Chapter 2
Lorentzian Manifolds

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In this chapter some basic notions from Lorentzian geometry will be reviewed. In particular causality relations will be explained, Cauchy hypersurfaces and the concept of global hyperbolic manifolds will be introduced. Finally the structure of globally hyperbolic manifolds will be discussed.

More comprehensive introductions can be found in [1] and [2].

2.1 Preliminaries on Minkowski Space

Let V be an n-dimensional real vector space. A *Lorentzian scalar product* on V is a nondegenerate symmetric bilinear form $\langle\langle\cdot,\cdot\rangle\rangle$ of index 1. This means one can find a basis $e_1, \dots, e_n$ of V such that

$$\langle\langle e_i, e_j\rangle\rangle = \begin{cases} -1 & \text{if } i = j = 1, \\ \;\;1 & \text{if } i = j = 2, \dots, n, \\ \;\;0 & \text{otherwise.} \end{cases}$$

The simplest example for a Lorentzian scalar product on $\mathbb{R}^n$ is the Minkowski product $\langle\langle\cdot,\cdot\rangle\rangle_0$ given by $\langle\langle x, y\rangle\rangle_0 = -x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$. In some sense this is the only example because from the above it follows that any n-dimensional vector space with Lorentzian scalar product $(V, \langle\langle\cdot,\cdot\rangle\rangle)$ is isometric to Minkowski space $(\mathbb{R}^n, \langle\langle\cdot,\cdot\rangle\rangle_0)$.

We denote the quadratic form associated with $\langle\langle\cdot,\cdot\rangle\rangle$ by

$$\gamma : V \to \mathbb{R}, \quad \gamma(X) := -\langle\langle X, X\rangle\rangle.$$

A vector $X \in V \setminus \{0\}$ is called *timelike* if $\gamma(X) > 0$, *lightlike* if $\gamma(X) = 0$ and $X \neq 0$, *causal* if timelike or lightlike, and *spacelike* if $\gamma(X) > 0$ or $X = 0$.

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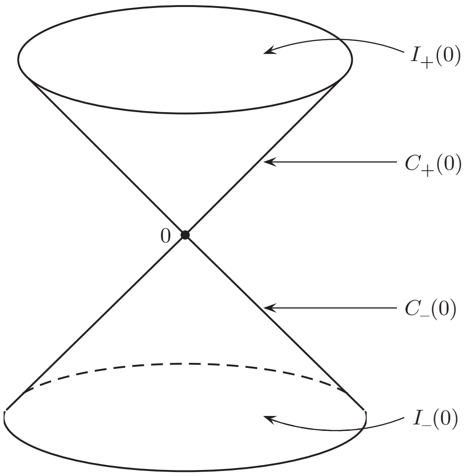
Fig. 2.1 Lightcone in Minkowski space



For $n \geq 2$ the set $I(0)$ of timelike vectors consists of two connected components. We choose a *time-orientation* on V by picking one of these two connected components. Denote this component by $I_+(0)$ and call its elements *future-directed*. We put $J_+(0) := \overline{I_+(0)}$, $C_+(0) := \partial I_+(0)$, $I_-(0) := -I_+(0)$, $J_-(0) := -J_+(0)$, and $C_-(0) := -C_+(0)$. Causal vectors in $J_+(0)$ (or in $J_-(0)$) are called *future-directed* (or *past-directed* respectively). (See Fig. 2.1.)

Remark 1. Given a positive number $\alpha > 0$ and a Lorentzian scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ on a vector space V one gets another Lorentzian scalar product $\alpha \cdot \langle\langle \cdot, \cdot \rangle\rangle$. One observes that $X \in V$ is timelike with respect to $\langle\langle \cdot, \cdot \rangle\rangle$ if and only if it is timelike with respect to $\alpha \cdot \langle\langle \cdot, \cdot \rangle\rangle$. Analogously, the notion lightlike coincides for $\langle\langle \cdot, \cdot \rangle\rangle$ and $\alpha \cdot \langle\langle \cdot, \cdot \rangle\rangle$, and so do the notions causal and spacelike.

Hence, for both Lorentzian scalar products one gets the same set $I(0)$. If $\dim(V) \geq 2$ and we choose identical time-orientations for $\langle\langle \cdot, \cdot \rangle\rangle$ and $\alpha \cdot \langle\langle \cdot, \cdot \rangle\rangle$, the sets $I_\pm(0)$, $J_\pm(0)$, $C_\pm(0)$ are determined independently whether formed with respect to $\langle\langle \cdot, \cdot \rangle\rangle$ or $\alpha \cdot \langle\langle \cdot, \cdot \rangle\rangle$.

2.2 Lorentzian Manifolds

A *Lorentzian manifold* is a pair (M, g) where M is an n-dimensional smooth manifold and g is a Lorentzian metric, i.e., g associates with each point $p \in M$ a Lorentzian scalar product g_p on the tangent space T_pM.

One requires that g_p depends smoothly on p: This means that for any choice of local coordinates $x = (x_1, \ldots, x_n) : U \to V$, where $U \subset M$ and $V \subset \mathbb{R}^n$ are open subsets, and for any $i, j = 1, \ldots, n$ the functions $g_{ij} : V \to \mathbb{R}$ defined by $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are smooth. Here $\frac{\partial}{\partial x_i}$ and $\frac{\partial}{\partial x_j}$ denote the coordinate vector fields as usual

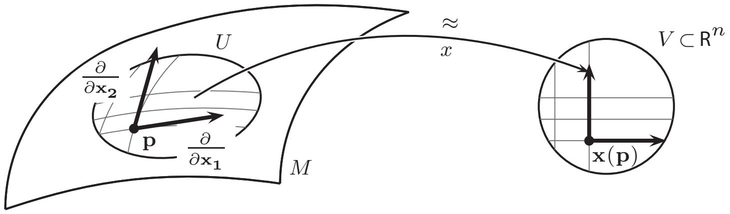


Fig. 2.2 Coordinate vectors $\frac{\partial}{\partial x_1}$, $\frac{\partial}{\partial x_2}$

(see Fig. 2.2). With respect to these coordinates one writes $g = \sum_{i,j} g_{ij}\, dx_i \otimes dx_j$ or shortly $g = \sum_{i,j} g_{ij}\, dx_i\, dx_j$.

Next we will give some prominent examples for Lorentzian manifolds.

Example 1. In cartesian coordinates $(x_1, \ldots, x_n)$ on $\mathbb{R}^n$ the *Minkowski metric* is defined by $g_{Mink} = -(dx_1)^2 + (dx_2)^2 + \cdots + (dx_n)^2$. This turns Minkowski space into a Lorentzian manifold.

Of course, the restriction of g_{Mink} to any open subset $U \subset \mathbb{R}^n$ yields a Lorentzian metric on U as well.

