Matrix Valued Brownian Motion and a Paper by Pólya

Philippe Biane

CNRS, Laboratoire d'Informatique Institut Gaspard Monge, Université Paris-Est
5 bd Descartes, Champs-sur-Marne, 77454 Marne-la-Vallée cedex 2, France
e-mail: Philippe.Biane@univ-mlv.fr

1 Introduction

This paper has two parts which are largely independent. In the first one I recall some known facts on matrix valued Brownian motion, which are not so easily found in this form in the literature. I will study three types of matrices, namely Hermitian matrices, complex invertible matrices, and unitary matrices, and try to give a precise description of the motion of eigenvalues (or singular values) in each case. In the second part, I give a new look at an old paper of G. Pólya [14], where he introduces a function close to Riemann's ξ function, and shows that it satisfies Riemann's hypothesis. As put by Marc Kac in his comments on Pólya's paper [11], "Although this beautiful paper takes you within a hair's breadth of Riemann's hypothesis it does not seem to have inspired much further work and reference to it in the mathematical litterature are rather scant". My aim is to point out that the function considered by Pólya is related in a more subtle way to Riemann's ξ function than it looks at first sight. Furthermore the nature of this relation is probabilistic, since these functions have a natural interpretation involving Mellin transforms of first passage times for diffusions. By studying infinite divisibility properties of the distributions of these first passage times, we will see that they are generalized gamma convolutions, whose mixing measures are related to the considerations in the first part of this note.

2 Matrix Brownian Motions

We will study three types of matrix spaces, and in each of these spaces consider a natural Brownian motion, and show that the motion of eigenvalues (or singular values) of this Brownian motion has a simple geometric description, using Doob's transform. The following results admit analogues in more general complex symmetric spaces, but for the sake of simplicity, discussion will be restricted to type A symmetric spaces. Actually the interesting case for us

C. Donati-Martin et al. (eds.), *Séminaire de Probabilités XLII*,
Lecture Notes in Mathematics 1979, DOI 10.1007/978-3-642-01763-6_7,

in the second part will be the simplest one, of rank one, but I think that this almost trivial case is better understood by putting it in the more general context. Some references for results in this section are [2], [4], [5], [7], [8], [9], [10], [13], [15], [16].

2.1 Hermitian Matrices

Consider the space of $n \times n$ Hermitian matrices, with zero trace, endowed with the quadratic form

$$\langle A, B \rangle = Tr(AB).$$

Let $M(t)$ be a Brownian motion with values in this space, which is simply a Gaussian process with covariance

$$E[Tr(AM(t))Tr(BM(s))] = Tr(AB) s \wedge t$$

for A, B traceless Hermitian matrices.

Let $\lambda_1(t) \geq \lambda_2(t) \geq \ldots \geq \lambda_n(t)$ be the eigenvalues of $M(t)$; they perform a stochastic process with values in the Weyl chamber

$$\mathcal{C} = \{(x_1, \ldots, x_n) \in \mathbf{R}^n \,|\, x_1 \geq x_2 \geq \ldots \geq x_n\} \cap H_n$$

where

$$H_n = \Big\{ (x_1, \ldots, x_n) \in \mathbf{R}^n \;\Big|\; \sum_{i=1}^n x_i = 0 \Big\}.$$

Let p_t^0 be the transition probability semi-group of Brownian motion killed at the boundary of the cone $\mathcal{C}$. This cone is a fundamental domain for the action of the symmetric group S_n, which acts by permutation of coordinates on $\mathbf{R}^n$. Using the reflexion principle, one shows easily that

$$p_t^0(x, y) = \sum_{\sigma \in S_n} \epsilon(\sigma) p_t(x, \sigma(y)) \qquad x, y \in \mathcal{C}$$

where $p_t(x, y) = (2\pi t)^{-(n-1)/2} e^{-|x-y|^2/2t}$ and $\epsilon(\sigma)$ is the signature of σ. Let h be the function

$$h(x) = \prod_{i>j} (x_i - x_j).$$

Proposition 1. *The function h is the unique (up to a positive multiplicative constant) positive harmonic function for the semigroup p_t^0, on the cone $\mathcal{C}$, which vanishes on the boundary.*

