2

Quantities of Spectroscopy

2.1 Radiometric Quantities

The total energy emitted by a plasma as electromagnetic radiation per unit time is called its *radiative loss* and plays a crucial role in all power balance considerations. As a physical quantity it is a *radiant flux* Φ (through the surface of the plasma), and its unit is watt (W). It is also called *radiant power*. *Radiant flux density* ϕ refers to the flux per unit area $\phi = \mathrm{d}\Phi/\mathrm{d}A$ with the unit $\mathrm{W\,m^{-2}}$, irrespective of whether the radiation is emitted from an area, crosses an imaginary surface in space, or falls onto an area A. In this latter case, it is customary to call this flux density at the surface *irradiance* E. The energy deposited per unit area during a given time is the *fluence* $H = \int E\mathrm{d}t$, with the unit $\mathrm{J\,m^{-2}}$.

The SI quantity *radiance* L is defined as the radiant flux per unit projected area per solid angle Ω [18]. Figure 2.1 illustrates the geometry, where θ is the angle between the normal of the surface element $\mathrm{d}A$ and the direction of the radiation.

$$L = \frac{\mathrm{d}^2\Phi}{\mathrm{d}A\ \cos\theta\ \mathrm{d}\Omega} \tag{2.1}$$

Its unit is $\mathrm{W\,m^{-2}\,sr^{-1}}$. Again, it also applies to real and imaginary surfaces. Although the radiance is the correct SI unit, the rather familiar word *intensity* is still being used instead in much of the plasma spectroscopy literature. Confusion is unavoidable when authors use intensity for the radiation flux density. It is advisable, therefore, to check the unit whenever "intensity" is encountered. The official SI quantity *radiant intensity* I is the flux per solid angle emitted from a source which makes it a property of a source rather than of a radiating surface: $I = \mathrm{d}\Phi/\mathrm{d}\Omega$, the unit being $\mathrm{W\,sr^{-1}}$. In the field of physical optics the word "intensity" refers to the magnitude of the Poynting vector.

For completeness the *radiant energy density* u (unit $\mathrm{J\,m^{-3}}$) should be mentioned, which is usually employed in atomic and thermodynamic

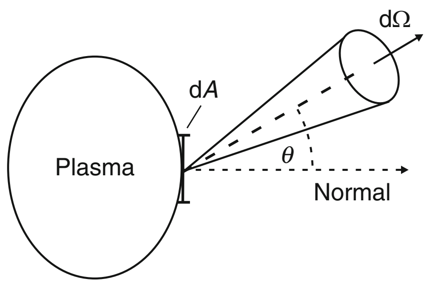

Fig. 2.1. Definition of radiance

considerations. In case of isotropic radiation, for example, it is given by relation (a), and for collimated radiation in the other limit by relation (b):

$$(a) \quad u = \frac{4\pi}{c}\, L \qquad (b) \quad u = \frac{1}{c}\, L \tag{2.2}$$

The *local* emission at the position $\boldsymbol{r}$ in the plasma is characterized by the *emission coefficient* $\varepsilon(\boldsymbol{r})$:

$$\varepsilon(\boldsymbol{r}) = \frac{\mathrm{d}^2\Phi(\boldsymbol{r})}{\mathrm{d}V\ \mathrm{d}\Omega}\ , \tag{2.3}$$

where $\mathrm{d}\Phi(\boldsymbol{r})$ is the radiant flux from the volume element $\mathrm{d}V$ at $\boldsymbol{r}$. This assumes that the emission is isotropic. It can, of course, also be non-isotropic, when a specific direction is given by electric or magnetic fields or by fast particle beams. The unit of the emission coefficient is $\mathrm{W\,m^{-3}\,sr^{-1}}$.

