Y) = 0$ for all $Y \in TM$, i.e., $\nabla_\xi\xi = 0$. This shows $\mathrm{Ric}(\xi) = 0$. From Bochner's formula for the Laplace operator on 1-forms (see e.g. [173, Cor. 8.3 p.156]) one deduces that $\nabla\xi = 0$, i.e., that ξ is parallel.

We now prove the limiting-case in Theorem 3.2.1. If ξ is parallel then the universal cover of M must be a Riemannian product of the form $\mathbb{R}\times N$. W.r.t. the pull-back spin structure the lift of φ to $\mathbb{R}\times N$ also satisfies (3.7) provided ξ is replaced by $\frac{\partial}{\partial t}$. Since each $\{t\}\times N$ sits totally geodesically in

$\mathbb{R} \times N$ the Gauss-type formula (1.21) implies that the induced spinor field on N is a real Killing spinor for one of the constants $\pm\frac{\lambda}{n-1}$. $\square$

Beware that the necessary condition for (3.5) to be an equality is not sufficient, since e.g. $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R}^2$ with flat metric carries non-zero parallel spinors whereas (flat) $\mathbb{T}^3 = \mathbb{Z}^3 \backslash \mathbb{R}^3$ only admits such spinors in case it carries the trivial spin structure (i.e., the spin structure induced by the trivial lift of the $\mathbb{Z}^3$-action to the spin level, see Proposition 1.4.2). In fact (3.5) is an equality if and only if there exists a $\pi_1(M)$-equivariant solution - in the sense of (1.24) - to (3.7) on the universal cover $\mathbb{R} \times N$ of M.

Although the (real) Killing-spinor-equation is completely understood (see Theorems A.4.2 and A.4.3), the list of all local Riemannian products on which (3.5) is sharp is not entirely known. B. Alexandrov, G. Grantcharov and S. Ivanov have shown in [5] that, under this assumption and if $n \neq 7$ is odd, then M is diffeomorphic - but not necessarily isometric - to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$.

It is moreover important to note that the hypothesis in Theorem 3.2.2 on the length being constant cannot be removed: C. Bär and M. Dahl showed in [46] that in dimension $n \geq 3$ Friedrich's inequality (3.1) cannot be improved with the help of topological assumptions. Namely there exists on any given compact spin manifold M^n admitting a metric with positive scalar curvature a smooth family of Riemannian metrics $(g_t)_{t>0}$ with $S_{g_t} \geq n(n-1)$ and

$$\frac{n^2}{4} \leq \lambda_1(D_{g_t}^2) \leq \frac{n^2}{4} + t,$$

where D_{g_t} stands for the Dirac operator to the metric g_t on M. In other words, one can get as close as one wants to the equality in Friedrich's inequality (3.1) on any such manifold. Note that the set of compact spin manifolds with positive first Betti number and admitting a metric with positive scalar curvature is non-empty since it contains e.g. $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, $n \geq 3$.

The generalization of Theorem 3.2.1 to locally reducible Riemannian manifolds was achieved by B. Alexandrov, extending earlier work by E.C. Kim [154]:

Theorem 3.2.3 (B. Alexandrov [4]) *Let (M^n, g) be an $n(\geq 2)$-dimensional closed Riemannian spin manifold with positive scalar curvature S. Assume that TM splits orthogonally into*

$$TM = \bigoplus_{j=1}^{k} T_j,$$

where T_j is a parallel distribution of dimension n_j and $n_1 \leq \ldots \leq n_k$. Then any eigenvalue λ of D satisfies

$$\lambda^2 \geq \frac{n_k}{4(n_k - 1)} \inf_M(S). \tag{3.8}$$

Moreover, if (3.8) *is an equality for some eigenvalue* λ*, then the universal cover of* M *is isometric to* $M_1 \times \ldots \times M_k$*, where* M_j *is a closed* n_j*-dimensional Riemannian spin manifold admitting a non-zero real non-parallel Killing spinor for* $j = k$*, a non-zero parallel spinor if* $n_j < n_k$ *and a non-zero real Killing spinor if* $n_j = n_k$*.*

Note that (3.8) contains both (3.1) and (3.5) and that $n_k > 1$ because of the assumption $S > 0$. Moreover, for any integers $1 \leq n_1 \leq \ldots \leq n_p < n_{p+1} = \ldots = n_k$, all Riemannian products of the form $M_1 \times \ldots \times M_k$, where M_j is an n_j-dimensional closed Riemannian spin manifold admitting a non-zero parallel spinor for $j \leq p$ and a non-zero real Killing spinor for $j \geq p+1$ which is furthermore non-parallel for $j = k$, satisfy the equality in (3.8) w.r.t. the product spin structure.

Sketch of proof of Theorem 3.2.3: The proof follows the lines of that of Theorem 3.2.1. Define the Penrose-like operator T: for any $\varphi \in \Gamma(\Sigma M)$ and any $X \in TM$,

$$T_X\varphi := \nabla_X\varphi + \sum_{j=1}^{k} \frac{1}{n_j}\pi_j(X) \cdot D_{[j]}\varphi,$$

where $\pi_j : TM \to T_j$ is the orthogonal projection, $D_{[j]}\varphi := \sum_{l=1}^{n_j} e_{l,j} \cdot \nabla_{e_{l,j}}\varphi$ and $(e_{1,j}, \ldots, e_{n_j,j})$ denotes a local orthonormal frame of T_j, for every $j \in \{1, \ldots, k\}$. A short computation gives

$$|T\varphi|^2 = |\nabla\varphi|^2 - \sum_{j=1}^{k} \frac{1}{n_j}|D_{[j]}\varphi|^2.$$

On the other hand, it is an exercise to show that $D^2 = \sum_{j=1}^{k} D_{[j]}^2$ and that $D_{[j]}$ is formally self-adjoint, so that, after integration and application of the Schrödinger-Lichnerowicz formula (1.15), one obtains

$$\|D\varphi\|^2 = \frac{n_k}{n_k - 1}\|T\varphi\|^2 + \sum_{j=1}^{k} \frac{n_k - n_j}{n_j(n_k - 1)}\|D_{[j]}\varphi\|^2 + \frac{n_k}{4(n_k - 1)}(S\varphi, \varphi). \quad (3.9)$$

Choosing φ to be an eigenvector for D associated to the eigenvalue λ leads to the inequality. If this inequality is an equality for some λ, then (3.9) implies that, for any non-zero eigenvector φ for D associated to the eigenvalue λ, one has $T\varphi = 0$, $D_{[j]}\varphi = 0$ as soon as $n_j < n_k$ and S is constant on $\{x \in M \,|\, \varphi(x) \neq 0\}$. In case $n_j < n_k$ one deduces that $\nabla_{\pi_j(X)}\varphi = 0$ for every X. It remains to prove that, on the universal cover of M, which is a Riemannian product of the form $M_1 \times \ldots \times M_k$ by assumption, the lift of φ induces a real Killing spinor on each M_j, which is parallel if $n_j < n_k$ and non-parallel for $j = k$. We refer to [4, Sec. 2] for the details. $\square$

For 2-forms, the canonical class of manifolds to be considered consists of that of Kähler manifolds, i.e., of triples (M^n, g, J) where (M^n, g) is a Riemannian manifold and J a parallel almost Hermitian structure on TM. Recall that $J \in \Gamma(\mathrm{End}(TM))$ is called almost Hermitian if and only if $J^2 = -\mathrm{Id}_{TM}$ and $g(J(X), J(Y)) = g(X, Y)$ for all $X, Y \in TM$. In this case n is even and the Kähler-form Ω is parallel, where Ω is defined by

$$\Omega(X, Y) := g(J(X), Y)$$

for all $X, Y \in TM$. K.-D. Kirchberg was the first to enhance Friedrich's inequality (3.1) on Kähler manifolds:

Theorem 3.2.4 (K.-D. Kirchberg [156]) *Any eigenvalue λ of D on an $n(\geq 4)$-dimensional closed* Kähler *spin manifold (M^n, g, J) satisfies*

$$\lambda^2 \geq \left| \begin{array}{ll} \frac{n+2}{4n} \inf_M(S) & \text{if } \frac{n}{2} \text{ is odd} \\ & \\ \frac{n}{4(n-2)} \inf_M(S) & \text{if } \frac{n}{2} \text{ is even} \end{array} \right. , \tag{3.10}$$

where S is the scalar curvature of M. Moreover, in the case where $S > 0$, (3.10) *is an equality for some eigenvalue λ if and only if there exists non-zero sections ψ, ϕ of ΣM satisfying*

$$\left| \begin{array}{l} \nabla_X \psi = -\frac{\lambda}{n+2}(X + iJ(X)) \cdot \phi \\ \nabla_X \phi = -\frac{\lambda}{n+2}(X - iJ(X)) \cdot \psi \end{array} \right. \tag{3.11}$$

for all $X \in TM$ if $\frac{n}{2}$ is odd and a non-zero section ψ of ΣM satisfying

$$\left| \begin{array}{ll} D^2\psi & = \lambda^2 \psi \\ \nabla_X \psi & = -\frac{1}{n}(X + iJ(X)) \cdot D\psi \\ \Omega \cdot \psi & = -2i\psi \\ \Omega \cdot D\psi & = 0 \end{array} \right. \tag{3.12}$$

for all $X \in TM$ if $\frac{n}{2}$ is even.