The harmonic function h corresponds to the unique point at infinity in the Martin compactification of $\mathcal{C}$. Consider now the Doob's transform of p_t^0, which is the semigroup given by

$$q_t(x,y) = \frac{h(y)}{h(x)} p_t^0(x,y).$$

It is a diffusion semigroup on $\mathcal{C}$ with infinitesimal generator

$$\frac{1}{2}\Delta + \langle \nabla \log h, \nabla \cdot \rangle.$$

Proposition 2. *The eigenvalue process of a traceless Hermitian Brownian motion is a Markov diffusion process in the cone $\mathcal{C}$, with semigroup q_t.*

We can summarize the last proposition by saying that the eigenvalue process is a Brownian motion in $\mathcal{C}$, conditioned (in Doob's sense) to exit the cone at infinity.

2.2 The Group $SL_n(\mathbf{C})$

This is the group of complex invertible matrices of size $n \times n$, with determinant 1. Its Lie algebra is the space $\mathfrak{sl}_n(\mathbf{C})$ of complex traceless matrices. Consider the Hermitian form

$$\langle A, B \rangle = Tr(AB^*)$$

on $\mathfrak{sl}_n(\mathbf{C})$ which is invariant by left and right action of the unitary subgroup $SU(n)$. This Hermitian form determines a unique Brownian motion with values in $\mathfrak{sl}_n(\mathbf{C})$. The Brownian motion g_t, on $SL_n(\mathbf{C})$, is the stochastic exponential of this Brownian motion, solution to the Stratonovich stochastic differential equation

$$dg_t = g_t dw_t$$

where w_t is a Brownian motion in $\mathfrak{sl}_n(\mathbf{C})$.

There are two remarkable decompositions of $SL_n(\mathbf{C})$, the Iwasawa and Cartan decompositions. The first one is $SL_n(\mathbf{C}) = NAK$ where K is the compact group $SU(n)$, A is the group of diagonal matrices with positive coefficients, and determinant one, and N is the nilpotent group of upper triangular matrices with 1's on the diagonal. Each matrix of $SL_n(\mathbf{C})$ has a unique decomposition as a product $g = nak$ of elements of the three subgroups N, A, K. This can be easily inferred from the Gram-Schmidt orthogonalization process. If g_t is a Brownian motion in $SL_n(\mathbf{C})$, one can consider its components n_t, a_t, k_t. In particular, denoting by $(e^{w_1(t)}, \dots, e^{w_n(t)})$ the diagonal components of a_t the following holds (cf [15]).

Proposition 3. *The process $\big(w_1(t), \dots, w_n(t)\big)$ is a Brownian motion with a drift $\rho = (-n+1, -n+3, \dots, n-1)$ in the subspace H_n.*

The other decomposition is the Cartan decomposition $SL_n(\mathbf{C}) = KA^+K$, where A^+ is the part of A consisting of matrices with positive nonincreasing coefficients along the diagonal. In order to get the Cartan decomposition of a matrix $g \in SL_n(\mathbf{C})$, take its polar decomposition $g = ru$ with r positive Hermitian, and u unitary, then diagonalize r which yields $g = vav'$ with v and v' unitary and a diagonal, with positive real coefficients which can be put in nonincreasing order along the diagonal. These coefficients are the singular values of the matrix g. This decomposition is not unique since the diagonal subgroup of $SU(n)$ commutes with A, but the singular values are uniquely defined. Call $(e^{a_1(t)}, \dots, e^{a_n(t)})$ the singular values of the Brownian motion g_t, with $a_1 \geq a_2 \geq \dots \geq a_n$. They form a process with values in the cone $\mathcal{C}$. Let us mention that this stochastic process can also be interpreted as the radial part of a Brownian motion with values in the symmetric space $SL_n(\mathbf{C})/SU(n)$. We will now give for the motion of singular values a similar description as the one of eigenvalues of the Hermitian Brownian motion. For this, consider a Brownian motion in H_n, with drift ρ, killed at the exit of the cone $\mathcal{C}$. This process has a semigroup given by

$$p_t^{0,\rho}(x,y) = e^{\langle \rho, y-x\rangle - t\langle \rho,\rho\rangle/2} p_t^0(x,y).$$

