Chapter IX
Generalized Entropy Formula

The entropy and the Lyapunov exponents provide two different ways of measuring the complexity of the dynamical behavior of a C^2 endomorphism $f: M \hookleftarrow$ associated with an invariant measure μ. Generally speaking, the entropy of the system (M, f, μ) is bounded up by the sum of positive Lyapunov exponents. This is the famous Ruelle inequality introduced in Chapter II. In some cases such as the invariant measure being absolutely continuous with respect to the Lebesgue measure (see Chapter VI), the inequality can become equality. The equality is the notable Pesin's entropy formula. As we have shown in Chapter VII, Pesin's entropy formula is equivalent to the SRB property of the invariant measure. Ledrappier and Young [43] presented a generalized entropy formula, which looks like and covers Pesin's entropy formula, for any Borel probability measure invariant under a C^2 diffeomorphism. This result is successfully generalized to random diffeomorphisms [70].

In this chapter we will extend Ledrappier and Young's result to the case of C^2 endomorphisms following the line of [71] (see Theorem IX.1.3).

IX.1 Related Notions and Statements of the Main Results

Let M be an m_0-dimensional smooth and compact Riemannian manifold without boundary and $f: M \hookleftarrow$ be a C^2 endomorphism. Throughout this chapter, it is always assumed that the invariant measure μ of f satisfies the integrability condition (V.V.1). We will resume the settings in Chapter V and employ the results in Chapters V and VII in an essential way.

Let $\lambda_1(x) > \lambda_2(x) > \cdots > \lambda_{r(x)}(x)$ be the Lyapunov exponents of f at x with multiplicities $m_1(x), \cdots, m_{r(x)}(x)$. Since condition (V.V.1) holds, $T_x f$ is invertible for μ-a.e. x. Denote by $(M^f, \theta, \tilde{\mu})$ the system (M^f, θ) associated with $\tilde{\mu}$. We can apply the Oseledec multiplicative ergodic theorem to $(M^f, \theta, \tilde{\mu})$ yielding the following. (See Appendix I, Proposition I.3.5.) There exists a θ-invariant Borel set $\widetilde{\Delta} \subset M^f$

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with full $\tilde{\mu}$-measure. For each $\tilde{x} \in \widetilde{\Delta}$, there is a splitting (which depends measurably on $\tilde{x}$) of $T_{x_0}M$

$$T_{x_0}M = E_1(\tilde{x}) \bigoplus E_2(\tilde{x}) \bigoplus \cdots \bigoplus E_{r(x_0)}(\tilde{x}) \tag{IX.1}$$

with $dimE_i(\tilde{x}) = m_i(x_0)$ for each i, such that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |T_0^n(\tilde{x})v| = \lambda_i(x_0)$$

for any $0 \neq v \in E_i(\tilde{x})$ $(i = 1, \cdots, r(x_0))$, where

$$T_0^n(\tilde{x}) \overset{\text{def}}{=} \begin{cases} T_{x_0}f^n, \text{ if } n > 0, \\ id, \text{ if } n = 0, \\ (T_{x_n}f^{-n})^{-1}, \text{ otherwise.} \end{cases}$$

For each $\tilde{x} \in \widetilde{\Delta}$, f is locally invertible along the full orbit $\tilde{x} = \{x_i\}_{i\in\mathbb{Z}}$. On M, the map $f_{\tilde{x}}^{-1}$ can be defined along a trajectory $\tilde{x}$ to be the " inverse" map of f which maps x_0 to x_{-1}, wherever it makes sense, i.e.,

$$f_{\tilde{x}}^{-1} \circ f = id \quad \text{and} \quad f \circ f_{\tilde{x}}^{-1} = id$$

hold true on certain neighbors of x_{-1} and x_0 respectively (see Chapter V.2). We write

$$f_{\tilde{x}}^{-n} \overset{\text{def}}{=} f_{\theta^{-n+1}\tilde{x}}^{-1} \circ \cdots \circ f_{\tilde{x}}^{-1}$$

for $n > 0$ with $f_{\tilde{x}}^0 \overset{\text{def}}{=} id_M$.

Let $u(x), c(x)$ and $s(x)$ be the number of positive, neutral and negative Lyapunov exponents at x respectively, i.e.,

$$u(x) \overset{\text{def}}{=} \#\{1 \leq j \leq r(x) : \lambda_j(x) > 0\}, \tag{IX.2}$$

$$c(x) \overset{\text{def}}{=} \#\{1 \leq j \leq r(x) : \lambda_j(x) = 0\}, \tag{IX.3}$$

$$s(x) \overset{\text{def}}{=} \#\{1 \leq j \leq r(x) : \lambda_j(x) < 0\}. \tag{IX.4}$$

When μ is an ergodic measure, all numbers $r(x), \lambda_i(x), m_i(x), u(x), c(x), s(x)$ will be constants for μ-a.e. x. In this case, when writing them we will just omit x's.

IX.1.1 Pointwise Dimensions and Transverse Dimensions

Definition IX.1.1 $\widetilde{W}^i(\tilde{x}) \overset{\text{def}}{=} \{\tilde{y} \in M^f : \limsup\limits_{n\to+\infty} \frac{1}{n} \log d(x_{-n}, y_{-n}) \leq -\lambda_i(x_0)\}$ is called the i^{th}-*unstable set* of f at $\tilde{x}$ in M^f, where $\tilde{x} \in \widetilde{\Delta}$ and $1 \leq i \leq u(x_0)$. $W^i(\tilde{x}) \overset{\text{def}}{=} p\widetilde{W}^i(\tilde{x})$ is called the i^{th}-*unstable manifold* of f at $\tilde{x}$ in M.

$W^i(\tilde{x})$'s are all $C^{1,1}$ immersed submanifolds of M tangent at x_0 to $\oplus_{j=1}^{i} E_j(\tilde{x})$ respectively (see Chapter V). Hence each $W^i(\tilde{x})$ inherits a Riemannian structure from M. This gives rise to a Riemannian metric, written $d^i_{\tilde{x}}(\cdot,\cdot)$, on each leaf of $W^i(\tilde{x})$.

Definition IX.1.2 A measurable partition η of M^f is said to be *subordinate to W^i-manifolds* if for $\tilde{\mu}$-a.e. $\tilde{x}$, $\eta(\tilde{x})$ has the following properties:

(1) $p|_{\eta(\tilde{x})} : \eta(\tilde{x}) \to p\,\eta(\tilde{x})$ is bijective;

(2) There exists a $\sum\limits_{k=1}^{i} m_k(x_0)$-dimensional C^1 embedded submanifold $W^i_{\tilde{x}}$ of M with $W^i_{\tilde{x}} \subset W^i(\tilde{x})$ such that $p\,\eta(\tilde{x}) \subset W^i_{\tilde{x}}$ and $p\,\eta(\tilde{x})$ contains an open neighborhood of x_0 in $W^i_{\tilde{x}}$, this neighborhood being taken in the topology of $W^i_{\tilde{x}}$ as a submanifold of M.

We have included in Section IX.2.2 an outline of the construction of such partitions. See also [73] for a similar construction.

Definition IX.1.3 A measurable partition η is said to be *increasing*, if $\theta^{-1}\eta > \eta$, and to be a *generator* , if $\mathscr{B}(\eta_0^n) \to \mathscr{B}(M^f), (\tilde{\mu} - \mathrm{mod}\ 0)$ as $n \to +\infty$, where $\eta_0^n \stackrel{\text{def}}{=} \bigvee\limits_{k=0}^{n} \theta^{-k}\eta$ and $\mathscr{B}(\eta)$ denotes the σ-algebra generated by measurable η-sets.

In what follows we will define notions of transverse dimensions along unstable manifolds.

Let $\varepsilon > 0$. For each $\tilde{x} \in \widetilde{\Delta}$, define

$$\widetilde{B}^i(\tilde{x};\varepsilon) \stackrel{\text{def}}{=} \{\tilde{y} \in \widetilde{W}^i(\tilde{x}) : d^i_{\tilde{x}}(x_0,y_0) < \varepsilon\}. \tag{IX.5}$$

Let $\eta_1 > \eta_2 > \cdots > \eta_u$ be a sequence of measurable partitions of M^f with each η_i subordinate to the corresponding W^i-manifolds. The canonical system of conditional measures of $\tilde{\mu}$ associated with η_i is denoted by $\{\tilde{\mu}^{\eta_i}_{\tilde{x}}\}$. We define the *lower* and *upper pointwise dimension of $\tilde{\mu}$ along W^i-manifolds* at $\tilde{x} \in \widetilde{\Delta}$ with respect to partition η_i by

$$\underline{\delta}_i(\tilde{x};\eta_i) \stackrel{\text{def}}{=} \liminf_{\varepsilon\to 0} \log \tilde{\mu}^{\eta_i}_{\tilde{x}}\big(\widetilde{B}^i(\tilde{x};\varepsilon)\big)/\log\varepsilon, \tag{IX.6}$$

$$\overline{\delta}_i(\tilde{x};\eta_i) \stackrel{\text{def}}{=} \limsup_{\varepsilon\to 0} \log \tilde{\mu}^{\eta_i}_{\tilde{x}}\big(\widetilde{B}^i(\tilde{x};\varepsilon)\big)/\log\varepsilon. \tag{IX.7}$$

Sometimes we denote $\underline{\delta}_i(\tilde{x};\eta_i)$ by $\underline{\delta}_i(\tilde{x};\eta_i,\tilde{\mu})$ to indicate the dependence of this quantity on $\tilde{\mu}$. Other notations have similar meanings.

The following proposition tells us that the lower and upper pointwise dimension of $\tilde{\mu}$ along W^i-manifolds are coincident and in fact well defined on M. We call this common value the *pointwise dimension of μ along W^i-manifolds*. It will be verified in Sections IX.4 and IX.5.

Proposition IX.1.1 *If η_i is an increasing generator subordinate to W^i-manifolds, then*

$$\underline{\delta}_i(\tilde{x};\eta_i) = \overline{\delta}_i(\tilde{x};\eta_i), \quad \tilde{\mu} - \text{a.e. } \tilde{x};$$

furthermore the common value, writing $\delta_i(\tilde{x})$, is θ-invariant and depends $\tilde{\mu}$-a.e. only on x_0, not on the choice of such η_i. Therefore $\delta_i(\tilde{x})$ is simply denoted by $\delta_i(x_0)$.

Proposition IX.1.2 *Let $\delta_i(x), 1 \le i \le u(x)$ be introduced as above, define*

$$\gamma_i(x) \stackrel{\text{def}}{=} \delta_i(x) - \delta_{i-1}(x)$$

with $\delta_0(x) \stackrel{\text{def}}{=} 0$ for μ-a.e. x and $1 \le i \le u(x)$, then

$$0 \le \gamma_i(x) \le m_i(x), \textit{ for } i = 1, \cdots, u(x).$$

The number $\gamma_i(x)$ is called the *transverse dimension of μ on $W^i(\tilde{x})/W^{i-1}$* at x.

IX.1.2 Statements of the Main Results

The main results of this chapter are the following theorems.

Theorem IX.1.3 *Let (M, f, μ) be given such that* $\log\left|\det(T_x f)\right| \in L^1(M,\mu)$. *Then entropy formula*

$$h_\mu(f) = \int \sum_i \lambda_i(x)^+ \gamma_i(x) d\mu \tag{IX.8}$$

holds true.

Remark IX.1. Theorem II.II.1.1, the notable Margulis-Ruelle inequality, follows directly from Theorem IX.1.3 and Proposition IX.1.2. So does Theorem VII.VII.1.1, since the validity of Pesin's entropy formula is equivalent to equations $\gamma_i(x) = m_i(x)$, $i = 1, \cdots, u(x)$, or equivalently $\delta_u(x) = \sum_{i=1}^u m_i(x)$ which is equivalent to the SRB property of the invariant measure μ.

We will prove Theorem IX.1.3 in Sections IX.2–IX.5. Before we start the proof, we make in advance the assumption that μ (and hence $\tilde{\mu}$) is ergodic for simplicity of presentation. It is used only in Sections IX.2–IX.4.

IX.2 Preliminaries

In this section, we state some preliminary results which will be very useful in the subsequent sections. First we state in Section IX.2.1 some propositions about unstable manifolds, each of which is analogous of certain statement in Chapter VII (see also [51, Chapter VI, Sections 3–4]) and can be proved following the same line. Hence the proofs are omitted. In view of these propositions we can compare the induced metric $d^i_{\tilde{x}}(\cdot,\cdot)$ on $W^i(\tilde{x})$ with the original metric $d(\cdot,\cdot)$ on M via Lyapunov charts. Then we include in Section IX.2.2 an outline of the construction of measurable partitions subordinate to W^i-manifolds. Some useful measurable partitions are also constructed. In Section IX.2.3 two types of transverse metrics are built on quotient spaces. Finally we present some entropy properties of the related partitions in Section IX.2.4.

IX.2.1 Some Estimations on Unstable Manifolds

We write $\mathbb{R}^{m_0} = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_r}$ and for each $z \in \mathbb{R}^{m_0}$, let $(z_1, z_2, \cdots, z_r)$ be its coordinates with respect to this splitting. The usual standard norm of Euclidean space will always be denoted by $\|\cdot\|$. Since we want to employ the Lyapunov charts introduced in Proposition VII.VII.4.2 to obtain estimations on the unstable and center unstable sets, for $i = 1, \cdots, c+u$ write

$$\bar{\mathbf{R}}^{(i)} \stackrel{\text{def}}{=} \mathbb{R}^{m_1+\cdots+m_i}, \quad \bar{\mathbf{R}}^{r-(i)} \stackrel{\text{def}}{=} \mathbb{R}^{m_{i+1}+\cdots+m_r}$$

and put

$$\begin{aligned}
\bar{\mathbf{R}}^{(i)}(\rho) &\stackrel{\text{def}}{=} \{z \in \mathbb{R}^{m_1+\cdots+m_i} : \|z\| \le \rho\},\\
\bar{\mathbf{R}}^{r-(i)}(\rho) &\stackrel{\text{def}}{=} \{z \in \mathbb{R}^{m_{i+1}+\cdots+m_r} : \|z\| \le \rho\},\\
\bar{\mathbf{R}}(\rho) &\stackrel{\text{def}}{=} \{z = (z_1, \cdots, z_r) \in \mathbb{R}^{m_0} : \|z_i\| \le \rho, 1 \le i \le r\}.
\end{aligned}$$

For each $z \in \mathbb{R}^{m_0}$ write $z^{(i)} \stackrel{\text{def}}{=} (z_1, \cdots, z_i)$ and $z^{r-(i)} \stackrel{\text{def}}{=} (z_{i+1}, \cdots, z_r)$; and define maximum norms $\|\cdot\|'$ and $\|\cdot\|'_i$ on $\mathbb{R}^{m_0} = \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_r}$ and $\mathbb{R}^{m_0} = \bar{\mathbf{R}}^{(i)} \times \bar{\mathbf{R}}^{r-(i)}$ respectively by

$$\begin{aligned}
\|z\|' &\stackrel{\text{def}}{=} \max_{1 \le i \le r} \|z_i\|,\\
\|z\|'_i &\stackrel{\text{def}}{=} \max(\|z^{(i)}\|, \|z^{r-(i)}\|).
\end{aligned}$$

Let $0 < \varepsilon < \min\{1, \Delta\lambda/100m_0\}$ and $e^{-\lambda_u + 10\varepsilon} + e^{5\varepsilon} < 2$, where $\Delta\lambda$ is defined by (VII.VII.18). Put $\lambda_0 \stackrel{\text{def}}{=} \max\{|\lambda_i| : 1 \le i \le r\} + 2\varepsilon$. By Proposition VII.VII.4.2, there exists a θ-invariant set $\Delta_2 \subset M^f$ of full $\tilde{\mu}$-measure such that $\{\Phi_{\tilde{x}}\}_{\tilde{x} \in \Delta_2}$ is a system of (ε, ℓ)-charts. For $i = 1, \cdots, u$, we introduce the local i^{th}-unstable manifold of (M, f, μ) at $\tilde{x}$ associated with $(\{\Phi_{\tilde{x}}\}_{\tilde{x} \in \Delta_2}, \delta)$, where $\delta \in (0, 1]$. It is defined to be the component of $W^i(\tilde{x}) \bigcap \Phi_{\tilde{x}} \bar{\mathbf{R}}(\delta \ell(\tilde{x})^{-1})$ that contains x_0. The $\Phi_{\tilde{x}}^{-1}$-image of this set is denoted by $W^i_{\tilde{x},\delta}(\tilde{x})$. The following proposition characterizes $W^i_{\tilde{x},\delta}(\tilde{x})$. See Lemmas VII.VII.5.1 and VII.VII.5.2 and also [51, pp. 146-147] for the proof of similar results.