Proof: We follow the proof given in [131, 133], see also [223, Sec. 3] or [150]. We may assume that $S > 0$ on M (otherwise the estimate is trivial). Set, for every $X \in TM$, $p_\pm(X) := \frac{1}{2}(X \mp iJ(X)) \in TM \otimes \mathbb{C}$. In the whole proof we shall redenote $m := \frac{n}{2}$. Given a pointwise orthonormal basis $(e_1, \ldots, e_n)$ of TM such that $e_{j+m} = J(e_j)$ for every $1 \leq j \leq m$, define $z_j := p_+(e_j)$ and $\overline{z}_j := p_-(e_j)$ for all $1 \leq j \leq m$. Then $(z_1, \ldots, z_m)$ and $(\overline{z}_1, \ldots, \overline{z}_m)$ are bases of $T^{1,0}M := p_+(TM)$ and $T^{0,1}M := p_-(TM)$ respectively satisfying

$$z_j \cdot z_k = -z_k \cdot z_j, \qquad \overline{z}_j \cdot \overline{z}_k = -\overline{z}_k \cdot \overline{z}_j, \qquad z_j \cdot \overline{z}_k + \overline{z}_k \cdot z_j = -\delta_{jk}$$

for all $1 \leq j, k \leq m$. With those notations, it is elementary to show that the ranked-2^m-vector bundle $(\oplus_{r=0}^m \Lambda^r T^{1,0}M) \cdot \overline{z}_1 \cdot \ldots \cdot \overline{z}_m$ becomes a non-trivial Clifford submodule of the Clifford algebra bundle, which is independent of the basis originally chosen. Therefore it can be identified with the spinor bundle ΣM itself. Moreover, setting $\Sigma_r M := (\Lambda^r T^{1,0}M) \cdot \overline{z}_1 \cdot \ldots \cdot \overline{z}_m$, it is an exercise to prove that, w.r.t. the Clifford action of the Kähler-form Ω,

$$\Sigma_r M = \mathrm{Ker}(\Omega \cdot -i(2r-m)\mathrm{Id})$$

and that the Clifford action by Ω is skew-Hermitian and parallel. As a consequence, one obtains the orthogonal and parallel decomposition

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M. \tag{3.13}$$

By construction $p_\pm(X) \cdot \Sigma_r M \subset \Sigma_{r\pm 1} M$, for every $X \in TM$, where we set $\Sigma_r M := 0$ as soon as $r \notin \{0, 1, \ldots, m\}$. In particular the Dirac operator D does not preserve (3.13): setting $D_\pm := \sum_{j=1}^n p_\pm(e_j) \cdot \nabla_{e_j}$, we have $D = D_+ + D_-$ with $D_\pm : \Gamma(\Sigma_r M) \to \Gamma(\Sigma_{r\pm1} M)$ for every $r \in \{0, 1 \ldots, m\}$. Nevertheless, a more precise study of $D_\pm$ shows that $D_+ \circ D_+ = D_- \circ D_- = 0$, so that (3.13) is preserved by $D^2 = D_+ \circ D_- + D_- \circ D_+$. Beware that the operators $D_\pm$ have nothing to do with the $D^\pm$ of Proposition 1.3.2.

For any $r \in \{0, 1 \ldots, m\}$ and $\varphi \in \Gamma(\Sigma_r M)$ define

$$T_X^{(r)}\varphi := \nabla_X \varphi + \frac{1}{2(r+1)} p_-(X) \cdot D_+ \varphi + \frac{1}{2(m-r+1)} p_+(X) \cdot D_- \varphi$$

for all $X \in TM$. In other words, $T^{(r)}\varphi$ is the orthogonal projection of $\nabla\varphi$ onto the kernel of the Clifford multiplication $\mu : T^*M \otimes \Sigma_r M \longrightarrow \Sigma_{r-1} M \oplus \Sigma_{r+1} M$. Elementary computations show that

$$\sum_{j=1}^n p_+(e_j) \cdot p_-(e_j) \cdot = i\Omega \cdot - m\mathrm{Id} \qquad \text{and} \qquad \sum_{j=1}^n p_-(e_j) \cdot p_+(e_j) \cdot = -i\Omega \cdot - m\mathrm{Id},$$

in particular $\sum_{j=1}^n p_+(e_j) \cdot p_-(e_j) \cdot \varphi = -2r\varphi$ and $\sum_{j=1}^n p_-(e_j) \cdot p_+(e_j) \cdot \varphi = -2(m-r)\varphi$. We deduce for the norms that

$$\begin{aligned} |T^{(r)}\varphi|^2 &= \sum_{j=1}^n |T_{e_j}^{(r)}\varphi|^2 \\ &= \sum_{j=1}^n |\nabla_{e_j}\varphi|^2 \end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n}\frac{1}{4(r+1)^2}|p_-(e_j)\cdot D_+\varphi|^2+\frac{1}{4(m-r+1)^2}|p_+(e_j)\cdot D_-\varphi|^2\\
&+2\sum_{j=1}^{n}\Re e\left(\frac{1}{2(r+1)}\langle\nabla_{e_j}\varphi,p_-(e_j)\cdot D_+\varphi\rangle\right)\\
&+2\sum_{j=1}^{n}\Re e\left(\frac{1}{2(m-r+1)}\langle\nabla_{e_j}\varphi,p_+(e_j)\cdot D_-\varphi\rangle\right)\\
&+2\sum_{j=1}^{n}\Re e\left(\frac{1}{4(r+1)(m-r+1)}\langle p_-(e_j)\cdot D_+\varphi,p_+(e_j)\cdot D_-\varphi\rangle\right)\\
=&\ |\nabla\varphi|^2\\
&-\frac{1}{4(r+1)^2}\sum_{j=1}^{n}\langle p_+(e_j)\cdot p_-(e_j)\cdot D_+\varphi,D_+\varphi\rangle\\
&-\frac{1}{4(m-r+1)^2}\sum_{j=1}^{n}\langle p_-(e_j)\cdot p_+(e_j)\cdot D_-\varphi,D_-\varphi\rangle\\
&-\frac{1}{r+1}|D_+\varphi|^2-\frac{1}{m-r+1}|D_-\varphi|^2\\
=&\ |\nabla\varphi|^2\\
&+\frac{1}{2(r+1)}|D_+\varphi|^2+\frac{1}{2(m-r+1)}|D_-\varphi|^2\\
&-\frac{1}{r+1}|D_+\varphi|^2-\frac{1}{m-r+1}|D_-\varphi|^2,
\end{aligned}
$$

that is,

$$
|T^{(r)}\varphi|^2=|\nabla\varphi|^2-\frac{1}{2(r+1)}|D_+\varphi|^2-\frac{1}{2(m-r+1)}|D_-\varphi|^2. \tag{3.14}
$$

Let $r\in\{0,1\ldots,m\}$ be the smallest integer for which $\mathrm{Ker}(D^2-\lambda^2\mathrm{Id})\cap\Gamma(\Sigma_r M)\neq 0$. Let $\psi\in\Gamma(\Sigma_r M)$ be a non-zero eigenvector for D^2 associated to the eigenvalue λ^2. Since $[D^2,D_\pm]=0$, both $D_+\psi_r$ and $D_-\psi_r$ lie in $\mathrm{Ker}(D^2-\lambda^2\mathrm{Id})$, in particular $D_-\psi_r=0$ by the choice of r. Independently, there exists on ΣM a parallel field j of complex antilinear automorphisms commuting with the Clifford multiplications by vectors (see e.g. [101, Lemma 1]), in particular $[\Omega\cdot,j]=[D,j]=0$, so that $j(\mathrm{Ker}(D^2-\lambda^2\mathrm{Id}))\subset\mathrm{Ker}(D^2-\lambda^2\mathrm{Id})$ and $j(\Sigma_l M)=\Sigma_{m-l}M$ for every $l\in\{0,\ldots,m\}$. Thus the existence of j imposes $r\leq m-r$, hence $r\leq\frac{m-1}{2}$ for m odd. If m is even, then $r=\frac{m}{2}$ cannot happen since otherwise $D_+\psi=D_-\psi=D\psi=0$ would hold, which would contradict (3.1) together with $S>0$, therefore $r\leq\frac{m-2}{2}$ for m even.

We are now ready to prove the estimate. Integrating (3.14) and using Schrödinger-Lichnerowicz' formula (1.15), we obtain

$$\begin{aligned}\|T^{(r)}\psi\|^2 &= (D^2\psi,\psi) - (\frac{S}{4}\psi,\psi) - \frac{1}{2(r+1)}\|D\psi\|^2\\ &= (\frac{2r+1}{2(r+1)}\lambda^2 - \frac{1}{4}\inf_M(S))\|\psi\|^2,\end{aligned}$$

from which one deduces that $\lambda^2 \geq \frac{2(r+1)}{4(2r+1)}\inf_M(S)$. The r.h.s. of that inequality decreases with r, so that it is bounded from below by the corresponding expression for $r = \frac{m-1}{2}$ in case m is odd and for $r = \frac{m-2}{2}$ in case m is even. Inequality (3.10) follows.

