Chapter 2
Simple Sharing Problems

2.1 Introduction

Simple sharing problems involve a group of n agents indexed $j = 1, \ldots, n$. The term "agent" is here broadly interpreted as persons, firms, departments, branches, products, etc. Each agent is characterized by some one-dimensional factor $q_j \in \mathbf{R}_+$ such as demand, claim, "stand-alone" cost, effort, surplus, etc.

Basically, we shall distinguish between two types of situations: First, the case where agents characteristics $q = (q_1, \ldots, q_n)$ do not influence the costs (or value) which has to be shared, say, a fixed amount E. Second, the case where agents characteristics $q = (q_1, \ldots, q_n)$ influence the amount of costs (or value). To be more specific, we shall assume that q is a demand vector and that costs are given either as the costs associated with the total demand $Q = q_1 + \ldots + q_n$ or as the costs associated with the highest demand $\mathcal{Q} = \max_i\{q_i\}$. In both cases ($z = Q$ and $z = \mathcal{Q}$), cost will be modeled by a (one-dimensional) non-decreasing cost function $C : \mathbf{R}_+ \to \mathbf{R}_+$ (or the value modeled by a one-dimensional non-decreasing value function $V : \mathbf{R}_+ \to \mathbf{R}_+$).

The following scenarios may be imagined:

1. A firm goes bankrupt with liquidation value E. "Agents" may here represent n creditors each characterized by their verifiable claim q_j. Hence, characteristics are *not* linked directly to the size of E. If total debt exceeds the liquidation value the problem may equivalently be construed as that of sharing a loss between n claimants. Such bankruptcy problems are treated in Sect. 2.2, but given a fixed-price setting, any excess demand will cause similar problems to arise and they will generally be referred to as rationing problems.
2. A community of n households wants to be connected to a local power plant and has to share the costs of establishing a connection. Assume, for example, that the households are located as a chain and are characterized by their individual distance q_j from the plant. Clearly, characteristics influence the total costs in this situation: as it is economically rational to link all households to the same line rather than establishing separate

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connections for each household the related cost function will be of the (decomposable) form $C(\max_j\{q_j\})$, since total costs will be determined by the most distant household. Alternatively, a decomposable cost function $C(\max_j\{q_j\})$ may be used to represent situations where a group of agents demand some amount of an excludable good which is consumed without rivalry (an excludable public good) and the total cost therefore is determined by the agent with the highest demand. Cost sharing rules that apply in such cases will be examined in Sect. 2.3.

3. Consider a firm where the total costs of operating a service department has to be allocated among n user departments or n products. That is, “agents” may refer to departments or products of the firm. If user departments are considered the characteristics q_j may simply be the amount of service demanded by department j which is, of course, directly related to the size of the total costs. If service can be considered as a homogeneous good, costs are generated by a (homogeneous) cost function $C(Q)$, where $Q = q_1 + \ldots + q_n$ is total demand. That is, $C(Q)$ has to be shared among the n user departments. Cost sharing rules that apply in such cases will be examined in Sect. 2.3. On the other hand, if index j refers to products the characteristics q_j may be the amount of working hours used to produce one unit of product j which is *not* linked directly to the size of total costs. Thus, total cost is regarded as a fixed amount E that has to be shared between products according to the characteristics vector q. Cost sharing rules that apply in such cases relate to the rules examined in Sect. 2.2.

In the present chapter we shall consider such simple sharing problems in further detail. First, the case of a fixed amount E will be analysed focusing on rationing problems and secondly, the case of a (one-dimensional) cost function will be analysed focussing on cost sharing problems.

2.2 Rationing Problems

As in (1) above, we consider a rationing problem where a given quantity $E \geq 0$ of money (or some other fully divisible good) has to be shared among n agents with non-negative *demands* $q = (q_1, \ldots, q_n)$ measured in monetary units (or in units of some other fully divisible good). Assume that E and q are measured in the same units and that individual demands are objectively determined, for example, as verifiable claims in case of bankruptcy problems. Moreover, assume for convenience that demands are increasingly ordered, i.e., $q_1 \leq \ldots \leq q_n$. Since the problem is that of rationing the *total demand* $Q = q_1 + \ldots + q_n$ exceeds the available quantity E, i.e., $Q \geq E \geq 0$.

Given a *rationing problem* (q, E), a *rationing rule* φ specifies a unique *vector of shares* $x = (x_1, \ldots, x_n) = \varphi(q, E)$ where $x_1 + \ldots + x_n = E$, and $0 \leq x_i \leq q_i$ for all $i = 1, \ldots, n$. The latter condition ensures that no agent gets a negative share or a share larger than what is demanded. Although this appears to be a rather natural constraint it actually excludes several well known sharing rules. Consider, for example the equal split rule where $x_j = E/n$ for all $j = 1, \ldots, n$ and the extreme priority rule where $x_j = E$ and $x_i = 0$ for $i \neq j$.

In the following we shall focus on rationing rules that are *order-preserving* in the sense that an agent with a higher demand than another agent will get both a larger share x_i and loss $q_i - x_i$, i.e.,

$$x_1 \leq \ldots \leq x_n, \quad \text{and} \quad q_1 - x_1 \leq \ldots \leq q_n - x_n, \tag{2.1}$$

and *resource monotonic* in the sense that the share of all agents, as given by $\varphi_i(q, E),\ i = 1, \ldots, n$, is non-decreasing in the quantity $E \in [0, Q]$. Denote by $\mathcal{R}$ the set of order-preserving and resource monotonic rationing rules. Order-preservation and resource monotonicity seem to be natural requirements with respect to rationing rules. Note, for example, that the extreme priority rule violates order-preservation.

As mentioned in (1) above, any rationing problem (q, E) may be construed either directly as the problem of sharing the quantity E given demands q or indirectly as the problem of sharing the loss $Q - E$ given demands q. Thus, for any rationing rule φ there is a *dual rule* φ^* defined by

$$\varphi^*(q, E) = q - \varphi(q, Q - E). \tag{2.2}$$

If $\varphi = \varphi^*$ the rationing rule φ is called *self-dual*, i.e., the resulting shares will be the same whether focus is on gains or losses.

Remark 2.1. Although we use the set-up of rationing it can be noted that in terms of *cost sharing* the model may be given the following interpretation: $C \geq 0$ is a fixed common cost that has to be shared among n agents (communities, institutions, departments, etc.) with non-negative *stand-alone costs* $c = (c_1, \ldots, c_n)$. Cooperation is assumed to be profitable in the sense that $0 \leq C \leq \sum_j c_j$. Given a cost sharing problem (c, C) a cost sharing rule φ specifies a unique vector of cost shares $x = \varphi(c, C)$ where $\sum_j x_j = C$ and $0 \leq x_j \leq c_j$. Here the latter condition appears to be a natural condition of individual rationality since no agent wants to participate in a joint project if it results in a share of the common cost that exceeds the agents stand-alone cost. △

2.2.1 Four Rationing Rules

Some rationing rules are particularly interesting due to their historical origins as well as wide applicability. Four such rules will now be further analysed. As we proceed we shall see that there are several good reasons for focussing on these particular rules.

Basically the rules relate to two different notions of fairness: *proportionality* and different versions of *equality*. The first three rules are:

- *The Proportional Rule* φ^P defined by shares

$$x_i^P = \frac{q_i}{Q}E, \quad i = 1, \ldots, n. \tag{2.3}$$

That is, E is shared in proportion to individual demands.
- *The Constrained Equal Gains Rule* φ^{CEG} defined by shares

$$x_i^{CEG} = \min\{q_i, \alpha\}, \quad i = 1, \ldots, n, \tag{2.4}$$

where α is chosen such that the shares add up to E. That is, E is shared equally provided that no one gets more than their individual demand.
- *The Constrained Equal Loss Rule* φ^{CEL} defined by shares

$$x_i^{CEL} = \max\{0, q_i - \beta\}, \quad i = 1, \ldots, n, \tag{2.5}$$

where β is chosen such that the shares add up to E. That is, the loss $Q-E$ is shared equally provided that no one gets a negative share.

Notice, that $\varphi^{CEG}(q, E) = q - \varphi^{CEL}(q, Q - E)$ implying that φ^{CEG} is the dual rule of φ^{CEL}, i.e., $\varphi^{CEG*} = \varphi^{CEL}$. Moreover, since $\varphi^P(q, E) = q - \varphi^P(q, Q - E)$, the proportional rule is self-dual, i.e., $\varphi^{P*} = \varphi^P$.

The fourth rule (the Talmud rule) is an amalgam of the constrained equality rules. One line of motivation is the following: First, the rule ought to be self-dual as gains and losses should be treated equally. Second, half of the demand can be construed as a psychological watershed for the individual agents. If $x < q/2$ the agent focuses on whatever he can get ("Less than half is like nothing"). If $x > q/2$ the agent is close to fulfilment of the demand and therefore focuses on his loss ("More than half is like the whole"). Fairness now prescribes that all agents ought to be on the same side of the watershed and should be treated equally. Hence, if $0 \leq E \leq Q/2$, focus is on gains, which shall be shared equally provided that no one get shares larger than half their demand and if $Q/2 < E \leq Q$, focus is on losses, which shall be shared equally provided that no one get shares smaller than half their demand – that is:

- *The Talmud Rule* φ^T is defined by shares

$$x_i^T(q, E) = \begin{cases} \min\{q_i/2, \alpha\} & \text{if} \quad 0 \leq E \leq Q/2 \\ \max\{q_i/2, q_i - \beta\} & \text{if} \quad Q/2 < E \leq Q \end{cases} \tag{2.6}$$

where α and β are chosen such that the shares add up to E.

Now, using the definition of the Constrained Equal Gains rule and the Constrained Equal Loss rule we get that the Talmud rule φ^T is defined as

$$\varphi^T(q, E) = \begin{cases} \varphi^{CEG}(q/2, E) & \text{if } 0 \leq E \leq Q/2 \\ q/2 + \varphi^{CEL}(q/2, E - Q/2) & \text{if } Q/2 < E \leq Q, \end{cases}$$

and consequently $\varphi^T(q, E) = q - \varphi^T(q, Q - E)$, i.e., the Talmud rule is self-dual. Further, note that $\varphi^T(q, Q/2) = \varphi^P(q, Q/2)$.

In the particular case of nested 2-agent problems, i.e., problems where $q_2 = E$ (q_2 being the highest demand) we can interpret the Talmud rule as the contested garment principle mentioned in Chap. 1. For example, if $E = 1$ and $q = (1/2, 1)$ we get that $Q/2 < E \leq Q$ and thereby $\varphi^T(q, E) = q/2 + \varphi^{CEL}(1/4, 1/2, 1/4) = (1/4, 3/4)$, i.e., the contested half is shared equally and the uncontested half goes to the agent demanding the entire piece of garment. As such the above definition of the Talmud rule (introduced in Aumann and Maschler 1985) is a modern extension of the contested garment principle to a larger domain of rationing problems.

Now, considering all four rules we are able to state the following well-known proposition.

Proposition 2.1. *The four rationing rules,* $\varphi^P, \varphi^{CEG}, \varphi^{CEL}$ *and* φ^T *are order-preserving and resource monotonic.*

Example 2.1. Consider a bankruptcy problem where $n = 5$ agents with claims $q = (50, 100, 150, 200, 250)$ must share an estate of value $E = 510$. First, we get that proportional sharing results in the vector of shares

$$x^P = (34, 68, 102, 136, 170),$$

whereas constrained equal sharing of gains results in

$$x^{CEG} = (50, 100, 120, 120, 120).$$

Since the total claim is $Q = 750$, the total loss is $750 - 510 = 240$. Thus, constrained equal sharing of the loss results in

$$x^{CEL} = (2, 52, 102, 152, 202),$$

and as $E = 510 \geq 375 = Q/2$ we have that $\varphi^T(q, E) = q/2 + \varphi^{CEL}(q/2, E - Q/2)$ and thereby that shares according to the Talmud rule are given by

$$x^T = (25, 50, 95, 145, 195).$$

Now, let the worth of the estate decrease such that $E = 240 \leq Q/2 = 375$. In this case we get:

$$\begin{aligned} x^P &= (16, 32, 48, 64, 80), \\ x^{CEG} &= (48, 48, 48, 48, 48), \\ x^{CEL} &= (0, 0, 30, 80, 130), \\ x^T &= (25, 50, 55, 55, 55). \end{aligned}$$

By resource monotonicity all agents receive weakly smaller shares than before and clearly order-preservation is confirmed in both cases. Moreover, the results also confirm that φ^P and φ^T are self-dual rules whereas φ^{CEL} is the dual rule of φ^{CEG}. △

Note that among the four rules, only φ^P is well defined if E and q are measured in different units. In fact, in such a case the conditions $0 \leq x_i \leq q_i$ for all i become irrelevant.

Remark 2.2. Young (1987) defines an interesting family of rationing rules comprising the four rules above: Let $f(q, \lambda)$ be a real-valued function of scalar variables q and λ where $q > 0$ and $\lambda \in [a, b] \subseteq [-\infty, +\infty]$. For each x, f is assumed to be weakly monotone increasing and continuous in λ with $f(q, a) = 0$ and $f(q, b) = q$. Given f the rationing rule φ is said to be *parametric with representation* f if, for every problem (q, E) that $x = \varphi(q, E)$ if and only if there exists a λ such that for all i,

$$x_i = f(q_i, \lambda) \quad \text{and} \quad x_1 + \ldots + x_n = E.$$

Note that by the assumptions on f, $0 \leq x_i \leq q_i$. Moreover, note that the proportional rule is given by $x_i = \lambda q_i,\ 0 \leq \lambda \leq 1$, where λ is chosen so that $x_1 + \ldots + x_n = E$; the constrained equal gains rule is given by $x_i = \min\{q_i, \lambda\},\ 0 \leq \lambda \leq \infty$, where λ is chosen so that $x_1 + \ldots + x_n = E$; and the constrained equal loss rule is given by $x_i = \max\{0, q_i - 1/\lambda\},\ 0 \leq \lambda \leq \infty$, where λ is chosen so that $x_1 + \ldots + x_n = E$. △

2.2.2 Inequality Comparisons

From Example 2.1 we see that the rules differ considerably with respect to how they distribute the shares. It seems that φ^{CEG} results in distributions with the smallest spread whereas φ^{CEL} results in shares with the largest spread. In case $E \leq Q/2$, shares given by the proportional rule seems more spread than shares given by the Talmud rule whereas when $E \geq Q/2$ it appears to be the other way around. In fact, such characterizations in terms of economic inequality comparisons can be formalized using the notion of *Lorenz-domination* (also known as majorization).