With the exception of bolometric measurements usually *spectral quantities* are measured and they are denoted by the pertinent subscript λ (or ν or ω). They are *derivative quantities* and they yield the total quantity when integrated over wavelength or frequency or an interval, respectively. For example:

$$\varepsilon(\boldsymbol{r}) = \int_o^\infty \varepsilon_\lambda(\boldsymbol{r},\lambda)\,\mathrm{d}\lambda \quad \text{or} \quad \varepsilon(\boldsymbol{r},\lambda_0,\Delta\lambda) = \int_{\lambda_0-\Delta\lambda/2}^{\lambda_0+\Delta\lambda/2} \varepsilon_\lambda(\boldsymbol{r},\lambda)\,\mathrm{d}\lambda\,. \tag{2.4}$$

2.2 Measured Quantities

In the most straightforward case radiation from a plasma (for simplicity we take a plane surface with surface element $\mathrm{d}A_p$) falls on a detector with surface element $\mathrm{d}A_\mathrm{d}$ positioned at a distance s, see Fig. 2.2. The quantity thus recorded is a flux Φ given according to (2.1) by

$$\Phi = \iint L\ \cos\theta\ \mathrm{d}A_p\ \mathrm{d}\Omega = \iint L\ \cos\theta\ \mathrm{d}A_p\ \frac{\cos\beta\ \mathrm{d}A_\mathrm{d}}{s^2}\ , \tag{2.5}$$

where integration is over plasma and detector surfaces. For small angles this flux reduces to

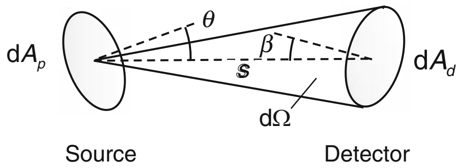

Fig. 2.2. Geometry of surface elements of source and detector

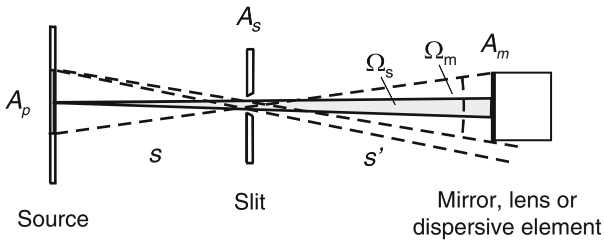

Fig. 2.3. Measured radiation

$$\Phi = L\ A_p\,\Omega_\mathrm{d} = L\ \frac{A_p\ A_\mathrm{d}}{s^2}\ . \tag{2.6}$$

Ω_d is the solid angle subtended by the detector area A_d.

For spectrally resolved measurements the radiation flux is limited by the area A_s of the entrance slit of a spectrographic instrument and by the size of a mirror, a lens, a dispersing element, or a real or virtual aperture (cross section A_m). Figure 2.3 illustrates this geometry, and it is obvious that the cross section A_m determines the area A_p of the source surface from which radiation is collected through the slit. Thus, the flux can be written

$$\Phi = L\ A_p\,\Omega_s = L\ A_p \frac{A_s}{s^2} = L\ A_s\ \frac{A_m}{s'^2} = L\ A_s \Omega_m\ . \tag{2.7}$$

The quantity $A_s\Omega_m$ is called the *throughput* or *étendue* of the instrument (it is conserved as we will see later), and the flux Φ is simply obtained as if the emitting surface with radiance L were in the plane of the entrance slit. (Keep in mind that this only holds if the emitting surface is larger than A_p.)

This is readily exploited in quantitative measurements: a standard radiator of known radiance L_0 is placed close to the entrance slit, or in principle at any distance where its size is still sufficient to fill the solid angle Ω_m. The flux $\Phi_\lambda(\lambda)\Delta\lambda = L_{0\lambda}(\lambda)\Delta\lambda\ A_s\Omega_m$ over a spectral interval $\Delta\lambda$ will produce a signal S_0 at the exit of a detector. Keeping all settings constant, the respective flux in the same spectral interval from an unknown radiator may give the signal S_x, hence

$$L_{x\lambda}(\lambda) = \frac{S_x}{S_0}\ L_{0\lambda}(\lambda)\ . \tag{2.8}$$

This is an extremely simple relation: the ratio of the signals yields the unknown spectral radiance $L_{x\lambda}(\lambda)$, and it is this quantity which is obtained in the measurement.

A plasma, on the other hand, emits throughout its volume, but the above considerations may also be applied directly, if we replace the radiating surface of Fig. 2.3 by a slice of plasma of thickness ds. $A_p(s)$ just has to be smaller than the cross section of the plasma at the distance s. With *no absorption* within the plasma (2.3) and (2.7) yield with $\mathrm{d}V = \mathrm{d}A_p(s)\,\mathrm{d}s$

$$\Phi = \iiint \varepsilon(s)\;\mathrm{d}A_p(s)\;\mathrm{d}s\;\mathrm{d}\Omega = \left(\int \varepsilon(s)\mathrm{d}s\right)\; A_p\Omega_s \;= L\;A_s\Omega_m \tag{2.9}$$

$$\text{with} \qquad L = \int \varepsilon(s)\mathrm{d}s\,, \tag{2.10}$$

where L is the radiance at the surface of the plasma and will also be again the effective radiance in the plane of the entrance slit.