Assume now (3.10) to be an equality for some eigenvalue λ. If $\inf_M(S) = 0$ then (M^n, g) has a non-zero parallel spinor, as already proved in Theorem 3.1.1. If $\inf_M(S) > 0$, then for any eigenvector ψ for D associated to the eigenvalue λ, one has on the one hand $\psi = \psi_{\frac{m-1}{2}} + \psi_{\frac{m+1}{2}}$ if m is odd and $\psi = \psi_{\frac{m-2}{2}} + \psi_{\frac{m}{2}} + \psi_{\frac{m+2}{2}}$ if m is even, on the other hand $T^{(r)}\psi_r = 0$ for $r = \frac{m\pm 1}{2}$ and $r = \frac{m\pm 2}{2}$ for m odd and even respectively. Redenoting $\psi_{\frac{m-1}{2}}$ by ψ and $\psi_{\frac{m+1}{2}}$ by ϕ, we obtain (3.11) in case m is odd. If m is even then redenoting $\psi_{\frac{m-2}{2}}$ by ψ one obtains (3.12). Conversely, mimiking the proof of Proposition A.4.1, it is elementary to show that, if (3.11) is satisfied by some non-zero (ψ, ϕ), then $\psi + \phi$ is an eigenvector for D associated to the eigenvalue λ and $S = \frac{4n\lambda^2}{n+2}$, therefore (3.10) is an equality. Similarly, if $\frac{n}{2}$ is even and (3.12) is satisfied by some non-zero ψ, then the scalar curvature of (M^n, g) is equal to $\frac{4(n-2)\lambda^2}{n}$, therefore (3.10) is an equality. □

A pair of spinors (ψ, ϕ) satisfying (3.11) for some non-zero real number λ is called a real Kählerian Killing spinor. As for Killing spinors (see Proposition A.4.1), it is not too difficult to show that, if a non-zero real Kählerian Killing spinor exist on a given complete Kähler spin manifold and associated to some (non-zero) real λ, then this manifold has odd complex dimension and is Einstein with positive scalar curvature (in particular it is closed). However, the precise classification of those Kähler spin manifolds carrying non-trivial real Kählerian Killing spinors is more technical, even if it turns out to provide simpler results. The idea to achieve it, due to A. Moroianu [197], can be summarised as follows: Show the existence of a suitable $\mathbb{S}^1$-bundle over such a manifold where the pull-back of the real Kählerian Killing spinor induces a non-zero real Killing spinor; then show that, among the possible holonomies listed in C. Bär's classification (Theorem A.4.3), only those associated to a so-called regular 3-Sasaki structure can occur on that $\mathbb{S}^1$-bundle. We refer to [197] for details and mention that, before [197] was published, partial results had been obtained by K.-D. Kirchberg [156, 157, 158] and O. Hijazi [131], see references in [197].

The even-complex-dimensional case turns out to be more involved since the underlying manifold is no more Einstein. In dimension $n = 4$, arguments from complex geometry and based on Kirchberg's work [159, Thm. 15] allowed T. Friedrich [87, Thm. 2] to prove that, if (M^4, g, J) carries a non-zero spinor ψ satisfying (3.12), then up to rescaling the metric (M^4, g, J) must be holomorphically isometric either to $\mathbb{S}^2 \times \mathbb{S}^2$ or to $\mathbb{S}^2 \times \mathbb{T}^2$, both with product metric and spin structure, where $\mathbb{T}^2$ carries a flat metric and the trivial spin structure. In higher dimensions, if a non-zero spinor ψ exists satisfying (3.12), then A. Moroianu showed [200] that the Ricci tensor of (M^n, g) is parallel and has exactly two eigenvalues. This implies that the universal cover of M is holomorphically isometric to the Riemannian product $N \times \mathbb{R}^2$, where N is a closed Kähler spin manifold admitting a non-zero real Kählerian Killing spinor, see [200] for details. This result had been formulated by A. Lichnerowicz [179] where there remained however gaps in the proof.

We formulate the precise statements on the characterization of the limiting-case of (3.10).

Theorem 3.2.5 *Let (M^n, g, J) be a closed $n(\geq 4)$-dimensional Kähler spin manifold with positive scalar curvature.*

1. *If $\frac{n}{2}$ is odd, then* (3.10) *is an equality for some eigenvalue λ of D if and only if (M^n, g, J) is holomorphically isometric to $\mathbb{C}\mathrm{P}^{\frac{n}{2}}$ in case $\frac{n}{2} \equiv 1\ (4)$ or to the twistor-space of a quaternionic Kähler manifold with positive scalar curvature in case $\frac{n}{2} \equiv 3\ (4)$* (K.-D. Kirchberg [157] for $n = 6$, A. Moroianu [197] for $n \geq 6$)*.*
2. *If $\frac{n}{2}$ is even, then* (3.10) *is an equality for some eigenvalue λ of D if and only if (M^n, g, J) is isometric to $_{\Gamma}\backslash^{N \times \mathbb{R}^2}$, where N is a simply-connected closed Kähler manifold admitting a non-zero real Kählerian Killing spinor (ψ, ϕ) and Γ is generated by (γ_j, τ_j), $j = 1, 2$, where the τ_j's are translations of $\mathbb{R}^2$ and the γ_j's are commuting holomorphic isometries of N preserving its spin structure and (ψ, ϕ)* (T. Friedrich [87] for $n = 4$, A. Moroianu [200] for $n \geq 8$)*.*

Beware that, in case $\frac{n}{2}$ is even, the Kähler manifold must not be holomorphically isometric to the quotient $_{\Gamma}\backslash^{N \times \mathbb{R}^2}$ endowed with the Kähler structure induced from that of N. A simple criterion for holomorphicity is given in [200, Lemma 7.6].

The last class of manifolds with a non-trivial parallel form having been handled is that of quaternionic-Kähler manifolds, which carry a canonical parallel 4-form. The following theorem was proved by O. Hijazi and J.-L. Milhorat [135, 136, 137] for $n = 8$, 12 and by W. Kramer, U. Semmelmann and G. Weingart in the general case [163, 164]:

Theorem 3.2.6 *Any eigenvalue λ of D on an $n(\geq 8)$-dimensional closed* quaternionic-Kähler *spin manifold (M^n, g) with positive scalar curvature satisfies*

$$\lambda^2 \geq \frac{n+12}{4(n+8)} S, \tag{3.15}$$

where S is the scalar curvature of M. Moreover, this inequality is an equality for some eigenvalue λ if and only if (M^n, g) is isometric to the quaternionic projective space $\mathbb{H}\mathrm{P}^{\frac{n}{4}}$.

Here it should be noticed that every quaternionic Kähler manifold of even quaternionic dimension is spin whereas only the quaternionic projective space is spin if $\frac{n}{4}$ is odd; moreover, every quaternionic Kähler manifold is Einstein, hence has constant scalar curvature, see references in [164].

Sketch of proof of Theorem 3.2.6: We follow the proof detailed in [164], which relies on the representation theory of $\mathrm{Sp}_1 \times \mathrm{Sp}_k$. Denote $\frac{n}{4}$ by m. A quaternionic structure on (M^n, g) is given by a triple (I, J, K) of parallel orthogonal endomorphisms of TM with $I^2 = J^2 = K^2 = -\mathrm{Id}_{TM}$ and $IJ = -JI = K$. Each of those endormorphisms is a Kähler structure on TM with associated Kähler form, so that one may define the so-called fundamental form

$$\Omega := \Omega_I \wedge \Omega_I + \Omega_J \wedge \Omega_J + \Omega_K \wedge \Omega_K$$

on TM. The 4-form Ω is parallel and can be shown to act on ΣM so as to split it into

$$\Sigma M = \bigoplus_{r=0}^{m} \Sigma_r M,$$

with $\Sigma_r M := \mathrm{Ker}(\Omega \cdot -(6m - 4r(r+2))\mathrm{Id}) \subset \Sigma M$ [136]. As in the Kähler case, the Clifford multiplication sends $T^*M \otimes \Sigma_r M$ into $\Sigma_{r-1} M \oplus \Sigma_{r+1} M$. Decomposing $\mathrm{Ker}(\mu)_{|_{T^*M \otimes \Sigma_r M}}$ into irreducible components under $\mathrm{Sp}_1 \times \mathrm{Sp}_{m-1}$, one obtains four twistor operators associated to the orthogonal projections of $\nabla^2 \varphi$ onto the irreducible components [164, p.745]. Taking $\varphi \in \Gamma(\Sigma_r M)$ to be an eigenvector for D^2 and applying Schrödinger-Lichnerowicz' formula (1.15) lead to the desired inequality, see [164, Sec. 4] for the rather technical proof where the authors determine all Weitzenböck formulas involving Dirac and twistor operators.