Formally, for two increasingly ordered n-vectors of real numbers $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, x is said to *Lorenz-dominate* y if:

(1) $x_1 + \ldots + x_k \geq y_1 + \ldots + y_k, \quad k = 1, \ldots, n-1.$
(2) $x_1 + \ldots + x_n = y_1 + \ldots + y_n.$

The partial ordering defined by (1) and (2) is written $x \succ_{LD} y$ and referred to as *Lorenz-domination* (note that $x \succ_{LD} x$ for any x), see, e.g., Marshall and Olkin (1979). In terms of economics, $x \succ_{LD} y$ can be interpreted as x being more equally distributed than y (less spread out).

Now, it turns out that the constrained equal gains rule φ^{CEG} is the unique Lorenz-maximising rationing rule and dually, that the constraint equal loss rule φ^{CEL} is the unique Lorenz-minimising rationing rule. In other words, there is no other rule that results in more equally distributed shares than the constraint equal gains rule and no other rule that results in less equally distributed shares than the constraint equal loss rule.

Theorem 2.1. *For any rationing rule φ and rationing problem (q, E),*

$$\varphi^{CEG}(q, E) \succ_{LD} \varphi(q, E) \succ_{LD} \varphi^{CEL}(q, E).$$

Proof. We argue that φ^{CEG} is the unique maximizer of $\succ_{LD}$ on the set of rationing methods and hence by Proposition 2.1, φ^{CEL} is the unique minimizer. Indeed, consider some arbitrary value E. By definition there exists a λ and a $k \in \{1, \ldots, n\}$ such that $x^{CEG} = (q_1, \ldots, q_k, \lambda, \ldots, \lambda)$. Now, suppose that there is some n-vector y originating from some allocation method where $x^{CEG} \not\succ_{LD} y$. Then there exists some smallest j where $k < j < n$ such that

$$\sum_{i=1}^{k} q_i + (j-k)\lambda = \sum_{i=1}^{j} x_i^{CEG} < \sum_{i=1}^{j} y_i,$$

and hence $y_j > \lambda$. However, since y is increasingly ordered it follows that $\sum_{i=1}^{n} x_i^{CEG} < \sum_{i=1}^{n} y_i = E$, a contradiction. □

In fact, for fixed E all four rules mentioned above are completely ordered by Lorenz-domination since for $0 \leq E \leq Q/2$ (or $Q/2 < E \leq Q$), then $x^T \succ_{LD} x^P$ (or $x^P \succ_{LD} x^T$).

Theorem 2.1 indicates that it is possible to construct a finite sequence of inequality monotone transfers from x^{CEL} to x^{CEG}, in the sense that the vector of shares gradually becomes more and more equal in terms of Lorenz-domination. It is well-known that such transfers are possible (see Marshall and Olkin 1979) but generally there is no upper bound on the number of transfers required. However, as shown in Hougaard and Thorlund-Petersen (2002) one needs at most $n-1$ transfers in order to go from x^{CEL} to x^{CEG} (and thereby x^T) as we shall demonstrate below. To be more specific, consider a sequence of transfers from "rich" to "poor" agents; first the agent with the largest share transfers value to all agents with smaller shares until his share is at the same level as that of the agent with the second largest share. Then the two agents with the highest value of shares transfer value to all agents with smaller shares until their share is at the same level as that of the agent with the third largest share, etc., i.e., $x^{CEL} + T\theta = x^{CEG}$ where T is an $n \times (n-1)$ matrix

$$T = \begin{bmatrix} 1 & 1 & \dots & n-1 \\ 1 & 1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{-(n-2)}{2} & \dots & -1 \\ -(n-1) & \frac{-(n-2)}{2} & \dots & -1 \end{bmatrix},$$

and $\theta = (\theta_1, \dots, \theta_{n-1})$.

Example 2.1 (continued). Let $n = 5$, $E = 510$ and $q = (50, 100, 150, 200, 250)$ as in Example 2.1 where $x^{CEL} = (2, 52, 102, 152, 202)$. Therefore the first sequence of transfers is given by $202 - 4\theta_1 = 152 + \theta_1 \;\Leftrightarrow\; \theta_1 = 10$ – that is, we go from x^{CEL} to the shares $x^1 = (12, 62, 112, 162, 162) \succ_{LD} x^{CEL}$. Next, we get that $162 - 1.5\theta_2 = 112 + \theta_2 \;\Leftrightarrow\; \theta_2 = 20$ – that is, we go from x^1 to the vector $x^2 = (32, 82, 132, 132, 132) \succ_{LD} x^1$. Finally, we solve $132 - 0.66\theta_3 = 82 + \theta_3 \;\Leftrightarrow\; \theta_3 = 30$ – that is, we go from x^2 to the vector $x^3 = (50, 100, 120, 120, 120) = x^{CEG} \succ_{LD} x^2$, in less than 4 ($= n-1$) inequality monotone steps. △

By Theorem 2.1 we can obtain a characterization of the Talmud rule in terms of Lorenz-domination.

Theorem 2.2 (Hougaard and Thorlund-Petersen 2002). *A rationing rule $\hat{\varphi}(q, E) \in \mathcal{R}$ is self-dual and satisfies $\hat{\varphi}(q, E) \succ_{LD} \varphi(q, E)$, for $0 \le E \le Q/2$, for any self-dual rule $\varphi \in \mathcal{R}$ if and only if $\hat{\varphi}(q, E) = \varphi^T(q, E)$.*

Proof. First, consider a self-dual rule $\hat{\varphi}(q, E) \in \mathcal{R}$. By self-duality

$$\hat{\varphi}(q, Q/2) = q/2,$$

and by resource monotonicity $\hat{\varphi}(q, E) \le q/2$ for $0 \le E \le Q/2$. Now, by Theorem 2.1 the unique maximizer of $\succ_{LD}$ on the set of order-preserving rationing rules is φ^{CEG}. Hence, $\hat{\varphi}(q, E) = \varphi^{CEG}(q/2, E) = \varphi^T(q, E)$ for $0 \le E \le Q/2$.

Second, it follows from Theorem 2.1 and the definition of the Talmud rule that φ^T is self-dual and satisfies $\hat{\varphi}(q, E) \succ_{LD} \varphi(q, E)$, for $0 \le E \le Q/2$. □

In other words, the Talmud rule is the unique order-preserving, resource monotonic and self-dual rule that maximises equality in gains or losses depending on E being smaller than or larger than half the total demand.

Now, it is natural to examine which rationing rules that preserve Lorenz-dominance in gains and in losses. We say that a rationing rule φ satisfies:

- *Lorenz-monotonicity in Gains:* If, for any E and $q' \succ_{LD} q$ that $\varphi(q', E) \succ_{LD} \varphi(q, E)$.
- *Lorenz-monotonicity in Losses:* If, for any E and $q' \succ_{LD} q$ that $q' - \varphi(q', E) \succ_{LD} q - \varphi(q, E)$.

Lorenz-monotonicity in gains ensures that shares become more equally distributed when the demands become more equally distributed. Likewise, Lorenz-monotonicity in losses ensures that the losses become more equally distributed when the demands become more equally distributed. The two concepts are related in the following way:

Proposition 2.2. *A rationing rule φ satisfies Lorenz-monotonicity in Gains if and only if its dual rule φ^* satisfies Lorenz-monotonicity in Losses.*

Proof. Let $q' \succ_{LD} q$. Recalling the definition of duality $\varphi^*(q, E) = q - \varphi(q, Q - E)$, and Lorenz-dominance, we get that for all $k = 1, \ldots, n$,

$$\sum_{i=1}^{k}(q'_i - [q'_i - \varphi_i(q', Q - E)]) \geq \sum_{i=1}^{k}(q_i - [q_i - \varphi_i(q, Q - E)]) \Leftrightarrow$$

$$\sum_{i=1}^{k}\varphi_i(q', Q - E) \geq \sum_{i=1}^{k}\varphi_i(q, Q - E).$$

Hence, clearly if φ satisfies Lorenz-monotonicity in Gains then φ^* satisfies Lorenz-monotonicity in Losses and vice versa. $\square$

We are now able to show that:

Proposition 2.3. *The Proportional rule φ^P satisfies Lorenz-monotonicity in both gains and losses. The Constrained Equal Gains rule φ^{CEG} satisfies Lorenz-monotonicity in gains whereas the Constrained Equal Loss rule φ^{CEL} satisfies Lorenz-monotonicity in losses.*

Proof. It is straight forward to see that φ^P satisfies both Lorenz-monotonicity in Gains and in Losses. By Proposition 2.2, it suffices to show that φ^{CEG} satisfies Lorenz-monotonicity in Gains. Hence, consider φ^{CEG}, and let $q' \succ_{LD} q$. Let q and q' be strictly increasing in demands. Let E be fixed, then clearly, $\sum_{i=1}^{h} q'_i \geq \sum_{i=1}^{h} q_i$, for $h = 1, \ldots, n-1$, implies that $\lambda(q') \leq \lambda(q)$ where $\lambda(q)$ is defined by $E = \sum_i \min\{q_i, \lambda(q)\}$.

Moreover, since $\succ_{LD}$ is a cone-ordering (see, e.g., Marshall and Olkin 1979) we may always replace q with a convex combination $\alpha q + (1-\alpha)q'$ such that we obtain solutions $\varphi^{CEG}(q', E) = (q'_1, \ldots, q'_h, \lambda(q'), \ldots, \lambda(q'))$ and $\varphi^{CEG}(q, E) = (q_1, \ldots, q_{h+1}, \lambda(q), \ldots, \lambda(q))$.

Thus, assume w.l.o.g. that Lorenz-monotonicity in Gains is violated for index $h + 1$:

$$\sum_{i=1}^{h} q'_i + \lambda(q') < \sum_{i=1}^{h} q_i + q_{h+1} \quad \Leftrightarrow \quad \sum_{i=1}^{h} q'_i - \sum_{i=1}^{h} q_i < q_{h+1} - \lambda(q').$$

Now, since

$$E = \sum_{i=1}^{h} q'_i + (n-h)\lambda(q') = \sum_{i=1}^{h} q_i + q_{h+1} + (n-h-1)\lambda(q)$$

we get that

$$\begin{aligned}\sum_{i=1}^{h} q'_i - \sum_{i=1}^{h} q_i \;&=\; q_{h+1} + (n-h-1)\lambda(q) - (n-h)\lambda(q') < q_{h+1} - \lambda(q') \\ &\Leftrightarrow \lambda(q) < \lambda(q'),\end{aligned}$$

a contradiction. □

It is easy to verify that the constraint equal gains rule φ^{CEG} violates Lorenz-monotonicity in Losses and that the constraint equal loss rule φ^{CEL} violates Lorenz-monotonicity in Gains. Moreover, it follows that the Talmud rule φ^{T} violates Lorenz-monotonicity in Losses when $0 \leq E \leq Q/2$ and Lorenz-monotonicity in Gains when $Q/2 < E \leq Q$. Consequently, the Talmud rule satisfies neither forms of Lorenz-monotonicity in general.

The concepts of Lorenz-monotonicity in Gains and Losses will further analyzed in Sect. 2.2.4 in relation to the issue of manipulation of resulting cost shares.

2.2.3 Axiomatic Characterizations

As a first natural property it seems that the way E is allocated should only be determined by agents demands q and not by who the agents are. In other words, if two agents have identical demands then a rationing rule ought to assign identical shares to these agents, i.e., agents with equal demand should be treated equally. If equal treatment is violated we either deliberately discriminate between agents or we should be able to reformulate the problem such that equal treatment can be met.

Formally, a rationing rule φ satisfies:

- *Equal Treatment of Equals*: If $x = \varphi(q, E)$ and $q_i = q_j$ implies that $x_i = x_j$.

Note that order-preservation implies Equal Treatment of Equals.

Second, it seems that when a group of agents agree to use some allocation principle then this agreement should not be influenced by the number of agents in the group. In other words, rationing rules ought to be consistent in the sense that reallocating the sum of shares for any subgroup of agents between the agents themselves should leave their original shares unchanged. If some agents were to gain by applying a given rationing rule on a subset of the original population (including themselves) they would have strong incentives to block any enlargement of such a group. Thereby consistency is closely related to the concept of population monotonicity stating that the addition of new agents should affect all original agents in the same direction (either all gain or all loose). In fact, as demonstrated in Chun, resource monotonicity together with consistency implies population monotonicity.

Formally, a rationing rule φ is:

- *Consistent:* If for all q, that $x = \varphi(q, E)$ implies that for all $i \in S$, $x_i = \varphi_i((q_i)_{i\in S}, \sum_{i\in S} x_i)$ for all $S \subseteq \{1, \ldots, n\}$ $(S \neq \emptyset)$.

Note that a rationing rule φ is consistent if and only if its dual rule φ^* is consistent.

All four rules defined in Sect. 2.2.1 satisfy equal treatment of equals and consistency. In fact, together with continuity of φ, Equal Treatment of Equals and Consistency characterize the entire family of parametric sharing methods defined in Remark 2.2.

Theorem 2.3 (Young 1987). *A continuous rationing rule φ satisfies Equal Treatment of Equals and Consistency if and only if it is representable by a continuous parametric function.*

Proof (sketch). Let φ be continuous and satisfy equal treatment of equals and consistency. First it is shown (by contradiction) that then φ is also resource monotonic. Suppose that φ is *not* resource monotonic. Then by consistency there exists a pair $(x_1, x_2) = \varphi((q_1, q_2), x_1 + x_2)$ and $(\bar{x}_1, \bar{x}_2) = \varphi((q_1, q_2), \bar{x}_1 + \bar{x}_2)$, where $x_1 + x_2 < \bar{x}_1 + \bar{x}_2$ and $x_1 < \bar{x}_1$, $x_2 > \bar{x}_2$. Now, choose n such that $x_1 + nx_2 > \bar{x}_1 + n\bar{x}_2$, and consider demand profile $\tilde{q} = (q_1, q_2, \ldots, q_2)$ with n times q_2. For all $E \in [0, q_1 + nq_2]$ define

$$\alpha(E) = \varphi_1(\tilde{q}, E) + \varphi_2(\tilde{q}, E),$$

which is continuous in E and $\alpha(0) = 0$. Moreover, by equal treatment $\alpha(q_1 + nq_2) = q_1 + q_2$. By continuity there exists $\bar{E} \in [0, q_1 + nq_2]$ such that $\alpha(\bar{E}) = \bar{x}_1 + \bar{x}_2$. By consistency and equal treatment $\varphi(\tilde{q}, \bar{E}) = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_2)$ with $\bar{E} = \bar{x}_1 + n\bar{x}_2$. Since $\alpha(0) = 0 \leq x_1 + x_2 \leq \alpha(\bar{E})$ continuity of α implies that there exists E such that $0 \leq E \leq \bar{E}$ and $\alpha(E) = x_1 + x_2$. By consistency and equal treatment $\varphi(\tilde{q}, E) = (x_1, x_2, \ldots, x_2)$ with $E = x_1 + nx_2$. By choice of n, $E = x_1 + nx_2 > \bar{x}_1 + n\bar{x}_2 = \bar{E}$ contradicting $E \leq \bar{E}$. We conclude that φ satisfies resource monotonicity.