Proposition IX.2.1 *Let $\{\Phi_{\tilde{x}}\}_{\tilde{x} \in \Delta_2}$ be a system of (ε, ℓ)-charts and $1 \le i \le u$.*

(1) *If $0 < \delta \le e^{-\lambda_0 - \varepsilon}$ and $\tilde{x} \in \Delta_2$, then*

(i) *$W^i_{\tilde{x},\delta}(\tilde{x})$ is the graph of a $C^{1,1}$ function*

$$g^i_{\tilde{x}} : \bar{\mathbf{R}}^{(i)}(\delta \ell(\tilde{x})^{-1}) \to \bar{\mathbf{R}}^{r-(i)}(\delta \ell(\tilde{x})^{-1})$$

with $g^i_{\tilde{x}}(0) = 0$ and $\mathrm{Lip}(g^i_{\tilde{x}}) < 1$;

(ii) *$W^1_{\tilde{x},\delta}(\tilde{x}) \subset \cdots \subset W^u_{\tilde{x},\delta}(\tilde{x}) \subset S^{cu}_\delta(\tilde{x})$;*

(2) *If $0 < \delta \le e^{-2\lambda_0 - 2\varepsilon}$ and $\tilde{x} \in \Delta_2$, then*

(i) *$H_{\tilde{x}} W^i_{\tilde{x},\delta}(\tilde{x}) \bigcap \bar{\mathbf{R}}(\delta \ell(\theta \tilde{x})^{-1}) = W^i_{\theta \tilde{x},\delta}(x_1)$;*

(ii) *$S^{cu}_\delta(\tilde{x}) \bigcap \Phi_{\tilde{x}}^{-1} W^i(\tilde{x}) = W^i_{\tilde{x},\delta}(\tilde{x})$.*

Fix $\tilde{x} \in \Delta_2$. Let $\delta \in (0, \frac{1}{4}]$. Consider now $\tilde{y} \in \widetilde{W}^{cu}_\delta(\tilde{x})$, where $\widetilde{W}^{cu}_\delta(\tilde{x})$ is defined by (VII.VII.19). Let $W^i_{\tilde{x},2\delta}(\tilde{y})$ be the $\Phi_{\tilde{x}}^{-1}$-image of the component of $W^i(\tilde{y}) \bigcap \Phi_{\tilde{x}}[\bar{\mathbf{R}}^{(i)}(2\delta \ell(\tilde{x})^{-1}) \times \bar{\mathbf{R}}^{r-(i)}(4\delta \ell(\tilde{x})^{-1})]$ containing y_0. Then $\Phi_{\tilde{x}} W^i_{\tilde{x},2\delta}(\tilde{y})$ contains an open neighborhood of y_0 in $W^i(\tilde{y})$ and is also referred to as a local i^{th}-unstable manifold of (M, f) at $\tilde{y}$ along $\tilde{x}$ (although in general $\Phi_{\tilde{x}} W^i_{\tilde{y},2\delta}(y_0) \ne \Phi_{\tilde{x}} W^i_{\tilde{x},2\delta}(\tilde{y})$). The following proposition holds analogue of Proposition IX.2.1 and can be proved following the line of the proof of Lemma VII.VII.5.3.

Proposition IX.2.2 *Let $\tilde{x} \in \Delta_2$ and $1 \le i \le u$.*

(1) *Let $0 < \delta \le \frac{1}{4} e^{-\lambda_0 - \varepsilon}$. If $\tilde{y} \in \widetilde{W}^{cu}_\delta(\tilde{x})$, then*

(i) *$W^i_{\tilde{x},2\delta}(\tilde{y})$ is a graph of a C^1 function*

$$g^i_{\tilde{x},\tilde{y}} : \bar{\mathbf{R}}^{(i)}(2\delta \ell(\tilde{x})^{-1}) \to \bar{\mathbf{R}}^{r-(i)}(4\delta \ell(\tilde{x})^{-1})$$

with $\mathrm{Lip}(g^i_{\tilde{x},\tilde{y}}) < 1$;

(ii) *$W^1_{\tilde{x},2\delta}(\tilde{y}) \subset \cdots \subset W^u_{\tilde{x},2\delta}(\tilde{y}) \subset S^{cu}_{4\delta}(\tilde{x})$;*

(2) *Let $0 < \delta \le \frac{1}{4} e^{-2\lambda_0 - 2\varepsilon}$. If $\tilde{y} \in \widetilde{W}^{cu}_\delta(\tilde{x})$ with $y_1 \in \Phi_{\theta \tilde{x}} S^{cu}_\delta(\theta \tilde{x})$, then*

$$H_{\tilde{x}} W^i_{\tilde{x},2\delta}(\tilde{y}) \bigcap [\bar{\mathbf{R}}^{(i)}(2\delta \ell(\theta \tilde{x})^{-1}) \times \bar{\mathbf{R}}^{r-(i)}(4\delta \ell(\theta \tilde{x})^{-1})] \subset W^i_{\theta \tilde{x},2\delta}(\theta \tilde{y});$$

(3) *Let* $0<\delta\le\frac{1}{4}e^{-2\lambda_0-2\varepsilon}$. *For* $\tilde\mu$*-a.e.* $\tilde x\in\Delta_2$

(i) *if* $\tilde y\in\widetilde W^{cu}_{\delta}(\tilde x)$, *then*

$$S^{cu}_{2\delta}(\tilde x)\bigcap\Phi_{\tilde x}^{-1}W^i(\tilde y)\subset W^i_{\tilde x,2\delta}(\tilde y)\subset S^{cu}_{4\delta}(\tilde x)\bigcap\Phi_{\tilde x}^{-1}W^i(\tilde y);$$

(ii) *if* $\tilde y\in\widetilde W^{cu}_{\delta}(\tilde x)$ *with* $y_0\in\Phi_{\tilde x}W^u_{\tilde x,\delta}(\tilde x)$, *then*

$$W^i_{\tilde x,2\delta}(\tilde y)\subset W^{i+1}_{\tilde x,2\delta}(\tilde y),\ i=1,\cdots,u-1;$$

(iii) *if* $\tilde y,\tilde z\in\widetilde W^{cu}_{\delta}(\tilde x)$ *with* $y_0,z_0\in\Phi_{\tilde x}W^u_{\tilde x,\delta}(\tilde x)$, *then either*

$$W^i_{\tilde x,2\delta}(\tilde y)=W^i_{\tilde x,2\delta}(\tilde z)$$

or otherwise the two terms in the above equation are disjoint.

The following proposition describes the actions $\{H^n_{\tilde x}\}_{n\in\mathbb Z}$.

Proposition IX.2.3 *Let* $0<\delta\le e^{-\lambda_0-\varepsilon}$. *For each* $\tilde x\in\Delta_2$ *and* $1\le i\le u$

(1) *If* $z,z'\in\bar{\mathbf R}(e^{-\lambda_1-3\varepsilon}\ell(\tilde x)^{-1})$, *then* $H_{\tilde x}z,H_{\tilde x}z'\in\bar{\mathbf R}(\ell(\theta\tilde x)^{-1})$ *and*

$$\|H_{\tilde x}z-H_{\tilde x}z'\|'\le e^{\lambda_1+2\varepsilon}\|z-z'\|';$$

(2) *If* $z,z'\in\bar{\mathbf R}(\delta\ell(\tilde x)^{-1})$ *and* $\|z-z'\|_i'=\|z^{(i)}-z'^{(i)}\|$, *then*

$$\|H_{\tilde x}z-H_{\tilde x}z'\|_i'=\|(H_{\tilde x}z)^{(i)}-(H_{\tilde x}z')^{(i)}\|\ge e^{\lambda_i-2\varepsilon}\|z-z'\|_i';$$

(3) $\|H_{\tilde x}^{-1}z-H_{\tilde x}^{-1}z'\|_i'\le e^{-\lambda_i+2\varepsilon}\|z-z'\|_i',\forall z,z'\in W^i_{\tilde x,\delta}(\tilde x)$;

(4) $\|H_{\tilde x}^{-1}z-H_{\tilde x}^{-1}z'\|_{c+u}'\le e^{2\varepsilon}\|z-z'\|_{c+u}',\forall z,z'\in S^{cu}_{\delta}(\tilde x)$;

(5) *Let* $0<\delta\le\frac14 e^{-2\lambda_0-2\varepsilon}$. *For* $\tilde\mu$*-a.e.* $\tilde x\in\Delta_2$, *if* $\tilde y\in\widetilde W^{cu}_{\delta}(\tilde x)$ *with* $y_0\in\Phi_{\tilde x}W^u_{\tilde x,\delta}(\tilde x)$, *then*

$$\|H_{\tilde x}^{-1}z-H_{\tilde x}^{-1}z'\|_i'\le e^{-\lambda_i+2\varepsilon}\|z-z'\|_i',\forall z,z'\in W^i_{\tilde x,2\delta}(\tilde y).$$

The following lemma says that the metrics $d^i_{\tilde x}(\cdot,\cdot)$ and $d(\cdot,\cdot)$ are locally equivalent within $\Phi_{\tilde x}W^i_{\tilde x,\delta}(\tilde x)$, a neighborhood of x_0 in $W^i(\tilde x)$.

Lemma IX.2.4 *If* $y,z\in\Phi_{\tilde x}W^i_{\tilde x,\delta}(\tilde x)$, *then*

$$d(y,z)\le d^i_{\tilde x}(y,z)\le 2K_0\|\Phi_{\tilde x}^{-1}y-\Phi_{\tilde x}^{-1}z\|_i'\le 2K_0\ell(\tilde x)d(y,z).$$

Proof. The first inequality is obvious. We prove the other two. By Proposition IX.2.1, we can assume

$$y=\Phi_{\tilde x}(v_0,g^i_{\tilde x}(v_0)),\quad z=\Phi_{\tilde x}(v_1,g^i_{\tilde x}(v_1))$$

for some $v_0, v_1 \in \bar{\mathbf{R}}^{(i)}(\delta\ell(\tilde{x}))$. Let $\{v(t)\}_{0\le t\le 1}$ be a smooth curve in $\bar{\mathbf{R}}^{(i)}(\delta\ell(\tilde{x}))$ connecting v_0 and v_1. Let $\mathscr{C}$ be a collection of such curves in $\bar{\mathbf{R}}^{(i)}(\delta\ell(\tilde{x}))$. Then

$$d^i_{\tilde{x}}(y,z) = \inf_{v(\cdot)\in\mathscr{C}} \int_0^1 |\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\tilde{x}}(v(t), g^i_{\tilde{x}}(v(t)))|\mathrm{d}t.$$

By Propositions VII.VII.4.2 and IX.2.1, we have

$$\begin{aligned}\inf_{v(\cdot)\in\mathscr{C}} \int_0^1 |\frac{\mathrm{d}}{\mathrm{d}t}\Phi_{\tilde{x}}(v(t), g^i_{\tilde{x}}(v(t)))|\mathrm{d}t &\le K_0 \inf_{v(\cdot)\in\mathscr{C}} \int_0^1 \|\frac{\mathrm{d}}{\mathrm{d}t}(v(t), g^i_{\tilde{x}}(v(t)))\|\mathrm{d}t \\ &\le 2K_0 \inf_{v(\cdot)\in\mathscr{C}} \int_0^1 \|\frac{\mathrm{d}}{\mathrm{d}t}v(t)\|\mathrm{d}t \\ &= 2K_0\|v_1 - v_0\| = 2K_0\|\Phi_{\tilde{x}}^{-1}y - \Phi_{\tilde{x}}^{-1}z\|'_i.\end{aligned}$$

This together with $\|\Phi_{\tilde{x}}^{-1}y - \Phi_{\tilde{x}}^{-1}z\|'_i \le \ell(\tilde{x})d(y,z)$ implies the last two inequalities in the lemma. □

IX.2.2 Related Partitions

First we include here an outline of the construction of partitions subordinate to W^i-manifolds, which follows the same line presented in [73]. It is simpler because $\tilde{\mu}$ is now assumed to be ergodic.

Fix an i with $1 \le i \le u$. Let l_0 be large enough such that $\triangle = \{\tilde{x} \in \Delta_2 : \ell(\tilde{x}) \le l_0\}$ has positive measure. Then there exists an increasing sequence $\{\triangle^k\}_{k\in\mathbb{Z}^+}$ of compact subsets of $\triangle$ such that $\tilde{\mu}(\triangle \setminus \bigcup_k \triangle^k) = 0$. Fix a k with $\tilde{\mu}(\triangle^k) > 0$. Let $\rho_0 > 0$ be as introduced in Chapter II.2.

Proposition IX.2.5 *Let $\{W^i_{loc}(\tilde{x})\}_{\tilde{x}\in\triangle^k}$ be a continuous family of $C^{1,1}$ embedded $\sum_{j=1}^i m_j$-dimensional disks described in Proposition V.V.4.5 with $\triangle^k$ in place of $\Delta_k^{(i)}$ and suitable α_k.*

(1) *For each $\tilde{x} \in \triangle^k$, $W^i_{loc}(\tilde{x}) \subset \Phi_{\tilde{x}} W^i_{\tilde{x},\delta}(\tilde{x})$, where $\delta = \frac{1}{4}e^{-\lambda_0+\varepsilon}$;*

(2) *There exists $A_k > 0$ such that for all $\tilde{y}, \tilde{z} \in M^f$ with $y_0, z_0 \in W^i_{loc}(\tilde{x})$ and $n \ge 0$*

$$d^i_{\theta^{-n}\tilde{x}}(y_{-n}, z_{-n}) \le A_k e^{-n(\lambda_i - 2\varepsilon)} d^i_{\tilde{x}}(y_0, z_0);$$

(3) *There exist $\hat{r} \in (0, \rho_0/4), \hat{\varepsilon} \in (0,1)$ and $\hat{d} \ge 2\hat{r}$ such that for all $\rho \in (0, \hat{r}]$ and $\tilde{x} \in \triangle^k$, if $\tilde{x}' \in B_{\triangle^k}(\tilde{x}; \hat{\varepsilon}\rho) \overset{\text{def}}{=} \{\tilde{y} \in \triangle^k : d(\tilde{x}, \tilde{y}) < \hat{\varepsilon}\rho\}$, then $W^i_{loc}(\tilde{x}') \bigcap B(x_0, \rho)$ is connected, its $d^i_{\tilde{x}'}$-diameter is less than $\hat{d}$ and the map*

$$\tilde{x}' \mapsto W^i_{loc}(\tilde{x}') \bigcap B(x_0, \rho)$$

is a continuous map from $B_{\triangle^k}(\tilde{x}; \hat{\varepsilon}\rho)$ to the space of subsets of $B(x_0, \rho)$ (endowed with the Hausdorff topology)*;*

(4) *Let* $\rho \in (0, \hat{r}]$ *and* $\tilde{x} \in \triangle^k$, *if* $\tilde{x}', \tilde{x}'' \in B_{\triangle^k}(\tilde{x}; \hat{\varepsilon}\rho)$, *then either*

$$W^i_{loc}(\tilde{x}') \bigcap B(x_0, \rho) = W^i_{loc}(\tilde{x}'') \bigcap B(x_0, \rho)$$

or otherwise the two terms in the above equation are disjoint. In the later case, if it is assumed moreover that $x''_0 \in W^i(\tilde{x}')$, *then*

$$d^i_{\tilde{x}'}(y, z) > \hat{d} > 2\hat{r}$$

for any $y \in W^i_{loc}(\tilde{x}') \bigcap B(x_0, \rho)$ *and* $z \in W^i_{loc}(\tilde{x}'') \bigcap B(x_0, \rho)$;

(5) *There exists* $\widehat{R} > 0$ *such that for each* $\tilde{x} \in \triangle^k$ *and* $y \in M$, *if* $\tilde{x}' \in B_{\triangle^k}(\tilde{x}; \hat{\varepsilon}\rho)$ *and* $y \in W^i_{loc}(\tilde{x}') \bigcap B(x_0, \hat{r})$, *then* $W^i_{loc}(\tilde{x}')$ *contains the closed ball of center* y *and* $d^i_{\tilde{x}'}$*-radius* $\widehat{R}$ *in* $W^i(\tilde{x}')$.

We now choose in $\triangle^k$ a density point $\tilde{x}^*$. For each $\rho \in [\hat{r}/2, \hat{r}]$, put

$$S_\rho \stackrel{\text{def}}{=} \bigcup_{\tilde{x} \in B_{\triangle^k}(\tilde{x}^*; \hat{\varepsilon}\rho)} \widetilde{W}^i_{loc}(\tilde{x}) \bigcap p^{-1}(B(x^*_0, \rho)),$$

where $\widetilde{W}^i_{loc}(\tilde{x}) \stackrel{\text{def}}{=} \widetilde{W}^{u,i}_{loc}(\tilde{x})$ is defined by (V.V.32). Let ξ_ρ denote the partition of M^f into all sets $\widetilde{W}^i_{loc}(\tilde{x}) \bigcap p^{-1}(B(x^*_0, \rho)), \tilde{x} \in B_{\triangle^k}(\tilde{x}^*; \hat{\varepsilon}\rho)$ and the set $M^f \setminus S_\rho$. We now define a measurable function $\beta_\rho : S_\rho \to \mathbb{R}^+$ by

$$\beta_\rho(\tilde{y}) \stackrel{\text{def}}{=} \inf_{n \geq 0} \{\widehat{R}, \frac{1}{2A_k} d(y_{-n}, \partial B(x^*_0, \rho)) e^{n(\lambda_i - 2\varepsilon)}, \frac{\rho}{A_k}\}.$$

By arguments analogous to those in the proof of Proposition IV.2.1 in [51], we know that there exists $r' \in [\hat{r}/2, \hat{r}]$ such that $\beta_{r'} > 0$ $\tilde{\mu}$-almost everywhere on $\widehat{S}_i \stackrel{\text{def}}{=} S_{r'}$. Put

$$\xi_i \stackrel{\text{def}}{=} \xi^+_{r'} = \bigvee_{n=0}^{+\infty} \theta^n \xi_{r'}.$$

Clearly, ξ_i is an increasing generator subordinate to W^i-manifolds of (M, f, μ).

