Chapter 3
Gradient and Divergence Operators

In this chapter we construct an abstract framework for stochastic analysis in continuous time with respect to a normal martingale $(M_t)_{t\in\mathbb{R}_+}$, using the construction of stochastic calculus presented in Section 2. In particular we identify some minimal properties that should be satisfied in order to connect a gradient and a divergence operator to stochastic integration, and to apply them to the predictable representation of random variables. Some applications, such as logarithmic Sobolev and deviation inequalities, are formulated in this general setting. In the next chapters we will examine concrete examples of operators that can be included in this framework, in particular when $(M_t)_{t\in\mathbb{R}_+}$ is a Brownian motion or a compensated Poisson process.

3.1 Definition and Closability

In this chapter, $(M_t)_{t\in\mathbb{R}_+}$ denotes a normal martingale as considered in Chapter 2. We let $\mathcal{S}$, $\mathcal{U}$, and $\mathcal{P}$ denote the spaces of random variables, simple processes and simple predictable processes introduced in Definition 2.5.2, and we note that $\mathcal{S}$ is dense in $L^2(\Omega)$ by Definition 2.5.2 and $\mathcal{U}$, $\mathcal{P}$ are dense in $L^2(\Omega\times\mathbb{R}_+)$ respectively from Proposition 2.5.3.
Let now

$$D : L^2(\Omega, d\mathbb{P}) \to L^2(\Omega\times\mathbb{R}_+, d\mathbb{P}\times dt)$$

and

$$\delta : L^2(\Omega\times\mathbb{R}_+, d\mathbb{P}\times dt) \to L^2(\Omega, d\mathbb{P})$$

be linear operators defined respectively on $\mathcal{S}$ and $\mathcal{U}$. We assume that the following duality relation holds.

Assumption 3.1.1. (Duality relation) The operators D and δ satisfy the relation

$$\mathbb{E}[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}] = \mathbb{E}[F\delta(u)], \qquad F\in\mathcal{S},\ u\in\mathcal{U}. \tag{3.1.1}$$

Note that $D\mathbf{1} = 0$ is equivalent to $\mathbb{E}[\delta(u)] = 0$, for all $u\in\mathcal{U}$. In the next proposition we use the notion of closability for operators in normed linear

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spaces, whose definition is recalled in Section 9.8 of the Appendix. The next proposition is actually a general result on the closability of the adjoint of a densely defined operator.

Proposition 3.1.2. *The duality assumption 3.1.1 implies that D and δ are closable.*

Proof. If $(F_n)_{n\in\mathbb{N}}$ converges to 0 in $L^2(\Omega)$ and $(DF_n)_{n\in\mathbb{N}}$ converges to $U \in L^2(\Omega\times\mathbb{R}_+)$, the relation

$$\mathbb{E}[\langle DF_n, u\rangle_{L^2(\mathbb{R}_+)}] = \mathbb{E}[F_n\delta(u)], \qquad u \in \mathcal{U},$$

implies

$$\begin{aligned}
&|\,\mathbb{E}[\langle U, u\rangle_{L^2(\mathbb{R}_+)}]|\\
&\leq |\,\mathbb{E}[\langle DF_n, u\rangle_{L^2(\mathbb{R}_+)}] - \mathbb{E}[\langle U, u\rangle_{L^2(\mathbb{R}_+)}]| + |\,\mathbb{E}[\langle DF_n, u\rangle_{L^2(\mathbb{R}_+)}|\\
&= |\,\mathbb{E}[\langle DF_n - U, u\rangle_{L^2(\mathbb{R}_+)}]| + |\,\mathbb{E}[F_n\delta(u)]|\\
&\leq \|DF_n - U\|_{L^2(\Omega\times\mathbb{R}_+)}\|u\|_{L^2(\Omega\times\mathbb{R}_+)} + \|F_n\|_{L^2(\Omega)}\|\delta(u)\|_{L^2(\Omega)},
\end{aligned}$$

hence as n goes to infinity we get $\mathbb{E}[\langle U, u\rangle_{L^2(\mathbb{R}_+)}] = 0$, $u \in \mathcal{U}$, i.e. $U = 0$ since $\mathcal{U}$ is dense in $L^2(\Omega\times\mathbb{R}_+)$. The proof of closability of δ is similar: if $(u_n)_{n\in\mathbb{N}}$ converges to 0 in $L^2(\Omega\times\mathbb{R}_+)$ and $(\delta(u_n))_{n\in\mathbb{N}}$ converges to $F \in L^2(\Omega)$, we have for all $G \in \mathcal{S}$:

$$\begin{aligned}
|\,\mathbb{E}[FG]| &\leq |\,\mathbb{E}[\langle DG, u_n\rangle_{L^2(\mathbb{R}_+)}] - \mathbb{E}[FG]| + |\,\mathbb{E}[\langle DG, u_n\rangle_{L^2(\mathbb{R}_+)}]|\\
&= |\,\mathbb{E}[G(\delta(u_n) - F)]| + |\,\mathbb{E}[\langle DG, u_n\rangle_{L^2(\Omega\times\mathbb{R}_+)}]|\\
&\leq \|\delta(u_n) - F\|_{L^2(\Omega)}\|G\|_{L^2(\Omega)} + \|u_n\|_{L^2(\Omega\times\mathbb{R}_+)}\|DG\|_{L^2(\Omega\times\mathbb{R}_+)},
\end{aligned}$$

hence $\mathbb{E}[FG] = 0$, $G \in \mathcal{S}$, i.e. $F = 0$ since $\mathcal{S}$ is dense in $L^2(\Omega)$. □

From the above proposition these operators are respectively extended to their closed domains $\mathrm{Dom}\,(D)$ and $\mathrm{Dom}\,(\delta)$, and for simplicity their extensions will remain denoted by D and δ.

3.2 Clark Formula and Predictable Representation

In this section we study the connection between D, δ, and the predictable representation of random variables using stochastic integrals.

Assumption 3.2.1. (Clark formula). Every $F \in \mathcal{S}$ can be represented as

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_tF|\mathcal{F}_t]dM_t. \qquad (3.2.1)$$

This assumption is connected to the predictable representation property for the martingale $(M_t)_{t\in\mathbb{R}_+}$, cf. Proposition 3.2.8 and Proposition 3.2.6 below.

