Chapter 2
Boundary Potential Theory for Schrödinger Operators Based on Fractional Laplacian

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2.1 Introduction

Precise boundary estimates and explicit structure of harmonic functions are closely related to the so-called Boundary Harnack Principle (**BHP**). The proof of **BHP** for classical harmonic functions was given in 1977-78 by H. Dahlberg in [65], A. Ancona in [3] and J.-M. Wu in [153] (we also refer to [99] for a streamlined exposition and additional results). The results were obtained within the realm of the analytic potential theory. A probabilistic proof of **BHP**, one which employs only elementary properties of the Brownian motion, was given in [11]. The proof encouraged subsequent attempts to generalize **BHP** to other processes, in particular to the processes of jump type.

BHP asserts that the ratio $u(x)/v(x)$ of nonnegative functions harmonic on a domain D which vanish outside the domain near a part of the domain's boundary, ∂D, is bounded inside the domain near this part of ∂D. The result requires assumptions on the underlying Markov process and the domain. For Lipschitz domains and harmonic functions of the isotropic α-stable Lévy process ($0 < \alpha < 2$), **BHP** was proved in [27]. Another proof, motivated by [11], was obtained in [31] and extensions beyond Lipschitz domains were obtained in [150] and [38]. In particular the results of [38] provide a conclusion of a part of the research in this subject, and offer techniques that may be used for other jump-type processes.

Lipschitz **BHP** leads to Martin representation of nonnegative α-harmonic functions on Lipschitz domains ([28] and [56]). Another important consequence of **BHP** are sharp estimates of the Green function of Lipschitz domains and the so-called **3G** Theorem (see (2.26) below). We give these applications in the first part of the chapter, along with a self-contained proof of **BHP**, following [27] and [38].

In the second part of the chapter we focus on the potential theory of Schrödinger-type perturbations, $\Delta^{\alpha/2}+q$, of the fractional Laplacian on subdomains of $\mathbb{R}^d$. The main result we discuss here is the Conditional Gauge Theorem (**CGT**), asserting comparability of the Green function of $\Delta^{\alpha/2}+q$

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with that of $\Delta^{\alpha/2}$, under an assumption of "non-explosion". Here $0 < \alpha < 2$, and the proof of **CGT** relies on the **3G** Theorem, thus on (Lipschitz) **BHP**. In presenting these results we generally follow the approach of papers [32] and [33]. The approach was modeled after [62], which deals with the Laplacian and its underlying process of the Brownian motion (see [64] for Schrödinger perturbations of elliptic partial differential operators of second order). For a different technique we refer to [54]. It should be noted that there are many algebraic similarities between the fractional Laplacian ($\alpha < 2$) and the Laplacian ($\alpha = 2$), but there are also deep analytical differences between these two cases, primarily due to the discontinuity of paths of the isotropic α-stable Lévy process for $0 < \alpha < 2$.

2.2 Boundary Harnack Principle

Below we freely mix ideas from [27], [31], [32], [150], and [38], with some didactic improvements and modifications aimed at the simplification of presentation. In particular we give perhaps the shortest existing proof of **BHP** for α-harmonic functions.

In what follows nonempty $D \subset \mathbb{R}^d$ is open. We intend to present the main ideas of the proof of **BHP** as given in [38] for arbitrary domains. However, for the simplicity of the discussion in the remainder of this chapter unless stated otherwise, *we will assume that D is a Lipschitz domain*, and we will concentrate on *finite nonnegative functions f on $\mathbb{R}^d$, which are represented on D as Poisson integrals of their values on D^c*:

$$f(x) = \int_{D^c} f(y) P_D(x,y)dy\,, \quad x \in D^c\,. \tag{2.1}$$

For instance, if (D is a Lipschitz domain and) $f \geqslant 0$ is bounded on $\overline{D}$, then $f = P_D[f]$ on D, see [27]. For a general discussion of the notion of α-harmonicity we refer the reader to [32, 38]. We should perhaps state a warning that some aspects of the notion are richer and even counter-intuitive when confronted with the properties of harmonic functions of local operators. In particular, non-negativeness of functions which are α-harmonic on D is useful only if assumed on the whole of $\mathbb{R}^d$ (rather than merely on D). For instance, if $|y| > r$, then the function

$$B_r \ni x \mapsto \left[\sup_{v \in B_r} P_{B_r}(v,y)\right] - P_{B_r}(x,y),$$

takes on the minimum of zero in a *interior* point of B_r, in stark contrast with the Harnack inequality. The reader may also want to consider (non-Lipschitz) domains with boundary of positive Lebesgue measure and domains

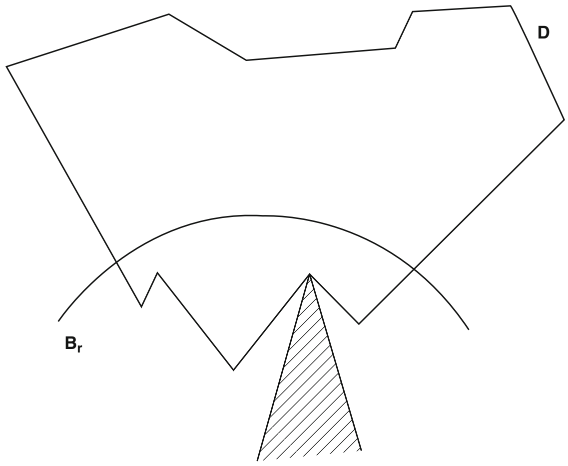


Fig. 2.1 D, B_r, and outer cone

with complement of zero Lebesgue measure but positive Riesz capacity, to apprehend the complexity of the boundary problems for α-harmonic functions.

For function $f \geqslant 0$ satisfying (2.1) we have $\Delta^{\alpha/2} f(x) = 0$ on D, see [32]. Furthermore, for *every* open $U \subset D$ we have

$$f(x) = \int_{U^c} f(y) \omega_U^x(dy) , \quad x \in U . \tag{2.2}$$

This follows from (1.51). We emphasize that for the above *mean value property* of Poisson integrals it is *not* necessary that $\overline{U}$ be a *compact subset* of D, and we to refer the reader to [38] for cautions needed to deal with the general nonnegative α-harmonic functions.

When $0 < r \leqslant 1$ we let $D_r = D \cap B_r$, a domain with the outer cone property, see Figure 2.1. We will often use (2.2) for $U = D_r$. We note that $\omega_{D_r}^x(\partial D_r) = 0$ for $x \in D_r$, in particular we can employ (1.53) for such U.

Consider $B = B_1$ and assume that

$$f = 0 \text{ on } B \setminus D . \tag{2.3}$$

Since $G_{D_r} \leqslant G_{B_r}$ (see (1.45), (1.46)), by the definition of Poisson kernel (1.49) we get

$$P_{D_r}(x, y) \leqslant P_{B_r}(x, y) , \quad x \in D_r , y \in B_r^c .$$

By the mean value property and the assumption (2.3) we obtain

$$f(x) \leqslant \int_{B_r^c} f(y) P_{B_r}(x, y) dy , \quad x \in B_r , \quad 0 < r \leqslant 1 . \tag{2.4}$$

The function $P_{B_r}(x,y)$ has a singularity at $|y|=r$. To remove this inconvenience, we will consider an analogue of volume averaging used on occasions in the classical potential theory. We fix a nonnegative function $\phi \in C_c^\infty((1/2,1))$ such that $\int_{1/2}^1 \phi(r)\,dr = 1$ and we define

$$\begin{aligned}\psi(x,y) &= \int_{1/2}^1 \phi(r)\, P_{B_r}(x,y)\,dr \\ &= C_\alpha^d\, |y-x|^{-d} \int_{|y|\wedge 1/2}^{|y|\wedge 1} \frac{(r^2-|x|^2)^{\alpha/2}}{(|y|^2-r^2)^{\alpha/2}}\,\phi(r)dr\,, \quad x,y\in\mathbb{R}^d\,.\end{aligned}$$

It is not difficult to check that

$$|\psi(x,y)| \leqslant \frac{C}{(1+|y|)^{d+\alpha}}\,, \quad |x|\leqslant 1/3\,,\ y\in\mathbb{R}^d\,. \tag{2.5}$$

By Fubini's theorem and (2.4) we obtain

$$f(x) \leqslant \int_{B_r^c} f(y)\psi(x,y)dy \leqslant C\int_{\mathbb{R}^d} f(y)(1+|y|)^{-d-\alpha}\,, \quad x\in B_{1/3}\,. \tag{2.6}$$

To obtain a reverse inequality for $x\in D_1 = D\cap B$ being not too close to ∂D_1 we note that $P_{B_r}(0,y) \geqslant C_\alpha^d r^\alpha |y|^{-d-\alpha}$, see (1.57). If $r_0>0$ and $B(2r_0,x_0)\in D_1$, then

$$f(x_0) = \int_{B^c(x_0,r_0)} P_{r_0}(0,y-x_0)f(y)dy \geqslant \int_{B^c(x_0,r_0)} C_\alpha^d r_0^\alpha |y-x_0|^{-d-\alpha} f(y)dy. \tag{2.7}$$

By the Harnack inequality for f on $B(x_0,r_0)$ we can enlarge the domain of integration so that

$$f(x_0) \geqslant c\int_{\mathbb{R}^d} (1+|y|)^{-d-\alpha} f(y)dy\,.$$

Here and in what follows the *constants* (c, C etc.) depend on d, α and D, in particular on r_0.

This and (2.6) yield the following Carleson-type estimate.

Corollary 2.1. *There is a constant C depending only on d, α, and x_0 such that*

$$f(x) \leqslant Cf(x_0)\,, \quad x,x_0\in D_{1/3}\,. \tag{2.8}$$

In what follows we will consider $D_{1/4}$ and will fix $x_0\in D_{1/5}$. We have

$$f(x) = \int_{D_{1/4}^c} f(y)P_{D_{1/4}}(x,y)dy = \int_{D_{1/4}} G_{D_{1/4}}(x,v)\kappa(v)dv\,, \tag{2.9}$$

where

$$\kappa(v) = \int_{D^c_{1/4}} \mathcal{A}_{d,-\alpha}|y-v|^{-d-\alpha} f(y)dy\,, \quad v \in D_{1/4}\,.$$

We thus have f expressed as the Green potential of the charge $\kappa(v)$ interpreted as the intensity of jumps of Y "to" f on D^c. Let

$$\kappa_1(v) = \int_{B^c_{1/3}} \mathcal{A}_{d,-\alpha}|y-v|^{-d-\alpha} f(y)dy\,, \quad v \in D_{1/4}\,,$$

$$\kappa_2(v) = \int_{B_{1/3}\setminus D_{1/4}} \mathcal{A}_{d,-\alpha}|y-v|^{-d-\alpha} f(y)dy\,, \quad v \in D_{1/4}\,,$$

and

$$f_i(x) = \int_{D_{1/4}} G_{D_{1/4}}(x,v)\kappa_i(v)dv\,, \quad i=1,2\,. \tag{2.10}$$

We note that f_i are α-harmonic, in fact Poisson integrals, on $D_{1/4}$. We observe that κ_1 is bounded, in fact *nearly constant* on $D_{1/4}$:

$$c^{-1}\kappa_1(v_2) \leqslant \kappa_1(v_1) \leqslant c\kappa_1(v_2)\,, \quad v_1, v_2 \in D_{1/4}\,, \tag{2.11}$$

because $|y-v|^{-d-\alpha}$ is nearly constant in $v \in D_{1/4}$ (uniformly in $y \in B^c_{1/3}$). Also, $\kappa_1(v) \leqslant cf(x_0)$, see (2.7). Thus

$$f_1(x) \leqslant cf(x_0)\int_{D_{1/4}} G_{D_{1/4}}(x,v)dv = cf(x_0)s_{D_{1/4}}(x)\,, \quad x \in D_{1/4}. \tag{2.12}$$