Example 2. Consider the unit circle $S^1 \subset \mathbb{R}^2$ with its standard metric $(d\theta)^2$. The *Lorentzian cylinder* is given by $M = S^1 \times \mathbb{R}$ together with the Lorentzian metric $g = -(d\theta)^2 + (dx)^2$.

Example 3. Let (N, h) be a connected Riemannian manifold and $I \subset \mathbb{R}$ an open interval. For any $t \in I$, $p \in N$ one identifies $T_{(t,p)}(I \times N) = T_t I \oplus T_p N$. Then for any smooth positive function $f : I \to (0, \infty)$ the Lorentzian metric $g = -dt^2 + f(t)^2 \cdot h$ on $I \times M$ is defined as follows: For any $\xi_1, \xi_2 \in T_{(t,p)}(I \times N)$ one writes $\xi_i = \left(\alpha_i \frac{d}{dt}\right) \oplus \zeta_i$ with $\alpha_i \in \mathbb{R}$ and $\zeta_i \in T_p N$, $i = 1, 2$, and one has $g(\xi_1, \xi_2) = -\alpha_1 \cdot \alpha_2 + f(t)^2 \cdot h(\zeta_1, \zeta_2)$. Such a Lorentzian metric g is called a *warped product metric* (Fig. 2.3).

This example covers *Robertson–Walker spacetimes* where one requires additionally that (N, h) is complete and has constant curvature. In particular *Friedmann cosmological models* are of this type. In general relativity they are used to discuss big bang, expansion of the universe, and cosmological redshift; compare [2, Chaps. 5 and 6] or [1, Chap. 12]. A special case of this is *deSitter spacetime* where $I = \mathbb{R}$, $N = S^{n-1}$, h is the canonical metric of S^{n-1} of constant sectional curvature 1, and $f(t) = \cosh(t)$.

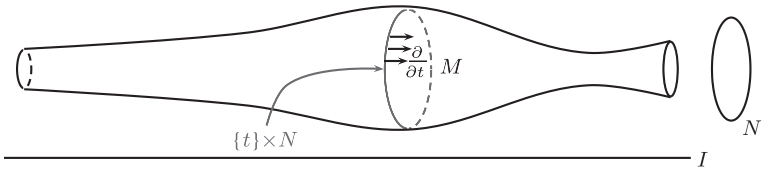


Fig. 2.3 Warped product

Example 4. For a fixed positive number $m > 0$ one considers the *Schwarzschild function* $h : (0, \infty) \to \mathbb{R}$ given by $h(r) = 1 - \frac{2m}{r}$. This function has a pole at $r = 0$ and one has $h(2m) = 0$. On both $P_I = \{(r, t) \in \mathbb{R}^2 \mid r > 2m\}$ and $P_{II} = \{(r, t) \in \mathbb{R}^2 \mid 0 < r < 2m\}$ one defines Lorentzian metrics by

$$g = -h(r) \cdot dt \otimes dt + \frac{1}{h(r)} \cdot dr \otimes dr,$$

and one calls (P_I, g) *Schwarzschild half-plane* and (P_{II}, g) *Schwarzschild strip*. For a tangent vector $\alpha \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial r}$ being timelike is equivalent to $\alpha^2 > \frac{1}{h(r)^2} \beta^2$. Hence, one can illustrate the set of timelike vector in the tangent spaces $T_{(r,t)} P_I$, resp., $T_{(r,t)} P_{II}$ as in Fig. 2.4.

The "singularity" of the Lorentzian metric g for $r = 2m$ is not as crucial as it might seem at first glance, by a change of coordinates one can overcome this singularity (e.g., in the so-called *Kruskal coordinates*).

One uses (P_I, g) and (P_{II}, g) to discuss the exterior and the interior of a static rotationally symmetric black hole with mass m, compare [1, Chap. 13]. For this one considers the two-dimensional sphere S^2 with its natural Riemannian metric can_{S^2}, and on both $N = P_I \times S^2$ and $B = P_{II} \times S^2$ one gets a Lorentzian metric by

$$-h(r) \cdot dt \otimes dt + \frac{1}{h(r)} \cdot dr \otimes dr + r^2 \cdot \mathrm{can}_{S^2}.$$

Equipped with this metric, N is called *Schwarzschild exterior spacetime* and B *Schwarzschild black hole*, both of mass m.

Example 5. Let $S^{n-1} = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid (x_1)^2 + \cdots + (x_n)^2 = 1\}$ be the n-dimensional sphere equipped with its natural Riemannian metric $\mathrm{can}_{S^{n-1}}$. The restriction of this metric to $S^{n-1}_+ = \{(x_1, \dots, x_n) \in S^{n-1} \mid x_n > 0\}$ is denoted by $\mathrm{can}_{S^{n-1}_+}$. Then, on $\mathbb{R} \times S^{n-1}_+$ one defines a Lorentzian metric by

$$g_{AdS} = \frac{1}{(x_n)^2} \cdot \left(-dt^2 + \mathrm{can}_{S^{n-1}_+}\right),$$

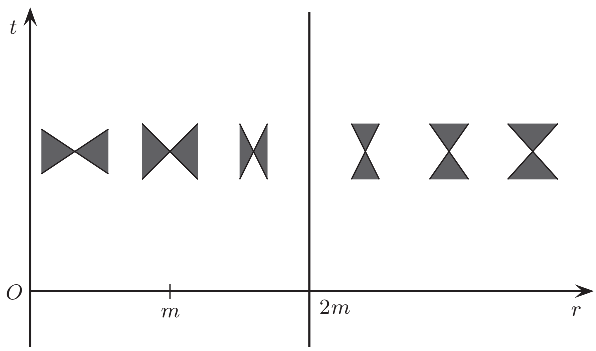


Fig. 2.4 Lightcones for Schwarzschild strip P_{II} and Schwarzschild half-plane P_I

and one calls $(\mathbb{R} \times S_+^{n-1}, g_{AdS})$ the n-dimensional *anti-deSitter spacetime*. This definition is not exactly the one given in [1, Chap. 8, p. 228f.], but one can show that both definitions coincide; compare [3, Chap. 3.5., p. 95ff.].

By Remark 1 we see that a tangent vector of $\mathbb{R} \times S_+^{n-1}$ is timelike (lightlike, spacelike) with respect to g_{AdS} if and only if it is so with respect to the Lorentzian metric $-dt^2 + \mathrm{can}_{S_+^{n-1}}$.

In general relativity one is interested in four-dimensional anti-deSitter spacetime because it provides a vacuum solution of Einstein's field equation with cosmological constant $\Lambda = -3$; see [1, Chap. 14, Example 41].

2.3 Time-Orientation and Causality Relations

Let (M, g) denote a Lorentzian manifold of dimension $n \geq 2$. Then at each point $p \in M$ the set of timelike vectors in the tangent space T_pM consists of two connected components, which cannot be distinguished intrinsically. A *time-orientation* on M is a choice $I_+(0) \subset T_pM$ of one of these connected components which depends continuously on p.