Definition 3.2.2. *Given $k \geq 1$, let $\mathbb{D}_{2,k}([a,\infty))$, $a > 0$, denote the completion of $\mathcal{S}$ under the norm*

$$\|F\|_{\mathbb{D}_{2,k}([a,\infty))} = \|F\|_{L^2(\Omega)} + \sum_{i=1}^{k} \left(\int_a^\infty |D_t^i F|^2 dt \right)^{1/2},$$

where $D_t^i = D_t \cdots D_t$ denotes the i-th iterated power of D_t, $i \geq 1$.

In other words, for any $F \in \mathbb{D}_{2,k}([a,\infty))$, the process $(D_t F)_{t\in[a,\infty)}$ exists in $L^2(\Omega \times [a,\infty))$. Clearly we have $\mathrm{Dom}\,(D) = \mathbb{D}_{2,1}([0,\infty))$. Under the Clark formula Assumption 3.2.1, a representation result for $F \in \mathbb{D}_{2,1}([a,\infty))$ can be stated as a consequence of the Clark formula:

Proposition 3.2.3. *For all $t \in \mathbb{R}_+ > 0$ and $F \in \mathbb{D}_{2,1}([t,\infty))$ we have*

$$\mathbb{E}[F|\mathcal{F}_t] = \mathbb{E}[F] + \int_0^t \mathbb{E}[D_s F|\mathcal{F}_s] dM_s, \qquad (3.2.2)$$

and

$$F = \mathbb{E}[F|\mathcal{F}_t] + \int_t^\infty \mathbb{E}[D_s F|\mathcal{F}_s] dM_s, \qquad t \in \mathbb{R}_+. \qquad (3.2.3)$$

Proof. This is a direct consequence of (3.2.1) and Proposition 2.5.7. □

By uniqueness of the predictable representation of $F \in L^2(\Omega)$, an expression of the form

$$F = c + \int_0^\infty u_t dM_t$$

where $c \in \mathbb{R}$ and $(u_t)_{t\in\mathbb{R}_+}$ is adapted and square-integrable, implies

$$u_t = \mathbb{E}[D_t F|\mathcal{F}_t], \qquad dt \times d\mathbb{P} - a.e.$$

The covariance identity proved in the next lemma is a consequence of Proposition 3.2.3 and the Itô isometry (2.5.5).

Lemma 3.2.4. *For all $t \in \mathbb{R}_+$ and $F \in \mathbb{D}_{2,1}([t,\infty))$ we have*

$$\mathbb{E}[(\mathbb{E}[F|\mathcal{F}_t])^2] = (\mathbb{E}[F])^2 + \mathbb{E}\left[\int_0^t (\mathbb{E}[D_s F|\mathcal{F}_s])^2 ds \right] \qquad (3.2.4)$$

$$= \mathbb{E}[F^2] - \mathbb{E}\left[\int_t^\infty (\mathbb{E}[D_s F|\mathcal{F}_s])^2 ds \right]. \qquad (3.2.5)$$

Proof. From the Itô isometry (2.5.4) and Relation 3.2.2 we have

$$\begin{aligned}\mathbb{E}[(\mathbb{E}[F|\mathcal{F}_t])^2] &= \mathbb{E}\left[\left(\mathbb{E}[F] + \int_0^t \mathbb{E}[D_sF|\mathcal{F}_s]dM_s\right)^2\right] \\ &= (\mathbb{E}[F])^2 + \mathbb{E}\left[\left(\int_0^t \mathbb{E}[D_sF|\mathcal{F}_s]dM_s\right)^2\right] \\ &= (\mathbb{E}[F])^2 + \mathbb{E}\left[\int_0^t (\mathbb{E}[D_sF|\mathcal{F}_s])^2 ds\right], \qquad t \in \mathbb{R}_+,\end{aligned}$$

which shows (3.2.4). Next, concerning (3.2.5) we have

$$\begin{aligned}\mathbb{E}[F^2] &= \mathbb{E}\left[\left(\mathbb{E}[F|\mathcal{F}_t] + \int_t^\infty \mathbb{E}[D_sF|\mathcal{F}_s]dM_s\right)^2\right] \\ &= \mathbb{E}\left[(\mathbb{E}[F|\mathcal{F}_t])^2\right] + \mathbb{E}\left[\mathbb{E}[F|\mathcal{F}_t]\int_t^\infty \mathbb{E}[D_sF|\mathcal{F}_s]dM_s\right] \\ &\quad + \mathbb{E}\left[\left(\int_t^\infty \mathbb{E}[D_sF|\mathcal{F}_s]dM_s\right)^2\right] \\ &= \mathbb{E}\left[(\mathbb{E}[F|\mathcal{F}_t])^2\right] + \mathbb{E}\left[\int_t^\infty \mathbb{E}[F|\mathcal{F}_t]\,\mathbb{E}[D_sF|\mathcal{F}_s]dM_s\right] \\ &\quad + \mathbb{E}\left[\int_t^\infty (\mathbb{E}[D_sF|\mathcal{F}_s])^2\,ds\right] \\ &= \mathbb{E}\left[(\mathbb{E}[F|\mathcal{F}_t])^2\right] + \mathbb{E}\left[\int_t^\infty (\mathbb{E}[D_sF|\mathcal{F}_s])^2\,ds\right], \qquad t \in \mathbb{R}_+,\end{aligned}$$

since from (2.5.7) the Itô stochastic integral has expectation 0, which shows (3.2.5). □

The next remark applies in general to any mapping sending a random variable to the process involved in its predictable representation with respect to a normal martingale.

Lemma 3.2.5. *The operator*

$$F \mapsto (\mathbb{E}[D_tF|\mathcal{F}_t])_{t\in\mathbb{R}_+}$$

defined on $\mathcal{S}$ *extends to a continuous operator from* $L^2(\Omega)$ *into* $L^2(\Omega \times \mathbb{R}_+)$.

Proof. This follows from the bound

$$\begin{aligned}\|\,\mathbb{E}[D_\cdot F|\mathcal{F}_\cdot]\|^2_{L^2(\Omega\times\mathbb{R}_+)} &= \|F\|^2_{L^2(\Omega)} - (\mathbb{E}[F])^2 \\ &\leq \|F\|^2_{L^2(\Omega)},\end{aligned}$$

that follows from Relation (3.2.4) with $t = 0$. □

As a consequence of Lemma 3.2.5, the Clark formula can be extended in Proposition 3.2.6 below as in the discrete case, cf. Proposition 1.7.2.