Let us introduce some more related partitions in order to make use of the geometry of Lyapunov charts in the evaluation of local entropy in Sections IX.3–IX.4. Let $\{\Phi_{\tilde{x}}\}_{\tilde{x} \in \Delta_2}$ be a system of (ε, ℓ)-charts and $\delta \in (0, \frac{1}{16} e^{-2(\lambda_0 + \varepsilon)})$ be a reduction factor. Then there exists a measurable partition $\mathscr{D}$ of M^f with $H_{\tilde{\mu}}(\mathscr{D}) < +\infty$ such that

(1) $p(\mathscr{D}^+(\tilde{x})) \subset \Phi_{\tilde{x}} S^{cu}_\delta(\tilde{x})$ for $\tilde{\mu}$-a.e. $\tilde{x}$, where $\mathscr{D}^+ \stackrel{\text{def}}{=} \bigvee_{n=0}^{+\infty} \theta^n \mathscr{D}$;

(2) $\{\widehat{S}_i, M^f \setminus \widehat{S}_i\} < \mathscr{D}$ for $i = 1, \cdots, u$;

(3) $\{\widehat{E}, M^f \setminus \widehat{E}\} < \mathscr{D}$, where $\widehat{E}$ will be specified later in Section IX.2.3.

We clearly have the following proposition.

Proposition IX.2.6 *The partition ξ_i is an increasing generator subordinate to W^i-manifolds of (M, f, μ). Furthermore, if $\mathscr{B}^i(M, f, \mu)$ denotes the σ-algebra of those Borel sets $A \subset M^f$ such that $A = \bigcup_{\tilde{x} \in A} \widetilde{W}^i(\tilde{x})$, then*

$$\mathscr{B}(\bigwedge_{n=0}^{+\infty} \theta^n \xi_i) = \mathscr{B}^i(M, f, \mu), \quad \tilde{\mu} - mod\, 0.$$

Now define $\eta_i \stackrel{\text{def}}{=} \xi_i \bigvee \mathscr{D}^+$ for $i = 1, \cdots, u$. We have the following proposition.

Proposition IX.2.7 $\{\eta_i\}_{i=1}^u$ *satisfy the following statements:*

(1) $\eta_1 > \eta_2 > \cdots > \eta_u$;
(2) η_i*'s are increasing generators and* $p(\eta_i(\tilde{x})) \subset \Phi_{\tilde{x}} W^i_{\tilde{x},\delta}(\tilde{x})$ *for* $\tilde{\mu}$*-a.e.* $\tilde{x}$;
(3) $h_{\tilde{\mu}}(\theta, \eta_i) = h_{\tilde{\mu}}(\theta, \xi_i)$ *for* $i = 1, \cdots, u$, *where* $h_{\tilde{\mu}}(\theta, \eta) \stackrel{\text{def}}{=} H_{\tilde{\mu}}(\eta \big| \theta \eta^-)$ *with* $\eta^- \stackrel{\text{def}}{=} \vee_{k=0}^{+\infty} \theta^k \eta$;
(4) *For* $\tilde{\mu}$*-a.e.* $\tilde{x}$ *and* $2 \leq i \leq u$, *if* $\tilde{y} \in \eta_i(\tilde{x})$ *with* $\tilde{y} \in \Delta_2$, *then*

$$\Phi_{\tilde{x}} W^{i-1}_{\tilde{x},2\delta}(\tilde{y}) \bigcap p(\eta_i(\tilde{x})) = p(\eta_{i-1}(\tilde{y}))$$

and

$$\theta^{-1}(\eta_{i-1}(\tilde{y})) = \eta_{i-1}(\theta^{-1}\tilde{y}) \bigcap \theta^{-1}(\eta_i(\tilde{x})).$$

The proof of item (3) in the above proposition is postponed later in subsection IX.2.4 (see the proof of Lemma IX.2.13); the other items are easy to be checked from the construction itself. We collect them all together for the convenience of the readers.

IX.2.3 Transverse Metrics on $\eta_i(\tilde{x})/\eta_{i-1}$ with $2 \leq i \leq u$

Let $\{\Phi_{\tilde{x}}\}_{\tilde{x} \in \Delta_2}$ be a system of (ε, l)-charts. Fix a point $\tilde{x} \in \Delta_2$. Let $1 \leq i \leq u$ and $\delta' = \frac{1}{4} e^{-\lambda_0 - \varepsilon}$. Denote by $L(\bar{\mathbf{R}}^{(i)}, \bar{\mathbf{R}}^{r-(i)})$ the space of all linear maps from $\bar{\mathbf{R}}^{(i)}$ to $\bar{\mathbf{R}}^{r-(i)}$. By Proposition IX.2.2(1)(i) we know that, if $\tilde{y} \in \widetilde{W}^{cu}_{\delta'}(\tilde{x})$, then there exists a unique $P^i_{\tilde{x},\tilde{y}} \in L(\bar{\mathbf{R}}^{(i)}, \bar{\mathbf{R}}^{r-(i)})$ with $\|P^i_{\tilde{x},\tilde{y}}\| < 1$ such that

$$T_{y_0} \Phi_{\tilde{x}}^{-1} E^i_{\tilde{y}} = \text{Graph}(P^i_{\tilde{x},\tilde{y}}).$$

Define

$$\begin{aligned} \mathscr{L}^i_{\tilde{x}} : \widetilde{W}^{cu}_{\delta'}(\tilde{x}) &\to L(\bar{\mathbf{R}}^{(i)}, \bar{\mathbf{R}}^{r-(i)}) \\ \tilde{y} &\mapsto P^i_{\tilde{x},\tilde{y}}. \end{aligned}$$

The following proposition says that the map $\mathscr{L}^i_{\tilde{x}}$ is Lipschitz for all $\tilde{x} \in \Delta_2$.

Proposition IX.2.8 *For each $\tilde{x} \in \Delta_2$ and $1 \leq i \leq u$, $\mathscr{L}_{\tilde{x}}^i$ is a Lipschitz map and*

$$\mathrm{Lip}(\mathscr{L}_{\tilde{x}}^i) \leq D_0 \ell(\tilde{x})^2$$

where $D_0 > 0$ is a number depending only on the exponents and ε.

The proof is similar to that of Lemma VII.VII.5.7 and hence is omitted.

Let η_i's be introduced as above. For $2 \leq i \leq u$, we now define two metrics on the factor-space $\eta_i(\tilde{x})/\eta_{i-1}$ for $\tilde{\mu}$-a.e. $\tilde{x}$. We shall actually deal with $\{\eta_i\}_i$ restricted to a certain measurable set with full $\tilde{\mu}$-measure. Now we choose a θ-invariant measurable set $\widetilde{\Delta_0'} \subset \Delta_2$ with $\tilde{\mu}(\widetilde{\Delta_0'}) = 1$ such that for each $\tilde{x} \in \widetilde{\Delta_0'}$ the requirements of Proposition IX.2.7 are satisfied. We then put

$$\eta_i' \stackrel{\text{def}}{=} \eta_i|_{\widetilde{\Delta_0'}}, i = 1, \cdots, u.$$

In what follows we define two transverse metrics on $\eta_i'(\tilde{x})/\eta_{i-1}'$ for $\tilde{\mu}$-a.e. $\tilde{x} \in \widetilde{\Delta_0'}$.

First we give a point-dependent definition. Let $\tilde{x} \in \widetilde{\Delta_0'}$. From Proposition IX.2.7, we know that for every $\tilde{y} \in \eta_i'(\tilde{x})$, $W_{\tilde{x},2\delta}^{i-1}(\tilde{y})$ intersects $\{0\} \times \bar{\mathbf{R}}^{r-(i-1)}$ at exactly one point. We denote the i^{th} coordinate of this point by $\zeta_{\tilde{y}}^i \in \mathbb{R}^{m_i}$. Clearly $\zeta_{\tilde{y}}^i =$ the i^{th} coordinate of the point $(0, g_{\tilde{x},\tilde{y}}^{i-1}(0))$. For $\tilde{y}, \tilde{y}' \in \eta_i'(\tilde{x})$, define

$$\hat{d}_{\tilde{x}}^i(\tilde{y}, \tilde{y}') \stackrel{\text{def}}{=} \|\zeta_{\tilde{y}}^i - \zeta_{\tilde{y}'}^i\|$$

By Proposition IX.2.7, $\hat{d}_{\tilde{x}}^i(\cdot,\cdot)$ induces a metric on $\eta_i'(\tilde{x})/\eta_{i-1}'$ for $i = 2, \cdots, u$.

To introduce a second metric on $\eta_i'(\tilde{x})/\eta_{i-1}'$ for $i = 2, \cdots, u$, we state the following lemma (**straightening out lemma**) without proof. Here $d \geq 2$ is a fixed integer. Let positive integers $n_1, \cdots, n_d$ and a number $0 < \rho < 1$ be given. Denote by $B^i(\rho)$ the closed disk centered at 0 of radius ρ in $\mathbb{R}^{n_i}$. Consider $B(\rho) \stackrel{\text{def}}{=} B^1(\rho) \times \cdots \times B^d(\rho)$ as a subset of $\mathbb{R}^{n_1+\cdots+n_d}$.

Lemma IX.2.9 *(See [43, Lemma 8.3.1].) For $i = 1, \cdots, d-1$, let F_i be a Lipschitz foliation with C^1 leaves on some subset of $\mathbb{R}^{n_1+\cdots+n_d}$ containing $B(\rho)$. Assume that each leaf of F_i is the graph of a function*

$$g^i : B^1(2\rho) \times \cdots \times B^i(2\rho) \to \mathbb{R}^{n_{i+1}+\cdots+n_d}$$

with $\|Dg^i\| \leq \frac{1}{3}$ and that the function $x \mapsto T_x F_i$ has Lipschitz constant smaller than some number C. Assume also that the F_i's are nested, i.e., if $F_i(x)$ denotes the leaf of F_i containing the point x, then

$$F_1(x) \subset F_2(x) \subset \cdots \subset F_{d-1}(x), \qquad \forall x \in B(\rho)$$

Define $\mathscr{O} = (\mathscr{O}_1, \cdots, \mathscr{O}_d) : B(\rho) \to \mathbb{R}^{n_1+\cdots+n_d}$ *as follows: for* $x = (x_1, \cdots, x_d) \in B(\rho)$*, let* $\mathscr{O}_1(x) = x_1$*, and let* $\mathscr{O}_i(x)$ *be the* i^{th} *coordinate of the unique point of intersection of* $F_{i-1}(x)$ *and* $\{0\} \times \cdots \times \{0\} \times \mathbb{R}^{n_i+\cdots+n_d}$ *for* $i = 2, \cdots, d$*. Then*

(1) $\mathscr{O}$ *is a homeomorphism between* $B(\rho)$ *and its image;*
(2) *For every* $x, y \in B(\rho)$*,* $\mathscr{O}_j(x) = \mathscr{O}_j(y)$ *for* $j = i+1, \cdots, d$ *if and only if* $y \in F_i(x)$*;*
(3) *Both* $\mathscr{O}$ *and* $\mathscr{O}^{-1}$ *are Lipschitz with Lipschitz constant depending only on* C*.*

In light of the above lemma, now we can express the metric $\hat{d}^i_{\tilde{x}}(\cdot,\cdot)$ in another way. Let

$$p^{(u)} : \mathbb{R}^{m_0} = \mathbb{R}^{m_1+\cdots+m_r} \to \bar{\mathbf{R}}^{(u)} = \mathbb{R}^{m_1+\cdots+m_u}$$

be the natural project map. Then for each $\tilde{x} \in \widetilde{\Delta}'_0$ fixed, $p^{(u)}|_{W^u_{\tilde{x},\delta}(\tilde{x})}$ is a lipeomorphism between $W^u_{\tilde{x},\delta}(\tilde{x})$ and its image. For $i = 1, \cdots, u-1$, define foliations by

$$F^i_{\tilde{x}} \stackrel{\text{def}}{=} \left\{ z \in p^{(u)} W^i_{\tilde{x},\delta}(\tilde{y}) : \tilde{y} \in \widetilde{\Delta}'_0 \bigcap \widetilde{W}^{cu}_\delta(\tilde{x}), y_0 \in \Phi_{\tilde{x}} W^u_{\tilde{x},\tau\delta}(\tilde{x}) \right\},$$

where $\tau \in (0, \frac{1}{4} e^{-2(\lambda_0+\varepsilon)})$. By Proposition IX.2.8, these foliations satisfy the requirements of Lemma IX.2.9 with $\rho = \delta \ell(\tilde{x})^{-1}$ and $C = D_0 \ell(\tilde{x})^2$, providing δ and ε small enough. Hence there exists a map $\mathscr{O}_{\tilde{x}} = (\mathscr{O}^1_{\tilde{x}}, \cdots, \mathscr{O}^u_{\tilde{x}}) : \bar{\mathbf{R}}^{(u)}(\rho) \to \bar{\mathbf{R}}^{(u)}$ such that

(1) $\mathscr{O}_{\tilde{x}}$ is a homeomorphism between $\bar{\mathbf{R}}^{(u)}(\rho)$ and its image;
(2) For every $z, z' \in \bar{\mathbf{R}}^{(u)}(\rho)$, $\mathscr{O}^j_{\tilde{x}}(z) = \mathscr{O}^j_{\tilde{x}}(z')$ for $j = i+1, \cdots, u$ iff $z' \in F^i_{\tilde{x}}(z)$ and
(3) Both $\mathscr{O}_{\tilde{x}}$ and $\mathscr{O}^{-1}_{\tilde{x}}$ are Lipschitz with Lipschitz constant depending only on C.

Let $\pi_{\tilde{x}} = (\pi^1_{\tilde{x}}, \cdots, \pi^u_{\tilde{x}}) : W^u_{\tilde{x},\tau\delta}(\tilde{x}) \to \bar{\mathbf{R}}^{(u)}$ be given by

$$\pi_{\tilde{x}} \stackrel{\text{def}}{=} \mathscr{O}_{\tilde{x}} \circ p^{(u)}.$$

Clearly $\zeta^i_{\tilde{y}} = \pi^i_{\tilde{x}} \circ \Phi^{-1}_{\tilde{x}}(y_0)$ for each $\tilde{y} \in \eta'_i(\tilde{x})$ with $2 \le i \le u$. We can conclude that $\pi_{\tilde{x}}$ is a lipeomorphism between $W^u_{\tilde{x},\tau\delta}(\tilde{x})$ and its image with $\mathrm{Lip}(\pi_{\tilde{x}}), \mathrm{Lip}(\pi^{-1}_{\tilde{x}}) \le N(\tilde{x})$, where $N(\tilde{x})$ depends only on $\ell(\tilde{x})$ and the Lyapunov exponents. Moreover, $\pi_{\tilde{x}}\big(W^i_{\tilde{x},\delta}(\tilde{y})\big)$ lies on a $\sum_{j=1}^{i} m_j$-dimensional plane parallel to $\bar{\mathbf{R}}^{(i)} \times \{0\} \times \cdots \times \{0\}$; and if $W^i_{\tilde{x},\delta}(\tilde{y}) \neq W^i_{\tilde{x},\delta}(\tilde{y}')$, then $\pi_{\tilde{x}}\big(W^i_{\tilde{x},\delta}(\tilde{y})\big)$ and $\pi_{\tilde{x}}\big(W^i_{\tilde{x},\delta}(\tilde{y}')\big)$ lie on distinct planes.

Though $\hat{d}^i_{\tilde{x}}(\cdot,\cdot)$ is a metric on $\eta'_i(\tilde{x})/\eta'_{i-1}$, in general $\hat{d}^i_{\tilde{x}}(\cdot,\cdot) \neq \hat{d}^i_{\tilde{x}'}(\cdot,\cdot)$ for $\tilde{x}' \in \eta'_i(\tilde{x})$ with $\tilde{x}' \neq \tilde{x}$. Now we need to rectify this situation to give a point-independent definition.