A time-orientation (Fig. 2.5) is given by a continuous timelike vector field τ on M which takes values in these chosen connected components: $\tau(p) \in I_+(0) \subset T_pM$ for each $p \in M$.

Definition 1. *One calls a Lorentzian manifold* (M, g) time-orientable *if there exists a continuous timelike vector field* τ *on* M*. A Lorentzian manifold* (M, g) *together with such a vector field* τ *is called* time-oriented*. In what follows time-oriented connected Lorentzian manifolds will be referred to as* spacetimes.

It should be noted that in contrast to the notion of orientability which only depends on the topology of the underlying manifold the concept of time-orientability depends on the Lorentzian metric.

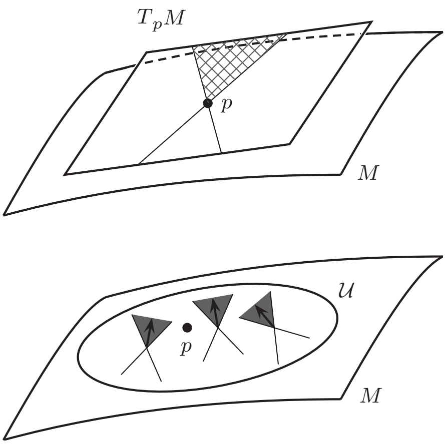


Fig. 2.5 Time-orientation

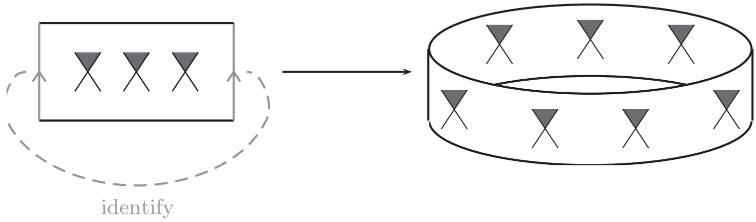


Fig. 2.6 Example for orientable and time-orientable manifold

If we go through the list of examples from Sect. 2.2, we see that all these Lorentzian manifolds are time-orientable (Fig. 2.7). Timelike vector fields can be given as follows: on Minkowski space by $\frac{\partial}{\partial x_1}$, on the Lorentzian cylinder by $\frac{\partial}{\partial \theta}$, on the warped product in Example 3 by $\frac{d}{dt}$, on Schwarzschild exterior spacetime by $\frac{\partial}{\partial t}$, on Schwarzschild black hole by $\frac{\partial}{\partial r}$, and finally on anti-deSitter spacetime by $\frac{\partial}{\partial t}$.

From now on let (M, g) denote a spacetime of dimension $n \geq 2$. Then for each point $p \in M$ the tangent space T_pM is a vector space equipped with the Lorentzian scalar product g_p and the time-orientation induced by the lightlike vector $\tau(p)$, and in (T_pM, g_p) the notions of timelike, lightlike, causal, spacelike, future-directed vectors are defined as explained in Sect. 2.1.

Definition 2. *A continuous piecewise C^1-curve in M is called* timelike, lightlike, causal, spacelike, future-directed*, or* past-directed *if all its tangent vectors are timelike, lightlike, causal, spacelike, future-directed, or past-directed, respectively.*

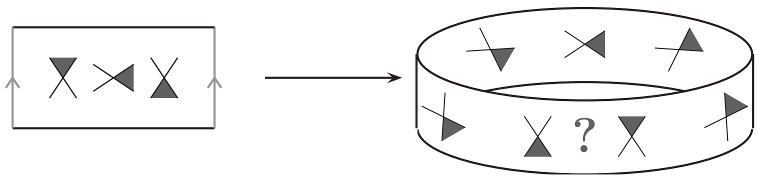


Fig. 2.7 Lorentzian manifold which is orientable, but not time-orientable

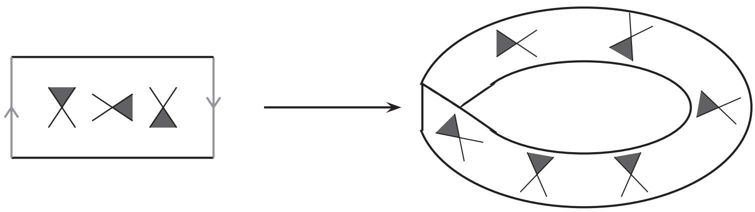


Fig. 2.8 Lorentzian manifold which is not orientable, but time-orientable

The *causality relations* on M are defined as follows: Let $p, q \in M$, then one has

$$\begin{aligned} p \ll q &:\Longleftrightarrow \text{there is a future-directed timelike curve in } M \text{ from } p \text{ to } q, \\ p < q &:\Longleftrightarrow \text{there is a future-directed causal curve in } M \text{ from } p \text{ to } q, \\ p \le q &:\Longleftrightarrow p < q \text{ or } p = q. \end{aligned}$$

These causality relations are transitive as two causal (timelike) curves in M, say the one from p_1 to p_2 and the other from p_2 to p_3, can be put together to a piecewise causal (timelike) C^1-curve from p_1 to p_3.

Definition 3. *The* chronological future $I_+^M(x)$ *of a point* $x \in M$ *is the set of points that can be reached from* x *by future-directed timelike curves, i.e.,*

$$I_+^M(x) = \{y \in M \mid x < y\}.$$

Similarly, the causal future $J_+^M(x)$*of a point* $x \in M$ *consists of those points that can be reached from* x *by future-directed causal curves and of* x *itself:*

$$J_+^M(x) = \{y \in M \mid x \le y\}.$$

The chronological future *of a subset* $A \subset M$ *is defined to be*

$$I_+^M(A) := \bigcup_{x \in A} I_+^M(x).$$

Similarly, the causal future *of* A *is*

$$J_+^M(A) := \bigcup_{x \in A} J_+^M(x).$$

The chronological past $I_-^M(x)$ *resp.* $I_-^M(A)$ *and the causal past* $J_-^M(x)$ *resp.* $J_-^M(A)$ *are defined by replacing future-directed curves by past-directed curves.*

For $A \subset M$ one also uses the notation

$$J^M(A) = J_+^M(A) \cup J_-^M(A).$$

Remark 2. Evidently, for any $A \subset M$ one gets the inclusion $A \cup I_+^M(A) \subset J_+^M(A)$.

Example 6. We consider Minkowski space $(\mathbb{R}^2, g_{Mink})$. Then for $p \in \mathbb{R}^2$ the chronological future $I_+^{\mathbb{R}^2}(p) \subset \mathbb{R}^2$ is an open subset, and for a compact subset A of the x_2-axis the causal past $J_-^{\mathbb{R}^2}(A) \subset \mathbb{R}^2$ is a closed subset, as indicated in Fig. 2.9.

Example 7. By Example 1 every open subset of Minkowski space forms a Lorentzian manifold. Let M be two-dimensional Minkowski space with one point removed. Then there are subsets $A \subset M$ whose causal past is not closed as one can see in Fig. 2.10.