Proposition 3.2.6. *The Clark formula*

$$F = \mathbb{E}[F] + \int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dM_t.$$

can be extended to all F in $L^2(\Omega)$.

Similarly, the results of Proposition 3.2.3 and Lemma 3.2.4 also extend to $F \in L^2(\Omega)$.
The Clark representation formula naturally implies a Poincaré type inequality.

Proposition 3.2.7. *For all $F \in \mathrm{Dom}\,(D)$ we have*

$$\mathrm{Var}\,(F) \leq \|DF\|^2_{L^2(\Omega \times \mathbb{R}_+)}.$$

Proof. From Lemma 3.2.4 we have

$$\begin{aligned}
\mathrm{Var}\,(F) &= \mathbb{E}[|F - E[F]|^2] \\
&= \mathbb{E}\left[\left(\int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dM_t\right)^2\right] \\
&= \mathbb{E}\left[\int_0^\infty (\mathbb{E}[D_t F | \mathcal{F}_t])^2 \, dt\right] \\
&\leq \mathbb{E}\left[\int_0^\infty \mathbb{E}[|D_t F|^2 | \mathcal{F}_t] dt\right] \\
&\leq \int_0^\infty \mathbb{E}\left[\mathbb{E}[|D_t F|^2 | \mathcal{F}_t]\right] dt \\
&\leq \int_0^\infty \mathbb{E}\left[|D_t F|^2\right] dt \\
&\leq \mathbb{E}\left[\int_0^\infty |D_t F|^2 dt\right],
\end{aligned}$$

hence the conclusion. □

Since the space $\mathcal{S}$ is dense in $L^2(\Omega)$ by Definition 2.5.2, the Clark formula Assumption 3.2.1 implies the predictable representation property of Definition 2.6.1 for $(M_t)_{t \in \mathbb{R}_+}$ as a consequence of the next corollary.

Corollary 3.2.8. *Under the Clark formula Assumption 3.2.1 the martingale $(M_t)_{t \in \mathbb{R}_+}$ has the predictable representation property.*

Proof. Definition 2.6.1 is satisfied because $\mathcal{S}$ is dense in $L^2(\Omega)$ and the process $(\mathbb{E}[D_t F \mid \mathcal{F}_t])_{t \in \mathbb{R}_+}$ in (3.2.1) can be approximated by a sequence in $\mathcal{P}$ from Proposition 2.5.3.

Alternatively, one may use Proposition 2.6.2 and proceed as follows. Consider a square-integrable martingale $(X_t)_{t\in\mathbb{R}_+}$ with respect to $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and let

$$u_s = \mathbb{E}[D_s X_{n+1}|\mathcal{F}_n], \qquad n \le s < n+1, \qquad t \in \mathbb{R}_+.$$

Then $(u_t)_{t\in\mathbb{R}_+}$ is an adapted process such that $u\mathbf{1}_{[0,T]} \in L^2(\Omega \times \mathbb{R}_+)$ for all $T > 0$, and the Clark formula Assumption 3.2.1 and Proposition 3.2.6 imply

$$\begin{aligned}
X_t &= \mathbb{E}[X_{n+1} \mid \mathcal{F}_t] \\
&= \mathbb{E}\left[X_0 + \int_0^{n+1} \mathbb{E}[D_s X_{n+1} \mid \mathcal{F}_s] dM_s \Big| \mathcal{F}_t\right] \\
&= X_0 + \int_0^t \mathbb{E}\left[D_s X_{n+1} \mid \mathcal{F}_s\right] dM_s \\
&= X_0 + \int_0^n \mathbb{E}\left[D_s X_{n+1} \mid \mathcal{F}_s\right] dM_s + \int_n^t \mathbb{E}\left[D_s X_{n+1} \mid \mathcal{F}_s\right] dM_s \\
&= X_n + \int_n^t \mathbb{E}[D_s X_{n+1} \mid \mathcal{F}_s] dM_s \\
&= X_n + \int_n^t u_s dM_s, \qquad n \le t < n+1, \quad n \in \mathbb{N},
\end{aligned}$$

where we used the Chasles relation (2.5.6), hence

$$X_t = X_0 + \int_0^t u_s dM_s, \qquad t \in \mathbb{R}_+, \tag{3.2.6}$$

hence from Proposition 2.6.2, $(M_t)_{t\in\mathbb{R}_+}$ has the predictable representation property. □

In particular, the Clark formula Assumption 3.2.1 and Relation (3.2.3) of Proposition 3.2.3 imply the following proposition.

Proposition 3.2.9. *For any $\mathcal{F}_T$-measurable $F \in L^2(\Omega)$ we have*

$$\mathbb{E}[D_t F|\mathcal{F}_T] = 0, \qquad 0 \le T \le t. \tag{3.2.7}$$

Proof. From from Relation (3.2.3) we have $F = \mathbb{E}[F|\mathcal{F}_T]$ if and only if

$$\int_T^\infty \mathbb{E}[D_t F|\mathcal{F}_t] dM_t = 0,$$

which implies $\mathbb{E}[D_t F|\mathcal{F}_t]$, $t \ge T$, by the Itô isometry (2.5.4), hence (3.2.7) holds as

$$\mathbb{E}[D_t F|\mathcal{F}_T] = \mathbb{E}[\mathbb{E}[D_t F|\mathcal{F}_t]|\mathcal{F}_T] = 0, \qquad t \ge T,$$

by the tower property of conditional expectations stated in Section 9.3. □

The next assumption is a stability property for the gradient operator D.

Assumption 3.2.10. (Stability property) For all $\mathcal{F}_T$-measurable $F \in \mathcal{S}$, $D_t F$ is $\mathcal{F}_T$-measurable for all $t \geq T$.

Proposition 3.2.11. *Let $T > 0$. Under the stability Assumption 3.2.10, for any $\mathcal{F}_T$-measurable random variable $F \in L^2(\Omega)$ we have $F \in \mathbb{D}_{[T,\infty)}$ and*

$$D_t F = 0, \qquad t \geq T.$$

Proof. Since F is $\mathcal{F}_T$-measurable, $D_t F$ is $\mathcal{F}_T$-measurable, $t \geq T$, by the stability Assumption 3.2.10, and from Proposition 3.2.9 we have

$$D_t F = \mathbb{E}[D_t F | \mathcal{F}_T] = 0, \qquad 0 \leq T \leq t.$$

□

3.3 Divergence and Stochastic Integrals

In this section we are interested in the connection between the operator δ and the stochastic integral with respect to $(M_t)_{t\in\mathbb{R}_+}$.