We will see that $s_{D_{1/4}}$ faithfully represents the asymptotics of $f = f_1 + f_2$ at $\partial D \cap B_{1/5}$. To this end we first note that by (2.8),

$$f_2(x) \leqslant Cf(x_0)\omega^x_{D_{1/4}}(B^c_{1/4})\,, \quad x \in D_{1/4}\,. \tag{2.13}$$

Lemma 2.2. *For every $p \in (0,1)$ there is a constant C such that if $D \subset B$ then*

$$\omega^x_D(B^c) \leqslant C\, s_D(x)\,, \quad x \in D_p\,.$$

Proof. Let $0 < p < 1$. We choose a function $\varphi \in C^\infty_c(\mathbb{R}^d)$ such that $0 \leqslant \varphi \leqslant 1$, $\varphi(y) = 1$ if $|y| \leqslant p$, and $\varphi(y) = 0$ if $|y| \geqslant 1$. Let $x \in D_p$. By (1.47) we have

$$\begin{aligned}\omega^x_D(B^c) &= \int_{B^c} (\varphi(x) - \varphi(y))\omega^x_D(dy) \leqslant \int_{D^c} (\varphi(x)-\varphi(y))\omega^x_D(dy) \\ &= -\int_D G_D(x,y)\Delta^{\alpha/2}\varphi(y)dy\,.\end{aligned}$$

It remains to observe that $\Delta^{\alpha/2}\varphi$ is bounded and the lemma follows. $\square$

By (2.13), scaling and Lemma 2.2 (with $p = 4/5$) we obtain that $f_2(x) \leqslant cf(x_0)s_{D_{1/4}}(x)$ for $x \in D_{1/5}$. This, and (2.12) yield the following improvement of Carleson estimate

$$c^{-1}f(x_0)s_{D_{1/4}}(x) \leqslant f(x) \leqslant cf(x_0)s_{D_{1/4}}(x)\,, \quad x \in D_{1/5}\,. \tag{2.14}$$

Indeed, the lower bound in (2.14) follows from the inequality

$$f(x) \geqslant \int_{D_{1/4}} G_{D_{1/4}}(x,v)\kappa_3(v)dv\,,$$

where

$$\kappa_3(v) = \int_{B(x',r')} f(y)\mathcal{A}_{d,-\alpha}|y-v|^{-d-\alpha}dy \geqslant cf(x_0)\;, \quad v \in D_{1/4}\,,$$

and $B(2r',x') \subset D_{1/4} \setminus D_{1/5}$ is a ball (if the set $D_{1/4} \setminus D_{1/5}$ is empty then $f_2 \equiv 0$, and we simply use (2.11) and (2.10)).

The following Boundary Harnack Principle is a direct analogue of (1.14).

Theorem 2.3 (BHP). *If functions f_1 and f_2 satisfy the above assumptions on f, then*

$$f_1(x)f_2(y) \leqslant Cf_1(y)f_2(x)\,, \quad x,y \in D_{1/5}\,.$$

Proof. We fix $x_0 \in D \cap B_{1/5}$. For $x, y \in D \cap B_{1/5}$ we obtain from (2.14)

$$f_1(x)f_2(y) \leqslant c^2 f_1(x_0)f_2(x_0)s_{D_{1/4}}(x)s_{D_{1/4}}(y)\,,$$

and

$$f_1(y)f_2(x) \geqslant c^{-2} f_1(x_0)f_2(x_0)s_{D_{1/4}}(y)s_{D_{1/4}}(x)\,.$$

The result, translation and scaling invariance of the class of α-harmonic functions, and the usual Harnack inequality, allow to estimate the growth of α-harmonic functions vanishing at a part of the domain's boundary up to this part of the boundary. The constant C in our present proof depends on D (and the choice of x_0), however a more delicate and technical proof shows that C may be so chosen to depend *only* on d and α. We refer the reader to [38] for this important strengthening of **BHP**. An important consequence of the domain-independent, or *uniform* **BHP** of [38] is given in the following statement

$$\lim_{D \ni x \to 0} f_1(x)/f_2(x) \;\text{ exists.} \tag{2.15}$$

BHP and (2.15) were given in [27] (see also [31]) for Lipschitz domains, generalized in [150] to the so-called κ-fat domains, and proved for *arbitrary* open sets in [38]. The proof of (2.15) seems too technical to be discussed here, but we will hopefully give some insight into its main idea, when discussing the uniqueness of the Martin kernel with the pole at infinity for cones.

Let $\rho(x) = \mathrm{dist}(x, D^c)$. Compared to **BHP**, the following local estimate for individual (nonnegative) Poisson integral on Lipschitz domains, if not sharp, is more explicit.

Lemma 2.4. *Let $\Gamma : \mathbb{R}^d \to \mathbb{R}$ satisfy* (1.12)*, and $\Gamma(0) = 0$. Let $D = D_\Gamma \cap B$, and $A = (0, 0, \ldots, 0, 1/2) \in D$. There are $C = C(d, \alpha, \lambda)$ and $\epsilon = \epsilon(d, \alpha, \lambda) \in (0, \alpha)$ such that*

$$C^{-1} f(A)\rho(x)^{\alpha-\epsilon} \leqslant f(x) \leqslant C f(A)\rho(x)^{\epsilon}\,, \quad x \in D_{1/2}\,. \tag{2.16}$$

The right hand side of (2.16) is a strengthening of the Carleson estimate, and it asserts a power-type decay of u at the boundary of D. This decay rate is related to the existence of outer cones for the boundary points of D, and steady escape of mass of the process when it approaches ∂D (see our discussion above of the fact that $\omega_D^x(\partial D) = 0$). For a class of domains including domains with the boundary defined by a C^2 function we have $\epsilon = \alpha/2$, which may be verified by a direct calculation involving the Green function of the ball, and of the complement of the ball, see [56], [109]. Then (2.16) becomes *sharp*, meaning that all sides of the inequality are in fact *comparable*. The exponent $\alpha/2$ is also related to the fact that

$$f(x) = x_+^{\alpha/2}\,, \quad x \in \mathbb{R}\,, \tag{2.17}$$

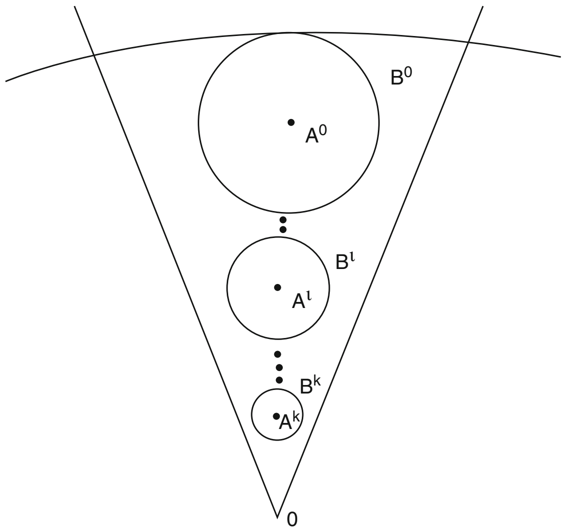
is α-harmonic on the half-line $\{x > 0\}$, see [30] for explicit calculations involving $\Delta^{\alpha/2}$.

For general Lipschitz domains the exponent ϵ on the right-hand side of (2.16) is usually not given explicitly. We like to note that $\epsilon > 0$ may be arbitrarily small, e.g. for the complement of cone with sufficiently small opening in dimension $d > 2$. For a more detailed study of the asymptotic behavior of α-harmonic functions in cones, and some open problems we refer the reader to [5] and [123].

We will briefly discuss the left hand side inequality in (2.16). We like to emphasize the fact that the power-type decay cannot be arbitrarily fast, a significant difference when compared with the classical harmonic functions in narrow cones. Indeed, $\epsilon > 0$ may be arbitrarily small (for very narrow cones), but we always have $\alpha - \varepsilon < \alpha$! This is a noteworthy contrast with the classical potential theory ($\alpha = 2$). For an explanation of this phenomenon we will consider exponentially shrinking disjoint balls $B^k = B(A^k, cr^k)$, where $0 < r < 1$ and c are such that $B^k \subset D_{r^k}$ ($k = 0, 1, \ldots$), see Figure 2.2. By the mean value property, we have

$$\begin{aligned} f(A^k) &\geqslant \sum_{l=0}^{k-1} \int_{B^l} f(y)\omega_{B^k}^{A^k}(dy) \\ &\geqslant C \sum_{l=0}^{k-1} \int_{B^l} f(A^l) r^{(k-l)\alpha}\,, \end{aligned} \tag{2.18}$$

Fig. 2.2 Exponentially shrinking balls



where we used the formula for the Poisson kernel of the ball. Thus, $\beta_k := f(A^k)r^{-k\alpha} \geqslant C\sum_{l=0}^{k-1}\beta_l$. By induction we see that $\beta_k \geqslant C(1+C)^k\beta_0$, which yields the exponent $\alpha - \varepsilon < \alpha$ on the left hand side of (2.16).

We note that the first term of the sum in (2.18) approximately equals $r^{k\alpha}f(A_0)$, which is much smaller than the whole sum. Thus a direct jump (say, to B_0) has a negligible impact on the values of the α-harmonic function on B^k. Instead, the many combined shorter jumps between the balls $\{B^l\}$ yield the main contribution. The geometry of Lipschitz domains plays a role here. Domains which are "thinner" at some boundary points may show a different decay rate of α-harmonic functions (i.e. that given by a few direct jumps may prevail, see [125]). This observation leads to a notion of *inaccessibility* developed in [38].

We want to point out after [38], that **BHP** can be studied as a property of the Poisson kernel and the Green function, without even referring to the notion of α-harmonicity. In fact, the main application of **BHP** is the following one, to $f_1(x) = G_D(x, x_1)$ and $f_2(x) = G_D(x, x_2)$, for x (in a Lipschitz subset of) $D \setminus \{x_1, x_2\}$. We fix an arbitrary reference point $x_0 \in D$ and we define the Martin kernel of D,

$$M_D(x, y) = \lim_{D \ni v \to y} \frac{G_D(x, v)}{G_D(x_0, v)}, \quad x \in \mathbb{R}^d, \ y \in \partial D. \tag{2.19}$$

Theorem 2.5. *The limit in* (2.19) *exists.* $x \mapsto M_D(x, y)$ *is up to constant multiples the only nonnegative* α*-harmonic function on* D *and equal to zero on* D^c *which continuously vanishes at* $D^c \setminus \{y\}$.

The existence part of the result follows easily from (2.15). The α-harmonicity of M_D, however, depends delicately on the Lipschitz geometry of the domain via the lower bound in (2.16), see [28]. We refer the reader to [28] for an elementary study of the properties of $M_D(\cdot, y)$ for Lipschitz domains. We also refer to [38] for the case of arbitrary open set and for the explanation of the role played by the *accessibility* of the point y from within the set.

It should be noted that $M_D(\cdot, y)$ is *not* of the form (2.1). Nonnegative α-harmonic functions *vanishing* on D^c are called *singular* α*-harmonic*. They resemble classical Poisson integrals of singular measures on the sphere (and also nonnegative martingales converging to zero almost surely).

We will cite after [28] the representation theorem for nonnegative α-harmonic functions on bounded Lipschitz domains D (for arbitrary nonempty open subsets of $\mathbb{R}^d$ see [38]).