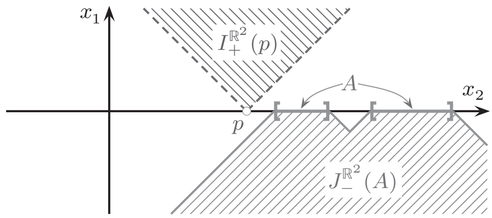


Fig. 2.9 Chronological future $I_+^{\mathbb{R}^2}(p)$ and causal past $J_-^{\mathbb{R}^2}(A)$

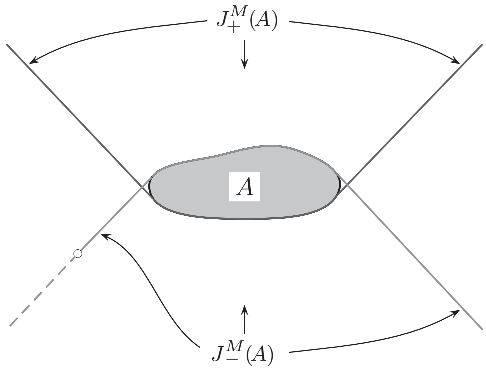


Fig. 2.10 Causal future and past of subset A of M

Example 8. If one unwraps Lorentzian cylinder $(M, g) = (S^1 \times \mathbb{R}, -d\theta^2 + d{x_1}^2)$ one can think of M as a strip in Minkowski space $\mathbb{R}^2$ for which the upper and lower boundaries are identified. In this picture it can easily be seen that $I_+^M(p) = J_+^M(p) = I_-^M(p) = J_-^M(p) = M$ for any $p \in M$; see Fig. 2.11.

Any connected open subset Ω of a spacetime M is a spacetime in its own right if one restricts the Lorentzian metric of M to Ω. Therefore $J_+^\Omega(x)$ and $J_-^\Omega(x)$ are well defined for $x \in \Omega$.

Definition 4. *A domain $\Omega \subset M$ in a spacetime is called* causally compatible *if for all points $x \in \Omega$ one has*

$$J_\pm^\Omega(x) = J_\pm^M(x) \cap \Omega.$$

Note that the inclusion "$\subset$" always holds. The condition of being causally compatible means that whenever two points in Ω can be joined by a causal curve in M this can also be done inside Ω (Fig. 2.12).

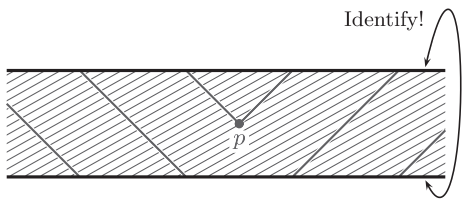


Fig. 2.11 $J_+^M(p) = M$

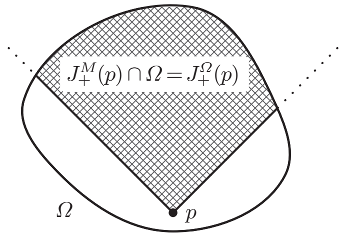


Fig. 2.12 Causally compatible subset of Minkowski space

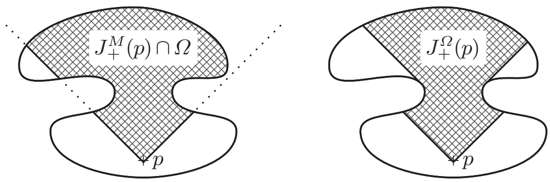


Fig. 2.13 Domain which is not causally compatible in Minkowski space

If $\Omega \subset M$ is a causally compatible domain in a spacetime, then we immediately see that for each subset $A \subset \Omega$ we have

$$J_\pm^\Omega(A) = J_\pm^M(A) \cap \Omega.$$

Note also that being causally compatible is transitive: If $\Omega \subset \Omega' \subset \Omega''$, if Ω is causally compatible in Ω', and if Ω' is causally compatible in Ω'', then so is Ω in Ω''.

Next, we recall the definition of the exponential map: For $p \in M$ and $\xi \in T_pM$ let c_ξ denote the (unique) geodesic with initial conditions $c_\xi(0) = p$ and $\dot{c}_\xi(0) = \xi$. One considers the set

$$\mathcal{D}_p = \left\{ \xi \in T_pM \,\middle|\, c_\xi \text{ can be defined at least on } [0,1] \right\} \subset T_pM$$

and defines the *exponential map* $\exp_p : \mathcal{D}_p \to M$ by $\exp_p(\xi) = c_\xi(1)$.

One important feature of the exponential map is that it is an isometry in radial direction which is the statement of the following lemma.

Lemma 1 (Gauss Lemma). *Let $\xi \in \mathcal{D}_p$ and $\zeta_1, \zeta_2 \in T_\xi(T_pM) = T_pM$ with ζ_1 radial, i.e., there exists $t_0 \in \mathbb{R}$ with $\zeta_1 = t_0\xi$, then*

$$g_{\exp_p(\xi)}\left(d\exp_p\big|_\xi(\zeta_1), d\exp_p\big|_\xi(\zeta_2) \right) = g_p(\zeta_1, \zeta_2).$$

A proof of the Gauss lemma can be found, e.g., in [1, Chap. 5, p. 126f.].

Lemma 2. *Let $p \in M$ and $b > 0$ and let $\widetilde{c} : [0,b] \longrightarrow T_pM$ be a piecewise smooth curve with $\widetilde{c}(0) = 0$ and $\widetilde{c}(t) \in \mathcal{D}_p$ for any $t \in [0,b]$. Suppose that $c := \exp_p \circ \widetilde{c} : [0,b] \longrightarrow M$ is a timelike future-directed curve, then*

$$\widetilde{c}(t) \in I_+(0) \subset T_pM \quad \textit{for any } t \in [0,b].$$

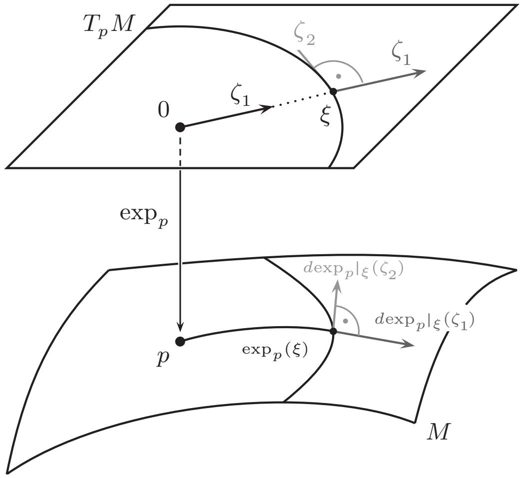


Fig. 2.14 In radial direction the exponential map preserves orthogonality

Proof. Suppose in addition that $\widetilde{c}$ is smooth. On T_pM we consider the quadratic form induced by the Lorentzian scalar product $\gamma : T_pM \longrightarrow \mathbb{R}$, $\gamma(\xi) = -g_p(\xi, \xi)$, and we compute $\operatorname{grad}\gamma(\xi) = -2\xi$. The Gauss lemma applied for $\xi \in \mathcal{D}_p$ and $\zeta_1 = \zeta_2 = 2\xi$ yields

$$g_{\exp_p(\xi)}\Big(d\exp_p\big|_\xi(\zeta_1), d\exp_p\big|_\xi(\zeta_2)\Big) = g_p(\zeta_1, \zeta_2) = -4\gamma(\xi).$$

Denote $P(\xi) = d\exp_p|_\xi(2\xi)$. Then by the above formula $P(\xi)$ is timelike whenever ξ is timelike.