Now suppose that φ satisfies *strict* resource monotonicity (only weak monotonicity was shown above) then the proof could continue like this: For every 2-agent problem and every $\lambda \in [0, 1]$ define $x_1 = f(q_1, \lambda)$ if and only if $(x_1, \lambda) = \varphi((q_1, 1), x_1 + \lambda)$. Continuity and strict monotonicity of φ imply that f is continuous and strictly monotonic in λ. Now, fix $x^* = \varphi(q, E^*)$ where $E^* \in [0, \sum_i q_i]$. Consider $(x, \lambda) = \varphi((q, 1), E)$ as E varies from 0 to $\sum_i q_i + 1$. By continuity there exists a $\bar{E}$ that renders agent 1 and 2 a total of $x_1^* + x_2^*$ which by consistency must be allocated as (x_1^*, x_2^*). Likewise there exists a $\tilde{E}$ that renders agent 1 and 3 a total of $x_1^* + x_3^*$ that must be allocated as (x_1^*, x_3^*). As φ is strictly monotonic and agent 1 receives the same amount in both cases $\bar{E} = \tilde{E}$. Continuing this argument there is a value E^* and a value λ^* such that $\varphi((q, 1), E^*) = (x^*, \lambda^*)$. Consistency implies that $(x_i^*, \lambda^*) = \varphi((q_i, 1), x_i^* + \lambda^*)$ so by definition of f, $x_i^* = f(x_i, \lambda^*)$ for all i.

Conversely, suppose that $f(q_i, \lambda) = x_i$ for some λ and all i. The above argument implies that there exists a λ' such that $f(q_i, \lambda') = x_i'$ where $\sum_i x_i' = \sum_i x_i$. Since f is monotonic in λ, $x_i' = x_i$ for all i and f is a parametric representation of φ.

For the exact proof the reader is referred to Young (1987) which also demonstrates that, in fact, only pairwise consistency is needed in the sense that consistency only has to be satisfied for all coalitions of cardinality two. □

Example 2.1 (continued). Consider the 5-agent problem of Example 2.1 with $q = (50, 100, 150, 200, 250)$ and $E = 510$. Here the Talmud rule resulted in the following allocation $x^T = (25, 50, 95, 145, 195)$. It is easily checked that for any subset of the agents, application of the Talmud rule results in consistency. For example the sub-problem of $q = (50, 100)$ and $E = 75$. Here $Q/2 = 75 = E$ and E is shared in proportion to demands or, interpreted along the lines of the contested garment principle; 50 is contested and hence shared equally whereas the residual of 25 is only claimed by agent 2 – hence the allocation becomes (25, 25+25). For any 2-agent rationing problem (q_1, q_2, E) the "contested-garment" principle can be defined (as in Aumann and Maschler 1985) by the shares,

$$x_1 = \frac{1}{2}\min\{q_1, E\} + \frac{1}{2}\max\{0, E - q_2\},$$
$$x_2 = \frac{1}{2}\min\{q_2, E) + \frac{1}{2}\max\{0, E - q_1\}.$$

As shown in Aumann and Maschler the Talmud rule is the unique consistent extension of the contested garment principle. △

In order to single out the rules of Sect. 2.2.1 further axioms are needed. For example an axiom of scale invariance to rule out any influence of the units of measurement. In case of a bankruptcy problem, for instance, it seems very natural to demand that the underlying allocation principle should be independent of whether we measure in Danish kroner, dollars or euro's.

Formally, a rationing rule φ satisfies:

- *Scale Invariance:* If for all (q, E) and $\lambda \in \mathbf{R}_+$,

$$\varphi(\lambda q, \lambda E) = \lambda \varphi(q, E).$$

Note that all four rules of Sect. 2.2.1 satisfy Scale Invariance.

The next two axioms may be interpreted along the following line: Assume that a group of agents have agreed to use a given rationing rule φ for some rationing problem (q, E'). However, it turns out that the true value of the amount that has to be shared is $E < E'$. Then using φ we should be able to take the solution $\varphi(q, E')$ and solve the problem $(\varphi(q, E'), E)$ instead of (q, E) – called Upper Composition. On the other hand, if $E > E'$ we should be able to determine $\varphi(q, E)$ by adding the solution of the problem $(q - \varphi(q, E'), E - E')$ to the solution $\varphi(q, E')$ – called Lower Composition.

Formally, a rationing rule φ satisfies:

- *Upper Composition:* If, for all (q, E) and $E' > E$,
$$\varphi(q, E) = \varphi(\varphi(q, E'), E).$$
- *Lower Composition:* If, for all (q, E) and $E' < E$,
$$\varphi(q, E) = \varphi(q, E') + \varphi(q - \varphi(q, E'), E - E').$$

Note that a rationing rule φ satisfies Lower Composition if and only if its dual rule φ^* satisfies Upper Composition. Moreover, note that the Proportional rule, the Constrained Equal Gains rule and the Constrained Equal Loss rule satisfy Upper and Lower Composition whereas the Talmud rule satisfies neither Upper nor Lower Composition.

Theorem 2.4 (Moulin 2000). *A rationing rule φ satisfies Equal Treatment of Equals, Consistency, Scale Invariance, Upper and Lower Composition if and only if $\varphi \in \{\varphi^P, \varphi^{CEG}, \varphi^{CEL}\}$.*

It has already been noted that all three rules $\{\varphi^P, \varphi^{CEG}, \varphi^{CEL}\}$ satisfy the axioms. To prove the converse, the reader is referred to the elaborate proof in Moulin (2000) or the alternative proof in Thomson (2006).

Finally, there are several alternative characterizations of the individual rules; a recent survey can be found in Thomson (2003).

2.2.4 Manipulation

When rationing rules are implemented in practice they may give rise to strategic reactions among the agents involved. In other words, agents (or some coalition of agents) may be able to manipulate the result of given rules to their own advantage. Since each individual demand q_i is considered to be verifiable, manipulation is not possible via strategic choice of q_i. Hence, manipulation can appear either by merging or splitting individual demands or by reallocating demand between groups of agents. Notice that by merging or splitting demands the dimension of the rationing problem is changed whereas by reallocation the dimension remains fixed.

Rationing rules are non-manipulable by merging and splitting if it is disadvantageous for any coalition of agents to merge and split their demands in the sense that the resulting aggregated or disaggregated shares are smaller than the shares resulting from the original problem of dimension n. That is, rationing rules φ must satisfy:

- *No Advantageous Merging:* Let E be given and let q' be determined by aggregating the demand of a subset M of the n agents, i.e., $q' = (\sum_{j \in M} q_j, (q_i)_{i \in N \setminus M})$ for $M \subset N = \{1, \ldots, n\}$. For all such coalitions $M \subset N$, $\varphi_M(q', E) \leq \sum_{j \in M} \varphi_j(q, E)$.

and

- *No Advantageous Splitting:* Let E be given and let $\hat{q}$ be determined by splitting the demand of agent j into s separate demands $(q_{ji})_{i=1}^s$ where $\sum_{i=1}^s q_{ji} = q_j$ – that is, $\hat{q} = ((q_{ji})_{i=1}^s, (q_l)_{l \in N \setminus j})$. For all such disaggregated demands $(q_{ji})_{i=1}^s$, $\sum_{i=1}^s \varphi_{ji}(\hat{q}, E) \leq \varphi_j(q, E)$.

Theorem 2.5 (Banker 1981; De Frutos 1999). *The proportional rule φ^P is the only rationing rule that satisfies both No Advantageous Merging and No Advantageous Splitting.*

Proof. It is easy to verify that φ^P satisfies No Advantageous Merging and Splitting. To prove the converse, let for given q, $q' = (q_i, Q - q_i)$ be the demand of an arbitrary agent i and coalition $M = N \setminus i$. First, note that No Advantageous Merging and Splitting implies that $\sum_{j \in M} \varphi_j(q, E) = \varphi_M(q', E)$ and by budget balance $\varphi_i(q, E) = \varphi_i(q', E)$. Secondly, note that No Advantageous Merging and Splitting implies Equal Treatment of Equals: Suppose not. Then there exists a pair (i, j) with $q_i = q_j = \bar{q}$ where $\varphi_i(q, E) > \varphi_j(q, E)$. By the above argument $\varphi_i((\bar{q}, Q - \bar{q}), E) > \varphi_j((\bar{q}, Q - \bar{q}), E)$. Now, let agent i split the demand into two equal amounts. Then

$$\begin{aligned} \varphi_i((\bar{q}, Q - \bar{q}), E) &= \varphi_{i'}((\bar{q}/2, \bar{q}/2, Q - \bar{q}), E) + \varphi_{i''}((\bar{q}/2, \bar{q}/2, Q - \bar{q}), E) \\ &> \varphi_j((\bar{q}, Q - \bar{q}), E), \end{aligned}$$

that is, agent j has incentive to split his demand – a contradiction.

To finish the proof, note that for any vector $q \in \mathbf{R}^n$, every element can be written as

$$q_i = \frac{a_i}{p} Q,$$

where $p > 0$ and a_i is non-negative number such that $a_1 + \ldots + a_n = p$. Now, let p agents each demand Q/p. Then Equal Treatment of Equals and budget balance implies that each of the p agents receives the share E/p. Moreover, as $\varphi_i = \varphi_{i'} + \varphi_{i''}$ when $q_i = q_{i'} + q_{i''}$ then by sequential merging or splitting of the demand of the p agents, coalitions i receive

$$\varphi_i = a_i \frac{E}{p} = \frac{q_i}{Q} E, \quad i = 1, \ldots, n.$$

□

Example 2.2 (continued). Consider the 5-agent problem of Example 2.1 with $q = (50, 100, 150, 200, 250)$ and $E = 510$. Here the Talmud rule resulted in the following allocation $\varphi^T(q, E) = (25, 50, 95, 145, 195)$. Now, assume that agent 1 and 2 merge such that $q_{\{1,2\}} = q_1 + q_2 = 150$. In this case the rationing problem is reduced to a 4-agent problem where $\varphi^T((150, 150, 200, 250), 510) = (90, 90, 140, 190)$ making it advantageous for agent 1 and 2 to merge as $\varphi_1^T(q, E) + \varphi_2^T(q, E) = 75 < 90$. Now, consider the problem (q, E) where $E = 240$. Here, $\varphi^T(q, E) = (25, 50, 55, 55, 55)$. Let agent 3 split his demand $q_3 = 150$ into $q_{3'} = 50$ and $q_{3''} = 100$. In this case the problem is extended to a 6-agent problem $((50, 50, 100, 100, 200, 250), 240)$ where $\varphi^T((50, 50, 100, 100, 200, 250), 240) = (25, 25, 47.5, 47.5, 47.5, 47.5)$ making it advantageous for agent 3 to split his demand as $\varphi_3^T(q, E) = 55 < 25 + 47.5 = 72.5$. Hence, the Talmud rule can be manipulated both by merging and by splitting. This is a consequence of the fact that the Constrained Equal Gains rule can be manipulated by splitting and that the Constrained Equal Loss rule can be manipulated by merging as clarified in Remark 2.3. △

Remark 2.3. Recall the definition of parametric rules with representation f in Remark 2.2. The representation f is said to be superadditive (subadditive) in demand if for all λ and all $q_i, q'_i \in \mathbf{R}_+$ that $f(q_i + q'_i, \lambda) \geq (\leq)\ f(q_i, \lambda) + f(q'_i, \lambda)$. In Ju (2003), it is shown that a parametric rule satisfies No Advantageous Merging if and only if the representation f is subadditive in q_i for each value of λ. Likewise a parametric rule satisfies No Advantageous Splitting if and only if the representation f is superadditive in q_i for each value of λ. As the Constrained Equal Gains rule has parametric representation $f(q_i, \lambda) = \min\{q_i, \lambda\}$, f is concave and hence subadditive in q_i, i.e., satisfies non-manipulability by merging. As the Constrained Equal Loss rule has parametric representation $f(q_i, \lambda) = \max\{0, q_i - 1/\lambda\}$, f is convex and hence superadditive in q_i, i.e., satisfies non-manipulability by splitting. Finally, note that the proportional rule has representation $f(q_i, \lambda) = \lambda q_i$ that is linear and hence both sub- and superadditive in q_i, i.e., satisfies non-manipulability by both merging and splitting. (On the other hand note, that there may be functions f that are sub- resp. superadditive and *not* concave resp. convex.) △

As mentioned above, there is another way to manipulate the resulting shares and that is by reallocating demand between groups of agents keeping the number of agents fixed. If such manipulation shall be prevented no coalition of agents shall be able to increase their total share by reshuffling their individual demands – that is, the rationing rule φ must satisfy:

- *No Advantageous Reallocation:* Let E be given. Then for every $S \subset N$ and $q, q' \in \mathbf{R}^n_+$, if $\sum_{i \in S} q_i = \sum_{i \in S} q'_i$ and $q_j = q'_j$ for all $j \in N \setminus S$, it implies that $\sum_{i \in S} \varphi_i(q, E) = \sum_{i \in S} \varphi_i(q', E)$.

Note that No Advantageous Reallocation is only meaningful in case $|N| \geq 3$. It is clear that the proportional rule φ^P satisfies No Advantageous Reallocation. The other three rules, however, do not as demonstrated by the following example.