Theorem 2.6. *For every function $u \geqslant 0$ which is α-harmonic in D there exists a unique finite measure $\mu \geqslant 0$ on ∂D, such that*

$$u(x) = \int_{D^c} P_D(x, y) u(y) dy + \int_{\partial D} M_D(x, y) \mu(dy) \,, \quad x \in D \,. \tag{2.20}$$

In view of the recent developments in [38] we like to make the following comments. First, $\int_{D^c} P_D(x, y) u(y) dy$ above may be generalized to Poisson integrals of nonnegative *measures*:

$$\int_{D^c} P_D(x, y) \lambda(dy) < \infty \,, \tag{2.21}$$

and it is legitimate to regard D^c as the "Martin boundary" of (bounded Lipschitz) D for $\Delta^{\alpha/2}$, with kernels $M_D(\cdot, y)$, $y \in \partial D$, and $P_D(\cdot, y) + \delta_y(\cdot)$, $y \in D^c \setminus \partial D$. Second, for *general* domains in arbitrary dimension, *inaccessible* points of the Euclidean boundary will contribute a Poisson kernel, rather than a Martin kernel. Third, for unbounded domains a Martin kernel may be attributed to the point at infinity (if accessible). For details we refer the reader to [38], which appears to finalize the problem of representing nonnegative α-harmonic functions, and offers notions and methods appropriate for handling more general Markov processes with jumps. To further encourage the interested reader, we want to point out that for bounded domains their "Martin boundary" decreases when the domain increases [38]. Comparing to Δ, we see that the potential theory of $\Delta^{\alpha/2}$ is more compatible with the Euclidean topology of $\mathbb{R}^d$.

We return to considering a Lipschitz domain $D \subset \mathbb{R}^d$ in dimension $d \geqslant 2$. For $y \in \partial D$, $M_D(x, y)$ is (up to constant multiples) the unique α-harmonic function continuously vanishing on $D^c \setminus \{y\}$ (and having a singularity at y, which "feeds" the function through (1.63)). As remarked above, a similar function can be constructed for the point at infinity, if D is unbounded:

$$M(x) = M_D(x,\infty) = \lim_{D\ni v, |v|\to\infty} \frac{G_D(x,v)}{G_D(x_0,v)}\,, \quad x \in \mathbb{R}^d\,. \tag{2.22}$$

In the case when D is an open *cone* $\mathcal{C} \subset \mathbb{R}^d$, the existence, uniqueness and homogeneity properties of M were studied [5] and [123]. Below we will give a flavor of the technique used in the study. We first note that the mean value property holds for such M for *every bounded* open subset U of $\mathcal{C}$, as the pole is so far away. Let $\mathbf{1} \neq 0$ be a point in $\mathbb{R}^d$ (say $\mathbf{1} = (0,\ldots,0,1)$). For $x \in \mathbb{R}^d\backslash\{0\}$, we denote by $\theta(x)$ the angle between x and $\mathbf{1}$. The right circular cone of angle $\Theta \in (0,\pi)$ is the Lipschitz domain

$$\mathcal{C} = \mathcal{C}_\Theta = \{x \in \mathbb{R}^d : \theta(x) < \Theta\}\,.$$

Clearly, for every $r > 0$ we have $r\mathcal{C} = \mathcal{C}$. In particular, by scaling, if u is α-harmonic on $\mathcal{C}$, then so is $x \mapsto u(rx)$. We will prove the uniqueness of M. To this end, we assume that there is another function $m \geqslant 0$ on $\mathbb{R}^d$ which vanishes on $\mathcal{C}^c$, satisfies $m(\mathbf{1}) = 1$ and

$$m(x) = \mathbb{E}_x m(Y_{\tau_B})\,, \quad x \in \mathbb{R}^d\,,$$

for every open bounded $B \subset \mathcal{C}$. By **BHP**,

$$C^{-1}m(x) \leqslant M(x) \leqslant Cm(x)\,,$$

for $x \in B \cap \mathcal{C}$. By scaling, this extends to all $x \in \mathcal{C}$ with the same constant. We let $a = \inf_{x\in\mathcal{C}} m(x)/M(x)$. For clarity, we note that $C^{-1} \leqslant a \leqslant 1$. Let $R(x) = m(x) - aM(x)$, so that $R \geqslant 0$ on $\mathbb{R}^d$. Assume (falsely) that $R(x) > 0$ for some, and therefore for every $x \in \mathcal{C}$. Then, by **BHP** and scaling,

$$R(x) \geqslant \varepsilon M(x)\,, \quad x \in \mathbb{R}^d,$$

for some $\varepsilon > 0$. We have

$$a = \inf_{x\in\mathcal{C}} \frac{m(x)}{M(x)} = \inf_{x\in\mathcal{C}} \frac{aM(x)+R(x)}{M(x)} \geqslant a + \varepsilon\,,$$

which is a contradiction. Thus $R \equiv 0$, $m = aM$, and the normalizing condition $m(\mathbf{1}) = M(\mathbf{1}) = 1$ yields $a = 1$. The uniqueness of M is verified.

We like to note that the existence of the limits of the ratios of nonnegative α-harmonic functions, (2.15), is proved by a similar argument, see [27, 38]. This *oscillation-reducing* mechanism of **BHP** is well known for local operators, e.g. Laplacian ([11]), but the non-local character of the fractional Laplacian seriously complicates such arguments, except in some special cases, like that of the cone. Some elements of the proof (of vanishing of oscillations of ratios of non-negative α-harmonic functions) are given in [27]. The complete details in the generality of arbitrary domains are given in [38].

To appreciate the importance of uniqueness, we return to the discussion of the Martin kernel with the pole at infinity for the cone. By scaling, for every $k > 0$ the function $M(kx)/M(k\mathbf{1})$ satisfies the hypotheses defining M. Thus it is equal to M, or

$$M(kx) = M(x)M(k\mathbf{1}) \quad x \in \mathbb{R}^d .$$

In particular, $M(kl\mathbf{1}) = M(l\mathbf{1})M(k\mathbf{1})$ for positive k, l. By continuity of α-harmonic functions on the domain of harmonicity, there exists β such that $M(k\mathbf{1}) = k^\beta M(\mathbf{1}) = k^\beta$ and

$$M(kx) = k^\beta M(x) , \quad x \in \mathbb{R}^d ,$$

or

$$M(x) = |x|^\beta M(x/|x|) , \quad x \neq 0 , \tag{2.23}$$

compare (2.16). By (2.14), M is locally bounded and tends to zero at the origin, thus

$$0 < \beta < \alpha . \tag{2.24}$$

It is known that β is close to α for very narrow cones, and it will be close to 0 for obtuse cones (for Θ close to π), at least in dimension $d \geqslant 2$. We refer the reader to [5], [123], [35] for more information and a few explicit values of β for specific cones (see (2.17) for the half-line).

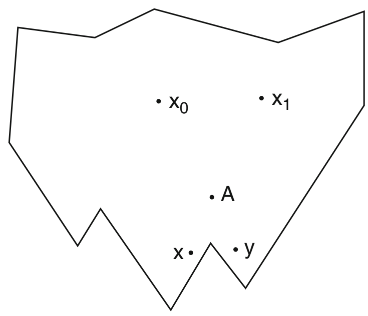
2.3 Approximate Factorization of Green Function

In this section we will consider a *bounded* Lipschitz domain $D \subset \mathbb{R}^d$, $d \geqslant 2$, with Lipschitz constant λ. To simplify formulas, we recall the notation $\approx$: we write $f(y) \approx g(y)$ for $y \in A$ if there exist constants C_1, C_2 not depending on y such that $C_1 f(y) \leqslant g(y) \leqslant C_2 f(y)$, $y \in A$.

Let $\delta(x) = \mathrm{dist}(x, D^c)$. We fix $x_0, x_1 \in D$, $x_0 \neq x_1$, and we let $\kappa = 1/(2\sqrt{1+\lambda^2})$. For $x, y \in D$ we denote $r = r(x,y) = \delta(x) \vee \delta(y) \vee |\, x - y\,|$. For *small* $r > 0$ we write $\mathcal{B}(x,y)$ for the set of points A such that $B(A, \kappa r) \subset D \cap B(x, 3r) \cap B(y, 3r)$, see Figure 2.3, and we put $\mathcal{B}(x,y) = \{x_1\}$ for *large* r [29]. The set $\mathcal{B}(x,y)$ is nonempty (see [98] or [29] for details). Informally speaking, $A \in \mathcal{B}(x,y)$ dominates x and y similarly as A of Lemma 2.4 dominates the points of $D_{1/2}$, see Figure 2.3. Let $G = G_D$, the Green function of D for the fractional Laplacian. We define

$$\phi(x) = G(x_0, x) \wedge c .$$

Fig. 2.3 $A \in \mathcal{B}(x,y)$



The following is a sharp, if not completely explicit, *approximate factorization* of $G(x,y)$.

$$C^{-1}\frac{\phi(x)\phi(y)}{\phi^2(A)}|x-y|^{\alpha-d} \leqslant G(x,y) \leqslant C\frac{\phi(x)\phi(y)}{\phi^2(A)}|x-y|^{\alpha-d}\,. \tag{2.25}$$

Here A is an arbitrary point of $\mathcal{B}(x,y)$. A proof of (2.25) is given in [98] (see also [29] for the case of $\alpha = 2$). We will sketch the proof.

If x and y are close to each other but far from the boundary, then (2.25) is equivalent to $G(x,y) \approx |x-y|^{\alpha-d}$, because the term subtracted in (1.45) is small.

Another case to consider is the situation of $|y-x|$ being large and $\delta(x)$, $\delta(y)$ being small. By symmetry, $G(x,y)$ is α-harmonic both in x, and in y (on $D\setminus\{y\}$ and $D\setminus\{x\}$, correspondingly). By **BHP** (and the usual Harnack inequality) $G(x,y)/G(x_0,y) \approx G(x,x_1)/G(x_0,x_1)$. Since $0 < G(x_0,x_1) < \infty$ is a constant, we obtain (2.25) in the considered case. If $|y-x|$, $\delta(x)$, and $\delta(y)$ are all small then we use **BHP** in a similar way, but *twice*. If $\delta(x)$ is small, and $\delta(y)$ is large, then $G(x,y) = G(y,x) \approx G(x_0,x)$ by the Harnack inequality.

We remark that $\phi(\cdot)$ may be replaced by $s_D(\cdot)$ in (2.25), compare Lemma 2.2, [38]. For bounded C^2 domains we may use $\phi(\cdot) = \delta^{\alpha/2}(\cdot)$, obtaining an estimate which is both sharp and explicit, [109, 56].