Proposition 3.3.1. *Under the duality Assumption 3.1.1 and the Clark formula Assumption 3.2.1, the operator δ applied to any square-integrable adapted process $(u_t)_{t\in\mathbb{R}_+} \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ coincides with the stochastic integral*

$$\delta(u) = \int_0^\infty u_t dM_t, \qquad u \in L^2_{ad}(\Omega \times \mathbb{R}_+), \tag{3.3.1}$$

of $(u_t)_{t\in\mathbb{R}_+}$ with respect to $(M_t)_{t\in\mathbb{R}_+}$, and the domain Dom (δ) *of δ contains $L^2_{ad}(\Omega \times \mathbb{R}_+)$.*

Proof. Let $u \in \mathcal{P}$ be a simple $\mathcal{F}_t$-predictable process. From the duality Assumption 3.1.1 and the fact (2.5.7) that

$$\mathbb{E}\left[\int_0^\infty u_t dM_t\right] = 0,$$

we have:

$$\begin{aligned}
\mathbb{E}\left[F\int_0^\infty u_t dM_t\right] &= \mathbb{E}[F]\,\mathbb{E}\left[\int_0^\infty u_t dM_t\right] + \mathbb{E}\left[(F - \mathbb{E}[F])\int_0^\infty u_t dM_t\right] \\
&= \mathbb{E}\left[(F - \mathbb{E}[F])\int_0^\infty u_t dM_t\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_t F|\mathcal{F}_t] dM_t \int_0^\infty u_t dM_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_0^\infty u_t \, \mathbb{E}[D_t F|\mathcal{F}_t] dt\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{E}[u_t D_t F|\mathcal{F}_t] dt\right] \\
&= \mathbb{E}\left[\int_0^\infty u_t D_t F dt\right] \\
&= \mathbb{E}[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}] \\
&= \mathbb{E}[F\delta(u)],
\end{aligned}
$$

for all $F \in \mathcal{S}$, hence by density of $\mathcal{S}$ in $L^2(\Omega)$ we have

$$
\delta(u) = \int_0^\infty u_t dM_t
$$

for all $\mathcal{F}_t$-predictable $u \in \mathcal{P}$. In the general case, from Proposition 2.5.3 we approximate $u \in L^2_{ad}(\Omega \times \mathbb{R}_+)$ by a sequence $(u^n)_{n\in\mathbb{N}} \subset \mathcal{P}$ of simple $\mathcal{F}_t$-predictable processes converging to u in $L^2(\Omega \times \mathbb{R}_+)$ and use the Itô isometry (2.5.4). □

As a consequence of the proof of Proposition 3.3.1 we have the isometry

$$
\|\delta(u)\|_{L^2(\Omega)} = \|u\|_{L^2(\Omega\times\mathbb{R}_+)}, \qquad u \in L^2_{ad}(\Omega \times \mathbb{R}_+). \tag{3.3.2}
$$

We also have the following partial converse to Proposition 3.3.1.

Proposition 3.3.2. *Assume that*

i) $(M_t)_{t\in\mathbb{R}_+}$ *has the predictable representation property, and*
ii) the operator δ *coincides with the stochastic integral with respect to* $(M_t)_{t\in\mathbb{R}_+}$ *on the space* $L^2_{ad}(\Omega\times\mathbb{R}_+)$ *of square-integrable adapted processes.*

Then the Clark formula Assumption 3.2.1 hold for the adjoint D *of* δ.

Proof. For all $F \in \mathrm{Dom}\,(D)$ and square-integrable adapted process u we have:

$$
\begin{aligned}
\mathbb{E}[(F - \mathbb{E}[F])\delta(u)] &= \mathbb{E}[F\delta(u)] \\
&= \mathbb{E}[\langle DF, u\rangle_{L^2(\mathbb{R}_+)}] \\
&= \mathbb{E}\left[\int_0^\infty u_t \, \mathbb{E}[D_t F|\mathcal{F}_t] dt\right] \\
&= \mathbb{E}\left[\int_0^\infty u_t dM_t \int_0^\infty \mathbb{E}[D_t F|\mathcal{F}_t] dM_t\right] \\
&= \mathbb{E}\left[\delta(u) \int_0^\infty \mathbb{E}[D_t F|\mathcal{F}_t] dM_t\right],
\end{aligned}
$$

hence

$$F - \mathbb{E}[F] = \int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dM_t,$$

since by (ii) we have

$$\left\{\delta(u) \ : \ u \in L^2_{ad}(\Omega \times \mathbb{R}_+)\right\} = \left\{\int_0^\infty u_t dM_t \ : \ u \in L^2_{ad}(\Omega \times \mathbb{R}_+)\right\},$$

which is dense in $\{F \in L^2(\Omega) \ : \ \mathbb{E}[F] = 0\}$ by (i) and Definition 2.6.1. □

3.4 Covariance Identities

Covariance identities will be useful in the proof of concentration and deviation inequalities. The Clark formula and the Itô isometry imply the following covariance identity, which uses the L^2 extension of the Clark formula, cf. Proposition 3.2.6.

Proposition 3.4.1. *For any $F, G \in L^2(\Omega)$ we have*

$$\mathrm{Cov}\,(F, G) = \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t]\, \mathbb{E}[D_t G | \mathcal{F}_t] dt\right]. \tag{3.4.1}$$

Proof. We have

$$\begin{aligned}
\mathrm{Cov}\,(F, G) &= \mathbb{E}[(F - \mathbb{E}[F])(G - \mathbb{E}[G])] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] dM_t \int_0^\infty \mathbb{E}[D_t G | \mathcal{F}_t] dM_t\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t]\, \mathbb{E}[D_t G | \mathcal{F}_t] dt\right].
\end{aligned}$$

□

The identity (3.4.1) can be rewritten as

$$\begin{aligned}
\mathrm{Cov}\,(F, G) &= \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t]\, \mathbb{E}[D_t G | \mathcal{F}_t] dt\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{E}[\mathbb{E}[D_t F | \mathcal{F}_t] D_t G | \mathcal{F}_t] dt\right] \\
&= \mathbb{E}\left[\int_0^\infty \mathbb{E}[D_t F | \mathcal{F}_t] D_t G dt\right],
\end{aligned}$$

provided $G \in \mathrm{Dom}\,(D)$.