Let $\tilde{x}^*$ be as introduced in Section IX.2.2 corresponding to $\triangle^k$. Then there exist positive numbers τ_0 and s_0 with $0 < \tau_0 < \frac{1}{4} e^{-2(\lambda_0+\varepsilon)}$ and a set

$$\widehat{E} \stackrel{\text{def}}{=} \widetilde{\Delta}'_0 \bigcap B_{\triangle^k}(\tilde{x}^*; s_0/2)$$

such that the following (a) and (b) hold true (where $\hat{\rho} = \tau_0 \delta l_0^{-1}$):

(a) Let $\tilde{x} \in \triangle^k$ with $x_0 \in \Phi_{\tilde{x}^*}\bar{\mathbf{R}}(\hat{\rho})$. For $i = 1, \cdots, u-1$, if $\tilde{y} \in \widetilde{\Delta}'_0 \bigcap \widetilde{W}^{cu}_{\delta}(\tilde{x})$ with $y_0 \in \Phi_{\tilde{x}} W^u_{\tilde{x},\tau_0\delta}(\tilde{x}) \bigcap \Phi_{\tilde{x}^*}\bar{\mathbf{R}}(\hat{\rho})$, then there exists a map

$$h^i_{\tilde{x},\tilde{y}} : \bar{\mathbf{R}}^{(i)}(2\hat{\rho}) \to \bar{\mathbf{R}}^{r-(i)}(2\hat{\rho})$$

with $\mathrm{Lip}(h^i_{\tilde{x},\tilde{y}}) < \frac{1}{3}$ and

$$\mathrm{Graph}(h^i_{\tilde{x},\tilde{y}}) = \Phi^{-1}_{\tilde{x}^*} \circ \Phi_{\tilde{x}} W^i_{\tilde{x},\delta}(\tilde{y}) \bigcap \bar{\mathbf{R}}(2\hat{\rho});$$

(b) For each $\tilde{x} \in \widehat{E}$ and $1 \le i \le u-1$, define a foliation by

$$F^i_{\tilde{x}^*,\tilde{x}} \stackrel{\text{def}}{=} \Big\{ p^{(u)}(\mathrm{Graph}(h^i_{\tilde{x},\tilde{y}})) : \tilde{y} \in \widetilde{\Delta}'_0 \bigcap \widetilde{W}^{cu}_{\delta}(\tilde{x}) \\ \text{with } y_0 \in \Phi_{\tilde{x}} W^u_{\tilde{x},\tau_0\delta}(\tilde{x}) \bigcap \Phi_{\tilde{x}^*}\bar{\mathbf{R}}(\hat{\rho}) \Big\}.$$

By Lemma IX.2.9, there exists a map

$$\mathscr{O}_{\tilde{x}^*,\tilde{x}} : \bar{\mathbf{R}}^{(u)}(\hat{\rho}) \to \bar{\mathbf{R}}^{(u)}$$

satisfying the requirements analogous of the above (1)-(3) for $\mathscr{O}_{\tilde{x}}$.

We then define a map $\tilde{\pi} = (\tilde{\pi}^1, \cdots, \tilde{\pi}^u) : \bigcup\limits_{n=0}^{+\infty} \theta^n \widehat{E} \to \bar{\mathbf{R}}^{(u)}$ as following: for each $\tilde{y} \in \widehat{E}$, suppose $y_0 \in \Phi_{\tilde{x}} W^u_{\tilde{x},\tau_0\delta}(\tilde{x}) \bigcap \Phi_{\tilde{x}^*}\bar{\mathbf{R}}(\hat{\rho})$ with $\tilde{x} \in \triangle^k$, put

$$\tilde{\pi}(\tilde{y}) \stackrel{\text{def}}{=} \mathscr{O}_{\tilde{x}^*,\tilde{x}} \circ p^{(u)} \circ \Phi^{-1}_{\tilde{x}^*} y_0$$

and in general, let $\tilde{\pi}(\tilde{y}) \stackrel{\text{def}}{=} \tilde{\pi}(\theta^{-n(\tilde{y})}\tilde{y})$, where $n(\tilde{y}) \stackrel{\text{def}}{=} \inf\{k \ge 0 : \theta^{-k}\tilde{y} \in \widehat{E}\}$. Thus for $i = 2, \cdots, u$ we can define a point-independent metric on $\eta'_i(\tilde{x})/\eta'_{i-1}$ by

$$\tilde{d}^i_{\tilde{x}}(\tilde{y},\tilde{y}') \stackrel{\text{def}}{=} \|\tilde{\pi}^i(\tilde{y}) - \tilde{\pi}^i(\tilde{y}')\|, \forall \tilde{y},\tilde{y}' \in \eta'_i(\tilde{x}).$$

Clearly the above metrics satisfy the following two propositions.

Proposition IX.2.10 *Let $z \in W^u_{\tilde{x},\tau\delta}(\tilde{x})$ with $0 < \tau < e^{-\lambda_1 - 3\varepsilon}$. Then for $1 \le i \le u$*

$$\|\pi^i_{\theta\tilde{x}} \circ H_{\tilde{x}} z\| \le e^{\lambda_i + 3\varepsilon} \|\pi^i_{\tilde{x}} z\|.$$

Proposition IX.2.11 *There exists $N_0 > 0$ such that for all $\tilde{y},\tilde{y}' \in \eta'_i(\tilde{x})$ with $\tilde{x} \in \widehat{E}$*

$$N_0^{-1} \hat{d}^i_{\tilde{x}}(\tilde{y},\tilde{y}') \le \tilde{d}^i_{\tilde{x}}(\tilde{y},\tilde{y}') \le N_0 \hat{d}^i_{\tilde{x}}(\tilde{y},\tilde{y}').$$

IX.2.4 Entropies of the Related Partitions

About the entropies of the above partitions, we have the following

Proposition IX.2.12 *Let ξ_i and ξ_i' be partitions subordinate to W^i-manifolds constructed following the procedure presented in Section IX.2.2. Then*

$$h_{\tilde{\mu}}(\theta^{-1}, \xi_i) = h_{\tilde{\mu}}(\theta^{-1}, \xi_i').$$

Proof. Let us first assume $h_\mu(f) = h_{\tilde{\mu}}(\theta) < +\infty$. It suffices to prove

$$h_{\tilde{\mu}}(\theta^{-1}, \xi_i \vee \xi_i') = h_{\tilde{\mu}}(\theta^{-1}, \xi_i).$$

Noting that for every $n \geq 1$

$$\theta^n(\xi_i \vee \xi_i') < \xi_i \vee \theta^n \xi_i' < \xi_i \vee \xi_i'$$

and

$$\begin{aligned}
H_{\tilde{\mu}}(\xi_i \vee \theta^n \xi_i' | \theta\big(\theta^n(\xi_i \vee \xi_i')\big)^+) &= H_{\tilde{\mu}}(\xi_i \vee \theta^n \xi_i' | \theta^{n+1}(\xi_i \vee \xi_i')) \\
&= H_{\tilde{\mu}}(\xi_i | \theta^{n+1}(\xi_i \vee \xi_i')) + H_{\tilde{\mu}}(\theta^n \xi_i' | \xi_i \vee \theta^{n+1} \xi_i') \\
&\leq H_{\tilde{\mu}}(\xi_i | \theta^{n+1} \xi_i) + H_{\tilde{\mu}}(\theta^n \xi_i' | \theta^{n+1} \xi_i') \\
&= (n+1) h_{\tilde{\mu}}(\theta^{-1}, \xi_i) + h_{\tilde{\mu}}(\theta^{-1}, \xi_i') \\
&\leq (n+2) h_{\tilde{\mu}}(\theta^{-1}) < +\infty \\
H_{\tilde{\mu}}(\xi_i \vee \xi_i' | \theta(\xi_i \vee \theta^n \xi_i')^+) &= H_{\tilde{\mu}}(\xi_i \vee \xi_i' | \theta \xi_i \vee \theta^{n+1} \xi_i') \\
&= H_{\tilde{\mu}}(\xi_i | \theta \xi_i \vee \theta^{n+1} \xi_i') + H_{\tilde{\mu}}(\xi_i' | \xi_i \vee \theta^{n+1} \xi_i') \\
&\leq H_{\tilde{\mu}}(\xi_i | \theta \xi_i) + H_{\tilde{\mu}}(\xi_i' | \theta^{n+1} \xi_i') \\
&= h_{\tilde{\mu}}(\theta^{-1}, \xi_i) + (n+1) h_{\tilde{\mu}}(\theta^{-1}, \xi_i') \\
&\leq (n+2) h_{\tilde{\mu}}(\theta^{-1}) < +\infty,
\end{aligned}$$

by [51, Theorem 0.5.2], we have

$$\begin{aligned}
h_{\tilde{\mu}}(\theta^{-1}, \xi_i \vee \xi_i') &= h_{\tilde{\mu}}(\theta^{-1}, \theta^n(\xi_i \vee \xi_i')) \\
&= h_{\tilde{\mu}}(\theta^{-1}, \xi_i \vee \theta^n \xi_i') \\
&= H_{\tilde{\mu}}(\xi_i \vee \theta^n \xi_i' | \theta \xi_i \vee \theta^{n+1} \xi_i') \\
&= H_{\tilde{\mu}}(\xi_i | \theta \xi_i \vee \theta^{n+1} \xi_i') + H_{\tilde{\mu}}(\xi_i' | \theta \xi_i' \vee \theta^{-n} \xi_i).
\end{aligned}$$

In view of Proposition IX.2.6, as $n \to +\infty$ we have

$$\theta \xi_i \vee \theta^{n+1} \xi_i' \searrow \theta \xi_i \vee (\wedge_{k=0}^{+\infty} \theta^k \xi_i') = \theta \xi_i.$$

Hence

$$H_{\tilde{\mu}}(\xi_i|\theta\xi_i \vee \theta^{n+1}\xi_i') \to H_{\tilde{\mu}}(\xi_i|\theta\xi_i) = h_{\tilde{\mu}}(\theta^{-1}, \xi_i)$$

as $n \to +\infty$. Also by Proposition IX.2.6, $\theta\xi_i' \vee \theta^{-n}\xi_i$ tends increasingly to the partition of M^f into single points. Thus

$$H_{\tilde{\mu}}(\xi_i'|\theta\xi_i' \vee \theta^{-n}\xi_i) \to 0$$

as $n \to +\infty$.

For the case $h_{\tilde{\mu}}(\theta) = +\infty$, since

$$\widetilde{M}_n := \{\tilde{x} \in M^f : |\lambda_i(\tilde{x})| \leq n, i = 1, \cdots, r(\tilde{x})\}$$

is θ-invariant and conditioned on $\widetilde{M}_n$ the entropy $h_{\tilde{\mu}_n}(\theta) \leq nm < +\infty$ by Ruelle's inequality (where $\tilde{\mu}_n$ is the conditional measure on $\widetilde{M}_n$), we have

$$h_{\tilde{\mu}_n}(\theta^{-1}, \xi_i) = h_{\tilde{\mu}_n}(\theta^{-1}, \xi_i') < +\infty.$$

The proof is finished by letting $n \to +\infty$. □

Proposition IX.2.13 *Let $\mathscr{D}$ be a measurable partition of M^f with $H_{\tilde{\mu}}(\mathscr{D}) < +\infty$ and let ξ_i be partitions subordinate to W^i-manifolds constructed following the procedure presented in Section IX.2.2. Then*

$$h_{\tilde{\mu}}(\theta^{-1}, \xi_i \vee \mathscr{D}^+) = h_{\tilde{\mu}}(\theta^{-1}, \xi_i).$$

Proof. The proof is similar to that of Proposition IX.2.12. One needs only to notice that

$$\theta^n(\xi_i \vee \mathscr{D}^+) < \xi \vee \theta\mathscr{D}^+ < \xi \vee \mathscr{D}^+$$

and

$$\xi < \xi \vee \mathscr{D} < \xi \vee \mathscr{D}^+$$

and check the conditions

$$H_{\tilde{\mu}}(\xi \vee \theta\mathscr{D}^+|\theta^{n+1}(\xi_i \vee \mathscr{D}^+)) < +\infty, \quad H_{\mu}(\xi \vee \mathscr{D}^+|\theta(\xi \vee \theta\mathscr{D}^+)) < +\infty$$

and

$$H_{\tilde{\mu}}(\xi \vee \mathscr{D}|\theta\xi) < +\infty.$$

Hence the proof is omitted. □

The following proposition is just Corollary VII.8.1.1 restated here.

Proposition IX.2.14 *For any partition ξ_u subordinate to W^u-manifolds of the type as constructed following the procedure presented in subsection IX.2.2, we have*

$$h_{\tilde{\mu}}(\theta^{-1}, \xi_u) = h_{\tilde{\mu}}(\theta^{-1}) = h_{\mu}(f).$$

IX.3 Definitions of Local Entropies along Unstable Manifolds

In this section, we will define quantities named *local entropy along unstable manifolds*. These quantities play an important role in our arguments. In ergodic case, the notion of local entropy along W^i-manifolds is described by a number h_i which measures the amount of randomness along the leaves of W^i-manifolds. As Ledrappier and Young have done in [43], though there are several equivalent definitions, we take a pointwise approach following [13].

Let $\varepsilon > 0$. For $\tilde{x} \in \Delta_2$ and $n \in \mathbb{Z}^+$, put

$$\widetilde{B^i}(\tilde{x}; n, \varepsilon) \stackrel{\text{def}}{=} \left\{ \tilde{y} \in \widetilde{W}^i(\tilde{x}) : d^i_{\theta^k \tilde{x}}(x_k, y_k) < \varepsilon \text{ for } 0 \le k \le n \right\}.$$

Let η be a measurable partition of M^f subordinate to W^i-manifolds. Define

$$\underline{h}_i(\tilde{x}; \varepsilon, \eta) \stackrel{\text{def}}{=} \liminf_{n \to +\infty} -\frac{1}{n} \log \tilde{\mu}^{\eta}_{\tilde{x}}(\widetilde{B^i}(\tilde{x}; n, \varepsilon)),$$
$$\overline{h}_i(\tilde{x}; \varepsilon, \eta) \stackrel{\text{def}}{=} \limsup_{n \to +\infty} -\frac{1}{n} \log \tilde{\mu}^{\eta}_{\tilde{x}}(\widetilde{B^i}(\tilde{x}; n, \varepsilon)).$$

One can easily show that these functions are indeed measurable. Furthermore, we define the *lower* and *upper local entropy along W^i-manifolds* at $\tilde{x}$ with lrespect to η by

$$\underline{h}_i(\tilde{x}; \eta) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \underline{h}_i(\tilde{x}; \varepsilon, \eta),$$
$$\overline{h}_i(\tilde{x}; \eta) \stackrel{\text{def}}{=} \lim_{\varepsilon \to 0} \overline{h}_i(\tilde{x}; \varepsilon, \eta).$$

These limits exist because $\underline{h}_i(\tilde{x}; \varepsilon, \eta)$ and $\overline{h}_i(\tilde{x}; \varepsilon, \eta)$ increase as $\varepsilon \downarrow 0$.

Proposition IX.3.1 *Let ξ_i be an increasing generator subordinate to W^i-manifolds. Then*

$$\underline{h}_i(\tilde{x}; \xi_i) = \overline{h}_i(\tilde{x}; \xi_i) =: h_i = H_{\tilde{\mu}}(\xi_i | \theta \xi_i),\ \tilde{\mu} - a.e.\ \tilde{x}.$$

The proposition above tells us that the lower and upper local entropy along W^i-manifolds with respect to ξ_i are coincident. From the proof below and the ergodic decompositions of μ and $\tilde{\mu}$ in Section 5, we know that in general h_i depends only on x_0 and is an f-invariant function independent of the choice of ξ_i or $\{\tilde{\mu}^{\xi_i}_{\tilde{x}}\}$ (see Propositions IX.2.12–IX.2.14). So we write $h_i = h_i(x_0)$ and call it the *local entropy along W^i-manifolds* at x_0. This completes the definition of h_i.

Let us first introduce some facts and postpone the proof of Proposition IX.3.1 at the end of this section.