We will give a short proof of the following celebrated inequality known as **3G** Theorem:

Theorem 2.7 (3G).

$$\frac{G(x,y)G(y,z)}{G(x,z)} \leqslant C\frac{|x-y|^{\alpha-d}|y-z|^{\alpha-d}}{|x-z|^{\alpha-d}}, \quad x,y,z \in D\,. \tag{2.26}$$

Proof. let $x,y,z \in D$ and $R \in \mathcal{B}(x,y)$, $S \in \mathcal{B}(y,z)$, $T \in \mathcal{B}(x,z)$. By (2.25),

$$\frac{G(x,y)G(y,z)}{G(x,z)} \leqslant C\frac{|x-y|^{\alpha-d}|y-z|^{\alpha-d}}{|x-z|^{\alpha-d}}\,W^2,$$

where $W = [\phi(y)\phi(T)]/[\phi(R)\phi(S)]$. We will verify the boundedness of W. Let $r_1 = \delta(x) \vee \delta(y) \vee |x-y|$, $r_2 = \delta(y) \vee \delta(z) \vee |y-z|$, and $r_3 = \delta(x) \vee \delta(z) \vee |x-z|$ because ϕ is bounded. If $R = x_1$ and $S = x_1$ then $W \leqslant C$. If $R \neq x_1$, that is if r_1 is small, then we choose $Q \in \partial D$ such that $\delta(y) = |y-Q|$. By the Carleson estimate $\phi(y) \leqslant C\phi(R)$. Consequently, if $S = x_1$, then $W \leqslant C$. The same holds true if $S \neq x_1$ and $R = x_1$. By symmetry, to complete the proof, we may assume that $r_1 \leqslant r_2$ are *small*. We have $r_3 \leqslant r_1 + r_2 \leqslant 2r_2$, so r_3 is also small. In fact $|T-Q| \leqslant |T-z|+|y-z|+|y-Q| < 3r_3+r_2+r_2 \leqslant 8r_2$, therefore by the Carleson estimate and the Harnack inequality $\phi(T) \leqslant C\phi(S)$. Recall that $\phi(y) \leqslant C\phi(R)$, thus W is bounded in this case, too. This finishes the proof. □

Since $|x-y|^{\alpha-d}|y-z|^{\alpha-d}/|x-z|^{\alpha-d} \leqslant 2^{d-\alpha}\left(|x-y|^{\alpha-d} + |y-z|^{\alpha-d}\right)$, we obtain the following version of **3G**:

$$\frac{G(x,y)G(y,z)}{G(x,z)} \leqslant C\left(K_\alpha(x-y) + K_\alpha(y-z)\right), \quad x, y \in D. \tag{2.27}$$

The definition of the Martin kernel yields

$$\frac{G(x,y)M_D(y,\xi)}{M_D(x,\xi)} \leqslant C\left(K_\alpha(x-y) + K_\alpha(y-\xi)\right), \quad x, y \in D,\ \xi \in \partial D. \tag{2.28}$$

As we have mentioned, the importance of **3G** in potential theory was observed in [64]. Below we will use a probabilistic framework of conditional processes to employ **3G** to construct and estimate the Green function of Schrödinger perturbations of $\Delta^{\alpha/2}$. Before we take our chances in this endeavor, however, we like to notice that a purely analytic approach to this problem also exists. The approach is based on the so called *perturbation series*, or Duhamel's formula, whose application is greatly simplified *if* **3G** is satisfied. We refer the reader to a self-contained exposition of this technique in [85] (see also [84]). Analogous consideration based on so-called *3P Theorem* of [36] yields comparability of the perturbed transition density with the original one. We refer the interested reader to [36] and [37] for these developments.

2.4 Schrödinger Operator and Conditional Gauge Theorem

We will focus on the potential theory of Schrödinger operators, $u \mapsto \Delta^{\alpha/2}u + qu$, on subdomains of $\mathbb{R}^d$, following the development of [32, 33, 62]. The class of admissible "potentials" q is tailor-made for the transition probability of $\{Y_t\}$ (and $\Delta^{\alpha/2}$).

Put in a general perspective we consider here “small” additive perturbations of the generator of a semigroup and we expect the potential-theoretic object to be similar before and after these perturbations. In particular, the conditional gauge function defined below is the ratio between the Green functions after and before the perturbation, and the Conditional Gauge Theorem (**CGT**) asserts that the function is bounded under certain assumptions. Originally, many authors considered the Laplace operator and bounded q, or q in a Kato class and smooth domains D, see the references in [64]. The paper [64] made an essential progress by including Lipschitz domains in the case of the Laplace operator. This direction of research is summarized in [62].

The paper [54] initiated in 1997 the study of **CGT** for rotation invariant stable Lévy and more general processes for Schrödinger and more general perturbations. The focus of [54] was on $C^{1,1}$ domains. **CGT** for the stable proceses in Lipschitz domains was proved in 1999 in [32]. We also like to note that there is a recent non-probabilistic approach to **CGT**, see [85] and [37].

A (Borel) function q on $\mathbb{R}^d$ is said to belong to the *Kato class* $\mathcal{J}_\alpha$ if

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_0^t E^x|q(Y_s)|\,ds = 0\,. \tag{2.29}$$

Thus, (2.29) is a statement of *negligibility* of q in (small) time, with respect to the given transition probability. To make (2.32) more explicit we recall the following well-known estimate (see (1.29)):

$$C^{-1}\left(\frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha}\right) \le p_t(x) \le C\left(\frac{t}{|x|^{d+\alpha}} \wedge t^{-d/\alpha}\right). \tag{2.30}$$

The estimate is proved by subordination (see, e.g., [39]). Noteworthy,

$$\frac{t}{|x|^{d+\alpha}} \le t^{-d/\alpha} \quad \text{iff} \quad t \le |x|^\alpha\,. \tag{2.31}$$

We easily see that

$$\int_0^t p_s(x)ds \approx \frac{t^2}{|x|^{d+\alpha}} \wedge \frac{1}{|x|^{\alpha-d}}\,.$$

By the definition of E^x, and Fubini-Tonelli, $q \in \mathcal{J}_\alpha$ if and only if

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \left[\frac{t^2}{|y-x|^{d+\alpha}} \wedge \frac{1}{|y-x|^{\alpha-d}}\right] |q(y)|dy = 0\,.$$

It follows that (2.29) is equivalent to the following condition of *negligibility* of q in (small) space with respect to the potential operator:

$$\lim_{\gamma \downarrow 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq \gamma} K_\alpha(x-y)\, |q(y)|\, dy = 0\,. \tag{2.32}$$

We also note that $L^\infty(\mathbb{R}^d) \subset \mathcal{J}_\alpha \subset L^1_{loc}(\mathbb{R}^d)$.

For $q \in \mathcal{J}_\alpha$ we define the *additive functional*

$$A_q(t) = \int_0^t q(Y_s)\, ds\,,$$

and the corresponding *multiplicative functional*

$$e_q(t) = \exp(A_q(t))\,.$$

We have

$$e_q(t+s) = e_q(t)\,(e_q(s) \circ \theta_t)\,, \quad t, s \geqslant 0\,.$$

Here θ_t is the usual shift operator acting on the process Y by the formula: $Y_s \circ \theta_t = Y_{t+s}$.

For an open bounded set D we define the killed Feynman-Kac semigroup T_t by the formula

$$T_t\, f(x) = E^x[t < \tau_D; e_q(t)\, f(Y_t)]\,. \tag{2.33}$$

T_t is a strongly continuous semigroup on $L_p(D)$, $1 \leqslant p < \infty$, and on $C(D)$– for regular D. For each $t > 0$, the operator T_t is determined by a symmetric transition density function $u_t(x,y)$ which is in $C_0(D \times D)$ for regular D. We should note that $\{T_t\,,\ t > 0\}$ is generated by $\Delta^{\alpha/2} + q$, see [32]. The next lemma is fundamental in the theory of Feynman-Kac semigroups– this is seen in the development of [62], which we will follow quite closely below.

Lemma 2.8. [Khasminski lemma] *Let τ be an optional time of Y such that*

$$\tau \leqslant t + \tau \circ \theta_t\,, \quad \textit{on}\ \{t < \tau\}\,,\ t > 0\,. \tag{2.34}$$

Suppose that $q \geqslant 0$ and $E^x A(\tau) < \infty$ for all $x \in \mathbb{R}^d$. Then for each integer $n \geqslant 0$ we have,

$$\sup_x E^x[A(\tau)^n] \leqslant n! \sup_x (E^x A(\tau))^n\,. \tag{2.35}$$

If $\sup_x E^x A(\tau) = \alpha < 1$ then

$$\sup_x E^x e^{A(\tau)} \leqslant (1-\alpha)^{-1}\,.$$

The condition (2.34) is satisfied if τ is constant or if $\tau = \tau_D$ for some $D \subseteq \mathbb{R}^d$.

Proof. Since $q \geqslant 0$, the functional $A(\cdot)$ is nonnegative and nondecreasing. By Fubini-Tonelli and (2.29), $A(\tau) < \infty$ a.s. We have

$$\frac{A(\tau)^{n+1}}{n+1} = \int_0^\tau [A(\tau) - A(t)]^n \, dA(t) \, .$$

For $t < \tau$, by (2.34),

$$A(\tau) - A(t) \leqslant A(t + \tau \circ \theta_t) - A(t) = \int_t^{t+\tau\circ\theta_t} q(Y_s) \, ds$$
$$= [\int_0^\tau q(Y_s) \, ds] \circ \theta_t = A(\tau) \circ \theta_t \, .$$

By Fubini's theorem,

$$\frac{E^x[A(\tau)^{n+1}]}{n+1} \leqslant E^x[\int_0^\tau [A(\tau) \circ \theta_t]^n \, dA(t)] = \int_0^\infty E^x[t < \tau; [A(\tau) \circ \theta_t]^n q(Y_t)] \, dt \, .$$

By the Markov property the last integral is equal to

$$\int_0^\infty E^x[t < \tau; E^{Y_t}[A(\tau)^n] \, q(Y_t)] \, dt \leqslant \sup_x E^x[A(\tau)^n] \int_0^\infty E^x[t < \tau; q(Y_t)] \, dt$$
$$= \sup_x E^x[A(\tau)^n] \, E^x A(\tau) \, .$$

It follows that

$$\sup_x E^x[A(\tau)^{n+1}] \leqslant (n+1) \sup_x E^x[A(\tau)^n] \, \sup_x E^x[A(\tau)] \, ,$$

hence (2.35) is proved by induction on n. The last assertion of the lemma is an immediate consequence of (2.35). □

We like to note that $A(t)$ increases where $q > 0$, and $e_q(t)$ may be interpreted as the *mass* of a particle moving along the trajectories of the process in the *potential well* given by q. If $q \leqslant 0$ then the mass is always bounded by 1 (subprobabilistic), which corresponds to Courrège's theorem, see (1.25).

The *gauge function* of D and q is defined as follows:

$$u(x) = E^x e_q(\tau_D) \, .$$

We can interpret $u(x)$ as the expected mass of the particle when it leaves the domain. We note that since τ_D is an unbounded random variable, the mass may be infinite if q is (say, positive and) large enough. When the gauge function satisfies $u(x) < \infty$ for (some, hence for all) $x \in D$, we call the pair (D, q) *gaugeable*.

We consider $u_t(x, y) = E^x[1_{t<\tau_D} e_q(t)|Y_t = y]$, the integral kernel of T_t. We define the Green function of the Schrödinger operator on D,

$$V(x, y) = \int_0^\infty u_t(x, y)\, dt\,.$$

The potential operator of the the Feynman-Kac semigroup T_t killed off D is, by definition

$$\begin{aligned} Vf(x) &= \int_0^\infty T_t f(x)\, dt = \int_0^\infty E^x[t < \tau_D; e_q(t)\, f(Y_t)]\, dt \\ &= E^x \int_0^{\tau_D} e_q(t)\, f(Y_t)\, dt = \int_D V(x, y)\, f(y)\, dy\,. \end{aligned}$$

Both functions V and u_t are symmetric in $(x, y) \in D \times D$ and u_t is continuous whenever D is regular.

The theorem below provides the fundamental property of the gauge and clarifies conditions on gaugeability. For the proof, we refer to [62] (see § 5.6 and Theorem 4.19), or [33, Theorem 4.2], where it can be seen that the result is analogous to the Harnack inequality.

Theorem 2.9 (Gauge Theorem). *Let D be a domain with $m(D) < \infty$ and let $q \in \mathcal{J}^\alpha$. If $u(x_0) < \infty$ for some $x_0 \in D$, then u is bounded in $\mathbf{R}^d$. Moreover, the following conditions are equivalent:*

(i) (D, q) is gaugeable;
(ii) The semigroup T_t satisfies $\int_0^\infty ||T_t||_\infty\, dt < \infty$;
(iii) $V\mathbf{1} \in L^\infty(\mathbf{R}^d)$;
(iv) $V|q| \in L^\infty(\mathbf{R}^d)$.