The equality case in the inequality is sharp for $\mathbb{H}\mathrm{P}^m$ [189]. Conversely, if it is sharp, then the spinor bundle of M carries a particular (non-zero) section called quaternionic Killing spinor [163, p.340]. This spinor induces a non-zero real Killing spinor on the total space of the SO_3-principal bundle associated to the quaternionic Kähler structure and for a suitable metric [163, p.344]. Then Bär's classification (Theorem A.4.3) of manifolds with real Killing spinors forces M to be isometric to $\mathbb{H}\mathrm{P}^m$, we refer to [163, Sec. 7] for the details. $\square$

3.3 Improving Friedrich's inequality in a conformal way

N. Hitchin [148] noticed in the Riemannian setting (as well as H. Baum [49] in the pseudo-Riemannian one) that the fundamental Dirac operator is *conformally covariant*, see Proposition 1.3.10. This was the starting point for the following result.

Theorem 3.3.1 (O. Hijazi [128]) *Let (M^n, g) be an $n(\geq 2)$-dimensional closed Riemannian spin manifold and $u \in C^\infty(M, \mathbb{R})$, then any eigenvalue λ of D on (M^n, g) satisfies*

$$\lambda^2 \geq \frac{n}{4(n-1)} \inf_M (\overline{S} e^{2u}), \tag{3.16}$$

where $\overline{S}$ is the scalar curvature of $(M^n, \overline{g} := e^{2u} g)$. Moreover, (3.16) *is an equality for some eigenvalue λ if and only if u is constant and (M^n, g) carries a non-zero real Killing spinor.*

Proof: We use the notations of Proposition 1.3.10. For any $\varphi \in \Gamma(\Sigma M)$ one has from (3.3) applied to $\overline{\psi} := e^{-\frac{n-1}{2}u}\overline{\varphi}$,

$$\int_M \left(|\overline{D\psi}|^2 - \frac{n}{4(n-1)} \overline{S}\, |\overline{\psi}|^2 \right) v_{\overline{g}} = \frac{n}{n-1} \int_M |\overline{P}\,\overline{\psi}|^2 v_{\overline{g}} \geq 0.$$

Proposition 1.3.10 states that $\overline{D\psi} = e^{-\frac{n+1}{2}u}\overline{D\varphi}$, so that choosing φ to be a non-zero eigenvector for D associated to the eigenvalue λ one obtains $\overline{D}\,\overline{\psi} = \lambda e^{-u}\overline{\psi}$ and the inequality (3.16). Furthermore, if (3.16) is an equality for some eigenvalue λ of D on (M^n, g), then for any non-zero eigenvector φ for D associated to λ, the identity $\overline{P}\,\overline{\psi} = 0$ holds, where $\overline{\psi} := e^{-\frac{n-1}{2}u}\overline{\varphi}$. This implies in turn

$$\overline{\nabla}_X \overline{\psi} = -\frac{\lambda e^{-u}}{n} X \,\overline{\cdot}\, \overline{\psi}$$

for every $X \in TM$. Elementary computations as in the proof of Proposition A.4.1 but carried out on $(M^n, \overline{g})$ (see e.g. [134, Prop. 5.12]) show that necessarily $du = 0$, thus u is constant and therefore φ is a real Killing spinor on (M^n, g). The converse statement follows from the characterization of the equality case in (3.1). This concludes the proof. $\square$

Corollary 3.3.2 *Any eigenvalue λ of D on an n-dimensional closed Riemannian spin manifold (M^n, g) satisfies:*

i) (C. Bär [35]) *For $n = 2$,*

$$\lambda^2 \geq \frac{2\pi\chi(M^2)}{\mathrm{Area}(M^2, g)}, \tag{3.17}$$

where $\chi(M^2)$ *is the Euler characteristic of* M^2*. Moreover,* (3.17) *is an equality for some eigenvalue* λ *of* D *if and only if* (M^2, g) *is isometric either to* $\mathbb{S}^2$ *with constant curvature metric or to* $\mathbb{T}^2$ *with flat metric and trivial spin structure.*

ii) (O. Hijazi [128]) *For* $n \geq 3$,

$$\lambda^2 \geq \frac{n}{4(n-1)}\mu_1, \tag{3.18}$$

where μ_1 *denotes the first eigenvalue of the scalar conformal Laplace operator* $4\frac{n-1}{n-2}\Delta + S$*. Moreover,* (3.18) *is an equality for some eigenvalue* λ *of* D *if and only if* (M^n, g) *carries a non-zero real Killing spinor.*

Proof: We deduce both (3.18) and (3.17) from (3.16) and from the following transformation formula for scalar curvature after conformal change of the metric:

$$\overline{S}e^{2u} = S + 2(n-1)\Delta u - (n-1)(n-2)|\mathrm{grad}(u)|^2, \tag{3.19}$$

for $\overline{g} := e^{2u}g$ and $u \in C^\infty(M, \mathbb{R})$. This is applied to a conformal metric $\overline{g}$ for which $\overline{S}e^{2u}$ is constant on M.

i) Let $u_0 \in C^\infty(M, \mathbb{R})$ solve $\Delta u_0 = \frac{\int_M S v_g}{2\mathrm{Area}(M^2,g)} - \frac{S}{2}$ (such a solution exists because the r.h.s. has vanishing integral). Since in dimension $n = 2$ the formula (3.19) reads $\overline{S}e^{2u_0} = S + 2\Delta u_0$ one obtains $\overline{S}e^{2u_0} = \frac{\int_M S v_g}{\mathrm{Area}(M^2,g)}$. Applying now the Gauss-Bonnet Theorem, (3.16) becomes

$$\lambda^2 \geq \frac{1}{2}\frac{\int_M S v_g}{\mathrm{Area}(M^2, g)} = \frac{2\pi\chi(M^2)}{\mathrm{Area}(M^2, g)},$$

which is (3.17). If now (3.17) is an equality for some eigenvalue λ of D, then by construction of u_0 the scalar curvature S must be constant and non-negative. Since we explicitly know the Dirac spectra of $\mathbb{S}^2$ (see Theorem 2.1.3) and of $\mathbb{T}^2$ (see Theorem 2.1.1), we can compare the smallest non-negative Dirac eigenvalue with the lower bound in (3.17) and state that equality always occurs for $S > 0$ and occurs on $\mathbb{T}^2$ only when fixing the trivial spin structure.

ii) We can assume that $\mu_1 > 0$. From Courant's nodal domain theorem (see e.g. [74, p.19]) every non-zero eigenfunction h_0 for $L := 4\frac{n-1}{n-2}\Delta + S$ associated to its smallest eigenvalue μ_1 cannot vanish, hence may be assumed to be positive. In particular $\mu_1 = h_0^{-1}Lh_0$. But in dimension $n \geq 3$ the formula (3.19) can be rewritten under the following form: for any positive smooth function h on M^n,

$$\overline{S}h^{\frac{4}{n-2}} = h^{-1}Lh,$$

where $\overline{S}$ is the scalar curvature of $(M^n, \overline{g} := h^{\frac{4}{n-2}}g)$. Thus, choosing $\overline{g}_0 := h_0^{\frac{4}{n-2}}g$ on M^n, one obtains $\overline{S}h_0^{\frac{4}{n-2}} = \mu_1$, which together with Theorem 3.3.1 implies (3.18). The characterization of the equality case in (3.18) follows from Theorem 3.3.1 as well. $\square$

Another proof of (3.18) involving Kato type inequalities can be found in [71].

Inequality (3.18) improves Friedrich's inequality (3.1) for $n \geq 3$ since obviously $\mu_1 \geq \inf_M (S)$. It also proves the existence in dimension $n \geq 3$ of an explicit conformal lower bound for the spectrum of D^2:

Corollary 3.3.3 (O. Hijazi [130]) *For any Riemannian metric g on a closed $n(\geq 3)$-dimensional spin manifold M^n,*

$$\lambda_1(D^2_{M,g})\mathrm{Vol}(M,g)^{\frac{2}{n}} \geq \frac{n}{4(n-1)} Y(M,[g]), \tag{3.20}$$

where $\lambda_1(D^2_{M,g})$ denotes the smallest non-negative eigenvalue of D^2 associated to the metric g and $Y(M,[g])$ is the Yamabe invariant of M w.r.t. the conformal class of g.