As is well known, if X is a real random variable and f, g are monotone functions then $f(X)$ and $g(X)$ are non-negatively correlated. Lemma 3.4.2, which is an immediate consequence of (3.4.1), provides an analog of this result for normal martingales, replacing the ordinary derivative with the adapted process $(\mathbb{E}[D_t F|\mathcal{F}_t])_{t\in[0,1]}$.

Lemma 3.4.2. *Let $F, G \in L^2(\Omega)$ such that*

$$\mathbb{E}[D_t F|\mathcal{F}_t] \cdot \mathbb{E}[D_t G|\mathcal{F}_t] \geq 0, \qquad dt \times d\mathbb{P} - a.e.$$

Then F and G are non-negatively correlated:

$$\mathrm{Cov}\,(F, G) \geq 0.$$

If $G \in \mathrm{Dom}\,(D)$, resp. $F, G \in \mathrm{Dom}\,(D)$, the above condition can be replaced by

$$\mathbb{E}[D_t F|\mathcal{F}_t] \geq 0 \quad \text{and} \quad D_t G \geq 0, \qquad dt \times d\mathbb{P} - a.e.,$$

resp.

$$D_t F \geq 0 \quad \text{and} \quad D_t G \geq 0, \qquad dt \times d\mathbb{P} - a.e..$$

Iterated versions of Lemma 3.2.4 can also be proved. Let

$$\Delta_n = \{(t_1, \ldots, t_n) \in \mathbb{R}_+^n \; : \; 0 \leq t_1 < \cdots < t_n\},$$

and assume further that

Assumption 3.4.3. (Domain condition) For all $F \in \mathcal{S}$ we have

$$D_{t_n} \cdots D_{t_1} F \in I\!D_{2,1}([t_n, \infty)), \qquad a.e. \; (t_1, \ldots, t_n) \in \Delta_n.$$

We denote by $I\!D_{2,k}(\Delta_k)$ the L^2 domain of D^k, i.e. the completion of $\mathcal{S}$ under the norm

$$\|F\|^2_{I\!D_{2,k}(\Delta_k)} = I\!E\left[F^2\right] + I\!E\left[\int_{\Delta_k} |D_{t_k} \cdots D_{t_1} F|^2 dt_1 \cdots dt_k\right].$$

Note the inclusion $I\!D_{2,k}(\Delta_k) \subset I\!D_{2,1}(\Delta_k)$, $k \geq 1$.

Next we prove an extension of the covariance identity of [56], with a shortened proof.

Theorem 3.4.4. *Let* $n \in \mathbb{N}$ *and* $F, G \in \bigcap_{k=1}^{n+1} \mathbb{D}_{2,k}(\Delta_k)$. *We have*

$$\mathrm{Cov}\,(F,G) = \sum_{k=1}^{n} (-1)^{k+1}\, \mathbb{E}\left[\int_{\Delta_k} (D_{t_k}\cdots D_{t_1}F)(D_{t_k}\cdots D_{t_1}G)dt_1\cdots dt_k\right] \quad (3.4.2)$$
$$+ (-1)^n\, \mathbb{E}\left[\int_{\Delta_{n+1}} D_{t_{n+1}}\cdots D_{t_1}F\, \mathbb{E}\left[D_{t_{n+1}}\cdots D_{t_1}G|\mathcal{F}_{t_{n+1}}\right] dt_1\cdots dt_{n+1}\right].$$

Proof. By polarization we may take $F = G$. For $n = 0$, ((3.4.2)) is a consequence of the Clark formula. Let $n \geq 1$. Applying Lemma 3.2.4 to $D_{t_n}\cdots D_{t_1}F$ with $t = t_n$ and $ds = dt_{n+1}$, and integrating on $(t_1,\ldots,t_n) \in \Delta_n$ we obtain

$$\begin{aligned}\mathbb{E}&\left[\int_{\Delta_n} \left(\mathbb{E}[D_{t_n}\cdots D_{t_1}F|\mathcal{F}_{t_n}]\right)^2 dt_1\cdots dt_n\right] \\ &= \mathbb{E}\left[\int_{\Delta_n} |D_{t_n}\cdots D_{t_1}F|^2 dt_1\cdots dt_n\right] \\ &\quad - \mathbb{E}\left[\int_{\Delta_{n+1}} \left(\mathbb{E}\left[D_{t_{n+1}}\cdots D_{t_1}F|\mathcal{F}_{t_{n+1}}\right]\right)^2 dt_1\cdots dt_{n+1}\right],\end{aligned}$$

which concludes the proof by induction. □

The variance inequality

$$\sum_{k=1}^{2n} (-1)^{k+1}\|D^kF\|^2_{L^2(\Delta_k)} \leq \mathrm{Var}\,(F) \leq \sum_{k=1}^{2n-1} (-1)^{k+1}\|D^kF\|^2_{L^2(\Delta_k)},$$

for $F \in \bigcap_{k=1}^{2n} \mathbb{D}_{2,k}(\Delta_k)$, is a consequence of Theorem 3.4.4, and extends (2.15) in [56]. It also recovers the Poincaré inequality Proposition 3.2.7 when $n = 1$.

3.5 Logarithmic Sobolev Inequalities

The logarithmic Sobolev inequalities on Gaussian space provide an infinite dimensional analog of Sobolev inequalities, cf. e.g. [77]. In this section logarithmic Sobolev inequalities for normal martingales are proved as an application of the Itô and Clark formulas. Recall that the entropy of a sufficiently integrable random variable $F > 0$ is defined by

$$\mathrm{Ent}\,[F] = \mathbb{E}[F\log F] - \mathbb{E}[F]\log\mathbb{E}[F].$$

Proposition 3.5.1. *Let $F \in \mathrm{Dom}\,(D)$ be lower bounded with $F > \eta$ a.s. for some $\eta > 0$. We have*

$$\mathrm{Ent}\,[F] \leq \frac{1}{2}\,\mathbb{E}\left[\frac{1}{F}\int_0^\infty (2 - \mathbf{1}_{\{\phi_t = 0\}})|D_t F|^2 dt\right]. \tag{3.5.1}$$