Proposition 4. *The function*

$$h^\rho(y) = \prod_{i>j} (1 - e^{2(y_j - y_i)})$$

is a positive harmonic function for the semigroup $p_t^{0,\rho}$*, in the cone* $\mathcal{C}$*, and vanishes at the boundary of the cone.*

It is not true that this function is the unique positive harmonic function on the cone; indeed the Martin boundary at infinity is now much larger and contains a point for each direction inside the cone, see [8]. The Doob-transformed semigroup

$$q_t^\rho(x,y) = \frac{h^\rho(y)}{h^\rho(x)} p_t^{0,\rho}(x,y)$$

is a Markov diffusion semigroup in the cone $\mathcal{C}$, with infinitesimal generator

$$\frac{1}{2}\Delta + \langle \rho, \cdot\rangle + \langle \nabla \log h^\rho, \nabla \cdot\rangle.$$

Note that it can also be expressed as

$$\frac{1}{2}\Delta + \langle \nabla \log \tilde{h}^\rho, \nabla \cdot\rangle$$

with

$$\tilde{h}^\rho(y) = \prod_{i>j} \sinh(y_j - y_i)$$

(see[10]).

Proposition 5. *The logarithms of the singular values of a Brownian motion in* $SL_n(\mathbf{C})$ *perform a diffusion process in the cone* $\mathcal{C}$*, with semigroup* q_t^ρ*.*

As in the preceding case, we can summarize by saying that the process of singular values is a Brownian motion with drift ρ in the cone $\mathcal{C}$, conditioned (in Doob's sense) to exit the cone at infinity, in the direction ρ.

2.3 Unitary Matrices

The Brownian motion with values in $SU(n)$ is obtained by taking the stochastic exponential of a Brownian motion in the Lie algebra of traceless anti-Hermitian matrices, endowed with the Hermitian form

$$\langle A, B\rangle = -Tr(AB).$$

Let $e^{i\theta_1}, \ldots, e^{i\theta_n}$ be the eigenvalues of a matrix in $SU(n)$, which can be chosen so that $\sum_i \theta_i = 0$, and $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_n$, $\theta_1 - \theta_n \leq 2\pi$. These conditions determine a simplex Δ_n in H_n, which is a fundamental domain for the action of the affine Weyl group on H_n. Recall that the affine Weyl group $\tilde{W}$ is the semidirect product of the symmetric group S_n, which acts by permutation of coordinates in H_n, and of the group of translations by elements of the lattice $(2\pi\mathbf{Z})^n \cap H_n$.

One can use the reflexion principle again to compute the semigroup of Brownian moton in this simplex killed at the boundary. One gets an alternating sum over the elements of $\tilde{W}$,

$$p_t^0(\theta, \xi) = \sum_{w\in\tilde{W}} \epsilon(w) p_t(\theta, w(\xi)).$$

The infinitesimal generator is $1/2 \times$ the Laplacian in the simplex, with Dirichlet boundary conditions. It is well known that this operator has a compact resolvent, and its eigenvalue with smallest module is simple, with an eigenfunction which can be chosen positive. Consider the function

$$h^u(\theta) = \prod_{j>k} (e^{i\theta_j} - e^{i\theta_k}).$$

Proposition 6. *The function h^u is positive inside the simplex Δ_n, it vanishes on the boundary, and it is the eigenfunction corresponding to the Dirichlet eigenvalue with smallest module on Δ_n. This eigenvalue is $\lambda = (n - n^3)/6$.*

The Doob-transformed semigroup

$$q_t^u(x, y) = \frac{h^u(y)}{h^u(x)} e^{-\lambda t} p_t^0(x, y)$$

is a Markov diffusion semigroup in Δ_n, with infinitesimal generator

$$\frac{1}{2}\Delta + \langle \nabla \log h^u, \nabla \cdot\rangle - \lambda.$$

Proposition 7. *The process of eigenvalues of a unitary Brownian motion is a diffusion with values in Δ_n with probability transition semigroup q_t^u.*

Again a good summary of this situation is that the motion of eigenvalues is that of a Brownian motion in the simplex Δ_n conditioned to stay forever in this simplex.