Lemma IX.3.2 *Let α be a measurable partition of M^f with $H_{\tilde{\mu}}(\alpha) < +\infty$ and let ξ be an increasing generator. Then*

$$\lim_{n\to+\infty} -\frac{1}{n}\log \tilde{\mu}_{\tilde{x}}^{\xi}([\alpha \bigvee \xi]_0^n(\tilde{x})) = H_{\tilde{\mu}}(\xi\big|\theta\xi), \quad \tilde{\mu} - \text{a.e. } \tilde{x} \in M^f.$$

Lemma IX.3.3 *There exists a measurable partition* α *of* M^f *with* $H_{\tilde{\mu}}(\alpha) < +\infty$ *such that*

$$[\alpha_0^n \bigvee \xi_i](\tilde{x}) \subset \widetilde{B}^i(\tilde{x}; n, \delta), \quad \forall n \geq n_0(\tilde{x})$$

for $\tilde{\mu}$*-a.e.* $\tilde{x}$*, where* $n_0 : M^f \to \mathbb{Z}^+$ *is a measurable function.*

Proof of Lemma IX.3.2. Define $I(\eta\big|\xi)(\tilde{x}) \stackrel{\text{def}}{=} -\log \tilde{\mu}_{\tilde{x}}^{\xi}(\eta(\tilde{x}))$. One has

$$\frac{1}{n} I([\alpha \bigvee \xi]_0^n\big|\xi)(\tilde{x}) = \frac{1}{n} I(\alpha\big|\xi)(\tilde{x}) + \frac{1}{n}\sum_{k=0}^{n} I(\alpha \bigvee \xi\big|\theta\xi \bigvee \alpha_{-k}^{-1})(\theta^k\tilde{x}), \qquad \text{(IX.9)}$$

where $\alpha_{-k}^{l} \stackrel{\text{def}}{=} \bigvee_{j=-k}^{l} \theta^{-j}\alpha$. Put $I_n(\tilde{x}) \stackrel{\text{def}}{=} I(\alpha \vee \xi\big|\theta\xi \vee \alpha_{-n}^{-1})(\tilde{x})$ and $I^*(\tilde{x}) \stackrel{\text{def}}{=} \sup_{n\geq 1} I_n(\tilde{x})$. One can prove that

$$\int I^*(\tilde{x})d\tilde{\mu} \leq H_{\tilde{\mu}}(\alpha \bigvee \xi\big|\theta\xi) + 1$$

and that $\{I_n, \mathscr{B}(\alpha_{-n}^0 \vee \xi)\}$ is a supermartingale. Therefore

$$I_n \quad \underset{\tilde{\mu} - \text{a.e.}}{\overset{L^1}{\longrightarrow}} \quad I_\infty.$$

Hence the second term in the right side of equation (IX.9) tends $\tilde{\mu}$-a.e. to a θ-invariant Borel function $F \in L^1$. Then by the ergodicity of $\tilde{\mu}$, one has

$$\lim_{n\to+\infty} \frac{1}{n} I([\alpha \bigvee \xi]_0^n\big|\xi)(\tilde{x}) = \int F d\tilde{\mu} = \int I_\infty d\tilde{\mu}.$$

So the limit function in (IX.9) is constant almost everywhere and is therefore equal to

$$\lim_{n\to+\infty} \frac{1}{n} H_{\tilde{\mu}}([\alpha \bigvee \xi]_0^n\big|\xi)$$

which can be written as

$$\lim_{n\to+\infty} \frac{1}{n} H_{\tilde{\mu}}(\xi_0^n\big|\xi) + \lim_{n\to+\infty} \frac{1}{n} H_{\tilde{\mu}}(\alpha_0^n\big|\theta^{-n}\xi).$$

The first term is equal to $H_{\tilde{\mu}}(\xi\big|\theta\xi)$. The second term goes to 0 since $\theta^{-n}\xi$ generates. □

Proof of Lemma IX.3.3. Without loss of generality, let ξ_i and $\widehat{S}_i$ be as introduced in Section IX.2 and $0 < \delta < \frac{1}{4}e^{-2\lambda_0 - 2\varepsilon}$. Let $\{\Phi_{\tilde{x}}\}_{\tilde{x}\in\Delta_2}$ be a system of (ε, ℓ)-charts. Put $S' \stackrel{\text{def}}{=} \widehat{S}_i \bigcap \{\tilde{x} \in \Delta_2 : \ell(\tilde{x}) \leq l_0\}$, where l_0 is large enough such that $\tilde{\mu}(S') > 0$.

First we define n_+, n_- and $n_0 : S' \to \mathbb{Z}^+$ by

$$n_+(\tilde{x}) \stackrel{\text{def}}{=} \inf\{n > 0 : \theta^n \tilde{x} \in S'\},$$
$$n_-(\tilde{x}) \stackrel{\text{def}}{=} \inf\{n > 0 : \theta^{-n} \tilde{x} \in S'\},$$
$$n_0(\tilde{x}) \stackrel{\text{def}}{=} \inf\{n \geq 0 : \theta^n \tilde{x} \in S'\}.$$

Then let $\psi : M^f \to \mathbb{R}$ be given by

$$\psi(\tilde{x}) \stackrel{\text{def}}{=} \begin{cases} \frac{\delta}{2K_0 l_0} e^{-(\lambda_0+\varepsilon)\max(n_+(\tilde{x}), n_-(\tilde{x}))}, & \text{if } \tilde{x} \in S', \\ \frac{\delta}{2K_0 l_0}, & \text{otherwise.} \end{cases}$$

Finally we define ψ_+ by replacing $\max(n_+, n_-)$ in the definition of ψ by n_+.

Since $\int -\log \psi d\tilde{\mu} < \infty$, there exists a measurable partition α of M^f with $H_{\tilde{\mu}}(\alpha) < \infty$ such that $p(\alpha(\tilde{x})) \subset B(x_0, \psi(\tilde{x}))$ for almost every $\tilde{x}$.

Let $\tilde{x} \in S'$. We will write $n_+ = n_+(\tilde{x})$ and $n_0 = n_0(\tilde{x})$ for simplicity of notation in the rest of this section. We assert that

Claim 1 If $\tilde{y} \in \widetilde{W}^i(\tilde{x})$ with $y_0 \in \Phi_{\tilde{x}} W^i_{\tilde{x},\delta}(\tilde{x})$ satisfies $\|\Phi_{\tilde{x}}^{-1} y_0\|_i' \leq l_0 \psi_+(\tilde{x})$, then

$$d^i_{\theta^j \tilde{x}}(x_j, y_j) < \delta \text{ for } 0 \leq j \leq n_+$$

and $y_{n_+} \in \Phi_{\theta^{n_+}\tilde{x}} W^i_{\theta^{n_+}\tilde{x},\delta}(\theta^{n_+}\tilde{x})$;

Claim 2 If $\tilde{y} \in \widetilde{W}^i(\tilde{x})$ with $y_0 \in p(\alpha_0^n(\tilde{x})) \bigcap \Phi_{\tilde{x}} W^i_{\tilde{x},\delta}(\tilde{x})$ for some $n \geq 0$, then

$$d^i_{\theta^j \tilde{x}}(x_j, y_j) < \delta \text{ for } 0 \leq j \leq n;$$

Claim 3 If $\tilde{y} \in [\alpha_0^{n_0} \bigvee \xi_i](\tilde{x})$, then

$$d^i_{\theta^j \tilde{x}}(x_j, y_j) < \delta \text{ for } 0 \leq j \leq n_0$$

and $y_{n_0} \in \Phi_{\theta^{n_0}\tilde{x}} W^i_{\theta^{n_0}\tilde{x},\delta}(\theta^{n_0}\tilde{x})$.

Let's postpone the proof of the above claims and first proceed the proof of Lemma IX.3.3. Consider now an arbitrary point $\tilde{x}$ with the property that $\theta^n \tilde{x} \in S'$ infinitely often as $n \to \pm\infty$. If $\tilde{y} \in [\alpha_0^n \bigvee \xi_i](\tilde{x})$ with $n \geq n_0$, then by Claim 3

$$d^i_{\theta^j \tilde{x}}(x_j, y_j) < \delta \text{ for } 0 \leq j \leq n_0$$

and $y_{n_0} \in \Phi_{\theta^{n_0}\tilde{x}} W^i_{\theta^{n_0}\tilde{x},\delta}(\theta^{n_0}\tilde{x})$. Then we can apply Claim 2 to $\theta^{n_0}\tilde{y}$ yielding that

$$d^i_{\theta^j \tilde{x}}(x_j, y_j) < \delta \text{ for } n_0 \leq j \leq n,$$

which implies $\tilde{y} \in \widetilde{B}^i(\tilde{x}; n, \delta)$ for all $\tilde{y} \in [\alpha_0^n \bigvee \xi_i](\tilde{x})$. $\square$

Proof of Claim 1. It follows from our assumptions on $\tilde{y}$ and Proposition IX.2.3(1) that $H_{\tilde{x}}^j \Phi_{\tilde{x}}^{-1} y_0 \in W^i_{\theta^j\tilde{x},\delta}(\theta^j\tilde{x})$ and (note that $\lambda_0 \geq \lambda_1 + 2\varepsilon$)

$$\|H_{\tilde{x}}^j \Phi_{\tilde{x}}^{-1} y_0\|_i' \leq e^{j\lambda_0}\|\Phi_{\tilde{x}}^{-1} y_0\|_i' \text{ for } j > 0$$

provided that $\|\Phi_{\tilde{x}}^{-1} y_0\|_i' \exp(k\lambda_0) \leq e^{-\lambda_1 - 3\varepsilon} l(\theta^k\tilde{x})^{-1}$ for all $0 \leq k < j$. This is guaranteed for $j \leq n_+$. Since $W^i_{\theta^j\tilde{x},\delta}(\theta^j\tilde{x})$ is a graph over $\bar{\mathbf{R}}^{(i)}(\delta l(\theta^j\tilde{x})^{-1})$ with slope< 1, one has

$$d^i_{\theta^j\tilde{x}}(x_j, y_j) \leq 2K_0\|\Phi_{\theta^j\tilde{x}}^{-1} y_j\|_i' < \delta \text{ for } 0 \leq j \leq n_+. \square$$

Proof of Claim 2. First if $\tilde{y} \in \alpha(\tilde{x})$, then $\|\Phi_{\tilde{x}}^{-1} y_0\| \leq l_0\psi(\tilde{x}) \leq l_0\psi_+(\tilde{x})$. So we have the desired conclusion for $0 \leq j \leq n_+$. Furthermore, if $n \geq n_+$ and $\tilde{y} \in \alpha_0^n(\tilde{x})$, then

$$y_{n_+} \in p\left(\alpha(\theta^{n_+}\tilde{x})\right)\bigcap \Phi_{\theta^{n_+}\tilde{x}} W^i_{\theta^{n_+}\tilde{x},\delta}(\theta^{n_+}\tilde{x})$$

and we can apply Claim 1 to $\theta^{n_+}\tilde{y}$ with $\theta^{n_+}\tilde{x}, x_{n_+}$ and y_{n_+} in place of $\tilde{x}, x_0$ and y_0 respectively. An inductive argument completes the proof of Claim 2. $\square$

Proof of Claim 3. To prove this claim, let us assume that $\tilde{x} \notin S'$ and for simplicity of notation write $k = n_0 - n_-(\theta^{n_0}\tilde{x})$. That is, k is the largest integer< 0 such that $\theta^k\tilde{y} \in S'$. Clearly, $n_+(\theta^k\tilde{x}) = n_0 - k = n_-(\theta^{n_0}\tilde{x})$. Since $\theta^{-n}\xi$ is increasing as $n \to +\infty$, we have $\theta^k\tilde{y} \in \xi(\theta^k\tilde{x})$ and $y_k \in \Phi_{\theta^k\tilde{x}} W^i_{\theta^k\tilde{x},\delta}(\theta^k\tilde{x})$ by our choice of $\widehat{S}_i$. Also ψ is chosen in such a way that $p[\theta^{-j}(\alpha(\theta^{n_0}\tilde{x}))]$ lies well inside the charts at x_{n_0-j} for $j = 1, 2, \cdots, n_0 - k$. Hence

$$\begin{aligned}
\|\Phi_{\theta^k\tilde{x}} y_k\|_i' &= \|H_{\tilde{x}}^k \Phi_{\tilde{x}}^{-1} y_0\|_i' \leq \|H_{\tilde{x}}^{k+1} \Phi_{\tilde{x}}^{-1} y_0\|_i' \leq \cdots \\
&\leq \|H_{\tilde{x}}^{n_0} \Phi_{\tilde{x}}^{-1} y_0\|_i' = \|\Phi_{\theta^{n_0}\tilde{x}}^{-1} y_{n_0}\|_i' \\
&\leq l_0\psi(\theta^{n_0}\tilde{x}) \leq l_0\psi_+(\theta^k\tilde{x}).
\end{aligned}$$

Therefore by Claim 1, Claim 3 holds true. $\square$

Proof of Proposition IX.3.1. It follows directly from the definition of $\overline{h}_i(\tilde{x};\xi_i)$ together with Lemmas IX.3.2 and IX.3.3 that $\overline{h}_i(\tilde{x};\xi_i) \leq H_{\tilde{\mu}}(\xi_i|\theta\xi_i)$ for $\tilde{\mu}$-a.e. $\tilde{x}$. What remains is to verify

$$\underline{h}_i(\tilde{x};\xi_i) \geq H_{\tilde{\mu}}(\xi_i|\theta\xi_i), \tilde{\mu} - \text{a.e.}\tilde{x} \in M^f \tag{IX.10}$$

We know that

$$H_{\tilde{\mu}}(\xi_i|\theta\xi_i) = H_{\tilde{\mu}}(\theta^{-1}\xi_i|\xi_i) = \int -\log\tilde{\mu}_{\tilde{x}}^{\xi_i}((\theta^{-1}\xi_i)(\tilde{x}))d\tilde{\mu},$$

where the item behind log is a conditional measure of the denoted set. Put

$$\begin{aligned}
g(\tilde{x}) &\stackrel{\text{def}}{=} -\log\tilde{\mu}_{\tilde{x}}^{\xi_i}((\theta^{-1}\xi_i)(\tilde{x})), \\
A_\delta &\stackrel{\text{def}}{=} \{\tilde{x} \in M^f : \widetilde{B}^i(\tilde{x};\delta) \subset \xi_i(\tilde{x})\}.
\end{aligned}$$

Since ξ_i is an increasing generator subordinate to W^i-manifolds, one has $A_\delta \uparrow$ and $\tilde{\mu}(A_\delta) \uparrow 1$ as $\delta \downarrow 0$. Hence given $\varepsilon > 0$, there exists $\delta' > 0$ such that for each $\delta \in (0, \delta')$

$$\int_{\theta^{-1}A_\delta} g(\tilde{x}) d\tilde{\mu} \geq H_{\tilde{\mu}}(\xi_i | \theta \xi_i) - \varepsilon.$$

Define $U^i(\tilde{x}; n, \delta) \overset{\text{def}}{=} \bigcap_{0 \leq k \leq n, \theta^k \tilde{x} \in A_\delta} (\theta^{-k}\xi_i)(\tilde{x})$. Then $\widetilde{B}^i(\tilde{x}; n, \delta) \subset U^i(\tilde{x}; n, \delta)$. Hence

$$-\log \tilde{\mu}_{\tilde{x}}^{\xi_i}(\widetilde{B}^i(\tilde{x}; n, \delta)) \geq -\log \tilde{\mu}_{\tilde{x}}^{\xi_i}(U^i(\tilde{x}; n, \delta)). \tag{IX.11}$$

Furthermore, we will prove that for $\tilde{\mu}$-a.e. $\tilde{x}$

$$-\log \tilde{\mu}_{\tilde{x}}^{\xi_i}(U^i(\tilde{x}; n, \delta)) \geq \sum_{k=0}^{n-1} (1_{\theta^{-1}A_\delta} \cdot g)(\theta^k \tilde{x}), \tag{IX.12}$$

which combined with inequality (IX.11) implies

$$-\frac{1}{n} \log \tilde{\mu}_{\tilde{x}}^{\xi_i}(\widetilde{B}^i(\tilde{x}; n, \delta)) \geq \frac{1}{n} \sum_{k=0}^{n-1} (1_{\theta^{-1}A_\delta} \cdot g)(\theta^k \tilde{x}).$$

Then applying Birkhoff's ergodic theorem, one has for each $\delta \in (0, \delta')$

$$\underline{h}_i(\tilde{x}; \delta, \xi_i) \geq \int_{\theta^{-1}A_\delta} g(\tilde{x}) d\tilde{\mu} \geq H_{\tilde{\mu}}(\xi_i | \theta \xi_i) - \varepsilon.$$

Now we return to the proof of the assertion (IX.12). Let

$$\tau(\tilde{x}; n) \overset{\text{def}}{=} \max\{0 \leq k \leq n : \theta^k \tilde{x} \in A_\delta\},$$

where $\max \emptyset \overset{\text{def}}{=} +\infty$. For $\tilde{\mu}$-a.e. $\tilde{x}$ fixed, $\tau(\tilde{x}; n)$ is a finite number with sufficiently large n by Poincaré's recurrence theorem. This together with the condition that ξ_i is an increasing generator yields

$$\begin{aligned}
-\log \tilde{\mu}_{\tilde{x}}^{\xi_i}(U^i(\tilde{x}; n, \delta)) &= \sum_{k=0}^{\tau(\tilde{x};n)-1} -\log \tilde{\mu}_{\tilde{x}}^{\theta^{-k}\xi_i}((\theta^{-k-1}\xi_i)(\tilde{x})) \\
&\geq \sum_{k=0}^{\tau(\tilde{x};n)-1} (1_{\theta^{-1}A_\delta} \cdot g)(\theta^k \tilde{x}) \\
&= \sum_{k=0}^{n-1} (1_{\theta^{-1}A_\delta} \cdot g)(\theta^k \tilde{x}). \square
\end{aligned}$$

IX.4 Estimates of Local Entropies along Unstable Manifolds

In this section we will estimate local entropy along unstable manifolds through Lyapunove exponents and transverse dimensions. To be explicit, we will prove in this section via Propositions IX.4.1, IX.4.3, IX.4.6 and IX.4.7 that

$$\overline{\delta}_i = \underline{\delta}_i =: \delta_i, 1 \le i \le u$$

where $\overline{\delta}_i = \overline{\delta}_i(\tilde{x})$ and $\underline{\delta}_i = \underline{\delta}_i(\tilde{x})$, and

$$\gamma_i \stackrel{\text{def}}{=} \delta_i - \delta_{i-1} = (h_i - h_{i-1})/\lambda_i \le m_i, 1 \le i \le u \tag{IX.13}$$

with $\delta_0 = 0$ and $h_0 = 0$. Thus identity

$$h_\mu(f) = \sum_{i=1}^{u} \lambda_i \gamma_i$$

follows from $h_u = h_{\tilde{\mu}}(\theta) = h_\mu(f)$.