Thus, for the sake of brevity, we can write $V\mathbf{1} \in L^\infty(\mathbf{R}^d)$ to indicate that (D, q) is gaugeable. In what follows we always assume that (D, q) is gaugeable indeed. We like to remark that gaugeability is difficult to express explicitly. However a useful connection exists of gaugeability to the existence of positive functions harmonic on D for $\Delta^{\alpha/2} + q$, which can be used to give natural and simple examples of the gauge function for some (not-so-natural) potentials q, see Figure 2.4 and [33].

The following estimate for the kernel $u_t(x, y)$ of the Feynman-Kac strengthens Lemma 4.7 in [32] and enables us to simplify the proof of **CGT**, compared to [32].

Theorem 2.10. *Let $D \subseteq \mathbb{R}^d$ be open with finite Lebesgue measure and $q \in \mathcal{J}_\alpha$. If (D, q) is gaugeable and $0 < \delta < 1$ then for $x, y \in \mathbb{R}^d$*

$$u_t(x, y) \leqslant C_1\, t^{-d/\alpha}\, [(t^{-1/\alpha}\, |x - y|)^{-d-\alpha} \wedge 1]^\delta \quad \textit{for} \quad 0 < t \leqslant t_0\,, \tag{2.36}$$

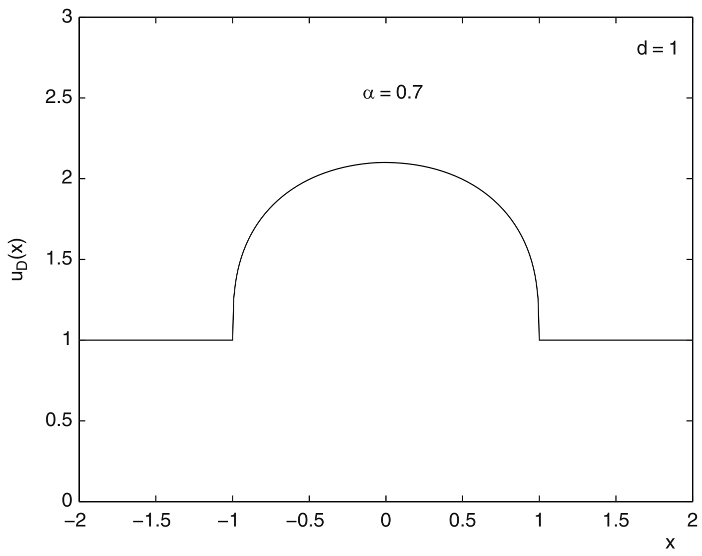


Fig. 2.4 A gauge function for $q \geqslant 0$, see [33]

where $t_0 = t_0(\delta, q, \alpha)$, $C_1 = C_1(d, \alpha)$ *and*

$$u_t(x, y) \leqslant C_2 \exp(-\beta t), \quad \textit{for} \quad t > t_0\,, \tag{2.37}$$

where $C_2 = C_2(D, d, \alpha)$, $\beta = \beta(\delta, q, \alpha)$. *Furthermore, if D is additionally bounded then*

$$V(x, y) \leqslant C_3\, |x - y|^{\alpha - d}\,. \tag{2.38}$$

Proof. Let $p > 1$ be fixed. Choose $t_0 > 0$ such that for $0 < t \leqslant t_0$

$$\sup_{x \in D} E^x[e_{2pq}(t)] \leqslant 2\,.$$

For $0 < t \leqslant t_0$ and $f \in L^2(D)$ define $S_t f(x) = E^x[t < \tau_D;\, e_{pq}(t)\, f(Y_t)]$. By Schwarz inequality, for $0 < t < t_0$ we have

$$\begin{aligned} |S_t f(x)|^2 &\leqslant E^x[t < \tau_D;\, e_{2pq}(t)]\, E^x[t < \tau_D;\, f(Y_t)^2] \leqslant 2\, E^x[f(Y_t)^2] \\ &= 2 \int f(y)^2\, p_t(x, y)\, dy \leqslant 2\, t^{-d/\alpha}\, ||f||_2^2 \sup_x p_1(x)\,. \end{aligned}$$

Thus, we have obtained

$$||S_t||_{2,\infty} \leqslant C\, t^{-d/2\alpha}\,.$$

Now, observe that for positive $f \in L^1$ and positive $\phi \in L^2$ we have

$$\int \phi\, S_t f\, dx = \int f\, S_t \phi\, dx \leqslant ||S_t \phi||_\infty \int f\, dm \leqslant ||S_t||_{2,\infty}\, ||\phi||_2\, ||f||_1\, .$$

This shows that $S_t f \in L^2$ and $||S_t||_{1,2} \leqslant ||S_t||_{2,\infty}$. If $f \in L^1$ we have $S_t f = S_{t/2} S_{t/2} f \in L^\infty$ so

$$||S_t||_{1,\infty} \leqslant ||S_{t/2}||_{1,2}\, ||S_{t/2}||_{2,\infty} \leqslant ||S_{t/2}||^2_{2,\infty} \leqslant C\, t^{-d/\alpha}\, .$$

Let B be a Borel subset of D. Then

$$\begin{aligned} T_t \mathbf{1}_B(x) &\leqslant E^x[t < \tau_D;\, e_{pq}(t)\, \mathbf{1}_B(Y_t)]^{1-\delta}\, E^x[t < \tau_D;\, \mathbf{1}_B(Y_t)]^\delta \\ &= (S_t \mathbf{1}_B(x))^{1-\delta}\, \left(P_t^D \mathbf{1}_B(x)\right)^\delta \\ &\leqslant ||S_t||^{1-\delta}_{1,\infty}\, m(B)^{1-\delta}\, [C_1\, t^{-d/\alpha}\, ((t^{-1/\alpha}\, \rho)^{-d-\alpha} \wedge 1)\, m(B)]^\delta \\ &\leqslant C\, t^{-(1-\delta)d/\alpha}\, m(B)\, [C_1\, t^{-d/\alpha}\, ((t^{-1/\alpha}\, \rho)^{-d-\alpha} \wedge 1)]^\delta \\ &\leqslant C_3\, t^{-d/\alpha}\, m(B)\, [(t^{-1/\alpha}\, \rho)^{-d-\alpha} \wedge 1]^\delta\, , \end{aligned}$$

where $\rho = \sup_{y\in B} |x-y|$ and we applied the fact that

$$\begin{aligned} p_t^D(x,y) &\leqslant p_t(x,y) = t^{-d/\alpha}\, p_1(t^{-1/\alpha}(x-y)) \\ &\leqslant C_1\, t^{-d/\alpha}\, [(t^{-1/\alpha}|x-y|)^{-d-\alpha} \wedge 1]\, , \quad t > 0\, ,\ x, y \in \mathbb{R}^d\, . \end{aligned}$$

Thus, we have obtained

$$\frac{1}{m(B)} \int_B u_t(x,y)\, dy \leqslant C_3\, t^{-d/\alpha}\, m(B)\, [(t^{-1/\alpha}\, \rho)^{-d-\alpha} \wedge 1]^\delta\, .$$

Consider $B = B(y_0, \delta)$ and $x, y_0 \in D$. Letting $\delta \downarrow 0$, we obtain

$$u_t(x,y) \leqslant C_3\, t^{-d/\alpha}\, [(t^{-1/\alpha}\, |x-y|)^{-d-\alpha} \wedge 1]^\delta\, ,$$

which gives (2.36). Since (D,q) is gaugeable, by Theorem 2.5 we obtain

$$||T_t||_1 \leqslant C\, e^{-\epsilon t}\, ,$$

for $t > t_0$ and for some $\epsilon > 0$. On the other hand

$$\frac{1}{m(B)} \int u_t(x,y)\, \mathbf{1}_B(y)\, dy = ||T_t||_{1,\infty}\, .$$

As before, this gives $u_t(x,y) \leqslant ||T_t||_{1,\infty}$. Since

$$||T_t||_{1,\infty} \leqslant ||T_{t_0}||_{1,\infty}\, ||T_{t-t_0}||_1 \leqslant ||T_{t_0}||_{1,\infty}\, C\, e^{-\epsilon(t-t_0)}\, ,$$

we obtain (2.37). To prove (2.38) we apply (2.36) to estimate

$$
\begin{aligned}
C_1^{-1}\int_0^{t_0} u_t(x,y)\,dt &\leqslant \int_0^{t_0} t^{-d/\alpha}[1\wedge(t^{-1/\alpha}|x-y|)^{-d-\alpha}]^\delta\,dt \\
&= \int_0^{t_0} t^{-d/\alpha}[1\wedge\left(\frac{t}{|x-y|^\alpha}\right)^{-d-\alpha}]^\delta\,dt \\
&= |x-y|^{-\delta(d+\alpha)}\int_0^{|x-y|^\alpha\wedge t_0} t^{\delta-(1-\delta)d/\alpha}\,dt \\
&\quad+\int_{|x-y|^\alpha\wedge t_0}^{t_0} t^{-d/\alpha}\,dt\,.
\end{aligned}
$$

In the first integral on the right hand-side we take a $\delta>0$ such that $d/\alpha<\frac{1+\delta}{1-\delta}$, and we then see that the first integral is convergent. We obtain the upper bound

$$
C_4|x-y|^{-\delta(d+\alpha)}|x-y|^{(1-d/\alpha+\delta(1+d/\alpha))^\alpha}+C_5|x-y|^{\alpha-d}=C\,|x-y|^{\alpha-d}\,.
$$

To finish the proof we observe that (2.37) yields

$$
\int_{t_0}^\infty u_t(x,y)\,dt\leqslant\beta^{-1}e^{-\beta t_0}\,,
$$

which, together with the observation that $|x-y|\leqslant\operatorname{diam}(D)$, concludes the proof of (2.38). □

We note that since $u_t(x,y)$ is continuous, the above estimate yields the continuity of $V(x,y)$ for $x,y\in\mathbb{R}^d$, $x\neq y$, under the assumption that D is regular (and bounded).

We should also mention that there exists a new method of estimating u_t based on the notion of *conditional smallness* of q which yields *comparability* of u_t and p_t in finite time, see [36].

The following lemma is a well-known but fundamental relationship between G_D and V, see [62], Ch. 6. For an analyst, the lemma is an instance of the (implicit) perturbation formula for V, compare [85].