Proof: Recall that the Yamabe invariant of M^n w.r.t. $[g]$ is the conformal invariant defined by

$$Y(M,[g]) := \inf_{f \in C^\infty(M,\mathbb{R})\setminus\{0\}} \Big\{ \frac{\int_M (4\frac{n-1}{n-2}\Delta_g f + S_g f) f v_g}{(\int_M f^{\frac{2n}{n-2}} v_g)^{\frac{n-2}{n}}} \Big\}.$$

Hölder's inequality gives $\int_M f^2 v_g \leq (\int_M f^{\frac{2n}{n-2}} v_g)^{\frac{n-2}{n}} \cdot \mathrm{Vol}(M^n,g)^{\frac{2}{n}}$. Assuming $Y(M,[g]) > 0$ (otherwise (3.20) is trivially satisfied), one obtains

$$\mu_1 \mathrm{Vol}(M^n,g)^{\frac{2}{n}} \geq Y(M,[g]),$$

which with (3.18) implies the result. □

Note however that (3.18) is not itself conformal. We also mention that inequality (3.18) can be combined with lower bounds of μ_1 to provide an estimate of $\lambda_1(D^2_{M,g})$ in terms of the total Q-curvature in dimension $n = 4$ and of the first eigenvalue of the so-called Branson-Paneitz operator in dimension $n \geq 5$, see [145, Sec. 4]. The *a priori* existence of a qualitative conformal lower bound for the Dirac spectrum was proved independently by J. Lott [181] using the boundedness of particular Sobolev embeddings. More precisely, if the Dirac operator of a given closed Riemannian spin manifold (M^n, g) is invertible, then there exists a positive constant c depending only on the conformal class of g such that [181, Prop. 1]

$$\lambda_1(D^2_{M,\overline{g}})\mathrm{Vol}(M,\overline{g})^{\frac{2}{n}} \geq c \tag{3.21}$$

for any metric $\overline{g}$ conformal to g on M^n.

C. Bär's estimate (3.17) gives a topologically invariant lower bound on the Dirac spectrum. Surprisingly enough this contrasts with the situation of the scalar Laplacian on $\mathbb{S}^2$ for which this invariant provides an upper bound for the first non-zero eigenvalue in one of the corresponding estimates established by J. Hersch (see reference in [42]) and which reads

$$\lambda_1(\Delta_{\mathbb{S}^2,g})\mathrm{Area}(\mathbb{S}^2,g) \leq 8\pi, \tag{3.22}$$

where $\lambda_1(\Delta_{\mathbb{S}^2,g})$ denotes the smallest positive eigenvalue of the scalar Laplace operator Δ on $(\mathbb{S}^2,g)$. For lower bounds in higher genus, where (3.17) is trivial, see Section 3.6.

As noticed in the proof of Corollary 3.3.2, one has from the Gauss-Bonnet Theorem in dimension 2 the following identity: $\frac{2\pi\chi(M^2)}{\mathrm{Area}(M^2,g)} = \frac{2}{4(2-1)}\frac{\int_{M^2} S v_g}{\mathrm{Area}(M^2,g)}$. Can one improve Friedrich's inequality (3.1) in dimension $n \geq 3$ by replacing $\inf_M(S)$ by $\frac{1}{\mathrm{Vol}(M,g)}\int_M S v_g$? B. Ammann and C. Bär showed [16] that this is not the case at all: on any compact spin manifold of dimension $n \geq 3$ and for any positive integer k there exists a sequence of Riemannian metrics for which the k^{th} Dirac eigenvalue remains bounded whereas the averaged total scalar curvature tends to infinity. We refer to [42] for a detailed and illustrated proof.

3.4 Improving Friedrich's inequality with the energy-momentum tensor

The main idea to prove inequality (3.1) was to split the spinorial Levi-Civita connection in a clever way so as to make the term which is dropped off after integration and application of the Schrödinger-Lichnerowicz formula as small as possible. This led to the introduction of the Penrose operator. In an equivalent way, this means deforming the Levi-Civita connection in the direction of Id_{TM}, i.e., defining $T_X\varphi := \nabla_X\varphi + fX\cdot\varphi$ for some real or complex-valued function f to be fixed later, see T. Friedrich's method of proof in Section 3.1. O. Hijazi's idea for the following result is to introduce a different Penrose-like operator, deforming the Levi-Civita connection in the direction of some symmetric 2-tensor tensor T^ψ associated to an eigenvector ψ:

Theorem 3.4.1 (O. Hijazi [132]) *Let λ be an eigenvalue of the fundamental Dirac operator on a closed $n(\geq 2)$-dimensional closed Riemannian spin manifold (M^n,g) and ψ be a non-zero eigenvector for D to the eigenvalue λ. Then*

$$\lambda^2 \geq \inf_{M_\psi}\left(\frac{S}{4} + |T^\psi|^2\right), \tag{3.23}$$

where $T^\psi(X,Y) := \frac{1}{2}\Re e\left(\langle X\cdot\nabla_Y\psi + Y\cdot\nabla_X\psi, \frac{\psi}{|\psi|^2}\rangle\right)$ for all $X,Y \in TM$ and $M_\psi := \{x \in M \mid \psi(x) \neq 0\}$. If furthermore (3.23) is an equality then ψ solves

$$\nabla_X\psi = -T^\psi(X)\cdot\psi \tag{3.24}$$

for all $X \in TM$.

Proof: Define the following modified connection $\widehat{\nabla}$ on ΣM (and outside the zero set of ψ, of which measure vanishes) by

$$\widehat{\nabla}_X\psi := \nabla_X\psi + T^\psi(X)\cdot\psi$$

for every $X \in TM$. We compute in a local orthonormal frame $\{e_j\}_{1\leq j\leq n}$:

$$\begin{aligned}
|\widehat{\nabla}\psi|^2 &= \sum_{j=1}^n |\widehat{\nabla}_{e_j}\psi|^2\\
&= \sum_{j=1}^n |\nabla_{e_j}\psi|^2 + |T^\psi(e_j)|^2|\psi|^2 + 2\Re e\left(\langle\nabla_{e_j}\psi, T^\psi(e_j)\cdot\psi\rangle\right)\\
&= |\nabla\psi|^2 + |T^\psi|^2|\psi|^2 - 2\sum_{j,k=1}^n T^\psi(e_j,e_k)\Re e\left(\langle e_k\cdot\nabla_{e_j}\psi,\psi\rangle\right)\\
&= |\nabla\psi|^2 - |T^\psi|^2|\psi|^2
\end{aligned}$$

since from its definition the tensor T^ψ is symmetric. Integrating and applying the Schrödinger-Lichnerowicz formula (1.15) leads straightforward to the inequality, of which limiting-case implies $\widehat{\nabla}\psi = 0$. This concludes the proof. $\square$

The tensor T^ψ is sometimes called the *energy-momentum tensor* associated to ψ, see e.g. [47, Sec. 6] for a justification of its name.

The lower bound in (3.23) for the eigenvalue λ has the obvious disadvantage to depend on the eigenvector ψ to λ, hence Theorem 3.4.1 does not directly provide a geometric lower bound for the Dirac spectrum. Note however that (3.23) improves Friedrich's inequality (3.1) whatever the tensor T^ψ could be: one can indeed write $T^\psi = (T^\psi)^0 + \frac{\mathrm{tr}_g(T^\psi)}{n}g$, where $(T^\psi)^0$ denotes the trace-free part of T^ψ. Since $D\psi = \lambda\psi$ one has in any local o.n.b. $\{e_j\}_{1\leq j\leq n}$ of TM:

$$\begin{aligned}
\mathrm{tr}_g(T^\psi) &= \sum_{j=1}^n \Re e\left(\langle e_j\cdot\nabla_{e_j}\psi, \frac{\psi}{|\psi|^2}\rangle\right)\\
&= \Re e\left(\langle D\psi, \frac{\psi}{|\psi|^2}\rangle\right)\\
&= \lambda,
\end{aligned}$$

so that $|T^\psi|^2 = |(T^\psi)^0|^2 + \frac{\lambda^2}{n}$ and

$$\lambda^2 \geq \inf_{M_\psi} \Big(\frac{n}{4(n-1)} S + \underbrace{\frac{n}{n-1} |(T^\psi)^0|^2}_{\geq 0} \Big),$$

which implies (3.1).

One can also remark that the proof of Theorem 3.4.1 only needs ψ to be eigen for D^2, which is weaker than ψ be eigen for D. However the comparison with Friedrich's inequality as just above is in this case not available.

Closed spin manifolds carrying a non-zero eigenvector ψ of D satisfying (3.24) have not been completely classified yet. From the above comparison with Friedrich's inequality (3.1) they contain all manifolds carrying non-zero real Killing spinors. Recent works [119, 104], where examples of manifolds are given where (3.23) is sharp but not (3.1) (e.g. Heisenberg manifolds, see [104, Ex. 6.4]), show that they form a strictly larger family.

Besides we mention that Theorem 3.4.1 was generalized by T. Friedrich and E.-C. Kim [93, Lemma 5.1] and by G. Habib [119, Thm. 2.2.1] (see also [122]). More recently, an analogous ansatz was successfully carried out by T. Friedrich and E.C. Kim [94, Thm. 1.1] where the lower bound for the Dirac spectrum depends on the spectrum of a Dirac-type operator associated to a so-called nondegenerate Codazzi tensor, we refer to [94] for details.

3.5 Improving Friedrich's inequality with other curvature components

In case the scalar curvature of a compact spin manifold is not everywhere positive one can try to look for lower eigenvalue bounds involving the Ricci and Weyl components of the curvature tensor. The proof of the following theorems relies on the application of the Schrödinger-Lichnerowicz formula (1.15) after a suitable choice of Penrose-like operator involving those tensors, see e.g. [161] for the highly technical details.