Example 2.1 (continued). Consider the 5-agent problem of Example 2.1 with $q = (50, 100, 150, 200, 250)$ and $E = 510$. Here the Constrained Equal Gains rule resulted in the following allocation

$$\varphi^{CEG}(q, E) = (50, 100, 120, 120, 120).$$

Assume now that agent 1, 2 and 3 form a coalition where they average their demands such that the reallocated demand vector becomes

$$q' = (100, 100, 100, 200, 250).$$

Using the Constrained Equal Gains rule we get that

$$\varphi^{CEG}(q', E) = (100, 100, 100, 105, 105)$$

making it advantageous for agent 1, 2 and 3 to perform their averaging operation. Likewise, consider the Constrained Equal Loss rule that resulted in the following allocation $\varphi^{CEL}(q, L) = (2, 52, 102, 152, 202)$. If the agents 1,2 and 3 now spread their demands such that the new demand vector becomes $q'' = (0, 100, 200, 200, 250)$, we get that $\varphi^{CEL}(q'', E) = (0, 40, 140, 140, 190)$ making it advantageous for agents 1,2 and 3 to reallocate as in q''. Since both the Constrained Equal Gains rule and the Constrained Equal Loss rule fail to satisfy No Advantageous Reallocation so does the Talmud rule by definition. $\triangle$

By Proposition 2.3 and order-preservation, it can generally be concluded that the Constraint Equal Gains rule can be manipulated by all lower coalitions $\{1, \ldots, k\}$ averaging their demands (as $q' \succ_{LD} q$ implies that $\sum_{i=1}^{k} \varphi_i^{CEG}(q', E) \geq \sum_{i=1}^{k} \varphi_i^{CEG}(q, E)$ for $k = 1, \ldots n-1$.). Likewise, it can generally be concluded that the Constraint Equal Loss rule can be manipulated by all lower coalitions $\{1, \ldots, k\}$ spreading their demands (as $q' \succ_{LD} q$ implies that $\sum_{i=1}^{k} (q_i' - \varphi_i^{CEG}(q', E)) \geq \sum_{i=1}^{k} (q_i - \varphi_i^{CEG}(q, E))$ for $k = 1, \ldots n-1$.).

In fact, it can be shown that the Proportional rule is the only rule that cannot be manipulated by reallocation of demands.

Theorem 2.6 (Moulin 1987). *The proportional rule φ^P is the only rationing rule that satisfies No Advantageous Reallocation.*

Proof (sketch). From Chun (1988) it is known that all rules that satisfy No Advantageous Reallocation (plus a weak continuity and symmetry property) are of the following form for all $i = 1, \ldots, n$,

$$\varphi_i(q, E) = \frac{q_i}{Q} E - \frac{1}{Q}[nq_i - Q] g(Q, E),$$

where $g : \mathbf{R}^2 \to \mathbf{R}$ is a continuous function. Since we require that a rationing rule must satisfy $0 \leq \varphi_i(q, E) \leq q_i$ for all i, we have that $g(Q, E) = 0$. □

Note that the equal split rule (which is not a rationing rule as defined above since E/n may exceed q_i for some i), satisfies No Advantageous Reallocation ($g(Q, E) = E/n$) but is manipulable by splitting. Likewise, equal split of the loss (that may result in negative shares) satisfies No Advantageous Reallocation ($g(Q, E) = (E - Q)/n$) but is manipulable by merging.

To demonstrate that there is a close connection between No-advantageous Reallocation and Lorenz-monotonicity, as defined in Sect. 2.2.2, we reconsider the Lorenz-monotonicity properties in light of manipulation. Indeed, due to order-preservation the Lorenz-monotonicity properties may be construed as follows: Suppose that some lower coalition of agents (ordered by the size of their demands) equalize their demands resulting in a new vector of demands that Lorenz-dominates the original demand vector. In this case, Lorenz-monotonicity in gains requires that such a reallocation is not disadvantageous for this lower coalition. Consequently, if a rationing method satisfies Lorenz-monotonicity in gains then it cannot be manipulated be any lower coalition spreading their demands (without changing the rank of agents according to demand). Likewise, Lorenz-monotonicity in losses concerns a spread of demands; If a rationing method satisfies Lorenz-monotonicity in losses then its resulting vector of shares cannot be manipulated by any lower coalition equalizing their demands.

Thus, if a rationing method satisfies Lorenz-monotonicity in both gains and losses it cannot be manipulated by any lower coalition of agents spreading or equalizing their demands. In fact, we are able to provide the following alternative characterization of the proportional rule based on Lorenz-monotonicity.

Theorem 2.7 (Hougaard and Østerdal 2005). *The Proportional rule φ^P is the only continuous and order-preserving rationing rule that satisfies Lorenz-monotonicity in both gains and losses.*

Note that Lorenz-monotonicity in Gains and Losses together are weaker than No-Advantageous Reallocation (on the other hand, Moulin's characterization – as in Theorem 2.6 – is not limited to order-preserving rules). In Proposition 2.3 it is shown that the Proportional Rule satisfies Lorenz-monotonicity in both gains and losses. For a proof of the converse claim the reader is referred to Hougaard and Østerdal (2005).

2.2.5 Comments

Before turning towards cost sharing with a common cost function as in scenarios (2) and (3) in Sect. 2.1, a few final comments concerning the rationing model will be made.

First, even though all claims are verifiable not all claims may be viewed as equal from the outset. For example, in cases concerning bankruptcy of firms Danish law states that in principle all claimants are equal and that the estate should be allocated according to the proportional rationing rule. However, some groups of claimants are favored: In case of unpaid salaries, employees have a so-called privileged claim that will be covered (or partly covered according to the size of the estate) before other claimants get their share of the estate. As such, agents may be ranked according to some prespecified list of priority and their claims handled accordingly. From a theoretical point of view there is an interesting generalization of the contested garment principle based on random priorities (O'Neill 1982): In particular if there are two agents, shares corresponding to the use of the contested garment principle can be found as the average of the shares in two situations – one where agent 1 has priority over 2 and one where 2 has priority over 1. In general, there are $n!$ possible orderings of n agents. For each such ordering let agents receive as much of their demand as possible, that is if $E > q_1$ then $x_1 = q_1$ and if $E - x_1 > q_2$ then $x_2 = q_2$, etc. Now, the random priority rule assigns shares which are then defined as the average over all such orderings for each agent. In general, concerning models of priority, each agent is described not only by their demand (or claim), but by a combination of their demand and type. Another well known type is "time of arrival" – here a rule could be the familiar "first to arrive on the spot is the first to be served" rule which clearly violates order-preservation in terms of demand. For further discussion of priority rules see, e.g., Moulin (2000, 2002).

Secondly, as noticed by Young (1987, 1988) the entire rationing model may alternatively be construed as a taxation problem where the sum of taxes $x_1+\ldots+x_n$ must equal a given revenue constraint E and q_i is the pre-tax income of agent i (post-tax incomes are hence given by $q_i - x_i$). Thus, the above results have a "dual" interpretation with respect to the taxation model (q, E). For example, in Theorem 2.7 it was demonstrated that the proportional rationing rule was the only (order-preserving) rule that satisfied both Lorenz-monotonicity in Gains and Losses. In terms of the taxation model this result reads: Proportional taxation – called a flat tax – is the only taxation rule that preserves equality in the sense that if pre-tax incomes become more equally distributed then both taxes and post-tax incomes become more equally distributed. Note that in case of taxation it could naturally be argued that the post-tax income of the agent i should be *independent* of the other agents pre-tax incomes q_{-i}. The distributional aspects of taxation rules with respect to such a taxation model is, for example, examined in Moyes (1989, 1994).

2.3 Cost Sharing with Joint Cost Function

Suppose that n agents are engaged in a joint project. Let $N = \{1, \ldots, n\}$ denote the set of agents. Moreover, let $q = (q_1, \ldots, q_n)$ be a vector of non-negative demands $q_i \in \mathbf{R}_+$ of each agent i for some homogeneous good (hence demand is not necessarily measured in monetary units). Assume for simplicity that these demands are increasingly ordered $q_1 \leq \ldots \leq q_n$. Since the demand of each agent refer to the same (homogeneous) good we shall focus on two particular cases: one where the joint cost is a function of total demand $Q = q_1 + \ldots + q_n$ (homogeneous cost functions), and one where the joint cost is a function of maximal demand $\mathcal{Q} = \max_i\{q_i\} = q_n$ (decomposable cost functions), see, e.g., scenarios (2) and (3) of the Introduction.

For fixed N, let (q, C) be a *cost sharing problem* where $C : \mathbf{R}_+ \to \mathbf{R}_+$ is a (one-dimensional) non-decreasing cost function with $C(0) = 0$ and denote by $\mathcal{D}$ the set of such cost sharing problems. For a given cost sharing problem $(q, C) \in \mathcal{D}$, a *cost sharing rule* ϕ specifies a unique vector of cost shares $x = (x_1, \ldots, x_n) = \phi(q, C)$ where the cost shares x_i add up to the total costs $C(Q)$ or $C(\mathcal{Q})$.

In practice, the cost function may be construed either as the costs of production or as a pricing scheme faced by the agents. In the latter case this pricing scheme can be used directly. In the former case, the cost function can be estimated using registered cost data.

Remark 2.4. Although we shall use the framework of cost sharing it can be noted that in terms of sharing some worth (surplus sharing) the model may be given the following equivalent interpretation: suppose that agents $N = 1, \ldots, n$ are engaged in a joint project. Let $q = (q_1, \ldots, q_n)$ be a vector of homogeneous characteristics $q_j \in \mathbf{R}_+$, for example, individual working hours supplied for the joint project. If working hours supplied by different agents are considered as homogeneous we may focus on the total number of working hours supplied $Q = q_1 + \ldots + q_n$ and thereby on a one-dimensional non-decreasing value function $V : \mathbf{R}_+ \to \mathbf{R}_+$ with respect to Q. Hence, (q, V) is a surplus sharing problem and ϕ is a surplus sharing rule specifying a unique vector of surplus shares $y = (y_1, \ldots, y_n) = \phi(q, V)$ where $y_1 + \ldots + y_n = V(Q)$. $\triangle$

2.3.1 Rules Based on Equality and Proportionality

Within the framework of cost sharing problems $(q, C) \in \mathcal{D}$, the proportional rule of the rationing model is known as

- *The Average Cost Rule* ϕ^{AC}, defined by cost shares

$$x_i^{AC} = \frac{q_i}{Q} C(z), \quad i = 1, \ldots, n, \tag{2.7}$$

where $z = Q$ in case of homogeneous costs and $z = \mathcal{Q}$ in case of decomposable costs.

The name refers to the fact that all agents pay the average price, $C(z)/Q$ for all units demanded. Thus, ϕ^{AC} is order-preserving, i.e., $x_1^{AC} \leq \ldots \leq x_n^{AC}$ when $q_1 \leq \ldots \leq q_n$.

In case of homogeneous cost functions $C(Q)$, the connection with the proportional rule of the rationing model implies that the average cost rule can be characterized by the same properties as the proportional rationing rule. In particular, the result of the average cost rule cannot be manipulated by neither reallocation nor by merging or splitting of demands.[1] However, one further general characterization is interesting since it relates to a monotonicity property that may be viewed as extending the resource monotonicity property of the rationing model to the present framework. This monotonicity property is defined as follows:

- *Monotonicity:* Let $C_1, C_2 \in \mathcal{D}$ and let $C_1(z) \leq C_2(z)$ for all z. Then $\phi_i(q, C_1) \leq \phi_i(q, C_2)$ for all i and all q.

In other words, Monotonicity states that all agents should benefit from a new technology that reduces costs. Cost sharing rules satisfying Monotonicity hence ensures that all agents have incentive to innovate and use cost reducing technologies. Now, this property proves rather powerful since together with a natural property related to linear cost functions (Constant Returns) it actually characterizes the average cost rule. The property of Constant Returns states that if the cost function is linear in total demand then there is a natural cost share per unit demanded, i.e., the constant average cost. Formally

- *Constant Returns:* If $C(z) = \lambda z$ for all $\lambda \geq 0$ then $\phi_i(q, C) = \lambda q_i$ for all i.

Indeed,

Theorem 2.8 (Moulin and Shenker 1994). *The Average Cost Rule ϕ^{AC} is the only cost sharing rule that satisfies Monotonicity and Constant Returns.*

The formal proof may be found in Moulin and Shenker (1994). At first sight, this result seems somewhat surprising since if we stick to the idea that everybody should be held responsible for their own demand and not share according to some degree of egalitarianism (which is basically the message of Constant Returns) then we cannot guarantee that all agents would gain from a general cost reduction using any alternative to average cost sharing. But as we shall see, this is closely linked to the fact the average cost rule only relates to the total cost $C(Q)$ and not to other parts of the cost function.

[1] Clearly, the average cost rule can be manipulated in case of decomposable cost functions, however, it is questionable whether there exists situations where a decomposable cost function is a proper description of the cost structure and where it makes sense to talk about agents equalizing or splitting their demands.

Example 2.2. If the agents know how the total cost of the group will be shared, it is easy to imagine a variety of situations where it would be natural for them to act strategically in their choice of demand. However, if the cost function is concave and they share costs using the Average Cost rule, a Nash equilibrium in the induced cost sharing game may not exist.

Consider the following example[2] where two agents jointly buy long-distance calls from AT&T at a (concave) two-part tariff "the One Rate 7c Plus", i.e., total cost is given by the function $C(q_1+q_2) = 0.07(q_1+q_2)+4.95$. Let the benefit $h(\cdot)$ from demanding quantity (minutes calling) q be given by,

$$h_1(q_1) = \begin{cases} 0.55q_1 & \text{if } q_1 \in [0,10) \\ 0.12q_1 + 4.3 & \text{if } q_1 \in [10,30) \\ 7.9 & \text{if } q_1 \in [30,\infty), \end{cases}$$

and

$$h_2(q_2) = \begin{cases} 0.17q_2 & \text{if } q_2 \in [0,30) \\ 5.1 & \text{if} q_2 \in [30,\infty), \end{cases}$$

respectively. Hence, using the average cost rule ϕ^{AC} and maintaining the assumption that both agents have quasi-linear utility functions, i.e., $u_i(q_i,q_j) = h_i(q_i) - \phi_i^{AC}$, induces a cost sharing game with pay-off's given by,

$$u_1(q_1,q_2) = h_1(q_1) - 0.07q_1 - \frac{q_1}{q_1+q_2}4.95,$$

and

$$u_2(q_1,q_2) = h_2(q_2) - 0.07q_2 - \frac{q_2}{q_1+q_2}4.95.$$

Now, this results in the following "best reply" correspondences for agent 1 and 2 respectively,

$$q_1^*(q_2) = \begin{cases} 30 & \text{if } q_2 \in [0,5.6] \\ 10 & \text{if } q_2 \in [5.6,53.4] \\ 30 & \text{if } q_2 \in [53.4,\infty), \end{cases}$$

and

$$q_2^*(q_1) = \begin{cases} 0 & \text{if } q_1 \in [0,19.5] \\ 30 & \text{if } q_1 \in [19.5,\infty). \end{cases}$$

Clearly, no equilibrium exists in this particular case since if agent 1 demands 10 then agent 2 will demand 0 and if agent 2 demands 0 then agent 1 will demand 30 – but if agent 1 demands 30 then agent 2 will demand 30 and in this case agent 1 will rather demand 10, etc.

---

[2] Kindly provided by Lars Thorlund-Petersen.