2.4 The Case of Rank 1

In the next section we will need the simplest case, that of 2×2 matrices. Consider first the case of Hermitian matrices. The process of eigenvalues is essentially a Bessel process of dimension 3, with infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} \qquad \text{on} \quad]0, +\infty[,$$

obtained from Brownian motion killed at zero, of infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2}$$

with Dirichlet boundary condition at 0, by a Doob transform with the positive harmonic function $h(x) = x$.

In the case of the group $SL_2(\mathbf{C})$, or the symmetric space $SL_2(\mathbf{C})/SU(2)$, which is the hyperbolic space of dimension 3, the radial process has infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \coth x \frac{d}{dx}$$

obtained from Brownian motion with a drift

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{d}{dx}$$

with Dirichlet boundary condition at 0, by a Doob's transform with the function $1 - e^{-2x}$.

Finally the last case is Brownian motion in $SU(2)$, where the eigenvalue process takes values in $[0, \pi]$ and has an infinitesimal generator

$$\frac{1}{2}\frac{d^2}{dx^2} + \cot x \frac{d}{dx}$$

obtained by a Doob transform at the bottom of the spectrum from

$$\frac{1}{2}\frac{d^2}{dx^2}$$

on $[0, \pi]$ with Dirichlet boundary conditions at 0 and π, by the function $\sin(x)$.

For these last two examples, we shall write a spectral decomposition of the generator $L_i, i = 1, 2$, of the form

$$f(x) = \int \Phi^i_\lambda(x)\Big[\int \Phi^i_\lambda(y) f(y)\, dm_i(y)\Big]\, d\nu_i(\lambda) \qquad i = 1, 2 \tag{1}$$

for every $f \in L^2(m_i)$, where m_i is measure for which L_i is selfadjoint in $L^2(m_i)$, and the functions Φ^i_λ are solutions to

$$L_i \Phi^i_\lambda + \lambda \Phi^i_\lambda = 0$$

and ν_i is a spectral measure for L_i.

For $L_1 = \frac{1}{2}\frac{d^2}{dx^2}$ on $[0, \pi]$ with Dirichlet boundary conditions, $m_1(dx)$ is Lebesgue measure on $[0, \pi]$, and L_1 is selfadjoint on $L^2([0, \pi)]$. Furthermore

$$\Phi^1_\lambda(x) = \sin(\sqrt{2\lambda} x)$$

and

$$\nu_1(d\lambda) = \frac{1}{\pi} \sum_{n=1}^{\infty} \delta_{n^2/2}(d\lambda). \tag{2}$$

For $L_2 = \frac{1}{2}\frac{d^2}{dx^2} + \frac{d}{dx}$ on $[0, +\infty[$, the measure $m_2(dx) = e^{2x} dx$, and

$$\Phi^2_\lambda(x) = e^{-x} \sin(\sqrt{2\lambda - 1} x) \qquad \lambda > 1/2.$$

The spectral measure is

$$\nu_2(d\lambda) = \frac{1}{\pi\sqrt{2\lambda - 1}} d\lambda \qquad \lambda > 1/2 \tag{3}$$

on $[1, +\infty[$. Of course formulas (1), (2), (3) are immediate consequences of ordinary Fourier analysis.

Note that the spectral decompositions, and in particular the measures ν_i, depend on the normalisation of the fonctions Φ_λ. We have made a natural choice, but it does not coincide with the usual normalisation of Weyl-Titchmarsh-Kodaira theory, see [6].