Theorem 3.5.1 (T. Friedrich and K.-D. Kirchberg [96]) *Any eigenvalue λ of D on an $n(\geq 2)$-dimensional closed Riemannian spin manifold* (M^n, g) with divergence-free curvature tensor, vanishing scalar curvature and nowhere-vanishing Ricci-curvature *satisfies:*

$$\lambda^2 > \frac{1}{4} \frac{\inf_M |\mathrm{Ric}|^2}{\sqrt{\frac{n-1}{n}} \inf_M |\mathrm{Ric}| - \kappa_0},$$

where κ_0 denotes the smallest eigenvalue of the Ricci tensor Ric *on M.*

Examples of closed Riemannian manifolds satisfying the assumptions of Theorem 3.5.1 and where the lower bound can be explicitly computed can be found among the following families, see [96, Ex. 1-4] and [160, Ex. 4.1 & 4.2]: (local) Riemannian products of Einstein manifolds, warped products of $\mathbb{S}^1$ with an Einstein manifold with positive scalar curvature, warped products on Riemannian surfaces, conformally flat manifolds. Note however that Einstein manifolds themselves or manifolds whose Ricci tensor vanishes somewhere cannot be handled by Theorem 3.5.1. This was the motivation of T. Friedrich and K.-D. Kirchberg for obtaining a lower bound involving the Weyl tensor only. The best result in this direction was obtained by K.-D. Kirchberg, generalizing an earlier one by T. Friedrich and himself [95, Thm. 3.1]:

Theorem 3.5.2 (K.-D. Kirchberg [161]) *Any eigenvalue λ of D on an $n(\geq 2)$-dimensional closed Riemannian spin manifold (M^n, g)* with divergence-free Weyl-tensor *and $\mu > 0$ satisfies:*

$$\lambda^2 \geq \frac{1}{8(n-1)}\left((2n-1)\inf_M(S) + \sqrt{\inf_M(S)^2 + \frac{n}{n-1}(\frac{4\nu_0}{\mu})^2}\right), \qquad (3.25)$$

where $\nu_0 \geq 0$ and μ are conformal invariants depending on the Weyl tensor only.

Recall that every Einstein Riemannian manifold has divergence-free Weyl tensor. In case $\inf_M(S) > 0$ inequality (3.25) obviously enhances Friedrich's inequality (3.1). In case $\inf_M(S) \leq 0$ it is easy to see that the lower bound in (3.25) is positive if and only if $\nu_0 > \frac{(n-1)\mu}{2}|\inf_M(S)|$. However, it is up to now not known if (3.25) can be an equality [161, Rem. 4.2.ii)].

Theorems 3.5.1 and 3.5.2 actually follow from a whole series of estimates [96, 161] involving curvature tensors and that can be applied to produce fine vanishing theorems for the kernel of the Dirac operator.

3.6 Improving Friedrich's inequality on surfaces of positive genus

C. Bär's inequality (3.17) does not give any information on the spectrum of D on compact Riemannian surfaces with nonpositive Euler characteristic, i.e., with positive genus. Estimates on such surfaces have to depend on the choice of spin structure, as the example of the 2-torus already shows: for its trivial spin structure (i.e., for the spin structure coming from the trivial lift of the lattice-action to the spin level, see Proposition 1.4.2) it admits harmonic spinors - for flat hence any metrics because of (1.16) - but not for any other spin structure [86].

The first estimate to have been proved is a qualitative one and dates back to J. Lott's work [181] providing lower bounds for general conformally covariant elliptic self-adjoint linear differential operators. In the case of surfaces it states that, if the Dirac operator of a given closed oriented surface (M^2, g) is invertible, then there exists a positive constant c such that, for any metric $\overline{g}$ conformal to g on M^2 (see (3.21)):

$$\lambda_1(D^2_{M,\overline{g}})\mathrm{Area}(M, \overline{g}) \geq c.$$

The constant c expresses the boundedness of particular Sobolev embeddings hence cannot be made explicit in general.

The first successful attempt in looking for a geometric estimate is due to B. Ammann [14]. His lower bound, which was proved for the 2-torus, involves the so-called *spinning systole* spin-sys(M) of a closed oriented surface M with positive genus which is defined to be the minimum of the lengths of all noncontractible loops (in our convention, loops are simply closed curves) along which the induced spin structure is non trivial. Recall that the systole of M is defined to be the minimum of the lengths of all noncontractible loops in M.

Theorem 3.6.1 (B. Ammann [14]) *Let g be an arbitrary Riemannian metric on the 2-torus $M := \mathbb{T}^2$ carrying a non-trivial spin structure. Assume that $\|K_g\|_{\mathrm{L}^1(\mathbb{T}^2,g)} < 4\pi$, where K_g ist the Gauss curvature of $(\mathbb{T}^2, g)$. Then there exists for each $p > 1$ a constant $C_p > 0$ depending on $\|K_g\|_{\mathrm{L}^1(\mathbb{T}^2,g)}$, $\|K_g\|_{\mathrm{L}^p(\mathbb{T}^2,g)}$, the area and the systole of $(\mathbb{T}^2, g)$ such that any eigenvalue λ of D satisfies*

$$\lambda^2 \geq \frac{\sup\limits_{p>1} C_p}{\text{spin-sys}(\mathbb{T}^2)^2}.$$

Moreover this inequality is an equality if and only if g is flat, the lattice is generated by an orthogonal pair and the spin structure is the $(1,0)$- or $(0,1)$-one.

Sketch of proof of Theorem 3.6.1: The proof of the inequality combines the following steps. First one chooses a flat metric $g_0 := e^{2u}g$ in the conformal class of g. Using the min-max principle (see e.g. Lemma 5.0.2) it can be easily proved that $\lambda^2 \geq e^{2\max(u)}\lambda_0^2$, where $\lambda_0 > 0$ is the smallest Dirac eigenvalue (in absolute value) on $(\mathbb{T}^2, g_0)$ and for the same spin structure. Now the Dirac spectrum of $(\mathbb{T}^2, g_0)$ for any spin structure is explicitly known (see Theorem 2.1.1), in particular the following equality holds

$$\lambda_0^2\mathrm{Area}(\mathbb{T}^2, g_0) = 4\pi^2\|\chi\|^2_{\mathrm{L}^2(\mathbb{T}^2,g_0)},$$

where $\chi \in H^1(\mathbb{T}^2, \mathbb{Z}_2)$ is the cohomology class representing the spin structure (it is non-zero if the spin structure is non-trivial). Obviously $\mathrm{Area}(\mathbb{T}^2, g) \geq e^{-2\min(u)}\mathrm{Area}(\mathbb{T}^2, g_0)$ so that one obtains a lower bound of λ in terms of the area of $(\mathbb{T}^2, g)$, of the L^2-norm of χ and of the so-called oscillation $\mathrm{osc}(u) := \max(u) - \min(u)$ of u on $\mathbb{T}^2$. On the other hand the L^2-norm of χ can be proved to be only dependent of the conformal class of

g and can be estimated against an expression involving the spinning systole, the area and $\mathrm{osc}(u)$ [14, Sec. 4]. What remains - the whole work - is to estimate $\mathrm{osc}(u)$ against the desired geometric data. For an illustrated proof of this Sobolev-type inequality we refer to [14, Sec. 6]. The limiting-case occurs if and only if $\mathrm{osc}(u) = 0$ (i.e., g is flat) and the estimate of $\mathrm{osc}(u)$ is sharp, which yields strong conditions on the lattice defining the flat metric g, see [14]. $\square$

Another and completely different approach was developed by B. Ammann and C. Bär in [17]. It aimed at obtaining a lower bound in terms of a geometric invariant called the *spin-cut diameter* $\delta(M)$. This is a positive number which is associated to the surface M and its spin structure. The idea is simple: apply (3.17) to the surface obtained from the genus g surface M by cutting g suitable loops out of M. Here "suitable" means the following: on the one hand one has to choose the loops such that the resulting surface $\widetilde{M}$ is diffeomorphic to an open subset of $\mathbb{S}^2$ - actually to a 2-sphere with $2g$ disks removed; this is the case as soon as the $\mathbb{Z}_2$-homology classes associated to those loops form a basis of $H_1(M, \mathbb{Z}_2)$. On the other hand the cut-out-process must also respect the spin structures in the sense that the restrictions of the original one and of the one from $\mathbb{S}^2$ have to coincide on $\widetilde{M}$. This however is only possible if the so-called *Arf-invariant* (which associates to each spin structure on M the number 1 or -1, see [17, Def. p.430]) of the spin structure of M is 1 [17, Cor. 3.3]. The spin-cut diameter can then be defined from the distances between the cut-out loops, see [17, Def. p.433]. For $M = \mathbb{T}^2$ the Arf-invariant of the trivial spin structure is -1 and it is 1 for the other ones. In the latter case, extending by means of suitable cut-offs an eigenvector on M to the $\mathbb{S}^2$ obtained by adding two disks to the gluing of a finite number of copies of $\widetilde{M}$ (which is then a cylinder) one can prove the following result [17, Sec. 5]:

Theorem 3.6.2 (B. Ammann and C. Bär [17]) *Let $M := \mathbb{T}^2$ be the 2-torus with arbitrary Riemannian metric and non-trivial spin structure. Then any eigenvalue λ of D satisfies*

$$|\lambda| \geq \sup_{\substack{k\in\mathbb{N}\\ k\neq 0}} \left(-\frac{2}{k\delta(\mathbb{T}^2)} + \sqrt{\frac{\pi}{k\mathrm{Area}(\mathbb{T}^2)} + \frac{2}{k^2\delta(\mathbb{T}^2)^2}} \right), \tag{3.26}$$

where $\delta(\mathbb{T}^2)$ is the spin-cut diameter of $\mathbb{T}^2$ associated to this spin structure.