From $\widetilde{c}(0) = 0$ and $(d/dt)\widetilde{c}(0) = d\exp_p|_0((d/dt)\widetilde{c}(0)) = \dot{c}(0) \in I_+(0)$ we get for a sufficiently small $\varepsilon > 0$ that $\widetilde{c}(t) \in I_+(0)$ for all $t \in (0, \varepsilon)$. Hence $P(\widetilde{c}(t))$ is timelike and future-directed for $t \in (0, \varepsilon)$.

For $\xi = \widetilde{c}(t)$, $\zeta_1 = 2\xi = -\operatorname{grad}\gamma(\xi)$ and $\zeta_2 = (d/dt)\widetilde{c}(t)$ the Gauss lemma gives

$$\frac{d}{dt}(\gamma \circ \widetilde{c})(t) = -g_p(\zeta_1, \zeta_2) = -g_{\exp_p(\xi)}\big(P(\widetilde{c}(t)), \dot{c}(t)\big).$$

If there were $t_1 \in (0, b]$ with $\gamma(\widetilde{c}(t_1)) = 0$, w.l.o.g. let t_1 be the smallest value in $(0, b]$ with $\gamma(\widetilde{c}(t_1)) = 0$, then one could find a $t_0 \in (0, t_1)$ with

$$0 = \frac{d}{dt}(\gamma \circ \widetilde{c})(t_0) = -g_{\exp_p(\xi)}\big(P(\widetilde{c}(t_0)), \dot{c}(t_0)\big).$$

On the other hand, having chosen t_1 minimally implies that $P(\widetilde{c}(t_0))$ is timelike and future-directed. Together with $\dot{c}(t_0) \in I_+^M(c(t_0))$ this yields $g_{\exp_p(\xi)}\big(P(\widetilde{c}(t_0)), \dot{c}(t_0)\big) < 0$, a contradiction.

Hence one has $\gamma(\widetilde{c}(t)) > 0$ for any $t \in (0, b]$, and the continuous curve $\widetilde{c}|_{(0,b]}$ does not leave the connected component of $I(0)$ in which it runs initially. This finishes the proof if one supposes that $\widetilde{c}$ is smooth. For the proof in the general case see [1, Chap. 5, Lemma 33]. $\square$

Definition 5. *A domain $\Omega \subset M$ is called* geodesically starshaped *with respect to a fixed point $p \in \Omega$ if there exists an open subset $\Omega' \subset T_pM$, starshaped with respect to* 0*, such that the Riemannian exponential map* $\exp_x$ *maps Ω' diffeomorphically onto Ω.*

One calls a domain $\Omega \subset M$ geodesically convex *(or simply* convex*) if it is geodesically starshaped with respect to all of its points.*

Remark 3. Every point of a Lorentzian manifold (which need not necessarily be a spacetime) possesses a convex neighborhood, see [1, Chap. 5, Prop. 7]. Furthermore, for each open covering of a Lorentzian manifold one can find a refinement consisting of convex open subsets, see [1, Chap. 5, Lemma 10].

Sometimes sets that are geodesically starshaped with respect to a point p are useful to get relations between objects defined in the tangent T_pM and objects defined on M. For the moment this will be illustrated by the following lemma.

Lemma 3. *Let M a spacetime and $p \in M$. Let the domain $\Omega \subset M$ be a geodesically starshaped with respect to p (Fig. 2.15). Let Ω' be an open neighborhood of* 0 *in T_pM such that Ω' is starshaped with respect to* 0 *and* $\exp_p|_{\Omega'} : \Omega' \longrightarrow \Omega$ *is a diffeomorphism. Then one has*

$$I_\pm^\Omega(p) = \exp_p\left(I_\pm(0) \cap \Omega'\right) \quad \textit{and}$$
$$J_\pm^\Omega(p) = \exp_p\left(J_\pm(0) \cap \Omega'\right).$$

Proof. We will only prove the equation $I_+^\Omega(p) = \exp_p\left(I_+(0) \cap \Omega'\right)$.

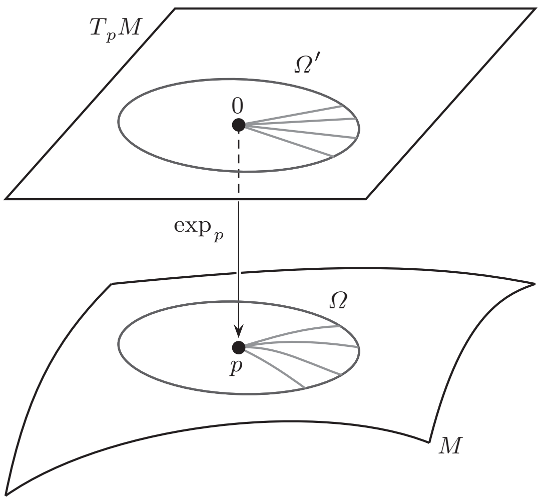


Fig. 2.15 Ω is geodesically starshaped with respect to p

For $q \in I_+^{\Omega}(p)$ one can find a future-directed timelike curve $c : [0, b] \to \Omega$ from p to q. We define the curve $\widetilde{c} : [0, b] \to \Omega' \subset T_pM$ by $\widetilde{c} = {\exp_p}^{-1} \circ c$ and get from Lemma 2 that $\widetilde{c}(t) \in I_+(0)$ for $0 < t \le b$, in particular ${\exp_p}^{-1}(q) = \widetilde{c}(b) \in I_+(0)$. This shows the inclusion $I_+^{\Omega}(p) \subset \exp_p\left(I_+(0) \cap \Omega'\right)$.

For the other inclusion we consider $\xi \in I_+(0) \cap \Omega'$. Then the map $t \longmapsto t \cdot \xi$ takes its values in $I_+(0) \cap \Omega'$ as $t \in [0, 1]$. Therefore $\exp_p(t\xi)$ gives a timelike future-directed geodesic which stays in Ω as $t \in [0, 1]$, and it follows that $\exp_p(\xi) \in I_+^{\Omega}(p)$.