For a correctly designed optical system, reduced to a lens of diameter D in Fig. 2.4, we have for object and image size y and y' with object and image distance s and s', respectively,

$$\frac{y/2}{y'/2} = \frac{s}{s'}\;, \tag{2.11}$$

and hence

$$\frac{y^2}{s^2}\,D^2 = \frac{y'^2}{s'^2}\,D^2 \qquad \Longrightarrow \qquad A_p\Omega_p = {A_p}'{\Omega_p}'\;. \tag{2.12}$$

The throughput is conserved. Since also the radiant flux Φ remains constant, (2.9) and (2.12) yield directly the well known law

$$L = L' \tag{2.13}$$

i.e., the radiance is an invariant of an optical system with no absorption.

In case the cross section of the plasma is smaller than the area A_p at the position s (Fig. 2.3), the full throughput is not utilized. In this case, the plasma should be imaged onto the entrance slit employing a lens or a spherical mirror. These considerations also apply when a large plasma is to be investigated spatially resolved. Figure 2.5 illustrates such an arrangement. The lens

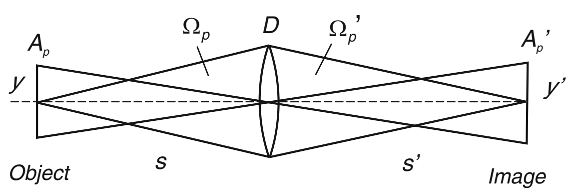

Fig. 2.4. Imaging by a lens

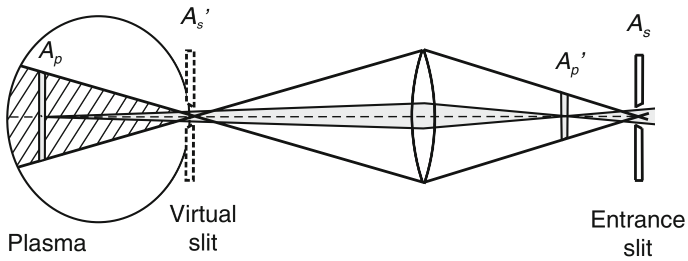

Fig. 2.5. Spatially resolved measurements

produces a virtual image (area A'_s) of the entrance slit, and only radiation from the shaded region is collected by the lens and the virtual slit and hence also by the entrance slit. This geometry corresponds exactly to that of Fig. 2.3 with a radiance $L = \int \varepsilon \mathrm{d}s$ at the position of the virtual slit and according to (2.13) also at the entrance slit: the quantity obtained is the *local* radiance at the surface of the plasma, given by the integral of the emission coefficient ε along the line of sight. One must, of course, select a lens of sufficient diameter, that the full throughput is used in the spectrographic instrument.

2.3 Local Quantities

2.3.1 Homogeneous Plasmas

The discussions in the preceding sections showed that the quantity measured is always the local radiance at the surface of the plasma, which is given by integration of the emission coefficient along the line of sight when absorption of radiation within the plasma is not important. Hence, only an average emission $\overline{\varepsilon}$ can be quoted

$$\overline{\varepsilon} = \frac{L}{s_2 - s_1} , \tag{2.14}$$

where $s_2 - s_1$ is the depth of the plasma along the line of sight. For a homogeneous plasma this corresponds to the true emission coefficient ε.