The supremum in the lower bound is attained for $k = \left[\frac{4(\sqrt{2}+1)\mathrm{Area}(\mathbb{T}^2)}{\pi\delta(\mathbb{T}^2)^2}\right]$ or $k = \left[\frac{4(\sqrt{2}+1)\mathrm{Area}(\mathbb{T}^2)}{\pi\delta(\mathbb{T}^2)^2}\right] + 1$. It is positive and for the boundary $\mathbb{T}^2_\varepsilon$ of an ε-tubular neighbourhood of a circle of radius $\frac{1}{\varepsilon}$ it is asymptotic to $\sqrt{\frac{\pi}{\mathrm{Area}(\mathbb{T}^2_\varepsilon)}}$ when ε tends to 0. Therefore Theorem 3.6.2 can be viewed as a generalization of Corollary 3.3.2.*ii*) for $\mathbb{T}^2$ with non-trivial spin structure.

In genus $g \geq 1$ one can apply the same argument to the $\mathbb{S}^2$ obtained by adding disks to a clever gluing of $2g+1$ copies of $\widetilde{M}$ and prove [17, Sec. 6]:

Theorem 3.6.3 (B. Ammann and C. Bär [17]) *Let M be a closed Riemannian surface of positive genus g with spin structure whose Arf-invariant equals 1. Then any eigenvalue λ of D satisfies*

$$|\lambda| \geq \frac{2}{2g+1} \cdot \sqrt{\frac{\pi}{\mathrm{Area}(M)}} - \frac{1}{\delta(M)}. \tag{3.27}$$

Although the lower bound need this time not be positive there exist examples for which it is: as above, consider an ε-tubular neighbourhood M_ε of a closed plane curve with exactly $g-1$ intersections and such that, w.r.t. any allowed spin structure, $\delta(M_\varepsilon) \underset{\varepsilon \to 0}{\sim} \frac{\mathrm{cst}}{\varepsilon}$ (fix for instance the diameter equal to $\frac{1}{\varepsilon}$). Then the lower bound is asymptotic to $\frac{2}{2g+1} \cdot \sqrt{\frac{\pi}{\mathrm{Area}(M_\varepsilon)}}$ for $\varepsilon \to 0$. In the case where $g=1$ the k-dependent expression in the r.h.s. of (3.26) is for $k=2$ greater than the r.h.s. of (3.27), so that (3.26) is better than (3.27).

Combining Theorems 3.6.2 and 3.6.3 with the extrinsic upper bound (5.19) for the smallest Dirac eigenvalue for surfaces embedded in $\mathbb{R}^3$ one obtains a lower bound of the Willmore functional, see [17, Thm. 7.1]. Besides, we mention that Theorems 3.6.2 and 3.6.3 can be extended to complete surfaces with finite area [17, Thm. 8.1].

3.7 Improving Friedrich's inequality on bounding manifolds

In case M bounds a compact spin manifold $\widetilde{M}$ the input of extrinsic geometrical data - such as the mean curvature of M in $\widetilde{M}$ - can improve Friedrich's inequality (3.1). The main step consists in solving a suitable boundary value problem, see also Chapter 4. The following theorem was proved by O. Hijazi, S. Montiel and X. Zhang in [142] for $c=0$ and by O. Hijazi, S. Montiel and A. Roldán in [140] for $c<0$. Recall that D_2 is the operator acting on the sections of $\Sigma := \Sigma M$ or $\Sigma := \Sigma M \oplus \Sigma M$ and which is defined by $D_2 := D$ for n even or $D_2 := D \oplus -D$ for n odd respectively, see Proposition 1.4.1.

Theorem 3.7.1 *Let $M^n = \partial\widetilde{M}$, where $\widetilde{M}$ is a compact Riemannian spin manifold. Assume that, for a constant $c \leq 0$, the scalar curvature $\widetilde{S}$ of $\widetilde{M}$ and the mean curvature H of M in $\widetilde{M}$ w.r.t. the inner normal satisfy $\widetilde{S} \geq (n+1)nc$ and $H \geq \sqrt{-c}$ respectively. Then for any eigenvalue λ of D,*

$$|\lambda| \geq \frac{n}{2} \inf_M (\sqrt{H^2 + c}). \tag{3.28}$$

Moreover (3.28) *is an equality if and only if H is constant, the manifold $(\widetilde{M}, g)$ admits a non-trivial $\frac{\sqrt{c}}{2}$- or $-\frac{\sqrt{c}}{2}$-Killing spinor and the eigenspace of D_2 to the eigenvalue $\frac{n}{2}\inf_M(\sqrt{H^2+c})$ coincides with $(\mathcal{K}_0)_{|M}$ for $c = 0$ and with*

$$\begin{array}{ll} \bigoplus_{j=0,1} (\mathrm{Id} + (-1)^j i(H - \sqrt{H^2+c})\nu\cdot)\mathcal{K}_{(-1)^j\frac{\sqrt{c}}{2}}(\widetilde{M}, g)_{|M} & \textit{if } H > \sqrt{-c} \\ \bigoplus_{j=0,1} \mathcal{K}_{(-1)^j\frac{\sqrt{c}}{2}}(\widetilde{M}, g)_{|M} & \textit{if } H = \sqrt{-c} \end{array}$$

for $c < 0$, where $\mathcal{K}_{\pm\frac{\sqrt{c}}{2}}(\widetilde{M}, g)$ denotes the space of $\pm\frac{\sqrt{c}}{2}$-Killing spinors on $(\widetilde{M}, g)$.

Proof in the case $c = 0$: Denote by $\widetilde{\nabla}$ the spinorial Levi-Civita connection of $\widetilde{M}$. The Schrödinger-Lichnerowicz formula for the Dirac operator $\widetilde{D}$ of $(\widetilde{M}, g)$ and elementary computations as in Section 1.3 show that, for any $\varphi \in \Gamma(\Sigma\widetilde{M})$,

$$\begin{aligned} |\widetilde{D}\varphi|^2 &= \Re e\left(\langle \widetilde{D}^2\varphi, \varphi\rangle\right) + \mathrm{div}_{\widetilde{M}}(V) \\ &\overset{(1.15)}{=} \Re e\left(\langle \widetilde{\nabla}^*\widetilde{\nabla}\varphi, \varphi\rangle\right) + \frac{\widetilde{S}}{4}|\varphi|^2 + \mathrm{div}_{\widetilde{M}}(V) \\ &= |\widetilde{\nabla}\varphi|^2 + \frac{\widetilde{S}}{4}|\varphi|^2 + \mathrm{div}_{\widetilde{M}}(V+W), \end{aligned} \tag{3.29}$$

where V and W are the vector fields on $\widetilde{M}$ defined by the relations $g(V, X) := \Re e\left(\langle X\cdot\widetilde{D}\varphi, \varphi\rangle\right)$ and $g(W, X) := \Re e\left(\langle \widetilde{\nabla}_X\varphi, \varphi\rangle\right)$ for all $X \in T\widetilde{M}$ respectively (remember that we denote by "$\cdot$" the Clifford multiplication of $\widetilde{M}$ and not that of M). Splitting $|\widetilde{\nabla}\varphi|^2$ as in (A.11) one comes to

$$\frac{\widetilde{S}}{4}|\varphi|^2 - \frac{n}{n+1}|\widetilde{D}\varphi|^2 = -|\widetilde{P}\varphi|^2 - \mathrm{div}_{\widetilde{M}}(V+W),$$

where $\widetilde{P}$ is the Penrose operator of $(\widetilde{M}^{n+1}, g)$, see Appendix A. Let ν be the inner unit normal vector field of M in $\widetilde{M}$. Integrating the last identity and applying Green's formula one obtains

$$\begin{aligned} \int_{\widetilde{M}} \left(\frac{\widetilde{S}}{4}|\varphi|^2 - \frac{n}{n+1}|\widetilde{D}\varphi|^2\right) v_g^{\widetilde{M}} &= -\int_{\widetilde{M}} |\widetilde{P}\varphi|^2 v_g^{\widetilde{M}} \\ &\quad - \int_{\widetilde{M}} \mathrm{div}_{\widetilde{M}}(V+W) v_g^{\widetilde{M}} \\ &= -\int_{\widetilde{M}} |\widetilde{P}\varphi|^2 v_g^{\widetilde{M}} - \int_M g(V+W, \nu) v_g \end{aligned}$$

$$
\begin{aligned}
&= -\int_{\widetilde{M}} |\widetilde{P}\varphi|^2 v_g^{\widetilde{M}} \\
&\quad -\int_M \Re e\left(\langle \nu\cdot\widetilde{D}\varphi,\varphi\rangle + \langle\widetilde{\nabla}_\nu\varphi,\varphi\rangle\right) v_g \\
&\overset{(1.22)}{=} -\int_{\widetilde{M}} |\widetilde{P}\varphi|^2 v_g^{\widetilde{M}} \\
&\quad +\int_M \Re e\left(\langle D_2\varphi,\varphi\rangle - \frac{nH}{2}|\varphi|^2\right) v_g.
\end{aligned}
$$