3 MacDonald's Function and Riemann's ξ Function

3.1 Pólya's Paper

In his paper [14], Pólya starts from Riemann's ξ function

$$\xi(s) = s(s - 1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

where ζ is Riemann's zeta function. Then ξ is an entire function whose zeros are exactly the nontrivial zeros of ζ.

Putting $s = 1/2 + iz$ yields

$$\xi(z) = 2\int_0^\infty \Phi(u) \cos(zu) du$$

with

$$\Phi(u) = 2\pi e^{5u/2} \sum_{n=1}^{\infty} (2\pi e^{2u} n^2 - 3) n^2 e^{-\pi n^2 e^{2u}} \tag{4}$$

and the function Φ is even, as follows from the functional equation for Jacobi θ function; furthermore

$$\Phi(u) \sim 4\pi^2 e^{9u/2 - \pi e^{2u}} \qquad u \to +\infty$$

so that

$$\Phi(u) \sim 4\pi^2 (e^{9u/2} + e^{-9u/2}) e^{-\pi(e^{2u} + e^{-2u})} \qquad u \to \pm\infty.$$

This lead Pólya to define a "falsified" ξ function

$$\xi^*(z) = 8\pi^2 \int_0^{\infty} (e^{9u/2} + e^{-9u/2}) e^{-\pi(e^{2u} + e^{-2u})} \cos(zu) du.$$

The main result of [14] is

Theorem 1. *The function ξ^* is entire, its zeros are real and simple. Let $N(r)$, (resp. $N^*(r)$) denote the number of zeros of $\xi(z)$ (resp. $\xi^*(z)$) with real part in the interval $[0, r]$, then $N(r) - N^*(r) = O(\log r)$.*

Recall that the same assertion about the zeros of the function ξ (without the statement about simplicity, beware also that $s = 1/2 + iz$) is Riemann's hypothesis. Recall also the well known estimate

$$N(r) = \frac{r}{2\pi} \log(r/2\pi) - \frac{r}{2\pi} + O(1).$$

Pólya's results rely on the intermediate study of the function

$$\mathfrak{G}(z, a) = \int_{-\infty}^{\infty} e^{-a(e^u + e^{-u}) + zu} du$$

from which ξ^* is obtained by

$$\xi^*(z) = 2\pi^2 (\mathfrak{G}(iz/2 - 9/4, \pi) + \mathfrak{G}(iz/2 + 9/4, \pi))$$

Pólya shows that $\mathfrak{G}(z, a)$ has only purely imaginary zeros, (as a function of z) and the number of these zeros with imaginary part in $[0, r]$ grows as $\frac{r}{\pi} \log \frac{r}{a} - \frac{r}{\pi} + O(1)$. The results on ξ^* are then deduced through a nice lemma which played a role in the history of statistical mechanics (the Lee-Yang theorem on Ising model), as revealed by M. Kac [11]. We shall now concentrate on $\mathfrak{G}(z, a)$. In particular, for $a = \pi$, the function $\tilde{\xi}(z) = \mathfrak{G}(iz/2, \pi)$ is another approximation of ξ which has many interesting structural properties.

3.2 MacDonald Functions

The function denoted $\mathfrak{G}(z,a)$ by Pólya is actually a Bessel function. Indeed, MacDonald's function, also called modified Bessel function (see e.g. [1]), given by

$$K_\mu(x) = \int_0^\infty t^{\mu-1} e^{-\frac{x}{2}(t+t^{-1})} dt \qquad x > 0,\ \mu \in \mathbf{C}.$$

satisfies $K_z(2x) = \mathfrak{G}(z,x)$. The function $\mathfrak{G}(z,a)$ is therefore essentially a MacDonald function, as noted by Pólya. MacDonald function is an even function of μ and satisfies

$$\frac{2\mu}{x} K_\mu(x) = K_{\mu+1}(x) - K_{\mu-1}(x) \tag{5}$$

$$-2\frac{d}{dx} K_\mu(x) = K_{\mu+1}(x) + K_{\mu-1}(x) \tag{6}$$

The first of these equations is used by Pólya in a very clever way to prove that the zeros (in z) of $\mathfrak{G}(z,x)$ are purely imaginary.