2.3.2 Axially and Spherically Symmetric Plasmas

In cases of axially symmetric plasma columns (cylinders), the derivation of local emission coefficients $\varepsilon(r)$ is possible if the radiance of the plasma is measured over the cross section of the column. Integration along a chord parallel to the x-axis yields, after substituting $x^2 = r^2 - y^2$ (Fig. 2.6), the one-dimensional profile of the radiance

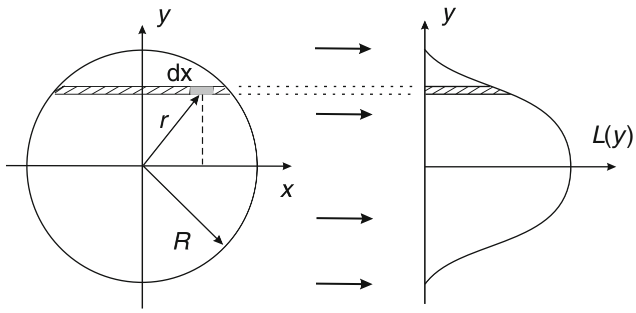

Fig. 2.6. Radiance parallel to the y-axis of a cylindrically symmetric plasma column

$$L(y) = 2\int_0^{x_{max}} \varepsilon(r)\,\mathrm{d}x = 2\int_0^{\sqrt{R^2-y^2}} \varepsilon(r)\,\mathrm{d}x = 2\int_y^R \frac{\varepsilon(r)\,r\,\mathrm{d}r}{\sqrt{r^2-y^2}}\,. \tag{2.15}$$

Equation (2.15) is of the Abel type, and the local emission coefficient $\varepsilon(r)$ is recovered by the Abel inversion which can be written analytically as

$$\varepsilon(r) = -\frac{1}{\pi}\int_r^R \frac{\mathrm{d}L(y)}{\mathrm{d}y}\frac{\mathrm{d}y}{\sqrt{y^2-r^2}}\,. \tag{2.16}$$

Since the spatial derivative $\mathrm{d}L/\mathrm{d}y$ enters the inversion, it is obvious that the results are very sensitive to noise and errors in the measured radiance $L(y)$. Unfortunately, the errors in the derivative accumulate over the integration interval $[r,R]$, and the resulting uncertainty is most severe on the axis $r = 0$. Hence, appropriate smoothing of the data can be essential for the quality of the inversion. An inversion algorithm without differentiation has been proposed by [19], and other improved inversion methods are mentioned in [8]. In cases of strongly spatially structured emission profiles, the finite solid angle of the radiation collecting optics has to be taken into account [20].

The two-dimensional radiance $L(y, z)$ in the y–z plane is readily analyzed only in the case of spherically symmetric plasmas. Because of the symmetry it usually suffices to analyze the radiance $L(y, 0)$ obtained by integration in the plane ($z = 0$), to which the above cylindrical solution can be applied. Physically, it corresponds to a one-dimensional analysis employing a narrow slit in front of the plasma.

In the X-ray region, the pinhole camera is the standard imaging system. It is simple, and the typical arrangement is shown in Fig. 2.7. Integration of the emission coefficient $\varepsilon(r)$ is now along differing tilted chords. The chordal heights p, or impact parameters, are calculated from the known geometric dimensions of the system, and the recorded radiances can then readily be transformed to the respective radiance distribution of parallel recording needed for the standard Abel inversion.

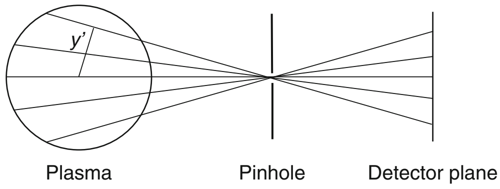

Fig. 2.7. Schematic arrangement of a pinhole camera

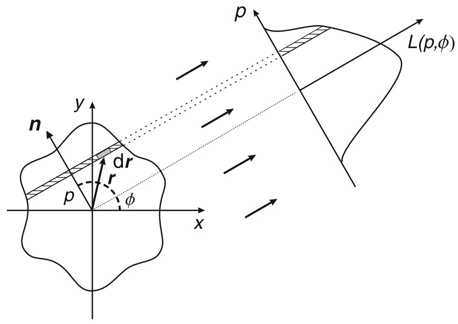

Fig. 2.8. Coordinate system for the Radon transform

2.3.3 Plasmas Without Symmetry

One special case of nonaxisymmetry exists in tokamaks which employ a limiter and have a circular poloidal cross-section: the magnetic flux surfaces and hence the density contour lines are very close to a set of nested eccentric circles, the shift of the circle centers being known as the Shafranov shift Δ. With the transformation $r^2 = (x+\Delta)^2+y^2$, Abel inversion as discussed in Sect. 2.3.2 can again be executed, and specific methods have been developed [21–23]. In [24] the technique was extended to plasmas of elliptic shape which is transformed mathematically to a concentric circular shape for the Abel inversion.