However, notice that if the agents announce their demand in a sequence and these announcements are observable by the other agents, then an equilibrium will exist. For example, let agent 1 determine his demand first and let this be observed by agent 2, who then determines his demand. In this case backward induction gives the (subgame-perfect) equilibrium $(q_1^* = 30, q_2^* = 30)$.

Moreover, implementation in Nash equilibrium is generally possible in case the cost function C is convex, see, e.g., Watts (1996). $\triangle$

Egalitarianism becomes a relevant alternative because, contrary to the scenario of the rationing model, the present framework does not initially exclude,

- *The Equal Split Rule*, ϕ^E, defined by cost shares,

$$x_i^E = \frac{C(z)}{n}, \quad i = 1, \ldots, n, \tag{2.8}$$

where $z = Q$ in case of homogeneous costs and $z = \mathcal{Q}$ in case of decomposable costs.

Clearly, the equal split rule ϕ^E is (trivially) order-preserving and satisfies Monotonicity but *not* Constant Returns.

Now, both the average cost rule and the equal split rule only relates to the total cost while the information contained by the rest of the cost function is "ignored". For example, it could be argued that the level of the stand-alone cost $C(q_i)$ for each agent i ought to influence the final allocation of costs. An immediate way to meet such a requirement could be to allocate costs in proportion to stand-alone costs instead of demands, i.e., to use cost shares,

$$x_i = \frac{C(q_i)}{\sum_{j \in N} C(q_j)} C(Q), \text{ for } i = 1, \ldots, n \tag{2.9}$$

(with the obvious changes for a decomposable cost function). Note that for homogeneous cost functions this version of proportional cost sharing satisfies Constant Returns but *not* Monotonicity. The problem with Monotonicity occurs because the rule exploits other parts of the cost function (the stand-alone costs) than just the total cost, while satisfying Constant Returns.

In the same spirit, cost could be allocated using constrained equal split

$$x_i^1 = \min\left\{C(q_i), \frac{C(Q)}{n}\right\} \text{ for } i = 1, \ldots, n \tag{2.10}$$

and adding an equal share of any resulting deficit, i.e., $1/n[C(Q) - \sum_{j \in N} x_j^1]$ (with the obvious changes for a decomposable cost function). Note, that for homogeneous cost functions this version of egalitarianism satisfies neither Constant Returns nor Monotonicity. Further, note that, except for (2.9) with respect to a decomposable cost function, none of these suggestions guarantee individual rationality, i.e., that no agent pays more than his stand-alone cost.

In case of decomposable cost functions $C(\mathcal{Q})$, however, there is a more direct way to ensure individual rationality. For example, we may define the *restricted equal split rule* by cost shares,

$$x_i^{RE} = \min\{C(q_i), \alpha\}, \quad i = 1, \ldots, n, \tag{2.11}$$

where α is chosen such that the cost shares add up to total costs $C(\mathcal{Q})$. This rule captures the spirit of the constrained equal gains rule of the rationing model. We may further define the *restricted average cost rule* by the following cost sharing scheme: First, calculate shares

$$x_i^1 = \min\left\{C(q_i), \frac{q_i}{Q}C(\mathcal{Q})\right\}, \quad i = 1, \ldots, n. \tag{2.12}$$

If some agents are bounded by their stand-alone cost the remaining agents must further share $C(\mathcal{Q}) - \sum_{i=1}^{n} x_i^1$ in proportion to their demand and so forth until total costs are fully allocated.

However, knowledge of the entire cost function opens up for the definition of types of rules that has not been treated so far since costs related to any subset of agents can be assessed. As argued such information, if accessible, may influence the way that costs (or value) should be shared. In the following we consider cost sharing rules based on two main principles: the *serial principle* and the *incremental principle*.

2.3.2 Rules Based on the Serial Principle

The serial principle basically states that agents with equal demand must be treated equally and that, according to a given ordering of demands, an agent's cost share should not depend on the demand of agents that appear after him in the ordering. The spirit of the serial principle is perhaps most clearly illustrated in case of decomposable cost functions.

2.3.2.1 Serial Cost Sharing: Decomposable Costs

We start out by demonstrating that sharing costs equally or in proportion to individual demand q_i may, in case of a (non-decreasing) decomposable cost function $C(\mathcal{Q})$, lead to violation of the stand-alone cost principle as illustrated by the following simple example.

Example 2.3. Let three agents have demands $q = (q_1, q_2, q_3) = (1, 2, 3)$ with associated stand-alone costs $C(q_1) = 100$, $C(q_2) = 800$ and $C(q_3) = 900$.

Since the cost function is decomposable, the total cost of a joint project is $C(q_3) = 900$. Now, if the total cost is shared equally, all agents pay $x^E = 300$. If, instead we use average cost sharing we get:

$$x^{AC} = \left(\frac{1}{6}900, \frac{1}{3}900, \frac{1}{2}900\right) = (150, 300, 450).$$

Notice, that in both cases, agent 1 will end up paying more than the stand-alone cost of 100. Moreover, notice that both methods only use individual demands q and the total costs $C(q_3)$ as relevant information whereas the information contained by the remaining part of the cost function is ignored. △

Although lack of individual rationality can be ensured using the restricted versions defined in (2.11) and (2.12), it seems natural to look for alternative cost sharing rules, where more of the information contained by the cost function is utilized. Indeed, one may suggest to use equal sharing but with respect to incremental costs, rather than total cost, following the serial principle.

Consider, for example, three agents with demands $q_1 \leq q_2 \leq q_3$, and total joint cost $C(q_3)$. If all agents had demanded q_1 the total cost of $C(q_1)$ should be split equally (according to "equal-treatment-of-equals"), i.e., all agents get cost share $1/3C(q_1)$. Now, the incremental cost in going from demand q_1 to q_2 should be split equally among agents 2 and 3 as they alone are responsible for this demand, i.e., both agents 2 and 3 further pay $1/2(C(q_2) - C(q_1))$. Finally, only agent 3 is responsible for the incremental demand going from q_2 to q_3 and should consequently cover the associated incremental costs alone, i.e., agent 3 further pays $C(q_3) - C(q_2)$. Thus, total cost is shared as

$$x_1 = \frac{C(q_1)}{3}, \quad x_2 = \frac{C(q_1)}{3} + \frac{C(q_2) - C(q_1)}{2},$$

$$x_3 = \frac{C(q_1)}{3} + \frac{C(q_2) - C(q_1)}{2} + C(q_3) - C(q_2).$$

As such, cost shares found using the serial principle can never exceed the stand-alone cost of any agent in case of decomposable cost functions.

In general, for a decomposable cost function we say that cost shares are associated with the *serial cost sharing rule* if they are determined as

$$x_j = \sum_{k=1}^{j} \frac{C(q_k) - C(q_{k-1})}{n - k + 1}, \quad j = 1, \ldots, n, \tag{2.13}$$

where $x_0 = C(q_0) = 0$.

Example 2.3 (continued). Use of the serial cost sharing rule will, in the case of Example 2.2., result in cost shares $x_1 = 33.33$, $x_2 = 33.33 + 350 = 383.33$ and $x_3 = 33.33 + 350 + 100 = 483.33$. Clearly, no agent pays more than their stand-alone cost by this method. Of course, we could have used restricted versions of equal split and average cost sharing to ensure that no agent pays more than their stand-alone cost (i.e., comply with individual rationality). In case of restricted equality, this would result in cost shares $x_1^{RE} = 100$, $x_2^{RE} = x_3^{RE} = 100 + 300 = 400$ whereas in case of restricted average cost sharing, resulting cost shares are given by $x_1^{RAC} = 100$, $x_2^{RAC} = 300 + (2/5)50 = 320$ and $x_3^{RAC} = 450 + (3/5)50 = 480$. △

2.3.2.2 Serial Cost Sharing: Homogeneous Costs

Consider now a (non-decreasing) homogeneous cost function $C(Q)$ and denote by $\hat{\mathcal{D}}$ and $\check{\mathcal{D}}$ the set of cost sharing problems with convex and concave homogeneous cost functions, respectively. Finally, denote by $\mathcal{D}^+ = \hat{\mathcal{D}} + \check{\mathcal{D}}$ the set of cost sharing problems where the homogeneous cost function equals a sum of a convex and a concave cost function.

The basic motivation behind serial cost sharing in case of homogeneous cost functions (as introduced in Shenker 1995 and Moulin and Shenker 1992) is given by the serial principle, i.e., agents with identical demand should be treated equally and no agent will be held responsible for the consumption of "greedier" agents even though they are associated with a joint project.

Consider, for example, three agents with individual demands $q_1 \leq q_2 \leq q_3$, and total cost $C(q_1 + q_2 + q_3)$. Cost shares according to serial cost sharing is found as follows: The agent with the smallest demand pays one-third (an equal share) of the total costs in case all agents had been as "modest" as agent 1 in their demands. The second agent further pays half (an equal share) of the incremental cost in going from a situation with total demand $3q_1$ to total demand $q_1 + 2q_2$ – that is, to a situation where agent 1 demands q_1 and the remaining agents are as "modest" as agent 2. Finally agent 3 further pays the incremental cost of going from a situation with total demand of $q_1 + 2q_2$ to a total demand of $q_1 + q_2 + q_3$. This leaves the agents with cost shares,

$$\begin{aligned}
x_1^{IS} &= \frac{1}{3}C\left(3q_1\right)\\
x_2^{IS} &= \frac{1}{3}C(3q_1) + \frac{1}{2}(C(q_1 + 2q_2) - C(3q_1))\\
x_3^{IS} &= \frac{1}{3}C(3q_1) + \frac{1}{2}(C(q_1 + 2q_2) - C(3q_1)) + C(q_1 + q_2 + q_3) - C(q_1 + 2q_2).
\end{aligned}$$

Clearly, $x_1^{IS} + x_2^{IS} + x_3^{IS} = C(q_1 + q_2 + q_3)$, and the cost share of agent i is independent of the demands of agents $j > i$.

Now, the serial rule has a natural mirror-image commencing with the agent having the largest demand instead of the agent having the smallest demand, as suggested in De Frutos (1998). Intuitively, no agent will be held responsible for the consumption of agents with smaller demands even though they are associated with a joint project. In this case the resulting cost shares will be,

$$
\begin{aligned}
x_3^{DS} &= \frac{1}{3}C(3q_3) \\
x_2^{DS} &= \frac{1}{3}C(3q_3) + \frac{1}{2}(C(q_3 + 2q_2) - C(3q_3)) \\
x_1^{DS} &= \frac{1}{3}C(3q_3) + \frac{1}{2}(C(q_3 + 2q_2) - C(3q_3)) + C(q_1 + q_2 + q_3) - C(q_3 + 2q_2).
\end{aligned}
$$

Again, clearly $x_1^{DS} + x_2^{DS} + x_3^{DS} = C(q_1 + q_2 + q_3)$, and the cost share of agent i is independent of the demands of agents $j < i$. However, in this case agents are not guaranteed non-negative cost shares for problems in $\mathcal{D}$. In general, non-negative cost shares are only guaranteed for problems in $\check{\mathcal{D}}$, i.e., for concave cost functions.

Remark 2.5. The cost shares of serial cost sharing are characterized by some degree of independence of other agents demands. At first sight, this seems to be in line with straightforward ideas of fairness. However, if consumption of the produced good involves externalities such an independence may seem less appealing. For example, in case the cost function is related to a vaccination program, agents with zero demand pay zero but potentially benefit from the consumption of agents demanding the vaccine. Hence, the presence of externalities calls for the use of alternative rules or further knowledge about the agents individual utility functions. $\triangle$

2.3.2.3 Increasing, Decreasing and Mixed Serial Rules

In general, consider the case of n agents. Let the vector of demand q define intermediate production levels given by vectors $r \in \mathbf{R}^n$ and $s \in \mathbf{R}^n$ as,

$$
\begin{bmatrix} r_1 \\ r_2 \\ r_3 \\ \vdots \\ r_n \end{bmatrix} = \begin{bmatrix} n & 0 & 0 & \dots & 0 \\ 1 & n-1 & 0 & \dots & 0 \\ 1 & 1 & n-2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ \vdots \\ q_n \end{bmatrix}.
$$

$$
\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ \vdots \\ s_n \end{bmatrix} = \begin{bmatrix} n & 0 & 0 & \dots & 0 \\ 1 & n-1 & 0 & \dots & 0 \\ 1 & 1 & n-2 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & 1 & 1 & \dots & 1 \end{bmatrix} \begin{bmatrix} q_n \\ q_{n-1} \\ q_{n-2} \\ \vdots \\ q_1 \end{bmatrix}.
$$

Since demands are increasingly ordered we get that, $r_1 \leq \ldots \leq r_n = Q = s_n \leq \ldots \leq s_1$. Now, define the Increasing resp. Decreasing Serial Cost Sharing Rule as follows:

Increasing Serial Cost Sharing ϕ^{IS} is defined by cost shares

$$x_i^{IS} = \sum_{k=1}^{i} \frac{C(r_k) - C(r_{k-1})}{n+1-k}, \quad i = 1, \dots, n, \tag{2.14}$$

where $r_0 = 0$ by definition.

Decreasing Serial Cost Sharing ϕ^{DS} is defined by cost shares

$$x_{n-j+1}^{DS} = \sum_{k=1}^{j} \frac{C(s_k) - C(s_{k-1})}{n+1-k}, \quad j = 1, \dots, n, \tag{2.15}$$

where $s_0 = 0$ by definition.

Both rules are *order-preserving*, i.e.,

$$x_1^{IS} \leq \ldots \leq x_n^{IS} \quad \text{and} \quad x_1^{DS} \leq \ldots \leq x_n^{DS}.$$

Moreover, if C is linear $x^{IS} = x^{DS}$ and clearly both rules satisfy Constant Returns. Consequently, by Theorem 2.8, neither increasing nor decreasing serial cost sharing satisfy Monotonicity. (In particular, note that ϕ^{IS} does not satisfy Monotonicity on $\hat{\mathcal{D}}$.)

It is possible to establish some bounds (in q_i) on the cost shares. If C is *convex* then cost shares resulting from increasing serial cost sharing are bounded from below by the stand-alone cost and from above by the unanimity cost, i.e., $C(q_i) \leq x_i^{IS} \leq C(nq_i)/n$. Cost shares resulting from decreasing serial cost sharing are bounded from above by the unanimity cost whereas there is no lower bound, i.e., $x_i^{DS} \leq C(nq_i)/n$. If C is *concave* then x^{IS} is bounded from below by the unanimity cost and from above by the stand-alone cost, i.e., $C(nq_i)/n \leq x_i^{IS} \leq C(q_i)$, whereas cost shares resulting from decreasing serial cost sharing are bounded from below by the unanimity cost, i.e., $C(nq_i)/n \leq x_i^{DS}$. Hence, the increasing serial cost sharing rule ϕ^{IS} has the following (universal) bounds on $\mathcal{D}$;

$$\frac{1}{n}C(q_i) \leq \phi^{IS}(q, C) \leq C(nq_i),$$

whereas the decreasing serial rule fails both these (universal) bounds (note that the average cost sharing rule ϕ^{AC} also fails both these universal bounds).