Proof. Let us assume that F is bounded and $\mathcal{F}_T$-measurable, and let

$$X_t = \mathbb{E}[F \mid \mathcal{F}_t] = X_0 + \int_0^t u_s dM_s, \qquad t \in \mathbb{R}_+,$$

with $u_s = \mathbb{E}[D_s F \mid \mathcal{F}_s]$, $s \in \mathbb{R}_+$. The change of variable formula Proposition 2.12.1 applied to $f(x) = x \log x$ shows that

$$\begin{aligned}
F \log F - \mathbb{E}[F] \log \mathbb{E}[F] &= f(X_T) - f(X_0) \\
&= \int_0^T \frac{f(X_{t^-} + \phi_t u_t) - f(X_{t^-})}{\phi_t} dM_t + \int_0^T i_t u_t f'(X_{t^-}) dM_t \\
&\quad + \int_0^T \frac{j_t}{\phi_t^2} \Psi(X_{t^-}, \phi_t u_t) dt + \frac{1}{2}\int_0^T i_t \frac{u_t^2}{X_t} dt,
\end{aligned}$$

with the convention $0/0 = 0$, and

$$\Psi(u, v) = (u + v)\log(u + v) - u \log u - v(1 + \log u), \qquad u, u + v > 0.$$

Using the inequality

$$\Psi(u, v) \leq v^2/u, \qquad u > 0, \quad u + v > 0,$$

and applying Jensen's inequality (9.3.1) to the convex function $(u, v) \to v^2/u$ on $\mathbb{R} \times (0, \infty)$ we obtain

$$\begin{aligned}
\mathrm{Ent}\,[F] &= \mathbb{E}\left[\int_0^T \frac{j_t}{\phi_t^2}\Psi(X_t, \phi_t u_t) dt + \frac{1}{2}\int_0^T i_t \frac{u_t^2}{X_t} dt\right] \\
&\leq \frac{1}{2}\,\mathbb{E}\left[\int_0^T (2 - i_t)\frac{u_t^2}{X_t} dt\right] \\
&\leq \frac{1}{2}\,\mathbb{E}\left[\int_0^T \mathbb{E}\left[(2 - i_t)\frac{|D_t F|^2}{F}\Big|\mathcal{F}_t\right] dt\right] \\
&= \frac{1}{2}\,\mathbb{E}\left[\frac{1}{F}\int_0^T (2 - i_t)|D_t F|^2 dt\right].
\end{aligned}$$

Finally we apply the above to the approximating sequence $F_n = F \wedge n$, $n \in \mathbb{N}$, and let n go to infinity. □

If $\phi_t = 0$, i.e. $i_t = 1$, $t \in \mathbb{R}_+$, then $(M_t)_{t\in\mathbb{R}_+}$ is a Brownian motion and we obtain the classical modified Sobolev inequality

$$\mathrm{Ent}_\pi[F] \le \frac{1}{2}\mathbb{E}_\pi\left[\frac{1}{F}\|DF\|^2_{L^2([0,T])}\right]. \tag{3.5.2}$$

If $\phi_t = 1$, $t \in \mathbb{R}_+$ then $i_t = 0$, $t \in \mathbb{R}_+$, $(M_t)_{t\in\mathbb{R}_+}$ is a standard compensated Poisson process and we obtain the modified Sobolev inequality

$$\mathrm{Ent}_\pi[F] \le \mathbb{E}_\pi\left[\frac{1}{F}\|DF\|^2_{L^2([0,T])}\right]. \tag{3.5.3}$$

More generally, the logarithmic Sobolev inequality (3.5.2) can be proved for any gradient operator D satisfying both the derivation rule Assumption 3.6.1 below and the Clark formula Assumption 3.2.1, see Chapter 7 for another example on the Poisson space.

3.6 Deviation Inequalities

In this section we assume that D is a gradient operator satisfying both the Clark formula Assumption 3.2.1 and the derivation rule Assumption 3.6.1 below. Examples of such operators will be provided in the Wiener and Poisson cases in Chapters 5 and 7.

Assumption 3.6.1. (Derivation rule) For all $F, G \in \mathcal{S}$ we have

$$D_t(FG) = FD_tG + GD_tF, \qquad t \in \mathbb{R}_+. \tag{3.6.1}$$

Note that by polynomial approximation, Relation (3.6.1) extends as

$$D_t f(F) = f'(F)D_tF, \qquad t \in \mathbb{R}_+, \tag{3.6.2}$$

for $f \in \mathcal{C}^1_b(\mathbb{R})$.

Under the derivation rule Assumption 3.6.1 we get the following deviation bound.

Proposition 3.6.2. *Let $F \in \mathrm{Dom}\,(D)$. If $\|DF\|_{L^2(\mathbb{R}_+,L^\infty(\Omega))} \le C$ for some $C > 0$, then*

$$\mathbb{P}(F - \mathbb{E}[F] \ge x) \le \exp\left(-\frac{x^2}{2C\|DF\|_{L^2(\mathbb{R}_+,L^\infty(\Omega))}}\right), \qquad x \ge 0. \tag{3.6.3}$$

In particular we have

$$\mathbb{E}[e^{\lambda F^2}] < \infty, \qquad \lambda < \frac{1}{2C\|DF\|_{L^2(\mathbb{R}_+,L^\infty(\Omega)}}. \tag{3.6.4}$$

Proof. We first consider a bounded random variable $F \in \mathrm{Dom}\,(D)$. The general case follows by approximating $F \in \mathrm{Dom}\,(D)$ by the sequence $(\max(-n, \min(F, n)))_{n\geq 1}$. Let

$$\eta_F(t) = \mathbb{E}_\mu[D_t F \mid \mathcal{F}_t], \qquad t \in [0, T].$$

Since F is bounded, the derivation rule (3.6.2) shows that

$$D_t \mathrm{e}^{sF} = s\mathrm{e}^{sF} D_t F, \qquad s, t \in \mathbb{R}_+,$$

hence assuming first that $\mathbb{E}[F] = 0$ we get

$$\begin{aligned}
\mathbb{E}[F\mathrm{e}^{sF}] &= \mathbb{E}\left[\int_0^T D_u \mathrm{e}^{sF} \cdot \eta_F(u) du\right] \\
&= s\,\mathbb{E}\left[\mathrm{e}^{sF} \int_0^T D_u F \cdot \eta_F(u) du\right] \\
&\leq s\,\mathbb{E}\left[\mathrm{e}^{sF} \|DF\|_{\mathbb{H}}\ \|\eta_F\|_{\mathbb{H}}\right] \\
&\leq s\,\mathbb{E}\left[\mathrm{e}^{sF}\right] \|\eta_F\|_{L^\infty(W,\mathbb{H})} \|DF\|_{L^2(\mathbb{R}_+, L^\infty(\Omega))} \\
&\leq sC\,\mathbb{E}\left[\mathrm{e}^{sF}\right] \|DF\|_{L^2(\mathbb{R}_+, L^\infty(\Omega))}.
\end{aligned}$$