The mathematical problems associated with the general reconstruction of the local emission $\varepsilon(\boldsymbol{r})$ throughout the plasma are essentially identical to those encountered in computer-aided tomography, drawing on the same mathematical techniques [25]. The radiance of the plasma has to be recorded in numerous directions, and the Radon transformation and its inverse then provide the mathematical basis for the reconstruction. We shall consider here only the two-dimensional Radon transform, the extension to three dimensions being straightforward. Figure 2.8 illustrates the geometry and the definition of the coordinates. Integration is along straight lines given by ($\boldsymbol{n} = \{\cos\phi, \sin\phi\}$ is a unit vector perpendicular to these lines)

$$\boldsymbol{r}\cdot\boldsymbol{n} - p = x\,\cos\phi + y\,\sin\phi - p = 0 \tag{2.17}$$

yielding the radiance

$$L(p,\phi) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varepsilon(x,y)\,\delta(x\cos\phi + y\sin\phi - p)\,\mathrm{d}x\mathrm{d}y\,, \tag{2.18}$$

where $\delta(.)$ is the Dirac delta function. Because of the symmetry $L(p,\phi) = L(-p,\phi+\pi)$ only angles 0 to π are needed. Mathematically, $L(p,\phi)$ is the Radon transform of $\varepsilon(x,y)$

$$L(p,\phi) = \mathcal{R}_2\{\varepsilon(x,y)\}\,. \tag{2.19}$$

For the execution of the inverse Radon transformation to obtain the emission $\varepsilon(x,y)$ within the plasma,

$$\varepsilon(x,y) = \mathcal{R}_2^{-1}\{L(p,\phi)\}\,, \tag{2.20}$$

a number of algorithms and numerous techniques are available, see for example [26]. The quality of the obtained reconstruction certainly depends on the number of views of the plasma (different angle ϕ), or precisely on the number of $L(p,\phi)$ values. Unfortunately, these are usually limited due to experimental constraints, as for example by the plasma vessel and its ports. The reconstruction is facilitated and improved, if additional information on the plasma shape or its overall structure is available and can be utilized.

2.4 Radiance of Plasmas with Re-Absorption

We now consider that some radiation is re-absorbed within the plasma and introduce a wavelength-dependent absorption coefficient $\kappa(x,\lambda)$. It is defined such that in the direction x the change of the spectral radiance $L_\lambda(x,\lambda)$ by a thin slab due to absorption is given by

$$\mathrm{d}L_\lambda(x,\lambda) = -\kappa(x,\lambda)\,L_\lambda(x,\lambda)\,\mathrm{d}x\,. \tag{2.21}$$

The unit of $\kappa(x,\lambda)$ is m^{-1}. The designation $\kappa_\lambda(x)$, which is sometimes found in the literature, should not be used since a subscript usually indicates differentiation. If we add the emission by the plasma slab (2.10), the total change of the radiance becomes

$$\mathrm{d}L_\lambda(x,\lambda) = \varepsilon_\lambda(x,\lambda)\,\mathrm{d}x - \kappa(x,\lambda)\,L_\lambda(x,\lambda)\,\mathrm{d}x\,. \tag{2.22}$$

Figure 2.9 illustrates the geometry. Division by dx leads to

$$\frac{\mathrm{d}L_\lambda(x,\lambda)}{\mathrm{d}x} = \varepsilon_\lambda(x,\lambda) - \kappa(x,\lambda)\,L_\lambda(x,\lambda)\,. \tag{2.23}$$

known as *equation of radiative transfer*.

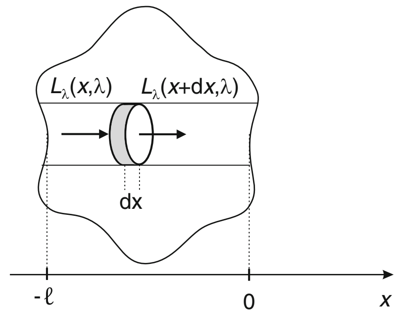

Fig. 2.9. Geometry of radiation transport

It is customary to introduce the *optical depth* τ measured back into the plasma by