Lemma 2.11. *Suppose that $q\in\mathcal{J}_\alpha$ and $V\mathbf{1}\in L^\infty(D)$. If $V\,|q|\,G_D\,|f|<\infty$ on D then*

$$
V f = G_D\,f + V\,q\,G_D\,f\,.
$$

Proof. By Fubini's theorem we obtain

$$
\begin{aligned}
V\,q\,G_D\,f(x) &= E^x\int_0^{\tau_D} e_q(t)\,q(Y_t)\,E^{Y_t}[\int_0^{\tau_D} f(Y_s)\,ds]\,dt \\
&= E^x\int_0^{\tau_D} e_q(t)\,q(Y_t)\int_t^{\tau_D} f(Y_s)\,ds\,dt
\end{aligned}
$$

$$= E^x \int_0^{\tau_D} f(Y_s) \int_s^{\tau_D} e_q(t)\, q(Y_t)\, dt\, ds$$
$$= E^x \int_0^{\tau_D} f(Y_s)\, [e_q(s) - 1]\, ds = V f(x) - G_D f(x)\,.$$

The application of Fubini's theorem is justified by the condition $V\,|q|\,G_D\,|f| < \infty$. □

An important consequence of the above lemma and Theorem 2.10 is the following

Lemma 2.12. *Suppose that $q \in \mathcal{J}_\alpha$ and $V\mathbf{1} \in L^\infty(D)$. Assume that D is bounded and regular. Then for every $x, y \in D$, $x \neq y$ we have*

$$V(x,y) = G_D(x,y) + \int_D V(x,w)\, q(w)\, G_D(w,y)\, dw\,. \tag{2.39}$$

Proof. Applying the preceding lemma we obtain that for every $x \in D$ the equation (2.39) holds y-almost everywhere. Assume that $|x-y| > \delta > 0$. Then either $|x-w| > \delta/2$ or $|w-y| > \delta/2$. Suppose that the first condition holds. Then, by Theorem 2.10 we obtain
$V(x,w) \leqslant C\,K_\alpha(x,w) \leqslant C\,K_\alpha(\delta/2)$ so

$$V(x,w)\,|q(w)|\,G_D(w,y) \leqslant C\,K_\alpha(\delta/2)\,|q(w)|\,K_\alpha(w-y)\,.$$

In the second case we obtain

$$V(x,w)\,|q(w)|\,G_D(w,y) \leqslant C\,K_\alpha(x-w)\,|q(w)|\,K_\alpha(\delta/2)\,.$$

Consequently, when $|x-y| > \delta$ we have

$$V(x,w)\,|q(w)|\,G_D(w,y) \leqslant C\,K_\alpha(\delta/2)\,|q(w)|\,[K_\alpha(w-y) + K_\alpha(x-w)]\,.$$

Since D is bounded, it follows that the set of functions

$$\{w \mapsto V(x,w)\,q(w)\,G_D(w,y); (x,y) \in D \times D, |x-y| > \delta\}$$

is uniformly integrable on D. On the other hand, for each $w \in D$, the function

$$(x,y) \mapsto V(x,w)\,q(w)\,G_D(w,y)$$

is continuous except possibly at $x = w$ or $y = w$. Therefore, the integral on the right-hand side of (2.39) is continuous in $(x,y) \in D \times D$, $|x-y| > \delta$. Since δ is arbitrary, both members of (2.39) are continuous in $(x,y) \in D \times D$, $x \neq y$. The proof is complete. □

Let h be an α-harmonic and positive on a bounded domain D. By $p_t^D(x, y)$ we denote the transition density function of (Y_t) killed on exiting D. For $x, y \in D$ and $t > 0$ we define time-homogeneous transition density (of Doob's h-process)

$$p_h(t; x, y) = h(x)^{-1} p_t^D(x, y) h(y) .$$

This defines a strong Markov process on $D_\partial = D \cup \{\partial\}$. Here ∂ is the absorbing state (cemetery) attached to the state space to accommodate for the loss of mass (the conditional process is generally subprobabilistic if considered on the original state space, [23]). The h-process is denoted also by Y_t, while the corresponding expectations and probabilities are denoted by E_h^x, P_h^x. We should note that even though we use the same generic notation for the conditional process, there is no pathwise correspondence between the original and the conditional processes, and theorems involving the conditional process are usually more difficult.

The definition of the h-process yields,

$$E_h^x[t < \tau_D; f(Y_t)] = h(x)^{-1} E^x[t < \tau_D; f(Y_t) h(Y_t)] . \tag{2.40}$$

Let D be a bounded Lipschitz domain; for fixed $\xi \in \partial D$ we put

$$h(y) = M_D(y, \xi) .$$

Here $M_D(\cdot, \xi)$ is Martin's kernel of D, which is α-harmonic in D [27, 38].

We also need another version of conditioning: for fixed $y \in D$ we let

$$h(y) = G_D(y, z) .$$

The function h above is α-harmonic in $D \setminus \{z\}$, and superharmonic in D, see [56]. In the sequel, we will use the notation E_ξ^x, P_ξ^x (E_z^x, P_z^x, respectively) to indicate conditioning by Martin kernel (Green function, respectively).

We redefine α-stable ξ-Lévy motion Y_t by putting $Y_s = \xi$ for $s \geqslant \tau_D$ to get the process on $D \cup \{\xi\}$. Analogously, α-stable z-Lévy motion Y_t is defined on D and $Y_s = z$ for $s \geqslant \tau_{D\setminus\{z\}}$.

For a stopping time $T \leqslant \tau_{D\setminus\{y\}}$ we obtain a specialization of the formula (2.40):

$$E_z^x[T < \tau_{D\setminus\{y\}}; f(Y_T)] = G_D(x, z)^{-1} E^x[T < \tau_D; f(Y_T) G_D(Y_T, z))] . \tag{2.41}$$

A similar formula holds true for the ξ-process.

As an instructive exercise we compute the Green function of α-stable z-Lévy motion.

Proposition 2.13 (Green function for conditional process). *Let D be a bounded Lipschitz domain in $\mathbb{R}^d$, $\alpha < d$, Y_t - α-stable z-Lévy motion. The Green function of D, as the function of y, computed for the process Y*

(starting at $x \in D$ and exiting at $z \in D$) has the following form:

$$\frac{G_D(x,y)\,G_D(y,z)}{G_D(x,y)}\,.$$

Proof. Indeed, we obtain

$$\begin{aligned}
E_z^x \int_0^{\tau_{D\setminus\{z\}}} f(Y_t)\,dt &= \int_0^\infty E_z^x[t < \tau_{D\setminus\{z\}}; f(Y_t)]\,dt \\
&= \int_0^\infty G_D(x,z)^{-1} E^x[t < \tau_{D\setminus\{z\}}; f(Y_t) G_D(Y_t,z)]\,dt \\
&= G_D(x,z)^{-1} E^x \int_0^{\tau_D} f(Y_t)\,G_D(Y_t,z)\,dt \\
&= G_D(x,z)^{-1} G_D(f(\cdot)\,G_D(\cdot,z))(x) \\
&= \int_D \frac{G_D(x,y)\,G_D(y,z)}{G_D(x,y)}\,f(y)\,dy\,.
\end{aligned}$$

□

By the above calculations, we obtain

$$\begin{aligned}
E_z^x \tau_{D\setminus\{z\}} &= E_z^x \int_0^{\tau_{D\setminus\{z\}}} \mathbf{1}(Y_t)\,dt = \int_D \frac{G_D(x,y)\,G_D(y,z)}{G_D(x,y)}\,dy \\
&\leqslant C \int_D [K_\alpha(x-y) + K_\alpha(y-z)]\,dy \leqslant 2C \int_{D\cap B(0,R)} K_\alpha(y)\,dy < \infty\,.
\end{aligned}$$

The calculations provide the proof of the second formula in Theorem 2.14 below (the proof of the first formula is similar and will be omitted).

Theorem 2.14.

$$\begin{aligned}
E_\xi^x \tau_D < \infty\,, &\quad P_\xi^x(\lim_{t\uparrow\tau_D} Y_t = \xi) = 1\,, \\
E_z^x \tau_{D\setminus\{z\}} < \infty\,, &\quad P_z^x(\lim_{t\uparrow\tau_{D\setminus\{z\}}} Y_t = z) = 1\,.
\end{aligned}$$

Theorem 2.14 shows that the behavior of the conditional process is dramatically different from that of the original process. In particular, the conditional process exits through the pole of the function h and does so in a continuous manner.

Lemma 2.15. *Let D, U be bounded regular (e.g. Lipschitz) domains such that $\overline{U} \subseteq D$ and $z \in U$. Put $D_0 = D \setminus \overline{U}$ and let $\zeta = \tau_{D\setminus\{z\}}$. Let $u \in D$, $u \neq z$ and $x \in D_0$. Then we have*

$$P_z^u\{\tau_{U\setminus\{z\}} = \zeta\} = \frac{G_U(u,z)}{G_D(u,z)}\,, \tag{2.42}$$

$$P_z^x\{\tau_{D_0} = \zeta\} = 0\,. \tag{2.43}$$

Let us remark that the second formula states that the conditional process cannot reach the point z jumping from outside a certain neighborhood of this point. The first formula gives the precise value of the probability of reaching the point z when starting from a point within a neighborhood of this point. The formulas are essential in proving **CGT** for the operator $\Delta^{\alpha/2}$, $0 < \alpha < 2$.

Proof. To prove the first formula we observe that

$$G_D(u,v) = K_\alpha(u-v) - E^u[K_\alpha(Y_{\tau_D} - v)]\,.$$

Consequently, we obtain

$$\begin{aligned} E^u[\tau_U < \tau_D; G_D(Y_{\tau_U}, z)] &= E^u[\tau_U < \tau_D; K_\alpha(Y_{\tau_U} - z)] \\ &\quad - E^u[\tau_U < \tau_D; E^{Y_{\tau_U}}[K_\alpha(Y_{\tau_D} - z)] \\ &= E^u[\tau_U < \tau_D; K_\alpha(Y_{\tau_U} - z)] \\ &\quad - E^u[\tau_U < \tau_D; E^u[K_\alpha(Y_{\tau_D} - z) \circ \theta_{\tau_U} | \mathcal{F}_{\tau_U}]] \\ &= E^u[\tau_U < \tau_D; K_\alpha(Y_{\tau_U} - z) - K_\alpha(Y_{\tau_D} - z)] \\ &= E^u[K_\alpha(Y_{\tau_U} - z) - K_\alpha(Y_{\tau_D} - z)] \\ &= G_D(u,z) - G_U(u,z)\,. \end{aligned}$$

Taking into account

$$P_z^u\{\tau_{U\setminus\{z\}} \neq \zeta\} = G_D(u,z)^{-1} E^u[\tau_U < \tau_D; G_D(Y_{\tau_U}, z)]\,,$$

we obtain the first formula. To prove the second formula we observe that $G_D(\cdot, z)$ is α-harmonic and bounded on $D_0 = D \setminus \overline{U}$ so

$$E^x G_D(Y_{\tau_{D_0}}, z) = G_D(x,z)$$

and

$$\begin{aligned} P_z^x\{\tau_{D_0} < \zeta\} &= G_D(x,z)^{-1} E^x[\tau_{D_0} < \tau_D; G_D(Y_{\tau_{D_0}}, z)] \\ &= G_D(x,z)^{-1} E^x G_D(Y_{\tau_{D_0}}, z) = G_D(x,z)^{-1} G_D(x,z) = 1\,. \end{aligned}$$

□

As a corollary we obtain (compare Lemma 4.4 in [64]):

Corollary 2.16. *Assume that $y \in D$ with $d(y, D^c) > 3\delta$. Put $U = B(y, 3\delta)$. Then we have*

$$\inf_{u \in \overline{B(y,\delta)}} P^u\{\tau_U = \tau_D\} \,>\, 0\,, \qquad \inf_{u \in \overline{B(y,\delta)} \setminus \{y\}} P_y^u\{\tau_{U\setminus\{y\}} = \zeta\} \,>\, 0\,. \tag{2.44}$$

Proof. We first prove the second part of (2.44). In view of Lemma 2.15 we have that

$$P_y^u\{\tau_{U\setminus\{y\}} = \zeta\} = \frac{G_U(u,y)}{G_D(u,y)} \geqslant \frac{G_U(u,y)}{K_\alpha(u,y)} = 1 - \mathcal{A}_{d,\alpha}^{-1}|u-y|^{d-\alpha} E^u K_\alpha(Y_{\tau_U}, y)\,.$$

Observe that we have $|u-y| \leqslant \delta$ for $u \in \overline{B(y,\delta)}$ and also $|Y_{\tau_U} - y| > 3\delta$ which yields $E^u K_\alpha(Y_{\tau_U}, y) \leqslant \mathcal{A}_{d,\alpha}(3\delta)^{\alpha-d}$. This completes the proof of the second part of (2.44). We now prove the first part. We denote $R = \mathrm{diam}(D)$. Then, by the explicit formula for Poisson kernel for balls (1.57), we obtain

$$\begin{aligned} P^u\{\tau_U = \tau_D\} &= P^u\{Y_{\tau_U} \in D^c\} \\ &= P^{u-y}\{Y_{\tau_{B(0,3\delta)}} \in D^c - y\} \geqslant P^{u-y}\{Y_{\tau_{B(0,3\delta)}} \in B(0,R)^c\} \\ &= C_\alpha^d \int_{|z|>R} \left(\frac{(3\delta)^2 - |u-y|^2}{|z|^2 - (3\delta)^2}\right)^{\alpha/2} \frac{dz}{|u-y-z|^d} \\ &\geqslant C_\alpha^d (8\delta)^{\alpha/2}\omega_d \int_R^\infty \frac{\rho^{d-1}\,d\rho}{\rho^\alpha\,(\rho+\delta)^d} \geqslant \frac{C_\alpha^d\,(8\delta)^{\alpha/2}\omega_d}{(7/6)^d\,\alpha\,R^\alpha}\,, \end{aligned}$$

because we have $\rho \geqslant R > 6\delta$ under the integral sign. By ω_d we denote the surface measure of the unit sphere in $\mathbf{R}^d$. □

The following lemma is a "conditional" version of Khasminski's lemma (see Lemma 2.8). The proof relies on **3G** Theorem (2.26) as in [64].