Since D_2 is formally self-adjoint (Proposition 1.3.4) one comes to

$$
\begin{aligned}
\int_{\widetilde{M}} \left(\frac{\widetilde{S}}{4}|\varphi|^2 - \frac{n}{n+1}|\widetilde{D}\varphi|^2\right) v_g^{\widetilde{M}} &= -\int_{\widetilde{M}} |\widetilde{P}\varphi|^2 v_g^{\widetilde{M}} \\
&\quad + \int_M \left(\langle D_2\varphi,\varphi\rangle - \frac{nH}{2}|\varphi|^2\right) v_g. \qquad (3.30)
\end{aligned}
$$

Let λ be an eigenvalue of D. The spectrum of D_2 being the symmetrized of that of D w.r.t. the origin (for n even it follows from (1.10) that the spectrum of D is already symmetric w.r.t. the origin) there always exists a non-zero eigenvector ψ for D_2 associated to the eigenvalue $|\lambda|$. The crucial point is now the existence of a smooth solution ϕ to the boundary value problem with APS-boundary condition

$$
\left|\begin{array}{lll} \widetilde{D}\phi & = 0 & \text{on } \widetilde{M} \\ \pi_{\geq 0}\phi & = \psi & \text{on } M, \end{array}\right. \qquad (3.31)
$$

where $\pi_{\geq 0} : \Gamma(\Sigma) \to \Gamma(\Sigma)$ denotes the L^2-orthogonal projection onto the eigenspaces of D_2 to nonnegative eigenvalues, see Section 1.5 and Chapter 4. Since $\widetilde{S} \geq 0$ and $H \geq 0$ the identity (3.30) with $\varphi := \phi$ implies

$$
\begin{aligned}
0 \leq \int_{\widetilde{M}} \frac{\widetilde{S}}{4}|\phi|^2 v_g^{\widetilde{M}} &= \int_{\widetilde{M}} \left(\frac{\widetilde{S}}{4}|\phi|^2 - \frac{n}{n+1}|\widetilde{D}\phi|^2\right) v_g^{\widetilde{M}} \\
&\leq \int_M \left(\langle D_2\phi,\phi\rangle - \frac{nH}{2}|\phi|^2\right) v_g \\
&\leq \int_M \left(\langle D_2\pi_{\geq 0}\phi,\pi_{\geq 0}\phi\rangle - \frac{nH}{2}|\pi_{\geq 0}\phi|^2\right) v_g \\
&= \int_M (|\lambda| - \frac{nH}{2})|\psi|^2 v_g
\end{aligned}
$$

from which the inequality follows.

In case the lower bound is attained the mean curvature H of M must be constant, $\phi = \pi_{\geq 0}\phi$ on M and $\widetilde{P}\phi = 0$ on $\widetilde{M}$, where ϕ is any section of $\Sigma\widetilde{M}$

solving (3.31) for any given eigenvector ψ of D_2 associated to the eigenvalue $|\lambda|$. In particular $\psi = \phi_{|M}$ with $\widetilde{\nabla}\phi = 0$ on $\widetilde{M}$, i.e., every eigenvector ψ of D_2 associated to the eigenvalue $|\lambda|$ must be the restriction on M of parallel spinor on $\widetilde{M}$ (note that the existence of a non-zero parallel spinor on $\widetilde{M}$ implies $\widetilde{S} = 0$, see Proposition A.4.1). Moreover (1.22) already implies that the restriction of any parallel spinor onto a hypersurface with constant mean curvature is an eigenvector of D to the eigenvalue $\frac{nH}{2}$. Therefore the eigenspace of D_2 associated to the eigenvalue $\frac{nH}{2}$ exactly coincides with $(\mathcal{K}_0)_{|M}$. The other implication is trivial.

For the proof in the case $c < 0$, which is based on the same argument for a Schrödinger operator associated to D, we refer to [140]. $\square$

In case $\widetilde{M} \subset \widetilde{M}^{n+1}(c)$, where $\widetilde{M}^{n+1}(c)$ is a spaceform with constant curvature $c \leq 0$, Gauss' equations imply in particular $(\frac{n}{2})^2(H^2 + c) \geq \frac{n}{4(n-1)}S$, hence (3.28) improves (3.1) under the supplementary assumption $H \geq \sqrt{-c}$.

There exists a conformal version of (3.28) in terms of the so-called Yamabe relative invariant, see [144].

The characterization of the equality case in (3.28) provides a short proof of Alexandrov's theorem (see reference in [142]) on constant mean curvature embedded hypersurfaces in the Euclidean and hyperbolic spaces respectively [142, Thm. 8]:

Theorem 3.7.2 (A.D. Alexandrov) *Every closed embedded hypersurface with constant mean curvature in $\mathbb{R}^{n+1}$ or $\mathbb{H}^{n+1}$ is a round geodesic hypersphere.*

Proof in the case $c = 0$: Let M be such a hypersurface. It is embedded so that on the one hand it bounds a compact domain $\widetilde{M}$ (in particular it is orientable hence spin, see Proposition 1.4.1); on the other hand, it can be shown that necessarily $H \geq 0$ by a result of S. Montiel and A. Ros (see reference in [142]). Moreover the assumption H constant implies that (3.28) is an equality, in which case every non-zero eigenvector of D_2 associated to the eigenvalue $\frac{nH}{2}$ must be the restriction onto M of a parallel spinor on $\widetilde{M}$ (Theorem 3.7.1). But considering the spinor field

$$x \longmapsto \varphi_x := \nu_x \cdot \phi + Hx \cdot \phi,$$

on $\widetilde{M}$, where ν is the inner unit normal and ϕ a parallel spinor on $\widetilde{M} \subset \mathbb{R}^{n+1}$, one notices that

$$\begin{aligned} D_2\varphi &= D_2(\nu \cdot \phi) + HD_2(x \cdot \phi) \\ &\overset{(1.19)}{=} -\nu \cdot D_2\phi + HD_2(x \cdot \phi) \\ &\overset{(1.22)}{=} -\nu \cdot D_2\phi + H(\frac{nH}{2}x \cdot \phi - \widetilde{\nabla}_\nu(x \cdot \phi) - \nu \cdot \widetilde{D}(x \cdot \phi)) \end{aligned}$$

$$
\begin{aligned}
&= -\frac{nH}{2}\nu\cdot\phi + H(\frac{nH}{2}x\cdot\phi - \nu\cdot\phi + (n+1)\nu\cdot\phi) \\
&= \frac{nH}{2}(\nu\cdot\phi + Hx\cdot\phi) \\
&= \frac{nH}{2}\varphi,
\end{aligned}
$$

i.e., φ is an eigenvector for D_2 associated to the eigenvalue $\frac{nH}{2}$. Therefore φ must be either identically zero or non-zero and parallel. The first possibility already implies that M must be a geodesic sphere. The second one means that, for every $X \in TM$,

$$0 = \widetilde{\nabla}_X\varphi = -A(X)\cdot\phi + HX\cdot\phi.$$

Since ϕ has no zero on $\widetilde{M}$ one deduces that $A = H\mathrm{Id}_{TM}$, i.e., that M must be totally umbilical in $\widetilde{M}$ hence in $\mathbb{R}^{n+1}$. This concludes the proof for $c = 0$. For $c < 0$ we refer to [140]. $\square$

Another clever application of Theorem 3.7.1 is:

Theorem 3.7.3 (O. Hijazi and S. Montiel [138]) *Let $(\widetilde{M}^{n+1}, g)$ be a complete Riemannian spin manifold with nonnegative Ricci curvature, mean convex boundary $\partial\widetilde{M}$ and nonnegative Einstein-tensor along the normal direction of $\partial\widetilde{M}$. Then $(\widetilde{M}^{n+1}, g)$ is isometric to a Euclidean ball.*

As a corollary, any Ricci-flat complete spin manifold with boundary isometric to the round sphere $\mathbb{S}^n$ is already isometric to a Euclidean ball. Recently rigidity results have been obtained by S. Raulot [213] under weaker assumptions on the boundary. We also mention that it remains open whether analogous estimates on the boundary of positively-curved domains can be obtained.

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