For a proof of $J_\pm^{\Omega}(p) = \exp_p\left(J_\pm(0) \cap \Omega'\right)$ we refer to [1, Chap. 14, Lemma 2]. □

For Ω and Ω' as in Lemma 3 we put $C_\pm^{\Omega}(p) = \exp_p(C_\pm(0) \cap \Omega')$.

Proposition 1. *On any spacetime M the relation "$\ll$" is open, this means that for every $p, q \in M$ with $p \ll q$ there are open neighborhoods U and V of p and q, respectively, such that for any $p' \in U$ and $q' \in V$ one has $p' \ll q'$ (Fig. 2.16).*

Proof. For $p, q \in M$ with $p \ll q$ there are geodesically convex neighborhoods $\widetilde{U}$, $\widetilde{V}$, respectively. We can find a future-directed timelike curve c from p to q. Then we choose $\widetilde{p} \in \widetilde{U}$ and $\widetilde{q} \in \widetilde{V}$ sitting on c such that $p \ll \widetilde{p} \ll \widetilde{q} \ll q$. As $\widetilde{U}$ is starshaped with respect to $\widetilde{p}$ there is a starshaped open neighborhood $\widetilde{\Omega}$ of 0 in $T_{\widetilde{p}}M$ such that $\exp_{\widetilde{p}} : \widetilde{\Omega} \to \widetilde{U}$ is a diffeomorphism. We set $U = I_-^{\widetilde{U}}(\widetilde{p})$, and Lemma 3 shows that $U = \exp_{\widetilde{p}}(I_-(0) \cap \widetilde{\Omega})$ is an open neighborhood of p in M. Analogously, one finds that $V = I_+^{\widetilde{V}}(\widetilde{q})$ is an open neighborhood of q. Finally, for any $p' \in U$ and $q' \in V$ one gets $p' \ll \widetilde{p} \ll \widetilde{q} \ll q'$ and hence $p' \ll q'$. □

Corollary 1. *For an arbitrary $A \subset M$ the chronological future $I_+^M(A)$ and the chronological past $I_-^M(A)$ are open subsets in M.*

Proof. Proposition 1 implies that for any $p \in M$ the subset $I_+^M(p) \subset M$ is open, and therefore $I_+^M(A) = \bigcup_{p \in A} I_+^M(p)$ is an open subset of M as well. □

On an arbitrary spacetime there is no similar statement for the relation "$\le$." Example 7 shows that even for closed sets $A \subset M$ the chronological future and past are

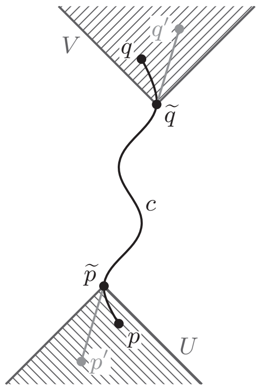


Fig. 2.16 The relation "$\ll$" is open

not always closed. In general one only has that $I_{\pm}^{M}(A)$ is the interior of $J_{\pm}^{M}(A)$ and that $J_{\pm}^{M}(A)$ is contained in the closure of $I_{\pm}^{M}(A)$.

Definition 6. *A domain Ω is called* causal *if its closure $\overline{\Omega}$ is contained in a convex domain Ω' and if for any $p, q \in \overline{\Omega}$ the intersection $J_{+}^{\Omega'}(p) \cap J_{-}^{\Omega'}(q)$ is compact and contained in $\overline{\Omega}$.*

Causal domains appear in the theory of wave equations: The local construction of fundamental solutions is always possible on causal domains provided their volume is small enough, see Proposition 3 on page 71.

Remark 4. Any point $p \in M$ in a spacetime possesses a causal neighborhood, compare [4, Theorem 4.4.1], and given a neighborhood $\widetilde{\Omega}$ of p, one can always find a causal domain Ω with $p \in \Omega \subset \widetilde{\Omega}$ (Fig. 2.17).

The last notion introduced in this section is needed if it comes to the discussion of uniqueness of solutions for wave equations:

Definition 7. *A subset $A \subset M$ is called* past-compact *if $A \cap J_{-}^{M}(p)$ is compact for all $p \in M$. Similarly, one defines* future-compact *subsets (Fig. 2.18).*

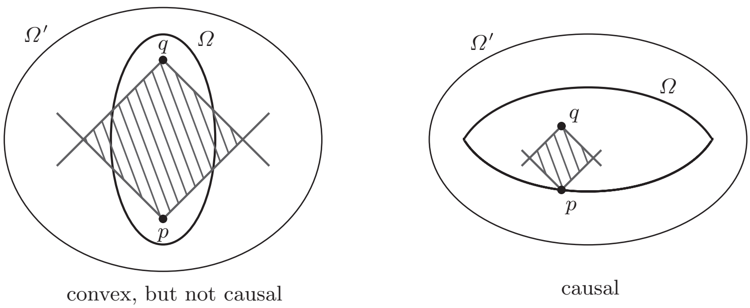


Fig. 2.17 Convexity versus causality

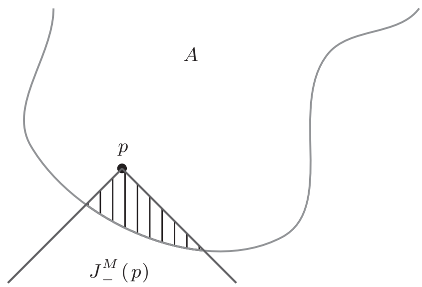


Fig. 2.18 The subset A is past-compact

2.4 Causality Condition and Global Hyperbolicity

In general relativity worldlines of particles are modeled by causal curves. If now the spacetime is compact something strange happens.

Proposition 2. *If the spacetime M is compact, there exists a closed timelike curve in M.*

Proof. The family $\{I_+^M(p)\}_{p\in M}$ is an open covering of M. By compactness one has $M = I_+^M(p_1) \cup \cdots \cup I_+^M(p_k)$ for suitably chosen $p_1, \ldots, p_k \in M$. We can assume that $p_1 \notin I_+^M(p_2) \cup \cdots \cup I_+^M(p_k)$, otherwise $p_1 \in I_+^M(p_m)$ for an $m \geq 2$ and hence $I_+^M(p_1) \subset I_+^M(p_m)$ and we can omit $I_+^M(p_1)$ in the finite covering. Therefore we can assume $p_1 \in I_+^M(p_1)$, and there is a timelike future-directed curve starting and ending in p_1. □

In spacetimes with timelike loops one can produce paradoxes as travels into the past (like in science fiction). Therefore one excludes compact spacetimes, for physically reasonable spacetimes one requires the causality condition or the strong causality condition (Fig. 2.19).

Definition 8. *A spacetime is said to satisfy the* causality condition *if it does not contain any closed causal curve.*

A spacetime M is said to satisfy the strong causality condition *if there are no almost closed causal curves. More precisely, for each point $p \in M$ and for each open neighborhood U of p there exists an open neighborhood $V \subset U$ of p such that each causal curve in M starting and ending in V is entirely contained in U.*

Obviously, the strong causality condition implies the causality condition.