3.3 Spectral Interpretation of the Zeros

From (5), (6)

$$(\tfrac{\mu}{x} - \tfrac{d}{dx})K_\mu = K_{\mu+1}$$

$$(-\tfrac{\mu}{x} - \tfrac{d}{dx})K_\mu = K_{\mu-1}$$

from which one gets

$$\begin{aligned} K_\mu &= (-\tfrac{\mu+1}{x} - \tfrac{d}{dx})(\tfrac{\mu}{x} - \tfrac{d}{dx})K_\mu \\ &= (\tfrac{d^2}{dx^2} + \tfrac{1}{x}\tfrac{d}{dx} - \tfrac{\mu^2}{x^2})K_\mu\ . \end{aligned}$$

This differential equation will give us a spectral interpretation of the zeros of $\mathfrak{G}(z,x)$. Change variable by $\psi_\mu(x) = K_\mu(e^x)$ to get

$$(-\frac{d^2}{dx^2} + e^{2x})\psi_\mu = -\mu^2\psi_\mu \tag{7}$$

Since K_μ vanishes exponentially at infinity, the spectral theory of Sturm-Liouville operators on the half-line (see e.g. [6], [12]) implies that the squares of the zeros of $\mu \mapsto \psi_\mu(y)$ are the eigenvalues of $\frac{d^2}{dx^2} - e^{2x}$ on the interval $[y, +\infty[$ with the Dirichlet boundary condition at y, the functions ψ_μ being the eigenfunctions. Since this operator is selfadjoint and negative the zeros are purely imaginary, and are simple.

This spectral interpretation of the zeros of MacDonald function is well known [17], I do not know why Pólya does not mention it.

3.4 $H = xp$

Equation (7) can be put into Dirac's form, indeed the equations

$$\left(\tfrac{d}{dx} + \tfrac{1}{2} + e^x\right) f = \gamma g$$

$$\left(-\tfrac{d}{dx} + \tfrac{1}{2} + e^x\right) g = \gamma f$$

imply

$$\left(-\frac{d^2}{dx^2} + e^{2x}\right) f = (\gamma^2 - \frac{1}{4}) f.$$

Using the change of variables $u = e^x$, we get

$$\left(u\tfrac{d}{du} + \tfrac{1}{2} + u\right) f = \gamma g$$

$$\left(-u\tfrac{d}{du} + \tfrac{1}{2} + u\right) g = \gamma f.$$

Remark that this Dirac system yields a perturbation of the Hamiltonian $H = xp$ considered by Berry et Keating [3], in relation with Riemann's zeta function.

3.5 Asymptotics of the Zeros

General results on Sturm-Liouville operators allow one to recover the asymptotic behaviour of the spectrum, thanks to a semiclassical analysis, see e.g. [12]. One can get a more precise result using the integral representation of $K_{i\mu}$. Pólya gives the asymptotic estimate

$$K_{x+iy}(2a) = \frac{1}{\sqrt{2\pi y}} e^{-\frac{\pi}{2}y + i\frac{\pi}{2}x} \left[\left(\frac{y}{a}\right)^x e^{i\Phi} + \left(\frac{y}{a}\right)^{-x} e^{-i\Phi} \right] + O(e^{-\frac{\pi}{2}y} y^{|x|-3/2})$$

in the strip $|x| \leq 1$ uniformly as $y \to \infty$, where

$$\Phi = y \log \frac{y}{a} - y - \frac{\pi}{4} \;.$$

This estimate can be obtained by the stationary phase method, writing

$$K_z(2a) = \int_{-\infty}^{\infty} e^{zt - 2a\cosh(t)} dt.$$

Making a contour deformation we get

$$\begin{aligned} K_z(2a) = {} & \textstyle\int_{-\infty}^{-A} e^{zt-2a\cosh(t)} dt + i \int_0^{\pi/2} e^{z(-A+it)-2a\cosh(-A+it)} dt \\ & + \textstyle\int_{-A}^{A} e^{z(t+i\frac{\pi}{2})-2ai\sinh(t)} dt - i \int_0^{\pi/2} e^{z(A-it+i\frac{\pi}{2})-2a\cosh(A-it+i\frac{\pi}{2})} dt \\ & + \textstyle\int_A^{\infty} e^{zt-2a\cosh(t)} dt \end{aligned}$$