Lemma 2.17. *For every $\varepsilon > 0$ there exists $\eta = \eta(\varepsilon, D, q)$ such that for every open set $U \subseteq D$ with $m(U) < \eta$ we have*

$$\sup_{u\in D, u\neq y} E_y^u \int_0^{\tau_{U\setminus\{y\}}} |q(Y_t)|\,dt < \varepsilon$$

and if $0 < \varepsilon < 1$ then $\exp(-\varepsilon) \leqslant E_y^u e_q(\zeta) \leqslant (1-\varepsilon)^{-1}$.

Proof. Let x, y be in D, $x \neq y$. Applying the definition of transition probability p_h^D of the process conditioned by the function $h(\cdot) = G(\cdot, y)$ and using Fubini's Theorem we obtain

$$\begin{aligned} E_y^x[\int_0^{\tau_U} |q(Y_t)|\,dt] &\leqslant E_y^x[\int_0^{\tau_D} \mathbf{1}_U(Y_t)|q(Y_t)|\,dt] \\ &= \int_0^\infty \int_U p^D_{G(\cdot,y)}(t;x,u)\,|q(u)|\,du\,dt \\ &= G(x,y)^{-1} \int_0^\infty \int_U p^D(t;x,u)\,|q(u)|\,G(u,y)\,du\,dt \\ &= G(x,y)^{-1} \int_U G(x,u)\,|q(u)|\,G(u,y)\,du\,. \end{aligned}$$

By **3G** Theorem, the last integral is estimated by

$$C \int_U [K_\alpha(x,u) + K_\alpha(u,y)]\;|q(u)|\,du\,,$$

with C depending only on D and q. However, by the properties of Kato class, we obtain

$$\sup_{x\in D}\int_U K_\alpha(x,u)\,|q(u)|\,du \longrightarrow 0\,,$$

as $m(U) \longrightarrow 0$. The last part of the lemma now follows from Khasminski's Lemma. □

By the above lemma, Lemma 2.8 and Corollary 2.44, we easily obtain the following result (compare Lemma 4.3 in [64]).

Lemma 2.18. *Under the notation of* Corollary 4.4 *there exist constants* C_1 *and* C_2 *such that for every* $u, v \in \overline{B(y,\delta)}$, $v \neq y$, *with* $\delta > 0$ *small enough we have*

$$C_1 \leqslant E^u[\tau_U = \tau_D;\, e_q(\tau_D)] \leqslant C_2\,, \quad C_1 \leqslant E^v_y[\tau_{U\setminus\{y\}} = \zeta;\, e_q(\zeta)] \leqslant C_2\,. \tag{2.45}$$

Proof. We prove the second part of (2.18). The other case is similar and is left to the reader. Applying Lemma 2.17 with $\varepsilon = 1/2$, we obtain $E^u_y[e_{|q|}(\tau_U)] \leqslant 2$. Denote the infimum in Corollary 2.16 by C. Then, in view of Jensen's inequality, we obtain

$$\begin{aligned}
E^u_y[e_q(\tau_U)|\tau_U = \zeta] &\geqslant \exp\{E^u_y[\int_0^{\tau_U} q(Y_t)\,dt|\tau_U = \zeta]\}\\
&\geqslant \exp\{-\,E^u_y[\int_0^{\tau_U} |q(Y_t)|\,dt|\tau_U = \zeta]\}\\
&\geqslant \exp\{-\frac{1}{C}\,E^u_y[\tau_U = \zeta;\, \int_0^{\tau_U} |q(Y_t)|\,dt]\}\\
&\geqslant \exp\{-\frac{1}{C}\,E^u_y[\int_0^{\tau_U} |q(Y_t)|\,dt]\} \geqslant \exp\{-\frac{1}{2\,C}\}\,.
\end{aligned}$$

Before stating the next lemma, we introduce some notation. For $y \in \mathbf{R}^d$, $|y| > 1$ let

$$I_1(y) = \int_{B(0,1)} \frac{\mathcal{A}_{d,\alpha}\,du}{|u-y|^{d+\alpha}|u|^{d-\alpha}}\,, \quad I_2(y) = \int_{B(0,1)} \frac{\mathcal{A}_{d,\alpha}\,du}{|u-y|^{d+\alpha}}\,. \tag{2.46}$$

Lemma 2.19. *For all* $y \in \mathbf{R}^d$ *such that* $|y| > 1$ *we have*

$$I_1(y) \approx I_2(y)\,.$$

Proof. Clearly, we have $I_1(y) \geqslant I_2(y)$. To show a reverse inequality, denote $A(y) = B(y/|y|, 1/2) \cap B(0,1)$, $B(y) = B(0,1) \setminus A(y)$ and $M(y) = \sup_{u\in B(y)} |y-u|^{-d-\alpha}$, $m(y) = \inf_{u\in B(y)} |y-u|^{-d-\alpha}$. It is not difficult to see that $M(y) \leqslant Cm(y)$. Consequently,

$$\int_{B(y)} \frac{du}{|u-y|^{d+\alpha}|u|^{d-\alpha}} \leqslant M(y) \int_{|u|<1} \frac{du}{|u|^{d-\alpha}} \leqslant C\frac{\omega_d}{\alpha} m(y)$$
$$\leqslant C\frac{\omega_d}{\alpha\,|B(y)|} \int_{B(y)} \frac{du}{|u-y|^{d+\alpha}}\,.$$

However, for $u \in A(y)$ we have $|u| > 1/2$, so

$$\int_{A(y)} \frac{du}{|u-y|^{d+\alpha}|u|^{d-\alpha}} \leqslant 2^{d-\alpha} \int_{A(y)} \frac{du}{|u-y|^{d+\alpha}}\,.$$

□

We define the *conditional gauge* as the gauge function for the conditional process:

$$u(x,y) = E^x_y e_q(\tau_{D\setminus\{y\}})\,, \quad x \in D\,, \quad y \in \overline{D}\,.$$

Recall the Ikeda-Watanabe formula (1.52): for a bounded domain D with the exterior cone property the density function of the P^x-distribution of Y_{τ_D} is given, for $x \in D$, by

$$\mathcal{A}_{d,-\alpha} \int_D \frac{G_D(x,v)}{|v-y|^{d+\alpha}}\,dv\,, \quad y \in D^c\,.$$

The following explains the role of the conditional gauge function (compare [62, Theorem 6.3]).

Lemma 2.20. *If (D,q) is gaugeable then*

$$V(x,y) = u(x,y)\,G_D(x,y)\,, \quad x,y \in D\,, \quad x \neq y\,. \tag{2.47}$$

Proof. Since $x \neq y$, by the proof of Lemma 2.12 we obtain that

$$V(x,\cdot)\,|q|(\cdot)\,G_D(\cdot,y) < \infty$$

on the set $\{(x,y) \in D \times D, |x-y| > \delta\}$, for a fixed $\delta > 0$. Applying Fubini's theorem,

$$\begin{aligned} E^x_y \int_0^\zeta e_q(t)\,q(Y_t)\,dt &= \int_0^\infty E^x_y[t<\zeta; e_q(t)\,q(Y_t)]\,dt \\ &= G_D(x,y)^{-1} \int_0^\infty E^x[t<\tau_D; e_q(t)\,q(Y_t)\,G_D(Y_t,y)]\,dt \\ &= G_D(x,y)^{-1} \int_D V(x,w)\,q(w)\,G_D(w,y)\,dw\,. \end{aligned}$$

Since

$$E^x_y \int_0^\zeta e_q(t)\,q(Y_t)\,dt = E^x_y \int_0^\zeta \frac{d\,e_q(t)}{dt}\,dt = E^x_y[e_q(\zeta)-1] = u(x,y)-1\,,$$

by Lemma 2.12 we obtain (2.47). □

The preceding lemma yields the following.

Lemma 2.21. *Let D be an open regular bounded subset of $\mathbb{R}^d$. Then the gauge function $u(x,y)$ is continuous and symmetric on $D \times D$, $x \neq y$.*

We thus arrive at the main conclusion of this section: Conditional Gauge Theorem (**CGT**).

Theorem 2.22 (CGT). *Let D be a bounded Lipschitz domain, $q \in \mathcal{J}^\alpha$. If (D,q) is gaugeable (i.e. $E^x e_q(\tau_D) < \infty$) then*

$$\sup_{x,y \in D} u(x,y) < \infty \,,$$

and, moreover, u has a symmetric continuous extension to $\overline{D} \times \overline{D}$.

Proof. The proof is carried out in several steps.

Step 1. For $\delta > 0$ we put $D_\delta = \{x \in D;\, d(x, D^c) > 3\delta\}$. We choose and fix throughout the proof δ and a Lipschitz domain U^δ such that $D \backslash D_\delta \subseteq U^\delta \subseteq D$ and for all $y \in D$

$$\sup_{u \in D, u \neq y} E_y^u [\int_0^\tau |q(Y_t)|\, dt] \;<\; 1/2 \,, \quad \sup_{u \in \mathbb{R}^d} E^u [\int_0^\tau |q(Y_t)|\, dt] \;<\; 1/2 \,,$$

with $\tau = \tau_{U^\delta \backslash \{y\}}$ or $\tau = \tau_{B(y,3\delta)\backslash\{y\}}$. By Lemma 2.8 and Lemma 2.17 we obtain

$$\sup_{u \in D, u \neq y} E_y^u e_{|q|}(\tau) \;\leqslant\; 2 \,, \quad \sup_{u \in \mathbb{R}^d} E^u e_{|q|}(\tau) \;\leqslant\; 2 \,.$$

We show for $x, y \in D_\delta$, $x \neq y$ the following:

$$u(x,y) \;< C \,, \;\text{ where } C = C(D, \alpha, q, \delta) \,.$$

Fix $x, y \in D_\delta$ and denote $D_0 = D \setminus \overline{B(y,\delta)}$, $U = B(y, 3\delta) \setminus \{y\}$ and

$$\begin{aligned} T_1 = \tau_{D_0} \,, &\quad T_n = S_{n-1} + \tau_{D_0} \circ \theta_{S_{n-1}} \,, \\ S_0 = 0 \,, &\quad S_n = T_n + \tau_U \circ \theta_{T_n} \,, \;\; n = 1, 2, ... \end{aligned}$$

Put $\zeta = \tau_{D \backslash \{y\}}$. Because of the second formula in Lemma 2.15, the (conditional) process exits from $D \setminus \{y\}$ first entering $B(y, \delta)$. Hence $\zeta = S_n$, for some n. Thus, we obtain

$$E_y^x e_q(\zeta) = \sum_{n=1}^\infty E_y^x [T_n < \zeta, S_n = \zeta;\, e_q(\zeta)] \,. \tag{2.48}$$