$$\mathrm{d}\tau = -\kappa(x,\lambda)\,\mathrm{d}x\,. \tag{2.24}$$

As new variable τ replaces x by

$$\tau(x,\lambda) = -\int_0^x \kappa(x',\lambda)\,\mathrm{d}x' \tag{2.25}$$

with $\tau = 0$ at the plasma surface ($x = 0$). The magnitude of the total optical depth (also called as optical thickness) $\tau(-\ell,\lambda)$ determines the significance of radiative transport at the wavelength λ. Equation (2.23) can now be written as

$$\frac{\mathrm{d}L_\lambda(x,\lambda)}{\mathrm{d}\tau} = L_\lambda(x,\lambda) - \frac{\varepsilon_\lambda(x,\lambda)}{\kappa(x,\lambda)} = L_\lambda(x,\lambda) - S_\lambda(x,\lambda). \tag{2.26}$$

Integration is along the line of sight (or more precisely, along an actual ray trajectory) into the plasma.

The ratio $\varepsilon_\lambda(x,\lambda)/\kappa(x,\lambda) = S_\lambda(x,\lambda)$ is known as *source function.* In calculating the source function in the most general cases, induced emission and scattering of radiation have to be considered. Fortunately, in many diagnostic applications scattering does not play a role.

The solution of (2.26) for the case of no radiation incident on the plasma at $x = -\ell$ gives the radiance at the plasma surface

$$L_\lambda(0,\lambda) = \int_0^{\tau(-\ell,\lambda)} S_\lambda(\tau,\lambda)\,\mathrm{e}^{-\tau}\,\mathrm{d}\tau\,. \tag{2.27}$$

The limit (2.10) of an optically thin plasma ($\tau \ll 1$) is recovered by expanding $\mathrm{e}^{-\tau}$ and keeping only the first term:

$$L_\lambda(0,\lambda) \cong \int_0^{\tau(-\ell,\lambda)} S_\lambda(\tau,\lambda)\,\mathrm{d}\tau\; = \int_{-\ell}^{0} \varepsilon_\lambda(x,\lambda)\,\mathrm{d}x\,. \tag{2.28}$$

An analytic solution of (2.26) is readily given for a homogeneous plasma, $S_\lambda(x,\lambda) = S_\lambda(\lambda)$:

$$L_\lambda(0,\lambda) = S_\lambda(\lambda)\left[1 - \mathrm{e}^{-\tau(-\ell,\lambda)}\right] . \tag{2.29}$$

For optically thick, $\tau(-\ell,\lambda) \gg 1$, inhomogeneous plasmas we make the transition $\tau(-\ell,\lambda) \to \infty$ and integrate (2.26) by parts:

$$\begin{aligned} L_\lambda(0,\lambda) &= -\int_0^\infty S_\lambda(\tau,\lambda)\,\mathrm{e}^{-\tau}\mathrm{d}\tau \\ &= S_\lambda(\tau=0,\lambda) + \frac{S_\lambda(\tau,\lambda)}{\mathrm{d}\tau}\Big|_{\tau=0} + \frac{\mathrm{d}^2 S_\lambda(\tau,\lambda)}{\mathrm{d}\tau^2}\Big|_{\tau=0} + \cdots . \end{aligned} \tag{2.30}$$

Alternatively, one can develop $S_\lambda(\tau,\lambda)$ as a MacLaurin power series and obtains [27]

$$S_\lambda(\tau,\lambda) = S_\lambda(\tau=0,\lambda) + \frac{S_\lambda(\tau,\lambda)}{\mathrm{d}\tau}\Big|_{\tau=0}(\tau-0) + \frac{1}{2}\frac{\mathrm{d}^2 S_\lambda(\tau,\lambda)}{\mathrm{d}\tau^2}\Big|_{\tau=0}(\tau-0)^2 + \cdots . \tag{2.31}$$

Provided the higher terms do not contribute too much, the comparison of both equations yields

$$L_\lambda(0,\lambda) \simeq S_\lambda(\tau=1,\lambda) , \tag{2.32}$$

i.e., the spectral radiance is about equal to the source function at the optical depth $\tau(\lambda) \simeq 1$. Since τ varies with wavelength, the observed spectral radiance corresponds to different positions in the plasma. In astronomy, (2.32) is known as Eddington–Barbier relation.

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Introduction to Plasma Spectroscopy
Kunze, H.-J.
2009, XII, 242 p. 87 illus., Hardcover
ISBN: 978-3-642-02232-6