IX.4.1 Estimate of Local Entropy h_1

The local entropy h_1 measures the amount of randomness along the leaves of W^1-manifolds and can be formulized as the following.

Proposition IX.4.1 *For $\tilde{\mu}$-a.e. $\tilde{x}$, $\overline{\delta}_1(\tilde{x}) = \underline{\delta}_1(\tilde{x}) =: \delta_1 =: \gamma_1$ and $h_1 = \lambda_1 \gamma_1$. By Lemma VII.VII.7.5 and from the definition of $\overline{\delta}_1$ and $\underline{\delta}_1$, it is clear that* $0 \le \gamma_1 \le m_1$.

Proof. Let $\delta > 0$ be a sufficiently small number. We divide the proof into two parts.

(i) First for each $\tilde{y} \in \widetilde{B}^1(\tilde{x}; \rho_n)$ with $\rho_n = \frac{1}{2} K_0^{-1} \ell(\tilde{x})^{-1} e^{-n(\lambda_1 + 2\varepsilon)} \delta$, one has $y_0 \in W^1_{loc}(\tilde{x})$ and $d^1_{\tilde{x}}(x_0, y_0) < \rho_n$. Hence by Proposition IX.2.3 and Lemma IX.2.4, for each $k \le n$, $y_k \in W^1_{loc}(\theta^k \tilde{x})$ and

$$\begin{aligned} d^1_{\theta^k \tilde{x}}(x_k, y_k) &\le 2K_0 \|\Phi^{-1}_{\theta^k \tilde{x}} \circ f^k x_0 - \Phi^{-1}_{\theta^k \tilde{x}} \circ f^k y_0\|'_1 = 2K_0 \|H^k_{\tilde{x}} \circ \Phi^{-1}_{\tilde{x}} y_0\|'_1 \\ &\le 2K_0 \ell(\tilde{x}) d(x_0, y_0) e^{k(\lambda_1 + 2\varepsilon)} \\ &\le 2K_0 \ell(\tilde{x}) d^1_{\tilde{x}}(x_0, y_0) e^{k(\lambda_1 + 2\varepsilon)} < \delta \end{aligned}$$

Therefore $\widetilde{B}^1(\tilde{x}; \rho_n) \subset \widetilde{B}^1(\tilde{x}; n, \delta)$ for $\tilde{\mu}$-a.e. $\tilde{x}$ and all $n \ge 0$. This implies $h_1 \le \lambda_1 \underline{\delta}_1$.

(ii) We then prove that $h_1 \ge \lambda_1 \overline{\delta}_1$. Let ξ_i's be as introduced in Section 3.3 and let α be as introduced in Lemma IX.3.3. By Lemma IX.3.3, one has

$$[\alpha_0^n \bigvee \xi_i](\tilde{x}) \subset \widetilde{B}^i(\tilde{x}; n, \delta), \quad \forall n \ge n_0(\tilde{x}).$$

Therefore for each $\tilde{y} \in (\alpha_0^n \bigvee \xi_i)(\tilde{x})$ with $n \geq n_0(\tilde{x})$, one has $y_n \in W^i_{loc}(\theta^n \tilde{x})$ and $d^i_{\theta^n \tilde{x}}(x_n, y_n) < \delta$. By Proposition IX.2.3(3)

$$\begin{aligned} d^i_{\tilde{x}}(x_0, y_0) &\leq 2K_0 \|\Phi_{\tilde{x}}^{-1} y_0\|'_i = 2K_0 \|H^{-n}_{\theta^n \tilde{x}} \circ \Phi^{-1}_{\theta^n \tilde{x}} y_n\|'_1 \\ &\leq 2K_0 e^{-n(\lambda_i - 2\varepsilon)} \|\Phi^{-1}_{\theta^n \tilde{x}} y_n\|'_1 \end{aligned}$$

where by Lemma IX.2.4 $\|\Phi^{-1}_{\theta^n \tilde{x}} y_n\|'_i \leq l(\theta^n \tilde{x}) d^i_{\theta^n \tilde{x}}(x_n, y_n) < \delta \ell(\tilde{x}) e^{n\varepsilon}$. Hence

$$d^i_{\tilde{x}}(x_0, y_0) < 2K_0 \delta e^{-n(\lambda_i - 3\varepsilon)} \ell(\tilde{x}).$$

This implies for $\tilde{\mu}$-a.e. $\tilde{x}$

$$[\alpha_0^n \bigvee \xi_i](\tilde{x}) \subset \widetilde{B}^i(\tilde{x}; 2K_0 \delta \ell(\tilde{x}) e^{-n(\lambda_i - 3\varepsilon)}), \quad \forall n \geq n_0(\tilde{x})$$

and hence

$$[\alpha_0^n \bigvee \xi_i](\tilde{x}) \subset \widetilde{B}^i(\tilde{x}; e^{-n(\lambda_i - 4\varepsilon)}), \quad \forall n \geq n'_0(\tilde{x}) \tag{IX.14}$$

with $n'_0(\tilde{x}) \stackrel{\text{def}}{=} \max\{n_0(\tilde{x}), [\frac{1}{\varepsilon} log(2K_0 \delta \ell(\tilde{x}))] + 1\}$. By Lemma IX.3.2, equation (IX.14) with $i = 1$ implies $h_1 \geq \lambda_1 \overline{\delta}_1$. □

IX.4.2 Estimate of Local Entropy h_i from Below with $2 \leq i \leq u$

Based on Proposition IX.4.1, we will then prove the coincidence of $\underline{\delta}_i$ and $\overline{\delta}_i$ for $i = 2, \cdots, u$. For this end, let us assume from here on that we have proved inductively the coincidence of $\underline{\delta}_j$ and $\overline{\delta}_j$ for $j = 1, 2, \cdots, i-1$, i.e.,

$$\underline{\delta}_j = \overline{\delta}_j =: \delta_j, \text{ for } j = 1, 2, \cdots, i-1.$$

Then, in view of (IX.14), Proposition IX.3.1 and Lemmas IX.3.2 and IX.3.3, the following lemma holds.

Lemma IX.4.2 *For each sufficiently small $\varepsilon > 0$, let α be as introduced in Lemma IX.3.3 (with $\delta > 0$ small enough). There exists a Borel function $\widehat{n} : M^f \to \mathbb{Z}^+$ satisfying the following for $\tilde{\mu}$-a.e. $\tilde{x}$ (where $2 \leq i \leq u$):*

(1) $[\xi_i \bigvee \alpha_0^n](\tilde{x}) \subset \widetilde{B}^i(\tilde{x}; e^{-n(\lambda_i - 4\varepsilon)})$ *for any* $n \geq \widehat{n}(\tilde{x})$*;*
(2) $-\frac{1}{n} \log \tilde{\mu}^{\xi_{i-1}}_{\tilde{x}}(\alpha_0^n(\tilde{x})) \geq h_{i-1} - \varepsilon$ *for any* $n \geq \widehat{n}(\tilde{x})$*;*
(3) $-\frac{1}{n} \log \tilde{\mu}^{\xi_i}_{\tilde{x}}(\alpha_0^n(\tilde{x})) \leq h_i + \varepsilon$ *for any* $n \geq \widehat{n}(\tilde{x})$*;*
(4) $L \stackrel{\text{def}}{=} \widetilde{B}^{i-1}(\tilde{x}; e^{-n(\lambda_i - 4\varepsilon)}) \subset \xi_{i-1}(\tilde{x})$ *for any* $n \geq \widehat{n}(\tilde{x})$*;*
(5) $\log \tilde{\mu}^{\xi_{i-1}}_{\tilde{x}}(L) / [-n(\lambda_i - 4\varepsilon)] \leq \overline{\delta}_{i-1} + \varepsilon$ *for any* $n \geq \widehat{n}(\tilde{x})$*;*
(6) $\log \tilde{\mu}^{\xi_i}_{\tilde{x}}(\widetilde{B}^i(\tilde{x}; 2e^{-n(\lambda_i - 4\varepsilon)})) / [-n(\lambda_i - 4\varepsilon)] \geq \overline{\delta}_i - \varepsilon$ *for infinitely many* $n \geq \widehat{n}(\tilde{x})$.

We then give a lower bound of the local entropy h_i in terms of Lyapunov exponents and pointwise dimensions.

Proposition IX.4.3 *For* $2 \le i \le u$ *and* $\tilde{\mu}$*-a.e.* $\tilde{x}$, $(h_i - h_{i-1})/\lambda_i \ge \overline{\delta}_i - \overline{\delta}_{i-1}$.

Below is the famous Borel density lemma on a manifold M.

Lemma IX.4.4 (***Borel Density Lemma****; See [7, Proposition 3] or [23].*) *Let* m *be a Borel probability measure on a manifold* M *and let* $A \subset M$ *be a measurable set with* $m(A) > 0$. *Then for* m*-almost every* $x \in A$

$$\lim_{\rho \to 0} \frac{m(A \bigcap B(x,\rho))}{m(B(x,\rho))} = 1.$$

Furthermore, for each $\delta > 0$ *there is a set* $\widehat{A} \subset A$ *with* $m(\widehat{A}) > m(A) - \delta$ *and a number* ρ^* *such that for all* $x \in \widehat{A}$ *and* $0 < \rho < \rho^*$ *one has*

$$m(A \bigcap B(x,\rho)) \ge \frac{1}{2} m(B(x,\rho)).$$

But in general, the inverse limit space M^f is not a finite dimensional manifold. In the proof of Proposition IX.4.3, one has to overcome this deficiency of a Borel Density Lemma on M^f; Thus we establish the following slight variant of Density Lemma.

Lemma IX.4.5 *Let* $A \subset M^f$ *be a measurable set with* $\tilde{\mu}(A) > 0$. *Then for* $\tilde{\mu}$*-almost every* $\tilde{x} \in A$

$$\lim_{\rho \to 0} \frac{\tilde{\mu}_{\tilde{x}}^{\xi_i}(A \bigcap \widetilde{B}^i(\tilde{x},\rho))}{\mu_{\tilde{x}}^{\xi_i}(\widetilde{B}^i(\tilde{x},\rho))} = 1, \tag{IX.15}$$

where $1 \le i \le u$ *and each* ξ *is a measurable partition subordinate to the corresponding* W^i*-manifolds. Furthermore, for each* $\delta > 0$ *there is a set* $\widehat{A} \subset A$ *with* $\tilde{\mu}(\widehat{A}) > \tilde{\mu}(A) - \delta$ *and a number* ρ^* *such that for all* $\tilde{x} \in \widehat{A}$ *and* $0 < \rho < \rho^*$ *one has*

$$\tilde{\mu}(A \bigcap \widetilde{B}^i(\tilde{x},\rho)) \ge \frac{1}{2} \mu(\widetilde{B}^i(\tilde{x},\rho)).$$

Proof. Let A' be the set consisting of those points in A satisfying (IX.15). It is clearly measurable. Fix a point $\tilde{x}$ with the properties described in Definition IX.1.2 and write $C = \xi_i(\tilde{x})$. The induced distance on pC will be denoted as $d_C(\cdot,\cdot)$, which is $d^i_{\tilde{x}}(\cdot,\cdot)$ restricted on pC. We write a ball in pC centered at y with radius ρ as

$$B_C(y,\rho) := \{z \in pC : d_C(y,z) < \rho\}.$$

In view of (IX.5), we have

$$pC \cap p\widetilde{B}^i(\tilde{y};\rho) = B_C(y_0,\rho), \quad \forall \tilde{y}.$$

Since $p_C := p|_C : C \to pC$ is bijective, we can define a measure $\widehat{\mu}$ on $pC \subset M$ by $\widehat{\mu} := p_C \tilde{\mu}_C$. $p_C : (C, \tilde{\mu}_C) \to (pC, \widehat{\mu})$ becomes a measure preserving bijection. Obviously

$$\tilde{\mu}_C(\widetilde{B}^i(\tilde{y};\rho) \cap A) = \widehat{\mu}(p\widetilde{B}^i(\tilde{y};\rho) \cap p_C(A)) = \widehat{\mu}(B_C(y_0,\rho) \cap p_C(A)).$$

Therefore Borel Density Lemma on M [23] gives

$$\lim_{\rho \to 0} \frac{\widehat{\mu}(B_C(y_0,\rho) \cap p_C(A))}{\widehat{\mu}(B_C(y_0,\rho))} = 1 \tag{IX.16}$$

for $\widehat{\mu}$-a.e. $y_0 \in p_C(A)$. For such y_0 define $\tilde{y} := p_C^{-1}(y_0) \in C$. Then (IX.16) is equivalent to

$$\lim_{\rho \to 0} \frac{\tilde{\mu}_C(\widetilde{B}^i(\tilde{y};\rho) \cap A)}{\tilde{\mu}_C(\widetilde{B}^i(\tilde{y};\rho))} = 1. \tag{IX.17}$$

Clearly each $\tilde{y} \in C \cap A$ satisfying the above equation must be a point in A'. Of course each $\tilde{y} \in C \cap A'$ satisfies (IX.17). The assertion that $\widehat{\mu}$-a.e. $y_0 \in p_C(A)$ satisfies (IX.16) is equivalent to

$$\widehat{\mu}(p_C(A')) = \widehat{\mu}(p_C(A)),$$

which can be rewritten as $\tilde{\mu}_C(A') = \tilde{\mu}_C(A)$. Since this holds for $\tilde{\mu}_{\xi_i}$-a.e. C, we have

$$\tilde{\mu}(A') = \int \tilde{\mu}_C(A') d\tilde{\mu}_{\xi_i}(C) = \int \tilde{\mu}_C(A) d\tilde{\mu}_{\xi_i}(C) = \tilde{\mu}(A),$$

which implies the validity of (IX.15) for $\tilde{\mu}$-a.e. $\tilde{x} \in A$. □

Proof of Proposition IX.4.3. Let

$$\Gamma_n := \{\tilde{x} : \widehat{n}(\tilde{x}) \leq n \text{ and } \tilde{x} \text{ satisfies the requirements (1)–(6) of Lemma IX.4.2}\}.$$

Clearly we have $\tilde{\mu}(\Gamma_n) \uparrow 1$ as n tends to $+\infty$. Therefore for any $\varepsilon' \in (0,1)$, there is an integer N_1 such that $\Gamma' := \Gamma_{N_1}$ has $\tilde{\mu}$-measure$\geq 1 - \varepsilon'/2$. Then by Lemma IX.4.5, there is another integer $N_2 \geq N_1$ and a subset $\widehat{\Gamma} \subset \Gamma'$ of $\tilde{\mu}$-measure $\geq 1 - \varepsilon'$ such that for any $\tilde{x} \in \widehat{\Gamma}$ fixed, we have

$$\tilde{\mu}_{\tilde{x}}^{\xi_{i-1}}(L \bigcap \Gamma') \geq \frac{1}{2} \tilde{\mu}_{\tilde{x}}^{\xi_{i-1}}(L)$$

for any $n \geq N_2$, where $L = \widetilde{B}^{i-1}(\tilde{x}; e^{-n(\lambda_i - 4\varepsilon)})$. Then according to Lemma IX.4.2(5) one has

$$\tilde{\mu}_{\tilde{x}}^{\xi_{i-1}}(L \bigcap \Gamma') \geq \frac{1}{2} \exp(-n(\lambda_i - 4\varepsilon)(\overline{\delta}_{i-1} + \varepsilon)).$$

By Lemma IX.4.2(2) it is clear that

$$\tilde{\mu}_{\tilde{x}}^{\xi_{i-1}}(\alpha_0^n(\tilde{y})) = \tilde{\mu}_{\tilde{y}}^{\xi_{i-1}}(\alpha_0^n(\tilde{y})) \le \exp(-n(h_{i-1}-\varepsilon))$$

for each $\tilde{y} \in L \bigcap \Gamma'$. Hence for any $\tilde{x} \in \widehat{\Gamma}$ and $n \ge N_2$

$$\begin{aligned}\#\{\alpha_0^n(\tilde{y}) : \tilde{y} \in L \bigcap \Gamma'\} &\ge \tilde{\mu}_{\tilde{x}}^{\xi_{i-1}}(L \bigcap \Gamma') \big/ \exp(-n(h_{i-1}-\varepsilon)) \\ &\ge \frac{1}{2}\exp\{n[h_{i-1}-\varepsilon-(\lambda_i-4\varepsilon)(\overline{\delta}_{i-1}+\varepsilon)]\}.\end{aligned}$$

