Preface

This volume contains the notes of the lectures delivered at the CIME course *Geometric Analysis and PDEs* during the week of June 11–16 2007 in Cetraro (Cosenza). The school consisted in six courses held by M. Gursky (*PDEs in Conformal Geometry*), E. Lanconelli (*Heat kernels in sub-Riemannian settings*), A. Malchiodi (*Concentration of solutions for some singularly perturbed Neumann problems*), G. Tarantello (*On some elliptic problems in the study of selfdual Chern-Simons vortices*), X.J. Wang (*The k-Hessian Equation*) and P. Yang (*Minimal Surfaces in CR Geometry*).

Geometric PDEs are a field of research which is currently very active, as it makes it possible to treat classical problems in geometry and has had a dramatic impact on the comprehension of three- and four-dimensional manifolds in the last several years. On one hand the geometric structure of these PDEs might cause general difficulties due to the presence of some invariance (translations, dilations, choice of gauge, etc.), which results in a lack of compactness of the functional embeddings for the spaces of functions associated with the problems. On the other hand, a geometric intuition or result might contribute enormously to the search for natural quantities to keep track of, and to prove regularity or a priori estimates on solutions. This two-fold aspect of the study makes it both challenging and complex, and requires the use of several refined techniques to overcome the major difficulties encountered. The applications of this subject are many while for the CIME course we had to select only a few, trying however to cover some of the most relevant ones, with interest ranging from the pure side (analysis/geometry) to the more applied one (physics/biology). Here is a brief summary of the topics covered in the courses of this school.

M. Gursky treated a class of elliptic equations from conformal geometry: the general aim is to deform conformally (through a dilation which depends on the point) the metric of a given manifold so that the new one possesses special properties. Classical examples are the uniformization problem of two-dimensional surfaces and the Yamabe problem in dimension greater or equal to three, where one requires the Gauss or the scalar curvature to become constant. After recalling some basic facts on these problems, which can be reduced to semilinear elliptic PDEs, Gursky turned to their fully nonlinear counterparts. These concern the prescription of the symmetric forms in the eigenvalues of the *Schouten tensor* (a combination of the Ricci tensor and the scalar curvature), and turn out to be elliptic under suitable conditions on their domain of definition (admissible functions). The solvability of these equations has concrete applications in geometry, since for example they might guarantee pinching conditions on the Ricci tensor, together with its geometric/topological consequences. After recalling some regularity estimates by Guan and Wang, existence was shown using blow-up analysis techniques. Finally, the functional determinant of conformally invariant operators in dimension four was discussed: the latter turns out to have a universal decomposition into three terms which respectively involve the scalar curvature, the Q-curvature and the Weyl tensor. Some conditions on the coefficients of these three terms guarantee coercivity of the functional, and in these cases existence of extremal metrics was obtained using a minimization technique.

E. Lanconelli covered some topics on existence and sharp estimates on heat kernels of subelliptic operators. Typically, in a domain or a manifold Ω of dimension n, k vector fields $X_1, \dots, X_k$ are given (with $2 \leq k < n$) which satisfy the *Hörmander condition*, namely their Lie brackets span all of the tangent spaces to Ω. One considers then linear operators L (or their parabolic counterpart) whose principal part is given by $\sum_{i=1}^{k} X_i^2$. During the lectures, existence and regularity (Hörmander) theory for such operators was recalled, and in particular the role of the *Carnot-Caratheodory distance*, measured through curves whose velocities belong to the linear span of the X_i's. This distance is not homogeneous (at small scales), and it is very useful to describe the degeneracy of the operators in the above form. One of the main motivations for this study is the problem of prescribing the *Levi curvature* of boundaries of domains in $\mathbb{C}^n$, which for graphs amounts to solving a fully nonlinear degenerate equation, whose linearization is of the form previously described. Gaussian bounds for heat kernels were then given, first for constant coefficient operators modeled on Carnot groups, and then for general operators using the method of the parametrix. Finally, applications to Harnack type inequalities were derived in terms of the heat kernel bounds.