and Pólya's estimate can be obtained by standard methods, which give also estimates for the derivatives of MacDonald's function. Finally the zeros of $y \to K_{iy}(2a)$ behave like the solutions to

$$y \log \frac{y}{a} - y - \frac{\pi}{4} = (n + \frac{1}{2})\pi \qquad n \text{ integer}$$

The number of zeros with imaginary part in $[0, T]$ is thus $\frac{T}{\pi} \log \frac{T}{a} - \frac{T}{\pi} + O(1)$.

4 Probabilistic Interpretations

We will now give interpretations of the functions ξ and $\tilde{\xi}$ using first passage times of diffusions.

4.1 Brownian Motion with a Drift

The first passage time at $x > 0$ of Brownian motion started at 0 follows a 1/2 stable distribution i.e.,

$$P(T_x \in dt) = x \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t^3}} dt$$

with Laplace transform

$$E[e^{-\lambda^2 T_x/2}] = e^{-\lambda x} \ .$$

Adding a drift $a > 0$ to the Brownian motion gives a first passage distribution

$$P^a(T_x \in dt) = x \frac{e^{-\frac{x^2}{2t}}}{\sqrt{2\pi t^3}} e^{ax - \frac{a^2 t}{2}} dt$$

with Laplace transform

$$E^a[e^{-\lambda^2 T_x/2}] = e^{-x\sqrt{\lambda^2 + a^2} + ax}.$$

This is a generalized inverse Gaussian distribution. In particular, its Mellin transform is

$$E^a[T_x^s] = (x/a)^s \frac{K_{-1/2+s}(ax)}{K_{-1/2}(ax)} = (x/a)^s \sqrt{\pi/ax}\, e^{ax} K_{-1/2+s}(ax)$$

which gives a probabilistic interpretation of MacDonald's function (as a function of s) as a Mellin transform of a probability distribution.

4.2 Three Dimensional Bessel Process

There exists a similar interpretation of the ξ function, which is discussed in details in [4], [5], for example, Consider the first passage time at $a > 0$ of a three dimensional Bessel process (i.e., the norm of a three dimensional Brownian motion) starting from 0. The Laplace transform of this hitting time is

$$E[e^{-\frac{\lambda^2}{2}S_a}] = \frac{\lambda a}{\sinh \lambda a} .$$

Let S'_a be an independent copy of S_a, and let

$$W_a = S_a + S'_a ;$$

then the density of the distribution of W_a is obtained by inverting the Laplace transform. One gets

$$P(W_a \in dx) = \sum_{n=1}^{\infty} (\pi^4 n^4 x/a^4 - 3\pi^2 n^2/a^2) e^{-\pi^2 n^2 x/2a^2} dx$$

from which one can compute the Mellin transform

$$E[W_a^s] = 2(2a^2/\pi)^s \xi(2s).$$

The function 2ξ thus has a probabilistic interpretation, as Mellin transform of $\sqrt{\frac{\pi}{2}W_1}$.

4.3 Infinite Divisibility

The distributions of T_x and W_a are infinitely divisible. Indeed

$$\begin{aligned}\log E^a[\exp(-\tfrac{\lambda^2}{2}T_x)] &= -x\sqrt{\lambda^2 + a^2} + ax \\ &= x\int_0^\infty (e^{-\frac{\lambda^2}{2}t} - 1)\frac{e^{-\frac{a^2}{2}t}}{\sqrt{2\pi t^3}}dt\end{aligned}$$