On the other hand, according to Lemma IX.4.2(1), one has

$$[\xi_i \bigvee \alpha_0^n](\tilde{y}) \subset \widetilde{B}^i(\tilde{y}; e^{-n(\lambda_i-4\varepsilon)}), \quad \forall \tilde{y} \in \Gamma'.$$

Clearly $d_{\tilde{x}}^i(x_0, y_0) \le d_{\tilde{x}}^{i-1}(x_0, y_0)$ for any $\tilde{y} \in L$. Therefore

$$[\xi_i \bigvee \alpha_0^n](\tilde{y}) \subset \widetilde{B}^i(\tilde{x}; 2e^{-n(\lambda_i-4\varepsilon)}), \quad \forall \tilde{y} \in L \bigcap \Gamma'.$$

Hence for any $\tilde{x} \in \widehat{\Gamma}$ and $n \ge N_2$

$$\begin{aligned}&\log \tilde{\mu}_{\tilde{x}}^{\xi_i}(\widetilde{B}^i(\tilde{x}; 2e^{-n(\lambda_i-4\varepsilon)})) \\ &\ge \log \#\{[\xi_i \bigvee \alpha_0^n](\tilde{y}) : \tilde{y} \in L \bigcap \Gamma'\} + \log \min_{\tilde{y}} \tilde{\mu}_{\tilde{x}}^{\xi_i}([\xi_i \bigvee \alpha_0^n](\tilde{y})) \\ &\ge \log \#\{\alpha_0^n(\tilde{y}) : \tilde{y} \in L \bigcap \Gamma'\} + \log \min_{\tilde{y}} \tilde{\mu}_{\tilde{x}}^{\xi_i}(\alpha_0^n(\tilde{y})) \\ &\ge -\log 2 - n[h_i - h_{i-1} + 2\varepsilon + (\lambda_i - 4\varepsilon)(\overline{\delta}_{i-1}+\varepsilon)].\end{aligned}$$

Comparing the above inequality with Lemma IX.4.2(6), we obtain

$$(\overline{\delta}_i - \overline{\delta}_{i-1} - 2\varepsilon)(\lambda_i - 4\varepsilon) \le \frac{1}{n}\log 2 + h_i - h_{i-1} + 2\varepsilon,$$

which implies Proposition IX.4.3 by letting $n \to +\infty$, $\varepsilon \to 0$ and finally $\varepsilon' \to 0$. □

IX.4.3 Estimate of Local Entropy h_i from Above with $2 \le i \le u$

Let number N_0, map $\tilde{\pi}^i : \eta_i(\tilde{x}) \to \mathbb{R}^{m_i}$ and metrics $\tilde{d}_{\tilde{x}}^i(\cdot,\cdot)$ and $\hat{d}_{\tilde{x}}^i(\cdot,\cdot)$ on $\eta_i(\tilde{x})/\eta_{i-1}$ be as introduced in Section IX.2. We denote $\tilde{\mu}_{\tilde{x}}^{\eta_i}$ by $\tilde{\mu}_{\tilde{x}}^i$ for simplicity of notations. Write $\tilde{\pi}_i \overset{\text{def}}{=} (\tilde{\pi}^1, \cdots, \tilde{\pi}^i) : \eta_i(\tilde{x}) \to \bar{\mathbf{R}}^{(i)}$ and put

$$\widehat{B}^i(\tilde{x}, \rho) \overset{\text{def}}{=} \{\tilde{y} \in \eta_i(\tilde{x}) : \tilde{d}_{\tilde{x}}^i(\tilde{x}, \tilde{y}) < \rho\}.$$

Then we define transverse dimensions as the following.

Definition IX.4.1 $\tilde{\gamma}_i(\tilde{x}) \stackrel{\text{def}}{=} \liminf_{\rho\to 0} \frac{\log \tilde{\mu}^i_{\tilde{x}}(\widehat{B}^i(\tilde{x};\rho))}{\log \rho}$ is called the *transverse dimension* of η_i/η_{i-1} at $\tilde{x}$.

Definition IX.4.2 For each $\tilde{y} \in \eta_i(\tilde{x})$, $\widehat{\gamma}_i(\tilde{y};\tilde{x}) \stackrel{\text{def}}{=} \liminf_{\rho\to 0} \frac{\log \nu(\{z\in\bar{\mathbf{R}}^{(i)}:\|z_i-\tilde{\pi}^i(\tilde{y})\|<\rho\})}{\log \rho}$ is called the *transverse dimension* of $W^i(\tilde{x})/W^{i-1}$ at $\tilde{y}$, where $\nu \stackrel{\text{def}}{=} \tilde{\mu}^i_{\tilde{x}} \circ \tilde{\pi}_i^{-1}$ is a Borel probability on $\bar{\mathbf{R}}^{(i)}$.

Now we introduce the main results for this subsection.

Proposition IX.4.6 *Given $\beta \in (0,1)$. One has for sufficiently small δ and $\tilde{\mu}$-a.e. $\tilde{x}$*

$$(\lambda_i+\beta)\tilde{\gamma}_i(\tilde{x}) \geq (1-\beta)(h_i-h_{i-1}-\beta)$$

and $0 \leq \tilde{\gamma}_i(\tilde{x}) \leq m_i$. Hence

$$h_i - h_{i-1} \leq \lambda_i m_i. \tag{IX.18}$$

Proposition IX.4.7 *Given $\beta \in (0,1)$. For $\tilde{\mu}^i_{\tilde{x}}$-a.e. $\tilde{y} \in \eta_i(\tilde{x})$, one has $\tilde{\gamma}_i(\tilde{x}) = \widehat{\gamma}_i(\tilde{y};\tilde{x})$ and*

$$\underline{\delta}_i - \underline{\delta}_{i-1} \geq \widehat{\gamma}_i(\tilde{y};\tilde{x}) \geq \frac{(1-\beta)(h_i-h_{i-1}-\beta)}{\lambda_i+\beta}. \tag{IX.19}$$

Hence by letting $\beta \to 0$ we obtain

$$(h_i-h_{i-1})/\lambda_i \leq \underline{\delta}_i - \underline{\delta}_{i-1}. \tag{IX.20}$$

In order to prove the above two proposition, we need the following results; since they are of pure measure theoretical nature and are simple consequences of the above Borel Density Lemma, we state them here without proof.

Lemma IX.4.8 (See [43, Lemma 11.3.1].) *Let μ be a probability measure on $\mathbb{R}^p \times \mathbb{R}^q$, $\pi : \mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^p$ the natural projection. Let $\{\mu_t\}$ be a canonical system of conditional measures of μ associated with $\{\{t\} \times \mathbb{R}^q : t \in \mathbb{R}^p\}$. Define*

$$\gamma(t) \stackrel{\text{def}}{=} \liminf_{\rho\to 0} \frac{\log \mu\left(\pi^{-1}B^p(t,\rho)\right)}{\log \rho}$$

and let $\delta \geq 0$ be such that at μ-a.e. (t,x)

$$\delta \leq \liminf_{\rho\to 0} \frac{\log \mu_t\left(B^q(x,\rho)\right)}{\log \rho}$$

holds true, where $B^p(t,\rho)$ is the open disk in $\mathbb{R}^p$ centered at t of radius ρ. Then at μ-a.e. $z = (t,x)$

$$\delta + \gamma(t) \leq \liminf_{\rho\to 0} \frac{\log \mu\left(B^{p+q}(z,\rho)\right)}{\log \rho}$$

holds true.

Lemma IX.4.9 (See [43, Lemma 11.3.2].) *Let* (Ω, P) *be an abstract probability space which is a Polish space. Let* μ *be a probability measure on* $\Omega \times \mathbb{R}^q$ *with marginal measure* P *on* Ω. *Let* $\tilde{\gamma} \geq 0$ *be such that at* μ*-a.e.* (ω, x)

$$\tilde{\gamma} \leq \liminf_{\rho \to 0} \frac{\log \mu_\omega\big(B^q(x,\rho)\big)}{\log \rho}$$

holds. Then at μ*-a.e.* (ω, x)

$$\tilde{\gamma} \leq \liminf_{\rho \to 0} \frac{\log \mu\big(\Omega \times B^q(x,\rho)\big)}{\log \rho}.$$

Proof of Proposition IX.4.6. Let $e^{-(\lambda_i+\beta)\varepsilon} N_0^{4\tilde{\mu}(\widehat{E})} < 1$ with $\beta \in (0,1)$ (this holds true provided $\tilde{\mu}(\widehat{E})$ small enough). We will prove that for $\tilde{\mu}$-a.e. $\tilde{x}$

$$(\lambda_i + \beta) \liminf_{\rho \to 0} \frac{\log \tilde{\mu}^i_{\tilde{x}}(\widehat{B}^i(\tilde{x};\rho))}{\log \rho} \geq (1-\beta)(h_i - h_{i-1} - \beta). \tag{IX.21}$$

The first conclusion then follows immediately from this together with Proposition IX.2.7. The second conclusion follows from the definition of $\tilde{\gamma}_i$ and Lemma VII.VII.7.5 since the point-independent metric $\tilde{d}^i(\cdot,\cdot)$ on $\eta_i(\tilde{x})/\eta_{i-1}$ makes it isometric to a subset of $(\mathbb{R}^{m_i}, \|\cdot\|)$.

Now we come to the proof of (IX.21). First fix $\varepsilon \in (0, \beta/3)$. Let $\widetilde{\Delta}'_0$ be a set as chosen in Section IX.2. Recalling that $\tilde{\mu}(\widetilde{\Delta}'_0) = 1$ and $\theta \widetilde{\Delta}'_0 = \widetilde{\Delta}'_0$, for the sake of presentation we may assume that $\widetilde{\Delta}'_0 = M^f$.

We divide the proof into four parts following the line presented in [51].

(A) Before proceeding with the main argument, we record some estimates analogous of those in [51, pp. 171]. For $\delta > 0$, define g, g_δ and $g_* : M^f \to \mathbb{R}$ by

$$\begin{aligned}
g(\tilde{y}) &\stackrel{\text{def}}{=} \tilde{\mu}^{i-1}_{\tilde{y}}((\theta^{-1}\eta_i)(\tilde{y})), \\
g_\delta(\tilde{y}) &\stackrel{\text{def}}{=} \frac{1}{\tilde{\mu}^i_{\tilde{y}}(\widehat{B}^i(\tilde{y};\delta))} \int_{\widehat{B}^i(\tilde{y};\delta))} \tilde{\mu}^{i-1}_{\tilde{z}}((\theta^{-1}\eta_i)(\tilde{y}))\, d\tilde{\mu}^i_{\tilde{y}}(\tilde{z}), \\
g_*(\tilde{y}) &\stackrel{\text{def}}{=} \inf_{\delta \in Q} g_\delta(\tilde{y}),
\end{aligned}$$

where $Q \stackrel{\text{def}}{=} \{e^{-(\lambda_i+\beta)l} N_0^{2j} : l, j \in \mathbb{Z}^+\}$.

According to Proposition IX.2.7, one has $g(\tilde{y}) = \tilde{\mu}^{i-1}_{\tilde{y}}((\theta^{-1}\eta_{i-1})(\tilde{y}))$ for $\tilde{\mu}$-a.e. $\tilde{y}$. For each $\delta > 0$, one can check that the functions $\tilde{y} \to \tilde{\mu}^i_{\tilde{y}}(\widehat{B}^i(\tilde{y};\delta))$ and $\tilde{y} \to \tilde{\mu}^{\theta^{-1}\eta_i}_{\tilde{y}}(\widehat{B}^i(\tilde{y};\delta))$ are measurable and $\tilde{\mu}^i_{\tilde{y}}(\widehat{B}^i(\tilde{y};\delta)) > 0$ for $\tilde{\mu}$-a.e. $\tilde{y}$. Since $H_{\tilde{\mu}}(\theta^{-1}\eta_i|\eta_i) < +\infty$, one has $\tilde{\mu}^i_{\tilde{y}}((\theta^{-1}\eta_i)(\tilde{y})) > 0$ for $\tilde{\mu}$-a.e. $\tilde{y}$ and

$$g_\delta(\tilde{y}) = \frac{\tilde{\mu}^i_{\tilde{y}}(\widehat{B}^i(\tilde{y};\delta)\bigcap(\theta^{-1}\eta_i)(\tilde{y}))}{\tilde{\mu}^i_{\tilde{y}}(\widehat{B}^i(\tilde{y};\delta))} = \frac{\tilde{\mu}^{\theta^{-1}\eta_i}_{\tilde{y}}(\widehat{B}^i(\tilde{y};\delta))}{\tilde{\mu}^i_{\tilde{y}}(\widehat{B}^i(\tilde{y};\delta))} \cdot \tilde{\mu}^i_{\tilde{y}}((\theta^{-1}\eta_i)(\tilde{y})).$$

g_δ is therefore measurable for each fixed $\delta > 0$. The measurability of g_* is obvious.

We assert that $g_\delta \to g$ $\tilde{\mu}$-a.e. on M^f when $\delta \in Q$ and $\delta \to 0$ and that $-\int \log g_* d\tilde{\mu} < +\infty$. To see this, first consider one element of η_i at a time. Fix $\tilde{x}$. Substitute $(\eta_i(\tilde{x}), \tilde{\mu}^i_{\tilde{x}})$ for (X,m) in Lemma VII.VII.7.4, let $\pi = \tilde{\pi}_i : \eta_i(\tilde{x}) \to \bar{\mathbf{R}}^{(i)}$ and let $\alpha = \theta^{-1}\eta_i|_{\eta_i}$. Then we can conclude that $g_\delta(\cdot) \to g(\cdot)$ $\tilde{\mu}^i_{\tilde{x}}$-a.e. as $\delta \in Q$ and $\delta \to 0$ and that for $\tilde{\mu}$-a.e. $\tilde{x}$

$$\begin{aligned} -\int \log g_*(\tilde{z})\tilde{\mu}^i_{\tilde{x}}(d\tilde{z}) &\le -\int \log(\inf_{\delta>0} g_\delta(\tilde{z}))\tilde{\mu}^i_{\tilde{x}}(d\tilde{z}) \\ &\le H_{\tilde{\mu}^i_{\tilde{x}}}(\theta^{-1}\eta_i|\eta_i) + \log c(\sum_{j=1}^{i} m_j) + 1 < +\infty, \end{aligned}$$

where $c(\cdot)$ is the multiplicity defined in Theorem VII.VII.7.1.

(B) The purpose of this step is to study the induced action of θ on

$$\theta^{-1}(\eta_i(\tilde{x}))/\eta_{i-1} \to \eta_i(\tilde{x})/\eta_{i-1}$$

with respect to the metrics $\tilde{d}^i_{\theta^{-1}\tilde{x}}(\cdot,\cdot)$ and $\tilde{d}^i_{\tilde{x}}(\cdot,\cdot)$. Consider $\tilde{x} \in M^f$. The point $\tilde{x}$ will be subjected to a finite number of $\tilde{\mu}$-a.e. assumptions. Let $t_0 < t_1 < \cdots$ be the successive times t when $\theta^t\tilde{x} \in \widehat{E}$ with $t_0 \le 0 < t_1$. Note that t_0 is constant on $\eta_i(\tilde{x})$. For large n and $0 \le k < n$, define $a(\tilde{x};k)$ as follows: if $t_j \le k < t_{j+1}$, then

$$a(\tilde{x};k) \stackrel{\text{def}}{=} \widehat{B}^i(\theta^k\tilde{x}; N_0^{2j} e^{-(\lambda_i+\beta)(n-t_j)}).$$

We now claim that

$$a(\tilde{x};k)\bigcap(\theta^{-1}\eta_i)(\theta^k\tilde{x}) \subset \theta^{-1}a(\tilde{x};k+1) \tag{IX.22}$$

In fact, if $k \ne t_j - 1$ for any j, then $\theta a(\tilde{x};k)\bigcap \eta_i(\theta^{k+1}\tilde{x}) = a(\tilde{x};k+1)$ automatically since $\tilde{d}^i_{\theta^k\tilde{x}}(\cdot,\cdot)$ and $\tilde{d}^i_{\theta^{k+1}\tilde{x}}(\cdot,\cdot)$ are defined by pulling back to $\widehat{E}$. The case when $k = t_j - 1$ for some j reduces to the following consideration: Let $\tilde{y} \in \widehat{E}$ and let $t > 0$ be the smallest integer such that $\theta^t\tilde{y} \in \widehat{E}$. Let $\tilde{z} \in (\theta^{-t}\eta_i)(\tilde{y})$. It suffices to show that

$$\tilde{d}^i_{\theta^t\tilde{y}}(\theta^t\tilde{y}, \theta^t\tilde{z}) \le N_0^2 e^{t(\lambda_i+\beta)}\tilde{d}^i_{\tilde{y}}(\tilde{y},\tilde{z}).$$

First $\hat{d}^i_{\tilde{y}}(\tilde{y},\tilde{z}) \le N_0\tilde{d}^i_{\tilde{y}}((\tilde{y}),\tilde{z})$. Then for $k = 1,2,\cdots,t$, Proposition IX.2.1 tells us that

$$\hat{d}^i_{\theta^k\tilde{y}}(\theta^k\tilde{y}, \theta^k\tilde{z}) \le e^{k(\lambda_i+\beta)}\hat{d}^i_{\tilde{y}}(\tilde{y},\tilde{z}).$$

We pick up another factor of N_0 when converting back to the $\tilde{d}^i$-metric at $\theta^t\tilde{y}$ (see Proposition IX.2.11). What we claimed above is thus proved.