Thus, in general with homogeneous costs there is no guarantee that using a serial rule we obtain individual rationality. This may not be surprising for convex cost functions but even if the cost function C is concave some groups of agents may end up paying more than their stand alone cost using the decreasing serial rule. Hougaard and Thorlund-Petersen (2000) provide a set of sufficient conditions for the decreasing serial rule to satisfy the stand alone requirements for all coalitions in case of concave cost functions.

Example 2.4. Assume that a group of n agents make a joint decision of renting a copying machine at a fixed cost of β whereafter each copy taken has a constant marginal cost of α. That is, for a given total demand Q the (concave) cost function can be written as $C(Q) = \alpha Q + \beta$ with $C(0) = 0$. Now, it could be argued that the cost share of agent i ought to be determined by $x_i = \alpha q_i + \beta/n$ since all agents are supposed to share the fixed cost equally and pay the marginal cost of each copy demanded. This is, in fact, also the result of using both increasing and decreasing serial cost sharing if all demands are strictly positive. However, if some agent demands 0 then according to decreasing serial cost sharing he will still be forced to pay his equal share of the fixed cost whereas using increasing serial cost sharing agents with zero demand avoid payment. Moreover, since all agents pay an equal share of the fixed cost, both rules works to the relative advantage of agents with high demands in the sense that agent specific unit prices x_i/q_i are decreasing in i. For comparison, note that average cost sharing results in shares, $x_i^{AC} = \alpha q_i + q_i\beta/Q$, where the fixed cost is shared in proportion to demand (also ensuring that zero-demand avoid payment) and agent specific unit prices are the same for all agents. △

It can be shown that if C is convex then the cost share of agent i using the increasing serial rule ϕ_i^{IS} is non-decreasing in the demand of agent j, q_j, for any $j \neq i$. If C is concave then the cost share of agent i using the increasing serial rule ϕ_i^{IS} (resp. the decreasing serial rule ϕ_i^{DS}) is non-increasing (resp. non-decreasing) in the demand of agent j, q_j, for any $j \neq i$.

In general, both rules have decreasing (resp. increasing) agent specific unit prices when the cost function is concave (resp. convex), i.e.,

$$\frac{x_1^{IS}}{q_1} \geq \ldots \geq \frac{x_n^{IS}}{q_n} \quad \text{and} \quad \frac{x_1^{DS}}{q_1} \geq \ldots \geq \frac{x_n^{DS}}{q_n},$$

for problems in $\check{\mathcal{D}}$ and

$$\frac{x_1^{IS}}{q_1} \leq \ldots \leq \frac{x_n^{IS}}{q_n} \quad \text{and} \quad \frac{x_1^{DS}}{q_1} \leq \ldots \leq \frac{x_n^{DS}}{q_n},$$

for problems in $\hat{\mathcal{D}}$. Thus, under concave cost functions (increasing returns in production) agents with modest demands are penalized whereas under convex cost functions (decreasing returns in production) agents with modest demands are favored by both rules.

Example 2.5. Assume that agents can choose their demand strategically and let costs be shared using Increasing Serial Cost Sharing ϕ^{IS}. For instance, assume that two departments demand some service in quantity q_i delivered at quadratic costs $C(q_1 + q_2) = (q_1 + q_2)^2$. Let both departments have linear utility in demand q_i and payment ϕ_i^{IS}, i.e., $u_i(q_i, \phi_i) = \alpha_i q_i - \phi_i^{IS}$, where $\alpha \in \mathbf{R}_{++}$. That is, dept. 1 chooses its demand for service q_1 by solving

$$\max_{q_1} \alpha_1 q_1 - 2q_1^2,$$

yielding $q_1^* = \alpha_1/4$. Note, that the optimal demand level of dept. 1 is independent of the demand of dept. 2 (since the cost share of dept. 1 does not depend on the demand of dept. 2). Dept. 2 chooses its demand for service by solving

$$\max_{q_2} \alpha_2 q_2 - \left[(q_1 + q_2)^2 - 2q_1^2\right],$$

and knowing that dept. 1 demands $q_1^* = \alpha_1/4$, we get that $q_2^* = (\alpha_2 - \alpha_1/2)/2$. Hence, in this case the cost sharing game induced by the increasing serial rule has got a unique Nash equilibrium in demands

$$(q_1^*, q_2^*) = \left(\frac{\alpha_1}{4}, \frac{2\alpha_2 - \alpha_1}{4}\right),$$

and total cost is shared as

$$(\phi_1^{IS}, \phi_2^{IS}) = \left(\frac{\alpha_1^2}{8}, \frac{2\alpha_2^2 - \alpha_1^2}{8}\right).$$

As indicated by the example, it turns out that (when the cost function is convex) the cost sharing game induced by ϕ^{IS} is dominance solvable and yields a unique (Strong) Nash equilibrium for any (convex and monotonic) preference profile, see Moulin and Shenker (1992). Recall, that in case of convex cost functions we disregard ϕ^{DS} as it may result in negative cost shares.

For comparison, assume that costs are shared using the Average Cost rule ϕ^{AC} instead. In this case it turns out to be important how we construe the process of announcing the demands (strategies). For instance, imagine that the departments simultaneously choose their level of demand. Then the corresponding induced (normal form) game has got at least one Nash equilibrium (Watts 1996): In the current example we get the unique equilibrium

$$(q_1^*, q_2^*) = \left(\frac{2\alpha_1 - \alpha_2}{3}, \frac{2\alpha_2 - \alpha_1}{3}\right).$$

However, if the departments make a sequential choice of demands making the induced cost sharing game dynamic (for example dept. 1 chooses first and that choice is observed by dept. 2, which then chooses) the resulting (subgame-perfect) Nash equilibrium becomes

$$(q_1^*, q_2^*) = \left(\frac{3\alpha_1 - 2\alpha_2}{4}, \frac{2\alpha_2 - \alpha_1}{2}\right).$$

For results on equilibrium existence in case of increasing returns (concave cost functions), see, e.g., Moulin (1996) concerning the Increasing Serial rule and De Frutos (1998) concerning Decreasing Serial rule.

A recent survey of main results concerning strategic games in cost sharing problems can be found in Koster (2009). △

Now, consider problems in $\mathcal{D}^+$ where the cost function equals a sum of a convex and a concave function. Such functions often appear in managerial economics where they are used to model that returns to scale may vary with the size of production. Clearly, the increasing serial cost sharing rule can be used directly on this domain. However, alternative rules may be defined using a decomposition of the cost function into a convex and a concave component.

Let a *decomposition rule* be defined by a mapping $\Gamma : \mathcal{D}^+ \to \hat{\mathcal{D}} \times \check{\mathcal{D}}$ where $C = R + S$ for $\Gamma(C) = (R, S)$ and normalized by the requirement that the right derivative of R at $Q = 0$ equals zero. In particular, we shall focus on the so-called *complementary-slack* (CS) decomposition which maximizes the role of the concave component, see Thon and Thorlund-Petersen (1986). Formally, let Γ_{CS} denote the complementary-slack (CS) decomposition where $\limsup_{\epsilon \to 0} \Delta^2_\epsilon R(Q) \Delta^2_\epsilon S(Q) = 0$ and $R'(0) = 0$ with $\Delta^2_\epsilon C(Q) = \epsilon^{-1}(C'(Q+\epsilon) - c'(Q))$ for $Q + \epsilon \geq 0$, $\epsilon \neq 0$ and C' being the right derivative. If C is twice continuously differentiable the conditions read $R''(Q)S''(Q) = 0$ and $R'(0) = 0$.

Example 2.6. In some cases there is a unique way to decompose a cost function into a convex and concave component. For example, consider the case where a good is sold at a price of \$1 per unit and a bundle of 10 goods is offered at a price of 80 cents per unit implying that the cost function is determined by

$$C(Q) = \begin{cases} Q & \text{if } Q < 8 \\ 8 & \text{if } 8 \leq Q < 10 \\ Q - 2 & \text{if } 10 \leq Q. \end{cases}$$

This cost function is uniquely decomposed into a sum of a convex and concave function as $C(Q) = R(Q) + S(Q) = \max\{0, Q - 10\} + \min\{Q, 8\}$. However, if a 10 cent excise tax is added then total costs equal $C(Q) + 0.1Q$ with CS-decomposition $\max\{0, Q - 10\} + \min\{1.1Q, 0.1Q + 8\}$ but this decomposition is not unique as, for example, another decomposition could be $\max\{0, 1.1Q - 11\} + \min\{1.1Q, 0.1Q + 8, 9\}$. △

Using the CS-decomposition rule to decompose the cost function into a convex and a concave component, we are able to introduce a mixture of increasing and decreasing serial cost sharing as suggested in Hougaard and Thorlund-Petersen (2001). Here, it is argued that the spirit of increasing serial cost sharing seems to fit best with convex cost functions (as agents with smaller demands should not be penalized by the fact that agents with larger demands cause the common cost to escalate) whereas the spirit of decreasing serial cost sharing fits well with concave cost functions (as decreasing marginal costs should penalize agents with small demands). Hence, the cost share paid by agent i should be determined as a sum of i's cost shares related to using ϕ^{IS} on the convex part and ϕ^{DS} on the concave part of the CS-decomposition respectively:

Mixed Serial Cost Sharing ϕ^{MS} is defined for problems in $\mathcal{D}^+$ where C has CS-decomposition $C = R + S$ as

$$\phi^{MS}(q, C) = \phi^{IS}(q, R) + \phi^{DS}(q, S). \tag{2.16}$$

Clearly, if C is convex (resp. concave) then mixed serial cost sharing coincides with increasing (resp. decreasing) serial cost sharing. Hence, for any problem in $\mathcal{D}^+$, mixed serial cost sharing results in non-negative cost shares.

Example 2.7. Consider three agents ($n = 3$) with demands $q = (1, 3, 5)$ and cost function $C(Q) = Q^2 + 64\sqrt{Q}$ which is concave on $[0, 4)$ and convex on $(4, \infty)$. This cost function C can be decomposed into a convex function $R^*(Q) = Q^2$ and a concave function $S^*(Q) = 64\sqrt{Q}$. Using the principle of mixing the serial cost sharing rules with respect to such a decomposition results in the following cost shares:

	Agent 1	Agent 2	Agent 3	Sum
$R^*(Q)$	$x_1^{IS} = 3$	$x_2^{IS} = 23$	$x_3^{IS} = 55$	81
$S^*(Q)$	$x_1^{DS} = 44.6$	$x_2^{DS} = 63.8$	$x_3^{DS} = 82.6$	192
Sum	$x_1^{IS} + x_1^{DS} = 47.6$	$x_2^{IS} + x_2^{DS} = 87.8$	$x_3^{IS} + x_3^{DS} = 137.6$	273

Alternatively, consider the CS-decomposition $C = \tilde{R} + \tilde{S}$ where

$$\tilde{R}(Q) = \begin{cases} 0 & \text{if } Q \leq 4 \\ Q^2 + 64\sqrt{Q} - 24Q - 48 & \text{if } Q > 4, \end{cases}$$

and,

$$\tilde{S}(Q) = \begin{cases} Q^2 + 64\sqrt{Q} & \text{if } Q \leq 4 \\ 24Q + 48 & \text{if } Q > 4. \end{cases}$$

Notice that for $Q \neq 4$ we have $\tilde{R}''(Q)\tilde{S}''(Q) = 0$. Using the CS-decomposition and the mixed serial cost sharing rule we obtain the following cost shares:

	Agent 1	Agent 2	Agent 3	Sum
$\tilde{R}(Q)$	$x_1^{IS} = 0$	$x_2^{IS} = 1.2$	$x_3^{IS} = 7.8$	9
$\tilde{S}(Q)$	$x_1^{DS} = 40$	$x_2^{DS} = 88$	$x_3^{DS} = 136$	264
Sum	$x_1^{IS} + x_1^{DS} = 40$	$x_2^{IS} + x_2^{DS} = 89.2$	$x_3^{IS} + x_3^{DS} = 143.8$	273

Notice that there is a significant difference in the resulting cost shares depending on the particular way that the cost function is decomposed. In particular, it appears that the cost shares resulting from mixed serial cost sharing (and the CS-decomposition) are Lorenz-dominated by the cost shares related a mixture of ϕ^{IS} and ϕ^{DS} but related to the alternative decomposition. In fact, this is no coincidence as it will be demonstrated in Theorem 2.12. △

2.3.2.4 Inequality Comparisons

Using the ordering of Lorenz-domination we can consider the relation between the serial rules and average cost sharing with respect to equality of the resulting cost shares. Such an ordering naturally depends on the specific domain of problems considered.

Proposition 2.4 (Hougaard and Thorlund-Petersen 2001). *For problems in $\hat{\mathcal{D}}$ (with convex cost functions) we have*

$$\phi^{AC} \succ_{LD} \phi^{IS} \succ_{LD} \phi^{DS}.$$

For problems in $\check{\mathcal{D}}$ (with concave cost functions) we have

$$\phi^{DS} \succ_{LD} \phi^{IS} \succ_{LD} \phi^{AC}.$$

Finally, for problems in $\mathcal{D}^+$ (with sums of convex and concave cost functions) we have

$$\phi^{MS} \succ_{LD} \phi^{IS} \ (\phi^{DS}).$$

Proof (sketch). First we note that $x_i^P = ACq_i$ where $AC = C(Q)/Q$. Hence by increasing (resp. decreasing) agent specific unit prices of ϕ^{IS} and ϕ^{DS} on $\hat{\mathcal{D}}$ (resp. $\check{\mathcal{D}}$,) we get that

$$\frac{x_1^{IS(DS)}}{x_1^{AC}} \leq \dots \leq \frac{x_n^{IS(DS)}}{x_n^{AC}} \quad \left(\frac{x_1^{IS(DS)}}{x_1^{AC}} \geq \dots \geq \frac{x_n^{IS(DS)}}{x_n^{AC}} \right).$$

Since, in general we have that if $u_1/v_1 \leq \dots \leq u_n/v_n$ for $v_1 > 0$ then $v \succ_{LD} u$ (see, e.g., Marshall and Olkin 1979) we obtain the desired result with respect to the relation between the serial rules and average cost sharing. The relation between ϕ^{IS} and ϕ^{DS} follows from Lemma 4 in Hougaard and Thorlund-Petersen (2001). □

In other words, Proposition 2.4 states that for problems with convex cost functions the cost shares of average cost sharing are more equally distributed than the cost shares of increasing serial cost sharing which are more equally distributed than the cost shares of decreasing serial cost sharing. For problems with concave cost functions the cost shares of decreasing serial cost sharing are more equally distributed than the cost shares of increasing serial cost sharing which are more equally distributed than the cost shares of average cost sharing, and finally, for problems where the cost function is a sum of convex and concave cost functions the cost shares of mixed serial cost sharing are more equally distributed than the cost shares of both increasing and decreasing serial cost sharing.