Example 9. In Minkowski space $(\mathbb{R}^n, g_{Mink})$ the strong causality condition holds. One can prove this as follows: Let U be an open neighborhood of $p=(p_1, \ldots, p_n) \in \mathbb{R}^n$. For any $\delta > 0$ denote the open cube with center p and edges of length 2δ by $W_\delta = (p_1-\delta, p_1+\delta)\times\cdots\times(p_n-\delta, p_n+\delta)$. Then there is an $\varepsilon > 0$ with $W_{2\varepsilon} \subset U$, and one can put $V = W_\varepsilon$. Observing that any causal curve $c = (c_1, \ldots, c_n)$ in $\mathbb{R}^n$ satisfies $(\dot{c_1})^2 \geq (\dot{c_2})^2 + \cdots + (\dot{c_n})^2$ and $(\dot{c_1})^2 > 0$, we can conclude that any causal curve starting and ending in $V = W_\varepsilon$ cannot leave $W_{2\varepsilon} \subset U$.

Remark 5. Let M satisfy the (strong) causality condition and consider any open connected subset $\Omega \subset M$ with induced Lorentzian metric as a spacetime. Then non-existence of (almost) closed causal curves in M directly implies non-existence of such curves in Ω, and hence also Ω satisfies the (strong) causality condition.

Example 10. In the Lorentzian cylinder $S^1 \times \mathbb{R}$ the causality condition is violated. If one unwraps $S^1 \times \mathbb{R}$ as in Example 8 it can be easily seen that there are closed timelike curves (Fig. 2.20).

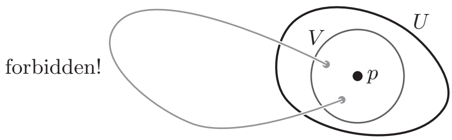


Fig. 2.19 Strong causality condition

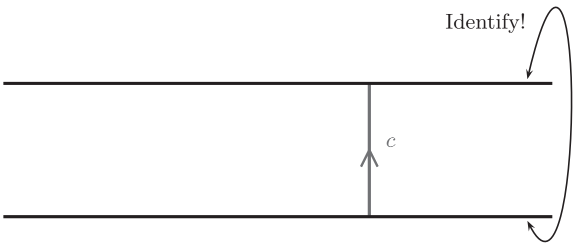


Fig. 2.20 Closed timelike curve c in Lorentzian cylinder

Example 11. Consider the spacetime M which is obtained from the Lorentzian cylinder by removing two spacelike half-lines G_1 and G_2 whose endpoints can be joined by a short lightlike curve, as indicated in Fig. 2.21. Then the causality condition holds for M, but the strong causality condition is violated: For any p on the short lightlike curve and any arbitrarily small neighborhood of p there is a causal curve which starts and ends in this neighborhood but is not entirely contained.

Definition 9. *A spacetime M is called a* globally hyperbolic manifold if it satisfies the strong causality condition and if for all $p, q \in M$ the intersection $J_+^M(p) \cap J_-^M(q)$ is compact.

The notion of global hyperbolicity has been introduced by J. Leray in [5]. Globally hyperbolic manifolds are interesting because they form a large class of spacetimes on which wave equations possess a very satisfying global solution theory; see Chap. 3.

Example 12. In Minkowski space $(\mathbb{R}^n, g_{Mink})$ for any $p, q \in \mathbb{R}^n$ both $J_+^{\mathbb{R}^n}(p)$ and $J_-^{\mathbb{R}^n}(q)$ are closed. Furthermore $J_+^{\mathbb{R}^n}(p) \cap J_-^{\mathbb{R}^n}(q)$ is bounded (with respect to Euclidean norm), and hence compact. In Example 9 we have already seen that for $(\mathbb{R}^n, g_{Mink})$ the strong causality condition holds. Hence, Minkowski space is globally hyperbolic.

Example 13. As seen before, the Lorentzian cylinder $M = S^1 \times \mathbb{R}$ does not fulfill the strong causality condition and is therefore not globally hyperbolic. Furthermore the compactness condition in Definition 9 is violated because one has $J_+^M(p) \cap J_-^M(q) = M$ for any $p, q \in M$.

Example 14. Consider the subset $\Omega = \mathbb{R} \times (0, 1)$ of two-dimensional Minkowski space $(\mathbb{R}^2, g_{Mink})$. By Remark 5 the strong causality condition holds for Ω, but there

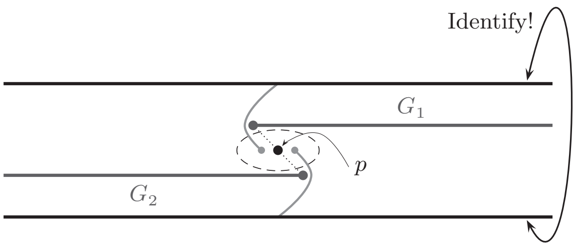


Fig. 2.21 Causality condition holds but strong causality condition is violated

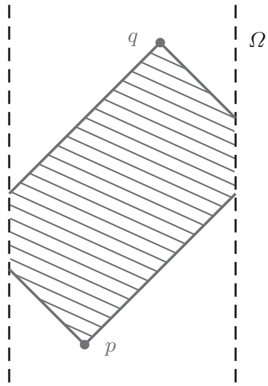


Fig. 2.22 $J_+^{\Omega}(p) \cap J_-^{\Omega}(q)$ is not always compact in the strip $\Omega = \mathbb{R} \times (0, 1)$

are points $p, q \in \Omega$ for which the intersection $J_+^{\Omega}(p) \cap J_-^{\Omega}(q)$ is not compact, see Fig. 2.22.

Example 15. The n-dimensional anti-deSitter spacetime $(\mathbb{R} \times S_+^{n-1}, g_{AdS})$ is not globally hyperbolic (Fig. 2.23). As seen in Example 5, a curve in $M = \mathbb{R} \times S_+^{n-1}$ is causal with respect to g_{AdS} if and only if it is so with respect to the Lorentzian metric $-dt^2 + \mathrm{can}_{S_+^{n-1}}$. Hence for both g_{AdS} and $-dt^2 + \mathrm{can}_{S_+^{n-1}}$. one gets the same causal futures and pasts. A similar picture as in Example 14 then shows that for $p, q \in M$ the intersection $J_+^M(p) \cap J_-^M(q)$ need not be compact.

In general one does not know much about causal futures and pasts in spacetime. For globally hyperbolic manifold one has the following lemma (see [1, Chap. 14, Lemma 22]).