For $n=1$, by strong Markov property, we obtain

$$\begin{aligned}
&G(x,y)\,E^x_y[T_1<\zeta,\, S_1=\zeta;\, e_q(\zeta)]\\
&=G(x,y)\,E^x_y[\tau_{D_0}<\zeta;\, e_q(T_1)\{\tau_U=\zeta;\, e_q(\tau_U)\}\circ\theta_{T_1}]\\
&=G(x,y)\,E^x_y[\tau_{D_0}<\zeta;\, e_q(T_1)\,E_y^{Y_{T_1}}[\tau_U=\zeta;\, e_q(\tau_U)]]\\
&=E^x[\tau_{D_0}<\tau_D;\, e_q(T_1)\,G(Y_{T_1},y)E_y^{Y_{T_1}}[\tau_U=\zeta;\, e_q(\tau_U)]]\,.
\end{aligned}$$

Since $\tau_{D_0}<\tau_D$ yields $Y_{T_1}\in\overline{B(y,\delta)}$, using Lemma 2.44 we obtain that the last term above is equivalent to

$$E^x[\tau_{D_0}<\tau_D;\, e_q(T_1)\,G(Y_{T_1},y)]\leqslant E^x[e_q(T_1)\,G(Y_{T_1},y)]\,.$$

Taking into account one term of the series (2.48), we get

$$\begin{aligned}
&G(x,y)\,E^x_y[T_n<\zeta,\, S_n=\zeta;\, e_q(\zeta)]\\
&=G(x,y)\,E^x_y[T_n<\zeta;\, e_q(T_n)\{\tau_U=\zeta;\, e_q(\tau_U)\}\circ\theta_{T_n}]\\
&=G(x,y)\,E^x_y[T_n<\zeta;\, e_q(T_n)\,E_y^{Y_{T_n}}[\tau_U=\zeta;\, e_q(\zeta)]\,]\\
&=E^x[T_n<\tau_D;\, e_q(T_n)\,G(Y_{T_n},y)E_y^{Y_{T_n}}[\tau_U=\zeta;\, e_q(\zeta)]\,]\,.
\end{aligned}$$

Since $Y_{T_n}\in\overline{B(y,\delta)}$, whenever $T_n<\tau_D$, so by Lemma 4.5 we obtain

$$\begin{aligned}
&E^x[T_n<\tau_D;\, e_q(T_n)\,G(Y_{T_n},y)E_y^{Y_{T_n}}[\tau_U=\zeta;\, e_q(\zeta)]\,]\\
&\approx E^x[T_n<\tau_D;\, e_q(T_n)\,G(Y_{T_n},y)]\\
&=E^x[S_{n-1}<\tau_D;\, e_q(S_{n-1})\{\tau_{D_0}<\tau_D;\, e_q(\tau_{D_0})\,G(Y_{\tau_{D_0}},y)\}\circ\theta_{S_{n-1}}]\\
&=E^x[S_{n-1}<\tau_D;\, e_q(S_{n-1})E^{Y_{S_{n-1}}}[\tau_{D_0}<\tau_D;\, e_q(\tau_{D_0})\,G(Y_{\tau_{D_0}},y)]\,]\\
&=E^x[S_{n-1}<\tau_D;\, e_q(S_{n-1})E^{Y_{S_{n-1}}}[e_q(\tau_{D_0})\,G(Y_{\tau_{D_0}},y)]\,]\,.
\end{aligned}$$

Using Ikeda-Watanabe formula for D_0 and Lemma 2.19, we have for $z\in D_0$

$$\begin{aligned}
E^z[e_q(\tau_{D_0})\,G(Y_{\tau_{D_0}},y)]&=\int_{D_0}\int_{D_0^c\cap D}\widetilde{u}(z,v)\,G(w,y)\frac{\mathcal{A}_{d,-\alpha}}{|v-w|^{d+\alpha}}\,G_{D_0}(z,v)\,dw\,dv\\
&\leqslant\int_{D_0}\int_{B(y,\delta)}\widetilde{u}(z,v)\,K_\alpha(w,y)\frac{\mathcal{A}_{d,-\alpha}}{|v-w|^{d+\alpha}}\,G_{D_0}(z,v)\,dw\,dv\\
&=\int_{D_0}\widetilde{u}(z,v)\,\mathcal{A}_{d,-\alpha}\,\delta^{-d}\,I_1\left(\frac{y-v}{\delta}\right)\,G_{D_0}(z,v)\,dv\\
&\approx\int_{D_0}\widetilde{u}(z,v)\,\mathcal{A}_{d,-\alpha}\,\delta^{\alpha-d}\,\delta^{-\alpha}\,I_2\left(\frac{y-v}{\delta}\right)\,G_{D_0}(z,v)\,dv\\
&\approx\delta^{\alpha-d}\int_{D_0}\int_{B(y,\delta)}\widetilde{u}(z,v)\,\frac{\mathcal{A}_{d,-\alpha}}{|v-w|^{d+\alpha}}\,G_{D_0}(z,v)\,dw\,dv\\
&\leqslant\delta^{\alpha-d}\int_{D_0}\int_{D_0^c}\widetilde{u}(z,v)\,\frac{\mathcal{A}_{d,-\alpha}}{|v-w|^{d+\alpha}}\,G_{D_0}(z,v)\,dw\,dv\\
&=\delta^{\alpha-d}\,E^z e_q(\tau_{D_0})\;\approx\;\delta^{\alpha-d}\,E^z e_q(\tau_D)\;\approx\delta^{\alpha-d}\,;
\end{aligned}$$

by gaugeability. Here $\widetilde{u}(z,v) = \widetilde{E}_v^z e_q(\tau_{D_0\setminus\{v\}})$ is the conditional gauge of the set D_0.

If $T_{n-1} < \tau_D$, for $n \geqslant 2$, then $Y_{T_{n-1}} \in \overline{B(y,\delta)}$ hence by (4) and Lemma 4.5 we obtain

$$\begin{aligned}
G(x,y)E_y^x[T_n < \zeta, S_n = \zeta; e_q(\zeta)] &\leqslant C\delta^{\alpha-d}E^x[S_{n-1} < \tau_D; e_q(S_{n-1})] \\
&= C\delta^{\alpha-d}E^x[T_{n-1} < \tau_D; e_q(T_{n-1})\{\tau_U < \tau_D; e_q(\tau_U)\} \circ \theta_{T_{n-1}}] \\
&= C\delta^{\alpha-d}E^x[T_{n-1} < \tau_D; e_q(T_{n-1})E^{Y_{T_{n-1}}}[\tau_U < \tau_D; e_q(\tau_U)]\,] \\
&\leqslant 2C\delta^{\alpha-d}E^x[T_{n-1} < \tau_D;\ e_q(T_{n-1})] \\
&\approx 2C\delta^{\alpha-d}E^x[T_{n-1} < \tau_D; e_q(T_{n-1})E^{Y_{T_{n-1}}}[\tau_U = \tau_D; e_q(\tau_U)]\,] \\
&= 2C\delta^{\alpha-d}E^x[T_{n-1} < \tau_D, S_{n-1} = \tau_D; e_q(\tau_D)]\,.
\end{aligned}$$

Thus

$$\begin{aligned}
G(x,y)\, E_y^x e_q(\zeta) &= G(x,y) \sum_{n=1}^{\infty} E_y^x[T_n < \zeta, S_n = \zeta;\ e_q(\zeta)] \\
&\leqslant\ C\delta^{\alpha-d}\,(1 + \sum_{n=2}^{\infty} E^x[T_{n-1} < \tau_D, S_{n-1} = \tau_D;\ e_q(\tau_D)]) \\
&\leqslant\ C\delta^{\alpha-d}\ (1 + E^x e_q(\tau_D))\ .
\end{aligned}$$

Recall that x, y satisfy the conditions $d(x, D^c) > 3\delta$, $d(y, D^c) > 3\delta$ and $|x-y| \leqslant \mathrm{diam}(D) < \infty$. We obtain (compare [62], Lemma 6.7)

$$G(x,y) \geqslant C'|x-y|^{\alpha-d} \geqslant C'\,(\mathrm{diam}(D))^{\alpha-d}\,,$$

with $C' = C'(D, \alpha, q, \delta)$. This clearly ends the proof of Step 1.

Step 2. In this step we remove the condition $d(x, D^c) > 3\delta$ imposed on $x \in D$ in (5).

To do this, assume that $y \in D_\delta$ but $d(x, D^c) \leqslant 3\delta$. Let U^δ be as in Step 1. Denote $U = U^\delta \setminus \{y\}$. Then we have

$$\begin{aligned}
u(x,y) \ &=\ E_y^x[\tau_U = \zeta;\ e_q(\tau_U)] \ +\ E_y^x[\tau_U < \zeta;\ e_q(\tau_U)\, u(Y_{\tau_U}, y)] \\
&\leqslant\ E_y^x e_{|q|}(\tau_U)\,(1 + \sup_{w \in D_\delta, w \neq y} u(w,y))\,.
\end{aligned}$$

By Step 1 and properties of U^δ, we obtain the conclusion.

Step 3. In this step we apply the symmetry of the function $u(x,y)$ in $x\,, y \in D$ to finish the proof of the boundedness of u.

Observe that the symmetry of u along with Step 2 settle the case when $x \in D_\delta$ and $y \in D_\delta^c$. It remains only the case when $x \neq y$, $x, y \in D_\delta^c$.

To resolve this case, we proceed exactly as in Step 2 to obtain

$$u(x,y) \,=\, E_y^x[\tau_U = \zeta;\, e_q(\tau_U)] \,+\, E_y^x[\tau_U < \zeta;\, e_q(\tau_U)\, u(Y_{\tau_U}, y)]\,.$$

If $\tau_U < \zeta$ then $d(Y_{\tau_U}, D^c) > 3\delta$ which reduces the proof to the case $x \in D_\delta$, $y \in D_\delta^c$. By Step 2 and symmetry of u, we obtain the conclusion. This completes the proof of the theorem. □

Concluding remarks. We like to note that in the proof of **CGT** for Δ ([62]) one first considers conditioning by the boundary (Martin kernel). The boundedness of the conditional gauge for interior points of the domain is then obtained as an easy corollary by considering the further evolution of the Brownian motion till it hits the boundary. For $\Delta^{\alpha/2}$ $(0 < \alpha < 2)$, due to the jumps of the process, the more important is the boundedness of the conditional gauge in the interior of D (conditioning by the Green function), and it cannot be obtained easily from the boundary behavior of the conditional gauge. Instead, we obtain the boundedness of the conditional gauge on the boundary as an easy corollary by approximation from within the domain.

We should also observe that the recent advances in the understanding of the role of the **3G** Theorem allow for *analytic* proofs of **CGT** in this and other settings, by using the perturbation series. We refer the reader to [85] for details, and to [84] for the general perspective on the role of **BHP** in proving **3G**. Such an approach has the advantage of being more explicit, algebraic, and *discrete*, paralleling the definition of the exponential function in terms of the power series, rather than differential equations. On the other hand, the probabilistic setting allows for intrinsic interpretations and verbalization of the proofs in terms of mass and trajectories of stochastic processes. The authors may only wonder which of these two approaches is more the reality, and which is more the language.

We want to conclude our discussion by mentioning a few directions of further research. First, it seems important to obtain an approximate factorization of the Green function for general (non-Lipschitz) domains, by using [38]. Second, it is of interest to study the asymptotics of the Martin kernel for narrow cones, and use the setup of [5] to complete the results of [111]. Third, it is of paramount importance to give sharp estimates for the *transition density of the killed process*. Fourth, it seems important to generalize the results discussed above to other stable Lévy processes ([40]), to more general jump type Markov processes, and to more general additive perturbations of their generators ([36, 52, 102, 82, 83]).

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