2.3.2.5 Axiomatic Characterization

Characterizing cost sharing rules in general, the structural property of additivity has drawn much attention: if a cost function is a sum of two separate cost functions then finding cost shares with respect to this aggregate function is tantamount to adding up the cost shares with respect to each separate cost function. In other words, cost shares should not depend on the way that the costs are categorized. Formally:

- *Additivity:* Let C_1 and C_2 be two cost functions then $\phi(q, C_1 + C_2) = \phi(q, C_1) + \phi(q, C_2)$.

Additivity is satisfied by increasing as well as decreasing serial cost sharing (and the average cost rule) but *not* by mixed serial cost sharing. Now, it turns out that Additivity together with Constant Returns and one further property of limited consistency is sufficient to single out increasing serial cost sharing. This limited consistency property (called Free Lunch) states that if the cost of serving n replica of a given agent's demand is zero then this agent i pays zero and cost shares of the remaining agents, $\phi_{-i}^{N\setminus i}$, are found removing the "zero-cost agent" and sharing the cost in the reduced problem $(q_{-i}, \tilde{C})$ where $\tilde{C}(z) = C(z + q_i)$ for agents in the set $N \setminus i$. Formally:

- *Free-Lunch:* If $C(nq_i) = 0$ then $\phi_i(q, C) = 0$ and

$$\phi_{-i}(q, C) = \phi_{-i}^{N\setminus i}(q_{-i}, \tilde{C}).$$

Clearly, Free-Lunch is violated by both the average cost rule and decreasing (and thereby also mixed) serial cost sharing since basically it states that if an agent can satisfy his demand by the free goods available to the group then he need not participate in the cost sharing exercise.

Theorem 2.9 (Moulin and Shenker 1994). *A continuous cost sharing rule ϕ on $\mathcal{D}$ satisfies Order-preservation, Constant Returns, Additivity and Free-Lunch if and only if it is the Increasing Serial Cost Sharing Rule ϕ^{IS}.*

It has already been noticed that Increasing Serial Cost Sharing satisfies Order-preservation, Constant Returns, Additivity and Free-Lunch. To prove the converse the reader is referred to the proof in Moulin and Shenker (1994). On the restricted domain of convex cost functions $\hat{\mathcal{D}}$, Moulin and Shenker (1994) note that ϕ^{IS} can be characterized by an axiom called Unanimity (Upper) Bound (stating that the cost share of agent i cannot exceed i's unanimity cost $C(nq_i)/n$) together with Continuity, Additivity and Free Lunch.

Now, a natural mirror image of the above characterization of the increasing serial rule can be provided in case of the decreasing serial cost sharing rule ϕ^{DS}.

Consider cost functions of the type $\Delta^t(z) = \min\{z, t\}$ where $t \geq 0$. These functions will be called plateau cost functions in the following, i.e., functions where total costs equal total demand z up to some threshold t from where the total cost remains fixed.

In case of plateau cost functions, the agent with the highest demand (agent n) should pay t/n if the total demand exceeds t in case all agents demanded the same quantity as agent n. This seems reasonable considering that average costs are decreasing and the group as a whole benefits from a large total demand. Now, having settled the cost share of the agent with the highest demand, this agent may now be removed from the set of agents and the cost shares of the remaining agents can be specified by imposing the same cost sharing rule on a reduced cost function (along the lines of the Free Lunch axiom). To formally define this property of Plateau Consistency we shall make use of the following definitions: For $S \subset N$, q^S is the projection of q on $\mathbf{R}^S$. Moreover, let $\alpha \geq 0$ be a demand and let $\beta \in \mathbf{R}$ be a cost share and define $C_{\alpha,\beta}(z) = (C(z+\alpha) - \beta)_+$ for $z \geq 0$ with $C_{\alpha,\beta}(0) = 0$, where for $z \in \mathbf{R}$, $(z)_+ = \max\{0, z\}$.

- *Plateau Consistency:* Let $n \geq 2$ and $C(z) = \min\{z, t\}$ and $nq_n \geq t$. Then $\phi_n(N, C, q) = t/n$ and $\phi_i(N, C, q) = \phi_i(N \backslash \{n\}, C_{q_n, t/n}, q^{N \backslash \{n\}})$ for $i \neq n$.

It is easy to see that Plateau Consistency is violated by increasing serial cost sharing: For example, let $N = \{1, 2\}$, $C(z) = \min\{z, t\}$ and $q = (0, t)$. Then according to the increasing serial rule $\phi_1^{IS}(\{1,2\}, C, q) = 0$ and $\phi_2^{IS}(\{1,2\}, C, q) = t$. According to the decreasing serial rule $\phi_1^{DS}(\{1,2\}, C, q) = \phi_2^{DS}(\{1,2\}, C, q) = t/2$ in line with Plateau Consistency. Actually, the plateau cost functions are extreme examples of how the increasing and decreasing serial rules differ on concave cost functions as specified in Proposition 2.4.

Now, it turns out to be convenient to extend $\mathcal{D}$ to the domain of all non-decreasing cost functions $\tilde{\mathcal{D}}$ (note that $C(0) = 0$ is not required here) since handling fixed costs becomes relevant.

Hence, we will introduce two additional axioms: (1) Fixed Cost, which states that in a situation of a fixed cost all agents have to share this cost equally, and (2) Zero Cost, which states that if it is free to provide n times the demand of agent n (the agent with the highest demand) then no agent pays. Formally:

- *Fixed Cost:* Let $\alpha > 0$ and $C(z) = \alpha$ for $z \geq 0$. Then, $\phi_i(N, C, q) = \alpha/n$, for all $i \in N$.
- *Zero Cost:* If $C(nq_n) = 0$ then $\phi_i(N, C, q) = 0$, for all $i \in N$.

Note that Zero Cost is satisfied by increasing serial rule while Fixed Cost obviously is not. Furthermore, it can be noted that Zero Cost together with Additivity and Plateau Consistency implies (a weak form of) Constant Returns.

Theorem 2.10 (Hougaard and Østerdal 2009). *A continuous cost sharing rule on $\tilde{\mathcal{D}}$ satisfies Fixed Cost, Zero Cost, Additivity and Plateau Consistency if and only if it is the Decreasing Serial Cost Sharing Rule ϕ^{DS}.*

Before proving Theorem 2.10, we make a useful observation (omitting the straightforward proof). In case of plateau cost functions Δ^t, the decreasing serial rule has a particularly simple structure: Agents i, for which s_{n+1-i} (i.e., the total demand in case all agents j for which $j < i$ also demand q_i) exceeds the threshold t, all pay t/n, while agents with smaller demands pay their share as if there was a common average cost, plus an equal share of the residual cost. Formally:

Lemma 2.1. *Let $t > 0$, and let (N, Δ^t, q) be a cost sharing problem. If $s_1 < t$, then $\phi_j^{DS}(N, \Delta^t, q) = q_j$ for $j = 1, \ldots, n$. If $s_1 \geq t$, let i be the smallest positive integer for which $s_{n+1-i} \geq t$. Then*

$$\phi_j^{DS}(N, \Delta^t, q) = \frac{t}{n}, \quad j = i, \ldots, n,$$

and

$$\phi_j^{DS}(N, \Delta^t, q) = q_j + \frac{\sum_{k=i}^{n} q_k - (n+1-i)t/n}{i-1}, \quad j = 1, \ldots, i-1.$$

Proof of Theorem 2.10 (sketch). It is simple to demonstrate that ϕ^{DS} satisfies the properties in question. Hence, consider the converse claim.

Using the definitions $L(z) = z, \Delta^t(z) = \min\{z, t\}$ and $\Lambda^t(z) = \max\{0, z - t\}$ then, since Zero Cost together with Additivity and Plateau Consistency implies (a weak form of) Constant Returns,

$$\phi_i(N, \Delta^{nq_n}, q) = \phi_i(N, L, q) - \phi_i(N, \Lambda^{nq_n}, q) = q_i - 0 = q_i,$$

for all i.

Observe that $\Lambda^t = L - \Delta^t$ for all t. Let $t \geq nq_n$. By Zero Cost and Additivity,

$$\phi_i(N, \Delta^t, q) = \phi_i(N, \Delta^{nq_n}, q) + \phi_i(N, \Delta^t - \Delta^{nq_n}, q) = \phi_i(N, \Delta^{nq_n}, q),$$

since $\Delta^t - \Delta^{nq_n}$ is non-decreasing and has value 0 at nq_n. Thus, for $t \geq nq_n = s_1$ we have $\phi_i(N, \Delta^t, q) = q_i$, for all i, and consequently, $\phi(N, \Delta^t, q) = \phi^{DS}(N, \Delta^t, q)$ by Lemma 2.1. In the remainder of the proof we assume that $t < s_1$.

By Plateau Consistency we have $\phi_n(N, \Delta^t, q) = t/n$. Now, consider an arbitrary $i \neq n$, and suppose that $\phi_j(N, \Delta^t, q) = \phi_j^{DS}(N, \Delta^t, q)$ for all $j = i+1, ..., n$. We will show that $\phi_i(N, \Delta^t, q) = \phi_i^{DS}(N, \Delta^t, q)$. For this, we consider two separate cases: (1) $t - \sum_{j=i+1}^{n} q_j \geq 0$, and (2) $t - \sum_{j=i+1}^{n} q_j < 0$.

Case (1). Repeated use of Fixed Cost, Additivity and Plateau Consistency gives

$$\begin{aligned}\phi_i(N,\Delta^t,q) &= \phi_i\left(N\backslash\{i+1,...,n\},\Delta^{t-\sum_{j=i+1}^n q_j},q^{N\backslash\{i+1,...,n\}}\right)\\ &\quad+\frac{q_n-t/n}{n-1}+\frac{q_{n-1}-\frac{t-q_n}{n-1}}{n-2}+...+\frac{q_{i+1}-\frac{t-\sum_{k=i+2}^n q_k}{i+1}}{i}\\ &= \phi_i\left(N\backslash\{i+1,...,n\},\Delta^{t-\sum_{j=i+1}^n q_j},q^{N\backslash\{i+1,...,n\}}\right)\\ &\quad+\frac{\sum_{k=i+1}^n q_k-(n-i)t/n}{i}.\end{aligned}$$

If $s_{n+1-i}\geq t$, then by Plateau Consistency we get

$$\phi_i(N\backslash\{i+1,...,n\},\Delta^{t-\sum_{j=i+1}^n q_j},q^{N\backslash\{i+1,...,n\}})=\frac{t-\sum_{k=i+1}^n q_k}{i},$$

hence

$$\begin{aligned}\phi_i(N,\Delta^t,q) &= \frac{\sum_{k=i+1}^n q_k-(n-i)t/n}{i}+\frac{t-\sum_{k=i+1}^n q_k}{i}\\ &= \frac{t}{n}.\end{aligned}$$

If $s_{n+1-i}<t$ we have

$$\phi_i(N\backslash\{i+1,...,n\},\Delta^{t-\sum_{j=i+1}^n q_j},q^{N\backslash\{i+1,...,n\}})=q_i$$

and consequently

$$\phi_i(N,\Delta^t,q)=q_i+\frac{\sum_{k=i+1}^n q_k-(n-i)t/n}{i}.$$

By Lemma 2.1, we conclude that $\phi_i(N,\Delta^t,q)=\phi_i^{DS}(N,\Delta^t,q)$. □

Case (2). By Lemma 2.1, we have $\phi_j^{DS}(N,\Delta^t,q)=\frac{t}{n}$ for $j=i+1,...,n$. By Plateau Consistency, $\phi_i(N,\Delta^t,q)=t/n$. Now, using Plateau Consistency the residual cost function becomes fixed hence using Fixed Cost we get that $\phi_j(N,\Delta^t,q)=\frac{t}{n}$ for all $j<i$. Thus, $\phi_i(N,\Delta^t,q)=\phi_i^{DS}(N,\Delta^t,q)$.

We hence conclude that $\phi(N,\Delta^t,q)=\phi^{DS}(N,\Delta^t,q)$.

This may be now be generalized to the entire domain of non-decreasing cost functions as demonstrated in Hougaard and Østerdal (2009). □

On the restricted domain of concave costs functions $\check{\mathcal{D}}$ it can be shown that ϕ^{DS} can be characterized by (continuity), Quasi-fixed Cost, Additivity, Plateau Consistency and Unanimity (Lower) Bound (stating that the cost share of any agent i cannot be smaller than the unanimity cost $C(nq_i)/n$).

Now, let $\nu_{\Gamma;\rho,\sigma}$ be a cost sharing rule on $\mathcal{D}^+$ using decomposition Γ and additive sub-rules ρ on $\hat{\mathcal{D}}$ and σ on $\check{\mathcal{D}}$. It turns out that mixed serial cost sharing can be characterized as the only rule that coincides with increasing and decreasing serial cost sharing on the respective domains $\hat{\mathcal{D}}$ and $\check{\mathcal{D}}$, is independent of cost levels below nq_1 and above nq_n, and additive with respect to fixed costs. Formally:

- *Extension:* The cost sharing rule $\nu_{\Gamma;\rho,\sigma}$ is identical to ϕ^{IS} on $\hat{\mathcal{D}}$ and ϕ^{DS} on $\check{\mathcal{D}}$.
- *Independence of irrelevant cost levels:* Let q be given. If two cost functions $C_1, C_2 \in \mathcal{D}^+$ coincide on the interval $[a, b]$ and $a \leq nq_1 < nq_n \leq b$, then $\nu_{\Gamma;\rho,\sigma}(q, C_1) = \nu_{\Gamma;\rho,\sigma}(q, C_2)$.
- *Fixed-cost additivity:* For any cost function $C \in \mathcal{D}^+$ and fixed-cost function $F \in \mathcal{D}^+$, $\nu_{\Gamma;\rho,\sigma}(q, C + F) = \nu_{\Gamma;\rho,\sigma}(q, C) + \nu_{\Gamma;\rho,\sigma}(q, F)$.