In the general case, letting

$$L(s) = \mathbb{E}[\exp(s(F - \mathbb{E}[F]))], \qquad s \in \mathbb{R}_+,$$

we obtain:

$$\begin{aligned}
\log\left(\mathbb{E}\left[\exp\left(t(F - \mathbb{E}[F])\right)\right]\right) &= \int_0^t \frac{L'(s)}{L(s)} ds \\
&\leq \int_0^t \frac{\mathbb{E}\left[(F - \mathbb{E}[F]) \exp\left(t(F - \mathbb{E}[F])\right)\right]}{\mathbb{E}\left[\exp\left(t(F - \mathbb{E}[F])\right)\right]} ds \\
&= \frac{1}{2} t^2 C \|DF\|_{L^2(\mathbb{R}_+, L^\infty(\Omega))}, \qquad t \in \mathbb{R}_+.
\end{aligned}$$

We now have for all $x \in \mathbb{R}_+$ and $t \in [0, T]$:

$$\begin{aligned}
\mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \mathrm{e}^{-tx}\,\mathbb{E}\left[\exp\left(t(F - \mathbb{E}[F])\right)\right] \\
&\leq \exp\left(\frac{1}{2} t^2 C \|DF\|_{L^2(\mathbb{R}_+, L^\infty(\Omega))} - tx\right),
\end{aligned}$$

which yields (3.6.3) after minimization in $t \in [0, T]$. The proof of (3.6.4) is completed as in Proposition 1.11.3. □

3.7 Markovian Representation

This subsection presents a predictable representation method that can be used to compute $\mathbb{E}[D_t F|\mathcal{F}_t]$, based on the Itô formula and the Markov property, cf. Section 9.6 in the appendix. It can applied to Delta hedging in mathematical finance, cf. Proposition 8.2.2 in Chapter 8, and [120]. Let $(X_t)_{t\in[0,T]}$ be a $\mathbb{R}^n$-valued Markov (not necessarily time homogeneous) process defined on Ω, generating a filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ and satisfying a change of variable formula of the form

$$f(X_t) = f(X_0) + \int_0^t L_s f(X_s)dM_s + \int_0^t U_s f(X_s)ds, \quad t \in [0,T], \qquad (3.7.1)$$

where L_s, U_s are operators defined on $f \in \mathcal{C}^2(\mathbb{R}^n)$. Let the (non homogeneous) semi-group $(P_{s,t})_{0\le s\le t\le T}$ associated to $(X_t)_{t\in[0,T]}$ be defined on $\mathcal{C}_b^2(\mathbb{R}^n)$ functions by

$$\begin{aligned} P_{s,t}f(X_s) &= \mathbb{E}[f(X_t) \mid X_s] \\ &= \mathbb{E}[f(X_t) \mid \mathcal{F}_s], \quad 0 \le s \le t \le T, \end{aligned}$$

with

$$P_{s,t} \circ P_{t,u} = P_{s,u}, \qquad 0 \le s \le t \le u \le T.$$

Proposition 3.7.1. *For any $f \in \mathcal{C}_b^2(\mathbb{R}^n)$, the process $(P_{t,T}f(X_t))_{t\in[0,T]}$ is an $\mathcal{F}_t$-martingale.*

Proof. By the tower property of conditional expectations, cf. Section 9.3, we have

$$\begin{aligned} \mathbb{E}[P_{t,T}f(X_t) \mid \mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[f(X_T) \mid \mathcal{F}_t] \mid \mathcal{F}_s] \\ &= \mathbb{E}[f(X_T) \mid \mathcal{F}_s] \\ &= P_{s,T}f(X_s), \end{aligned}$$

$0 \le s \le t \le T$. □

Next we use above the framework with application to the Clark formula. When $(\phi_t)_{t\in[0,T]}$ is random the probabilistic interpretation, of D is unknown in general, nevertheless it is possible to explicitly compute the predictable representation of $f(X_T)$ using (3.7.1) and the Markov property.

Lemma 3.7.2. *Let $f \in \mathcal{C}_b^2(\mathbb{R}^n)$. We have*

$$\mathbb{E}[D_t f(X_T) \mid \mathcal{F}_t] = (L_t(P_{t,T}f))(X_t), \quad t \in [0,T]. \qquad (3.7.2)$$

Proof. We apply the change of variable formula (3.7.1) to $t \mapsto P_{t,T}f(X_t) = \mathbb{E}[f(X_T) \mid \mathcal{F}_t]$, since $P_{t,T}f$ is $\mathcal{C}^2$. Using the fact that the finite variation term vanishes since $(P_{t,T}f(X_t))_{t\in[0,T]}$ is a martingale, (see e.g. Corollary 1, p. 64 of [119]), we obtain:

$$P_{t,T}f(X_t) = P_{0,T}f(X_0) + \int_0^t (L_s(P_{s,T}f))(X_s)dM_s, \quad t \in [0,T],$$

with $P_{0,T}f(X_0) = \mathbb{E}[f(X_T)]$. Letting $t = T$, we obtain (3.7.2) by uniqueness of the representation (4.2.2) applied to $F = f(X_T)$. □