The course by A. Malchiodi on singularly perturbed Neumann problems dealt with elliptic nonlinear equations where a small parameter (the singular perturbation) is present in front of the principal term (the Laplacian). The study is motivated by considering a class of reaction-diffusion systems (in particular the Gierer-Meinhardt model) and the (focusing) nonlinear Schrödinger equation. First a finite-dimensional reduction technique, which incorporates the variational structure of the problem, was presented: by means of this method, existence of solutions concentrating at points of the boundary of the domain was studied. Here the geometry of the boundary is significant, as concentration occurs at critical points of the mean curvature. After this, existence of solutions concentrating at the whole boundary was proved: the phenomenon here is rather different, since the latter family has a diverging Morse index when the singular perturbation parameter tends to zero. Initially, accurate approximate solutions were constructed (depending on the second fundamental form of the domain), and then the invertibility of the linearized equation was shown (primarily using Fourier analysis), which made it possible to prove existence using local inversion arguments.

G. Tarantello's course focused on self-dual vortices in Chern-Simons theory. The physical phenomenon of superconductivity is described by a system of coupled gauge-field equations whose components stand for the wave function and the electromagnetic potential. A relatively well understood model is the abelian-Higgs (corresponding in a non-relativistic context to the Ginzburg-Landau) variant, for which much has been accomplished even away from the self-dual regime. A more sophisticated alternative is the Chern-Simons model, which compared to the previous one has the advantage of predicting the fact that gauge vortices carry electrical charge in addition to magnetic flux, although its mathematical description is at the moment less complete. After describing the main features of these models, Tarantello presented the approach of Taubes to the selfdual regime for the abelian-Higgs case, which reduces the system to a semilinear elliptic problem with exponential nonlinearities. This method partially extends to the Chern-Simons case, where some natural requirements on solutions can be proved, like their asymptotic behavior at infinity, integrability properties and the decay of their derivatives. The structure of C-S vortices is however more rich (and, as we remarked, far from being completely characterized) in comparison to the abelian-Higgs ones: in addition to the *topological solutions*, which have a well defined winding number at infinity, there are also non-topological solutions, which display different asymptotic behavior.

X.J. Wang treated k-Hessian equations, a class of fully nonlinear equations related to the problem of prescribing the Gauss curvature of a hypersurface, and to the Monge-Ampére equation, which is of interest in complex geometry. First the class of *admissible functions* was defined, where the equations are elliptic, and then existence for Dirichlet boundary value problems was obtained by means of a priori estimates and a continuation argument. Next, interior gradient estimates were derived, which imply Harnack inequalities, plus Sobolev-type inequalities for admissible functions which vanish on the boundary of the domain: the embedding which follows from the latter inequality possesses compactness properties analogous to the classical ones, and makes it possible to derive L^∞ estimates for solutions of equations with sufficiently integrable right-hand sides. These estimates make it possible to treat equations with nonlinear (subcritical or critical) reaction terms, where min-max methods can be applied. After this, the notion of k-admissibility was extended to non smooth functions using the concept of hessian measure, and applied to the existence of weak solutions and to potential-theoretical results. Finally, parabolic equations and several examples were treated.

The course by P. Yang concerned minimal surfaces in three-dimensional CR manifolds, which possess a subriemannian structure modeled on the Heisenberg group $\mathbb{H}^1$. In this setting the volume form of M is naturally defined in terms of the contact form θ as $\theta \wedge d\theta$: the *p-area* and the *p-mean curvature* (p standing for *pseudo*) of a two-dimensional surface were defined looking at the first and second variation of the volume form. p-minimal (regular) graphs in $\mathbb{H}^1$ were then considered, showing that they are ruled surfaces. A study of the *singular set* (where the tangent plane to the surface coincides with the contact plane, the kernel of θ) was then performed, and it was shown that it consists either of isolated points or of smooth curves: applications were given to the classification of entire p-minimal graphs. Existence of weak solutions (minimizers) to boundary value problems (of Plateau type) was considered, showing the uniqueness, comparison principles and geometric properties of the solutions. The regularity issue was then discussed, which is a rather delicate one since the global C^2 regularity of minimizers might fail in general.

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Alice S.Y. Chang
Antonio Ambrosetti
Andrea Malchiodi

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