Theorem 2.11 (Hougaard and Thorlund-Petersen 2001). *A decomposition based cost sharing rule $\nu_{\Gamma;\rho,\sigma}$ (with additive sub-rules ρ and σ) on $\mathcal{D}^+$, satisfies Extension, Independence of irrelevant cost levels and Fixed-cost additivity if and only if it is the Mixed Serial Cost Sharing Rule φ^{MS}.*

Proof. It is easily verified that Mixed Serial Cost Sharing satisfies Extension, Independence of irrelevant cost levels and Fixed-cost additivity. To prove the converse: It suffices to consider a piecewise affine cost function C which is determined by some subdivision $0 < Q_1 < Q_2 \ldots$ of $[0, \infty)$. If C equals a convex angle function $C^{\alpha,\beta}(Q) = \max\{0, \alpha Q - \beta\}, \alpha, \beta > 0$, then C only has decomposition $C = C + 0$. Thus, by Extension $\nu_{\Gamma;\rho,\sigma}(q, C^{\alpha,\beta}) = \phi^{IS}(q, C^{\alpha,\beta})$. Since every piecewise affine convex cost function equals a finite sum of convex angle functions on any bounded interval and by assumption the sub-rule ρ is additive we get that $\rho = \phi^{IS}$. Similarly, considering concave angle functions we get $\sigma = \phi^{DS}$.

Now, consider a cost function C decomposed by Γ and demand q confined by the interval $[a, b]$ in the sense that $a \leq nq_1 < nq_n \leq b$. First, if C is convex on $[a, b]$ then there exists a convex function C^* and a fixed-cost function F^* such that $C^* + F^*$ coincides with C on $[a, b]$. Hence, by Extension and Fixed-cost additivity

$$\nu_{\Gamma;\phi^{IS},\phi^{DS}}(q, C^* + F^*) = \nu_{\Gamma;\phi^{IS},\phi^{DS}}(q, C^*) + \phi^{DS}(q, F^*),$$

and since $\nu_{\Gamma;\phi^{IS},\phi^{DS}}(q, C^*) = \phi^{IS}(q, R + S - F^*)$ we get that S must be affine on $[a, b]$. Likewise we can show that if C is concave on $[a, b]$ then R must be affine on this interval. Consequently, the piecewise affine function is decomposed according to the CS-decomposition. $\square$

Note, that even though ϕ^{IS} and ϕ^{DS} both are additive rules, mixed serial cost sharing does not satisfy Additivity but only Fixed-cost Additivity.

Now, among all cost sharing rules $\nu_{\Gamma;\phi^{IS},\phi^{DS}}$ on $\mathcal{D}^+$ using decomposition Γ and additive sub-rules ϕ^{IS} on $\hat{\mathcal{D}}$ and ϕ^{DS} on $\check{\mathcal{D}}$, mixed serial cost sharing can be characterized as resulting in the most unequal allocation of costs. A somewhat striking result since the CS-decomposition maximizes the concave component and thereby intuitively maximizes the role of ϕ^{DS} which might be expected to lead to more equally distributed cost shares. On the other hand, it follows from Proposition 2.4 that using increasing serial cost sharing on the entire domain of $\mathcal{D}^+$ results in more equal distributions than using mixed serial cost sharing.

Theorem 2.12 (Hougaard and Thorlund-Petersen 2001). *For any decomposition* Γ*,*

$$\nu_{\Gamma;\phi^{IS},\phi^{DS}} \succ_{LD} \phi^{MS}.$$

Proof. Let $\Gamma(C) = R^* + S^*$ and $\Gamma_{CS}(C) = R + S$ then it must be shown that for any $k = 1, \ldots, n$

$$\sum_{i=1}^{k} (\phi_i^{IS}(q, R) + \phi_i^{DS}(q, S)) \leq \sum_{i=1}^{k} (\phi_i^{IS}(q, R^*) + \phi_i^{DS}(q, S^*)).$$

Now, let $G = R^* - R = S - S^*$ then $\sum_{i=1}^{k} \phi_i^{DS}(q, G) \leq \sum_{i=1}^{k} \phi_i^{IS}(q, G)$ and since it can be shown that G is an increasing convex function (see, e.g., Thon and Thorlund-Petersen 1986) the theorem follows from $\phi^{IS} \succ_{LD} \phi^{DS}$ on $\hat{\mathcal{D}}$ cf. Proposition 2.4. $\square$

2.3.2.6 Manipulation

In case of homogeneous cost functions the only non-manipulable cost sharing rule is the average cost rule. Hence, all serial rules can be manipulated. In particular, it can be demonstrated that with *convex* cost functions, i.e., for problems in $\hat{\mathcal{D}}$, both increasing and decreasing serial cost sharing can be manipulated by coalitions *equalizing* their demand (for a fixed number of agents) and by agents *splitting* their demand (variable number of agents).

Example 2.8. Let $N = \{1, 2, 3\}$ and $q = (1, 2, 3)$. Then for the convex homogeneous cost function

$$C(Q) = \max\{0, Q - 4.5\},$$

we get the following cost shares using increasing serial cost sharing,

$$x^{IS} = (0, 0.25, 1.25).$$

Now, let agents 1 and 2 equalize their demand such that the new resulting demand vector becomes $\hat{q} = (1.5, 1.5, 3)$. In this case increasing serial cost sharing results in cost shares

$$\hat{x}^{IS} = (0, 0, 1.5),$$

making agents 1 and 2 better off as a group. Also, let agent 2 split his demand into two new demands such that the new demand vector becomes $\tilde{q} = (1, 1, 1, 3)$. In this case increasing serial cost sharing yield the following cost shares

$$\tilde{x}^{IS} = (0, 0, 0, 1.5),$$

making agent 2 better off. △

Using a similar type of example it can be demonstrated that with *concave* cost functions, i.e., for problems in $\check{\mathcal{D}}$, both increasing and decreasing serial cost sharing can be manipulated by coalitions *spreading* their demand (for a fixed number of agents) and by agents *merging* their demand (variable number of agents).

2.3.3 Rules Based on the Incremental Principle

According to the serial principle above, demands are increasingly ordered and incremental costs (given that ordering) are shared *equally* between the relevant group of agents. Now, consider any ordering of demands: according to the idea of the *incremental principle* agent i *alone* must cover the incremental cost associated with satisfying i's demand given the demand of all agents prior to i in the ordering. In other words, agent i must pay the additional costs connected with a joint operation involving agents with demands prior to i's in the ordering.

Formally, let $\pi : \mathbf{R}^n_+ \to \mathbf{R}^n_+$ be an ordering of demands and let $S_{\pi,i} = \{j \in N | \pi(j) \leq i\}$ denote the set of indices of the first i demands given the ordering π. Now, for a *decomposable* cost function C the incremental principle results in cost shares,

$$x_i^\pi = C\left(\max_{j \in S_{\pi,i}} \{q_j\}\right) - C\left(\max_{j \in S_{\pi,i-1}} \{q_j\}\right). \tag{2.17}$$

Likewise, for a *homogeneous* cost function C the incremental principle results in cost shares,

$$x_i^\pi = C\left(\sum_{j \in S_{\pi,i}} q_j\right) - C\left(\sum_{j \in S_{\pi,i-1}} q_j\right). \tag{2.18}$$

To illustrate the principle, consider the simple case with three agents $N = \{1, 2, 3\}$ and increasingly ordered demands $q_1 \leq q_2 \leq q_3$. For a decomposable cost function the resulting cost shares according to the incremental principle become,

$$x_1 = C(q_1), \quad x_2 = C(q_2) - C(q_1), \quad x_3 = C(q_3) - C(q_2).$$

Likewise, the resulting cost shares for a homogeneous cost function become,

$$x_1 = C(q_1), \quad x_2 = C(q_2 + q_1) - C(q_1), \quad x_3 = C(q_3 + q_2 + q_1) - C(q_2 + q_1).$$

Clearly, the cost share of individual agents depends crucially on the ordering of demands and it is questionable whether such cost shares are fair to all agents for a given ordering (in particular in case of decomposable cost functions). In fact, note that there is a close connection between the incremental principle and the priority rules of the rationing model discriminating between agents.

However, since there are n agents there are $n!$ different orderings of demand vector q. Denote by $\Pi = \{\pi^j | j = 1, \ldots, n!\}$ the set of all such orderings. Consequently, there is a set of $n!$ possible cost shares for every agent, which are in accordance with the incremental principle. Just as the random priority rule of the rationing model was found by taking the average of awards over all the $n!$ possible priorities (see Sect. 2.2.5) we may, in the present model, define a cost sharing rule that takes the average of the incremental cost shares over the set of $n!$ different orderings of demand. Such a cost sharing rule will be called the Shapley rule due to its obvious relation to the Shapley value of a cooperative game, see, e.g., Shubik (1962).

- *The Shapley Cost Sharing Rule* ϕ^{Sh} is defined by cost shares,

$$x_i^{Sh} = \frac{1}{n!} \sum_{\pi \in \Pi} x_i^{\pi} \tag{2.19}$$

for all $i = 1, \ldots, n$.

Further analysis of incremental cost shares as well as the Shapley cost sharing rule is postponed to the next chapter where we consider cost allocation as cooperative games, that is, for the case where the cost function is defined for binary demands $C : \{0,1\}^N \to \mathbf{R}_+$.

Remark 2.6. For decomposable cost functions $C(\mathcal{Q})$ it can be noted that the Shapley cost sharing rule as defined in (2.19) coincides with the serial rule as defined in (2.14). △

Presently, an example will be used to compare the results of the different allocation rules of Sect. 2.3.

Example 2.9. Recall the situation of Example 2.6 where three agents ($n = 3$) with demands $q = (1, 3, 5)$ are operating under a homogeneous cost function $C(Q) = Q^2 + 64\sqrt{Q}$. Since the total cost is $C(9) = 273$, the equal split rule results in the cost shares $x^E = (91, 91, 91)$, but since agent 1's stand-alone cost of $C(1) = 65$ is smaller than x_1^E the restricted equal split rule results in the allocation $x^{RE} = (65, 104, 104)$. Now, cost shares using average cost sharing, increasing serial cost sharing and mixed serial cost sharing are respectively,

$x^{AC} = (30.33, 91, 151.67)$,
$x^{IS} = (39.95, 89.19, 143.86)$,
$x^{MS} = (40, 89.2, 143.8)$.

Clearly, the shares of mixed serial cost sharing are more equally distributed than the shares of increasing serial cost sharing (a general property according to Proposition 2.4) and in this case also more equally distributed than average cost sharing.

Finally, using the Shapley cost sharing rule, cost shares can be found by using the incremental principle with respect to all possible orderings of demand and take the average. Since we get that,

$x^{\pi^1} = (65, 79, 129)$,
$x^{\pi^2} = (65, 80.23, 127.76)$,
$x^{\pi^3} = (24.15, 119.85, 129)$,
$x^{\pi^4} = (27.98, 119.85, 125.15)$,
$x^{\pi^5} = (24.66, 80.23, 168.10)$,
$x^{\pi^6} = (27.98, 76.91, 168.10)$,

the resulting cost shares of the Shapley rule become,

$$x^{Sh} = (39.13, 92.68, 141.19).$$

△

2.3.4 Comments

As mentioned in Moulin (2002) there is a clear formal relationship (a linear isomorphism) between order-preserving rationing rules and additive cost sharing rules. For example, the proportional rationing rule corresponds to the average cost rule, the random priority rule corresponds to the Shapley cost sharing rule, etc.

In terms of application, average cost sharing is very appealing in its simplicity and underlying notion of fairness (in proportion). Hence, it is often applied in practical cases along with its egalitarian counter part, the serial principle. For example, as mentioned in Sect. 1.3.2, Aadland and Kolpin (1998) record that both average cost sharing and the serial principle are used for sharing irrigation costs among farmers. Moreover, Herzog et al. (1997) demonstrate that the serial principle can be applied sharing multicast cost in computer networks according to the "Equal Link Split Downstream" method. Both papers provide an axiomatic analysis of the specific methods applied.

2.4 Summary

In this chapter we considered simple sharing problems where a group of agents, characterized by a one-dimensional individual factor such as a claim or a demand, should share a common cost (or value) which is either fixed or varies with the level of the characteristic.

In case of sharing a fixed cost (or value) rationing problems were considered. Basically, allocation rules are build around two principles of fairness; *proportionality* and *equality*. However, equality is not well-defined within the model since a rationing problem can be construed either as sharing what is left or as sharing what is lost. The Talmud allocation rule (building on the notion of equality) can be seen as an answer to this problem since this rule is constructed to be self-dual (like the proportional rule), i.e., independent of whether focus is on gains or losses.

It was shown that allocation rules based on both types of fairness satisfied fundamental principles like Equal Treatment of Equals and Consistency. However, only the proportional allocation rule were non-manipulable. Consequently, in cases where manipulation in the form of merging, splitting or reallocation of claims is a potential possibility, non-manipulability is a strong argument in favor of choosing the proportional allocation rule. Moreover, the proportional allocation also relates to the notion of equality although in a more indirect sense: the proportional rule were actually the only rule that preserved equality in the distribution of shares, that is, if characteristics (e.g., claims) become more equally distributed so does the allocated shares (and dually; losses).

In case of sharing a cost (or value) which varies with the level of characteristics, homogeneous (and decomposable) cost sharing problems were considered. Here, principles of proportionality and equality are still relevant but knowledge of the entire cost function introduces the possibility of constructing new allocation principles such as the *serial* and the *incremental* principle. In the present chapter we focussed on the serial principle while further analysis of the incremental principle is postponed to later chapters.

In both cases, however, the principles are based on a pre-specified ordering of agents characteristics (demand). Using the serial principle, the two natural orderings of increasing and decreasing demand defines two distinct allocation rules; increasing and decreasing serial cost sharing. As argued in the text the increasing serial rule seems most relevant on the domain of convex cost functions while the decreasing serial rules seems most relevant on the domain of concave cost functions. Generalizing this observation to the domain of all cost functions that can be decomposed into sums of convex and concave functions we analyzed a non-additive allocation rule called mixed serial cost sharing. On its relevant domain the cost shares resulting from mixed serial cost sharing were more equally distributed than those of both increasing and decreasing serial cost sharing.

Contrary to the proportional rule (average cost sharing), rules based on the serial principle may all be manipulated. Moreover, the proportional rule further satisfies a natural requirement of increasing cost shares for all agents when common costs are increasing – a property which is violated by the serial (and incremental) rules. The serial (and incremental) rules, however, all have the advantage that they are better in reflecting the underlying cost structure than average cost sharing where only information about the cost of total demand is utilized.

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