In practice we can use Proposition 3.2.6 to extend $(\mathbb{E}[D_t f(X_T) \mid \mathcal{F}_t])_{t\in[0,T]}$ to a less regular function $f : \mathbb{R}^n \to \mathbb{R}$.
As an example, if ϕ_t is written as $\phi_t = \varphi(t, M_t)$, and

$$dS_t = \sigma(t, S_t)dM_t + \mu(t, S_t)dt,$$

we can apply Proposition 2.12.2, with $(X_t)_{t\in[0,T]} = ((S_t, M_t))_{t\in[0,T]}$ and

$$\begin{aligned} L_t f(S_t, M_t) &= i_t\sigma(t, S_t)\partial_1 f(S_t, M_t) + i_t\partial_2 f(S_t, M_t) \\ &+\frac{j_t}{\varphi(t, M_t)}(f(S_t + \varphi(t, M_t)\sigma(t, S_t), M_t + \varphi(t, M_t)) - f(S_t, M_t)), \end{aligned}$$

where $j_t = \mathbf{1}_{\{\phi_t \neq 0\}}$, $t \in \mathbb{R}_+$, since the eventual jump of $(M_t)_{t\in[0,T]}$ at time t is $\varphi(t, M_t)$. Here, ∂_1, resp. ∂_2, denotes the partial derivative with respect to the first, resp. second, variable. Hence

$$\begin{aligned} \mathbb{E}[D_t f(S_T, M_T) \mid \mathcal{F}_t] &= i_t\sigma(t, S_t)(\partial_1 P_{t,T}f)(S_t, M_t) + i_t(\partial_2 P_{t,T}f)(S_t, M_t) \\ &+\frac{j_t}{\varphi(t, M_t)}(P_{t,T}f)(S_t + \varphi(t, M_t)\sigma(t, S_t), M_t + \varphi(t, M_t)) \\ &-\frac{j_t}{\varphi(t, M_t)}(P_{t,T}f)(S_t, M_t). \end{aligned}$$

When $(\phi_t)_{t\in\mathbb{R}_+}$ and $\sigma(t, x) = \sigma_t$, are deterministic functions of time and $\mu(t, x) = 0$, $t \in \mathbb{R}_+$, the semi-group $P_{t,T}$ can be explicitly computed as follows.
In this case, from (2.10.4), the martingale $(M_t)_{t\in\mathbb{R}_+}$ can be represented as

$$dM_t = i_t dB_t + \phi_t(dN_t - \lambda_t dt), \qquad t \in \mathbb{R}_+, \qquad M_0 = 0,$$

with $\lambda_t = j_t/\phi_t^2$, $t \in \mathbb{R}_+$, where $(N_t)_{t\in\mathbb{R}_+}$ is an independent Poisson process with intensity λ_t, $t \in \mathbb{R}_+$. Let

$$\Gamma_t(T) = \int_t^T \mathbf{1}_{\{\phi_s=0\}}\sigma_s^2 ds, \qquad 0 \le t \le T,$$

denote the variance of $\int_t^T i_s\sigma_s dB_s = \int_t^T \mathbf{1}_{\{\phi_s=0\}}\sigma_s dB_s$, $0 \le t \le T$, and let

$$\Gamma_t(T) = \int_t^T \lambda_s ds, \qquad 0 \le t \le T,$$

denote the intensity parameter of the Poisson random variable $N_T - N_t$.

Proposition 3.7.3. *We have for $f \in \mathcal{C}_b(\mathbb{R})$*

$$P_{t,T}f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{\mathrm{e}^{-\Gamma_t(T)}}{k!} \int_{-\infty}^{\infty} \mathrm{e}^{-t_0^2/2} \int_{[t,T]^k} \lambda_{t_1}\cdots\lambda_{t_k}$$
$$f\left(x\mathrm{e}^{-\frac{\Gamma_t(T)}{2}+\sqrt{\Gamma_t(T)}t_0 - \int_t^T \phi_s\lambda_s\sigma_s ds} \prod_{i=1}^{k}(1+\sigma_{t_i}\phi_{t_i}) \right) dt_1\cdots dt_k dt_0.$$

Proof. We have $P_{t,T}f(x) = \mathbb{E}[f(S_T)|S_t = x] = \mathbb{E}[f(S_{t,T}^x)]$, and

$$P_{t,T}f(x) = \exp(-\Gamma_t(T)) \sum_{k=0}^{\infty} \frac{(\Gamma_t(T))^k}{k!} \mathbb{E}\left[f(S_{t,T}^x) \Big| N_T - N_t = k \right]$$

$k \in \mathbb{N}$. It can be shown (see e.g. Proposition 6.1.8 below) that the time changed process $\left(N_{\Gamma_t^{-1}(s)} - N_t \right)_{s\in\mathbb{R}_+}$ is a standard Poisson process with jump times $(\tilde{T}_k)_{k\ge 1} = (\Gamma_t(T_{k+N_t}))_{k\ge1}$. Hence from Proposition 2.3.7, conditionally to $\{N_T - N_t = k\}$, the jump times $(\tilde{T}_1, \ldots, \tilde{T}_k)$ have the law

$$\frac{k!}{(T-t)^k}\mathbf{1}_{\{0<t_1<\cdots<t_k<T-t\}} dt_1\cdots dt_k.$$

over $[0, T-t]^k$. Consequently, conditionally to $\{N_T - N_t = k\}$, the k first jump times $(T_1, \ldots, T_k)$ of $(N_s)_{s\in[t,T]}$ have the distribution

$$\frac{k!}{(\Gamma_t(T))^k}\mathbf{1}_{\{t<t_1<\cdots<t_k<T\}}\lambda_{t_1}\cdots\lambda_{t_k} dt_1\cdots dt_k.$$

We then use the identity in law between $S_{t,T}^x$ and

$$xX_{t,T}\exp\left(-\int_t^T \phi_s\lambda_s(1+\phi_s\psi_s)\sigma_s ds \right) \prod_{k=1+N_t}^{N_T}(1+\sigma_{T_k}\phi_{T_k}),$$

where $X_{t,T}$ has same distribution as

$$\exp\left(W\sqrt{\Gamma_t(T)} - \Gamma_t(T)/2 \right),$$

and W a standard Gaussian random variable, independent of $(N_t)_{t\in[0,T]}$, which holds because $(B_t)_{t\in[0,T]}$ is a standard Brownian motion, independent of $(N_t)_{t\in[0,T]}$. $\square$

3.8 Notes and References

Several examples of gradient operators satisfying the hypotheses of this chapter will be provided in Chapters 4, 5, 6, and 7, on the Wiener and Poisson space and also on Riemannian path space. The Itô formula has been used for the proof of logarithmic Sobolev inequalities in [4], [6], [151] for the Poisson process, and in [22] on Riemannian path space, and Proposition 3.5.1 can be found in [111]. The probabilistic interpretations of D as a derivation operator and as a finite difference operator has been studied in [116] and will be presented in more detail in the sequel. The extension of the Clark formula presented in Proposition 3.2.6 is related to the approach of [88] of [142]. The covariance identity (3.4.1) can be found in Proposition 2.1 of [59]. See also [7] for a unified presentation of the Malliavin calculus based on the Fock space.

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