(C) It is easy to see that there exists a θ-invariant Borel set $\widetilde{\triangle} \subset M^f$ with full $\tilde{\mu}$-measure such that, if $\tilde{x} \in \widetilde{\triangle}$, then $\tilde{\mu}^i_{\tilde{x}}(\widehat{B}^i(\tilde{x};\delta)) > 0$ for all $\delta \in Q$. We now estimate $\tilde{\mu}^i_{\tilde{x}}(\widehat{B}^i(\tilde{x};e^{-(\lambda_i+\beta)(n-t_0(\tilde{x}))})) = \tilde{\mu}^i_{\tilde{x}}(a(\tilde{x};0))$ for $\tilde{x} \in \widetilde{\triangle}$ which will be subjected to a finite number of a.e. assumptions. Write

$$\tilde{\mu}^i_{\tilde{x}}(a(\tilde{x};0)) = \prod_{k=0}^{T-1} \frac{\tilde{\mu}^i_{\theta^k\tilde{x}}(a(\tilde{x};k))}{\tilde{\mu}^i_{\theta^{k+1}\tilde{x}}(a(\tilde{x};k+1))} \cdot \tilde{\mu}^i_{\theta^T\tilde{x}}(a(\tilde{x};T))$$

where $T = [n(1-\varepsilon)]$ (here $[x]$ denotes the integer part of x). First note that the last term ≤ 1. For each $0 \leq k < T$, by the θ-invariance of $\tilde{\mu}$ and by uniqueness of conditional measures one has

$$\frac{\tilde{\mu}^i_{\theta^k\tilde{x}}(a(\tilde{x};k))}{\tilde{\mu}^i_{\theta^{k+1}\tilde{x}}(a(\tilde{x};k+1))} = \tilde{\mu}^i_{\theta^k\tilde{x}}(a(\tilde{x};k)) \frac{\tilde{\mu}^i_{\theta^k\tilde{x}}(\theta^{-1}(\eta_i(\theta^{k+1}\tilde{x})))}{\tilde{\mu}^i_{\theta^k\tilde{x}}(\theta^{-1}a(\tilde{x};k+1))}.$$

This is

$$\leq \frac{\tilde{\mu}^i_{\theta^k\tilde{x}}(a(\tilde{x};k))}{\tilde{\mu}^i_{\theta^k\tilde{x}}((\theta^{-1}\eta_i)(\theta^k\tilde{x}) \bigcap a(\tilde{x};k))} \cdot \tilde{\mu}^i_{\theta^k\tilde{x}}((\theta^{-1}\eta_i)(\theta^k\tilde{x})) \tag{IX.23}$$

by (IX.22). If g_δ is defined as in (A), the first quotient in (IX.23) is equal to

$$[g_{\delta(\tilde{x};n,k)}(\theta^k\tilde{x})]^{-1},$$

where $\delta(\tilde{x};n,k) = e^{-(\lambda_i+\beta)(n-t_{j_k}(\tilde{x}))}N_0^{2j_k}$ with $j_k \stackrel{\text{def}}{=} \#\{0 < i \leq k : \theta^i\tilde{x} \in \widehat{E}\}$.

Write $I(\tilde{x}) \stackrel{\text{def}}{=} -\log\tilde{\mu}^i_{\tilde{x}}((\theta^{-1}\eta_i)(\tilde{x}))$. Then the second term in (IX.23) is equal to $e^{-I(\theta^k\tilde{x})}$. Hence

$$\log\tilde{\mu}^i_{\tilde{x}}(\widehat{B}^i(\tilde{x};e^{-(\lambda_i+\beta)(n-t_0(\tilde{x}))})) \leq -\sum_{k=0}^{T-1}\log g_{\delta(\tilde{x};n,k)}(\theta^k\tilde{x}) - \sum_{k=0}^{T-1} I(\theta^k\tilde{x}).$$

Multiplying by $-\frac{1}{n}$ and taking $\liminf$ on both sides of this inequality,

$$\begin{aligned}(\lambda_i+\beta)\liminf_{\rho\to 0}\frac{\log\tilde{\mu}^i_{\tilde{x}}(\widehat{B}^i(\tilde{x};\rho))}{\log\rho} &= (\lambda_i+\beta)\liminf_{n\to+\infty}\frac{\tilde{\mu}^i_{\tilde{x}}(\widehat{B}^i(\tilde{x};e^{-(\lambda_i+\beta)(n-t_0(\tilde{x}))}))}{\log e^{-(\lambda_i+\beta)n}} \\ &\geq \liminf_{n\to+\infty}\sum_{k=0}^{[n(1-\varepsilon)]}\log g_{\delta(\tilde{x};n,k)}(\theta^k\tilde{x}) \\ &\quad + \lim_{n\to+\infty}\frac{1}{n}\sum_{k=0}^{[n(1-\varepsilon)]} I(\theta^k\tilde{x}).\end{aligned}$$

The last limit $= (1-\varepsilon)H_{\tilde{\mu}}(\theta^{-1}\eta_i|\eta_i) = (1-\varepsilon)h_i$ by Birkhoff's ergodic theorem. Hence Proposition IX.4.6 is proved if we show that

$$\limsup_{n\to+\infty} -\frac{1}{n}\sum_{k=0}^{[n(1-\varepsilon)]} \log g_{\delta(\tilde{x};n,k)}(\theta^k\tilde{x}) \le (1-\varepsilon)(h_{i-1}+2\varepsilon). \tag{IX.24}$$

(D) We now prove the last assertion (IX.24). It follows from (A) that there is a measurable function $\delta : M^f \to \mathbb{R}^+$ such that for $\tilde{\mu}$-a.e. $\tilde{x}$, if $\delta \in Q$ and $\delta \le \delta(\tilde{x})$, then $-\log g_\delta(\tilde{x}) \le -\log g(\tilde{x}) + \varepsilon$. Since $\int -\log g_* d\tilde{\mu} < +\infty$, there is a number δ_* such that if $A = \{\tilde{x} : \delta(\tilde{x}) > \delta_*\}$, then $\int_{M^f\setminus A} -\log g_* d\tilde{\mu} \le \varepsilon$.

We claim that for $\tilde{\mu}$-a.e. $\tilde{x}$, if n is sufficiently large, then $\delta(\tilde{x};n,k) \le \delta_*$ for all $k \le n(1-\varepsilon)$. First by Birkhoff ergodic theorem, there is a positive integer $N_1(\tilde{x})$ such that for $n \ge N_1(\tilde{x})$, $\#\{0 \le i < n : \theta^i\tilde{x} \in \widehat{E}\} \le 2n\tilde{\mu}(\widehat{E})$. If $n \ge N_1(\tilde{x})$, then for each $k \le n(1-\varepsilon)$ one has $t_{j_k}(\tilde{x}) \le k \le n(1-\varepsilon)$ and

$$\delta(\tilde{x};n,k) = e^{-(\lambda_i+\beta)(n-t_{j_k}(\tilde{x}))}N_0^{2j_k} \le e^{-(\lambda_i+\beta)\varepsilon n}N_0^{4n\tilde{\mu}(\widehat{E})}.$$

Since $e^{-(\lambda_i+\beta)\varepsilon}N_0^{4\tilde{\mu}(\widehat{E})} < 1$, $\delta(\tilde{x};n,k)$ is less than δ_* for sufficiently large n. Thus

$$\sum_{k=0}^{[n(1-\varepsilon)]} -\log g_{\delta(\tilde{x};n,k)}(\theta^k\tilde{x}) \le \sum_{\substack{k=0\\ \theta^k\tilde{x}\in A}}^{[n(1-\varepsilon)]} (-\log g(\theta^k\tilde{x})+\varepsilon) + \sum_{\substack{k=0\\ \theta^k\tilde{x}\notin A}}^{[n(1-\varepsilon)]} -\log g_*(\theta^k\tilde{x})$$

and the lim sup we wish to estimate in (IX.24) is bounded above by

$$(1-\varepsilon)[\int -\log g d\tilde{\mu} + \varepsilon + \int_{M^f\setminus A} -\log g_* d\tilde{\mu}].$$

Recalling that $g(\tilde{x}) = \tilde{\mu}_{\tilde{x}}^{i-1}((\theta^{-1}\eta_{i-1})(\tilde{x}))$ for $\tilde{\mu}$-a.e. $\tilde{x}$,

$$\int -\log g d\tilde{\mu} = H_{\tilde{\mu}}(\theta^{-1}\eta_{i-1}|\eta_{i-1}) = H_{\tilde{\mu}}(\eta_{i-1}|\theta\eta_{i-1}) = h_{i-1}.$$

This completes the proof. □

Proof of Proposition IX.4.7. It follows from the Lipschitz property of $\tilde{\pi}_i$ together with the definition of $\tilde{\gamma}_i$ and $\widehat{\gamma}_i$ that for $\tilde{\mu}$-a.e. fixed $\tilde{x}$ and $\tilde{\mu}_{\tilde{x}}^i$-a.e. $\tilde{y} \in \eta_i(\tilde{x})$

$$\widehat{\gamma}_i(\tilde{y};\tilde{x}) = \tilde{\gamma}_i(\tilde{x}),$$

if we assume $\tilde{\mu}_{\tilde{x}}^i(\bigcup_{n\ge 0}\theta^n\widehat{E}) = 1$. Hence from Proposition IX.4.6 we may assume that

$$\widehat{\gamma}_i(\tilde{\pi}_i^{-1}z,\tilde{x}) \ge \frac{(1-\beta)[h_i-h_{i-1}-\beta]}{\lambda_i+\beta}, \quad \nu-\text{a.e. } z \in \bar{\mathbf{R}}^{(i)}.$$

Consider now the partition of $\tilde{\pi}_i(\eta_i(\tilde{x})) \subset \bar{\mathbf{R}}^{(i)} = \bar{\mathbf{R}}^{(i-1)} \times \mathbb{R}^{m_i}$ into planes of the form $\{z = (z_1, z_2, \cdots, z_i) \in \bar{\mathbf{R}}^{(i)} : z_i = \text{const}\}$. Using the Lipschitz property of $\tilde{\pi}_i$ and the definition of $\underline{\delta}_i$ and ν, one can easily verify that at ν-a.e. $z \in \bar{\mathbf{R}}^{(i)}$

$$\underline{\delta}_{i-1}(\tilde{x}) \leq \liminf_{\rho \to 0} \frac{\log \nu_{z_i}(\{w \in \bar{\mathbf{R}}^{(i-1)} : \|w - z^{(i-1)}\| < \rho\})}{\log \rho}$$

and

$$\underline{\delta}_i(\tilde{x}) = \liminf_{\rho \to 0} \frac{\log \nu(\{w \in \bar{\mathbf{R}}^{(i)} : \|w - z\| < \rho\})}{\log \rho},$$

where ν_c is the conditional measure of ν on $\{z = (z_1, \cdots, z_i) \in \bar{\mathbf{R}}^{(i)} : z_i = c\}$ (hence ν_c can be viewed as a Borel probability measure on $\bar{\mathbf{R}}^{(i-1)}$). Lemma IX.4.8 then tells us that inequality (IX.19) holds for $\tilde{\mu}^i_{\tilde{x}}$-a.e. $\tilde{y}$. $\square$

IX.5 The General Case: without Ergodic Assumption

Now we prove Theorem IX.1.3 in the general case without ergodic assumption via ergodic decompositions of μ and $\tilde{\mu}$. If μ is not ergodic, then according to Rokhlin [75], there exists a (μ-mod 0) unique measurable partition ζ_0 of M such that $f^{-1}C = C$ and $f|_C : (C, \mu|_C) \hookleftarrow$ is ergodic for μ_{ζ_0}-a.e. $C \in \zeta_0$. Let $\zeta = p^{-1}\zeta_0$. Then $\theta\widetilde{C} = \widetilde{C}$ for μ_{ζ_0}-a.e. $C \in \zeta_0$, where $\widetilde{C} \stackrel{\text{def}}{=} p^{-1}C \in \zeta$. $\tilde{\mu}_{\widetilde{C}} \stackrel{\text{def}}{=} p^{-1}\mu_C$ is an ergodic measure on $\widetilde{C}$ for μ_{ζ_0}-a.e. $C \in \zeta_0$.

Since ξ_u is a partition of M^f subordinate to W^u-manifolds of f, ξ_u refines ζ by Corollary 3.1.1 in [73]. Hence the transitivity of conditional measures implies that

$$\tilde{\mu}^{\xi_i}_{\tilde{x}} = (\tilde{\mu}_{\widetilde{C}})^{\xi_i}_{\tilde{x}}, \text{ for } \tilde{\mu}_\zeta - \text{a.e. } \widetilde{C} \text{ and } \tilde{\mu}_{\widetilde{C}} - \text{a.e. } \tilde{x}.$$

Then results in Sections IX.3 and IX.4 tell us that for $\tilde{\mu}_\zeta$-a.e. $\widetilde{C} \in \zeta$ and $\tilde{\mu}_{\widetilde{C}}$-a.e. $\tilde{x} \in \widetilde{C}$

$$\underline{h}_i(\tilde{x}; \xi_i, \tilde{\mu}) = \underline{h}_i(\tilde{x}; \xi_i|_{\widetilde{C}}, \tilde{\mu}_{\widetilde{C}}) = h_{\tilde{\mu}_{\widetilde{C}}}(\theta|_{\widetilde{C}}, \xi_i|_{\widetilde{C}}) = \overline{h}_i(\tilde{x}; \xi_i|_{\widetilde{C}}, \tilde{\mu}_{\widetilde{C}}) = \overline{h}_i(\tilde{x}; \xi_i, \tilde{\mu})$$

and

$$\underline{\delta}_i(\tilde{x}; \xi_i, \tilde{\mu}) = \underline{\delta}_i(\tilde{x}; \xi_i|_{\widetilde{C}}, \tilde{\mu}_{\widetilde{C}}) = \overline{\delta}_i(\tilde{x}; \xi_i|_{\widetilde{C}}, \tilde{\mu}_{\widetilde{C}}) = \overline{\delta}_i(\tilde{x}; \xi_i, \tilde{\mu}) =: \delta_i(\tilde{x}; \xi_i, \tilde{\mu}).$$

Since $\gamma_i(\tilde{x}; \tilde{\mu}) \stackrel{\text{def}}{=} \delta_i(\tilde{x}; \xi_i, \tilde{\mu}) - \delta_{i-1}(\tilde{x}; \xi_{i-1}, \tilde{\mu})$, it follows from equation (IX.13) that

$$\gamma_i(\tilde{x}; \tilde{\mu}) = \gamma_i(\tilde{x}; \tilde{\mu}_{\widetilde{C}}) = \frac{h_{\tilde{\mu}_{\widetilde{C}}}(\theta|_{\widetilde{C}}, \xi_i|_{\widetilde{C}}) - h_{\tilde{\mu}_{\widetilde{C}}}(\theta|_{\widetilde{C}}, \xi_{i-1}|_{\widetilde{C}})}{\lambda_i(x_0)}$$

holds true for $\tilde{\mu}_\zeta$-a.e. $\widetilde{C} \in \zeta$ and $\tilde{\mu}_{\widetilde{C}}$-a.e. $\tilde{x} \in \widetilde{C}$. Hence γ_i's are a.e. θ-invariant functions. It is easy to see that γ_i's are indeed a.e. functions well defined on M.

Furthermore, the entropy map $h_{\tilde{\mu}}(\theta)$ is affine with respect to $\tilde{\mu}$, i.e.

$$h_{\tilde{\mu}}(\theta) = \int h_{\tilde{\mu}_{\widetilde{C}}}(\theta) d\tilde{\mu}_\zeta(\widetilde{C}).$$

Hence by $h_\mu(f) = h_{\tilde{\mu}}(\theta)$ and $h_{\tilde{\mu}_{\widetilde{C}}}(\theta) = h_{\tilde{\mu}_{\widetilde{C}}}(\theta|_{\widetilde{C}}, \xi_u|_{\widetilde{C}})$, Theorem IX.1.3 holds. □

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