which shows that T_x is a subordinator with Lévy measure

$$\frac{e^{-\frac{a^2}{2}t}}{\sqrt{2\pi t^3}}dt.$$

Similarly

$$\begin{aligned}\log E[\exp(-\tfrac{\lambda^2}{2}W_a)] &= 2\log(\lambda a/\sinh(\lambda a)) \\ &= 2\int_0^\infty (e^{-\frac{\lambda^2}{2}t} - 1)\sum_{n=1}^\infty e^{-\pi^2 n^2 t/a^2} dt\end{aligned}$$

therefore the variable W_a has the distribution of a subordinator, with Lévy measure

$$2\sum_{n=1}^{\infty} e^{-\pi^2 n^2 t/a^2} dt,$$

taken at time 1. Observe however that the process $(W_a)_{a\geq 0}$ is not a subordinator.

4.4 Generalized Gamma Convolution

The gamma distributions are

$$P(\gamma_{\omega,c} \in dt) = \frac{c^{-\omega}}{\Gamma(\omega)} t^{\omega-1} e^{-t/c} dt = \Gamma_{\omega,c}(dt)$$

where ω and c are > 0 parameters. The Laplace transform is

$$E[e^{-\lambda\gamma_{\omega,c}}] = (1+\lambda/c)^{-\omega}.$$

The gamma distributions form a convolution semigroup with respect to the parameter ω, i.e.,

$$\Gamma_{\omega_1,c} * \Gamma_{\omega_2,c} = \Gamma_{\omega_1+\omega_2,c} .$$

The Lévy exponent of the gamma semigroup is

$$\psi_c(\lambda) = \log(1+\lambda/c) = \int_0^\infty (1-e^{-\lambda t}) \frac{e^{-ct}}{t} dt$$

so that this is the semigroup of a subordinator with Lévy measure $e^{-ct}/t\,dt$.

The generalized gamma convolutions are the distributions of linear combinations, with positive coefficients, of independent gamma variables, and their weak limits.

One can also characterize the generalized gamma convolutions as the infinitely divisible distributions with a Lévy exponent of the form

$$\psi(\lambda) = \int_0^\infty \psi_c(\lambda) d\nu(c)$$

for some positive measure ν which integrates $1/c$ at ∞. This measure is called the Thorin measure of the generalized gamma distribution. The variables T_x and W_a of the preceding paragraph are generalized gamma convolutions. Indeed it is easy to check, using the computations of section 4.3, that W_a has a generalized gamma convolution as distribution, with Thorin measure

$$\nu(dc) = 2\sum_{n=1}^{\infty} \delta_{n^2/a^2}(dc) \; ; \tag{8}$$

whereas T_x is distributed as a generalized gamma convolution with Thorin measure

$$\nu(dc) = \frac{dc}{\pi\sqrt{c-a^2/2}} \qquad c > a^2/2 \tag{9}$$

since

$$\frac{e^{-a^2t/2}}{\sqrt{\pi t^3}} = \int_{a^2/2}^\infty e^{-ct} \frac{dc}{\pi\sqrt{c-a^2/2}}.$$

4.5 Final Remarks

We can now make a connection between the preceding considerations and those of the first part of the paper. Indeed, the Thorin measures associated with the variables T_x and W_a can be expressed as spectral measures associated with the generators of Brownian motion on matrix spaces. The hitting times of Brownian motion with drift are related with the radial part of Brownian motion in the symmetric space $SL_2(\mathbf{C})/SU(2)$, whereas the hitting times of the Bessel three process are related with the Brownian motion on the unitary group $SU(2)$. The precise relations are contained in formulas (2), (3), (8), (9). Thus the Riemann ξ function, which is the Mellin transform of a hitting time of the Bessel three process, as in section 4.2, and the Polya $\tilde{\xi}$ function from section 3.1, which appears as Mellin transform of hitting time of Brownian motion with drift, are related in this non obvious way.

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Séminaire de Probabilités XLII
Donati-Martin, C.; Émery, M.; Rouault, A.; Stricker, C. (Eds.)
2009, XIII, 449 p., Softcover
ISBN: 978-3-642-01762-9