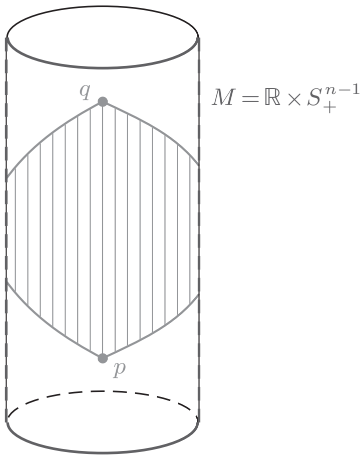


Fig. 2.23 $J_+^M(p) \cap J_-^M(q)$ is not compact in anti-deSitter spacetime

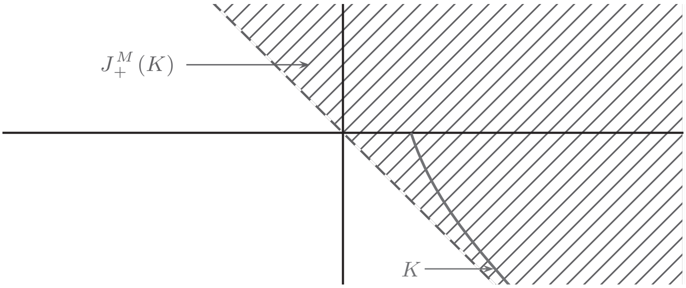


Fig. 2.24 For K is closed $J_+^M(K)$ need not be open

Lemma 4. *In any globally hyperbolic manifold M the relation "$\leq$" is closed, i.e., whenever one has convergent sequences $p_i \to p$ and $q_i \to q$ in M with $p_i \leq q_i$ for all i, then one also has $p \leq q$.*

Therefore in globally hyperbolic manifolds for any $p \in M$ and any compact set $K \subset M$ one has that $J_\pm^M(p)$ and $J_\pm^M(K)$ are closed.

If K is only assumed to be closed, then $J_\pm^M(K)$ need not be closed. In Fig. 2.24 a curve K is shown which is closed as a subset and asymptotic to to a lightlike line in two-dimensional Minkowski space. Its causal future $J_+^M(K)$ is the open half-plane bounded by this lightlike line.

2.5 Cauchy Hypersurfaces

We recall that a piecewise C^1-curve in M is called *inextendible*, if no piecewise C^1-reparametrization of the curve can be continuously extended beyond any of the end points of the parameter interval.

Definition 10. *A subset S of a connected time-oriented Lorentzian manifold is called* achronal *(or* acausal*) if and only if each timelike (or causal, respectively) curve meets S at most once.*

A subset S of a connected time-oriented Lorentzian manifold is a Cauchy hypersurface *if each inextendible timelike curve in M meets S at exactly one point.*

Obviously every acausal subset is achronal, but the reverse is wrong. However, every achronal spacelike hypersurface is acausal (see [1, Chap. 14, Lemma 42]). Any Cauchy hypersurface is achronal. Moreover, it is a closed topological hypersurface and it is hit by each inextendible causal curve in at least one point. Any two Cauchy hypersurfaces in M are homeomorphic. Furthermore, the causal future and past of a Cauchy hypersurface is past- and future-compact, respectively. This is a consequence of, e.g., [1, Chap. 14, Lemma 40].

Example 16. In Minkowski space $(\mathbb{R}^n, g_{Mink})$ consider a spacelike hyperplane A_1, hyperbolic spaces $A_2 = \{x = (x_1, \dots, x_n) \mid \langle\langle x, x\rangle\rangle = -1 \text{ and } x_1 > 0\}$ and $A_3 = \{x = (x_1, \dots, x_n) \mid \langle\langle x, x\rangle\rangle = 0, x_1 \geq 0\}$. Then all A_1, A_2, and A_3 are achronal, but only A_1 is a Cauchy hypersurface; see Fig. 2.25.

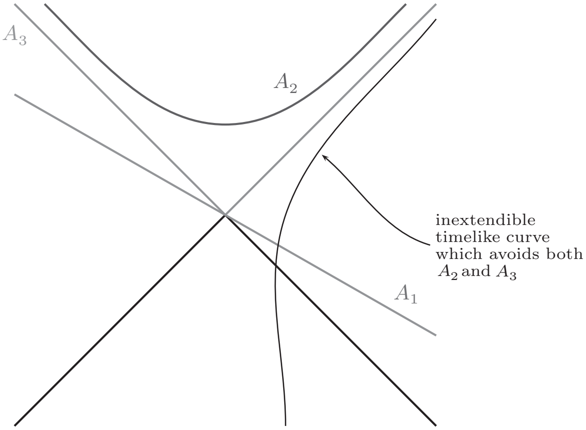


Fig. 2.25 Achronal subsets A_1, A_2, and A_3 in Minkowski space

Example 17. Let (N, h) be a connected Riemannian manifold, let $I \subset \mathbb{R}$ be an open interval and $f : I \to (0, \infty)$ a smooth function. Consider on $M = I \times N$ the warped product metric $g = -dt^2 + f(t) \cdot h$. Then $\{t_0\} \times N$ is a Cauchy hypersurface in (M, g) for any $t_0 \in I$ if and only if the Riemannian manifold (N, h) is complete (compare [3, Lemma A.5.14]).

In particular, in any Robertson–Walker spacetime one can find a Cauchy hypersurface.

Example 18. Let N be exterior Schwarzschild spacetime N and B Schwarzschild black hole, both of mass m, as defined in Example 4. Then for any $t_0 \in \mathbb{R}$ a Cauchy hypersurface of N is given by $(2m, \infty) \times \{t_0\} \times S^2$, and in B one gets a Cauchy hypersurface by $\{r_0\} \times \mathbb{R} \times S^2$ for any $0 < r_0 < 2m$.

Definition 11. *The* Cauchy development *(Fig. 2.26) of a subset S of a spacetime M is the set $D(S)$ of points of M through which every inextendible causal curve in M meets S, i.e.,*

$$D(S) = \left\{p \in M \,\middle|\, \textit{every inextendible causal curve passing through } p \textit{ meets } S\right\}.$$

Remark 6. It follows from the definition that $D(D(S)) = D(S)$ for every subset $S \subset M$. Hence if $T \subset D(S)$, then $D(T) \subset D(D(S)) = D(S)$.

Of course, if S is achronal, then every inextendible causal curve in M meets S at most once. The Cauchy development $D(S)$ of every *acausal* hypersurface S is open, see [1, Chap. 14, Lemma 43].

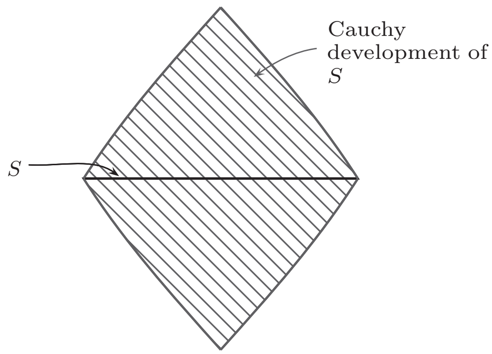
If $S \subset M$ is a Cauchy hypersurface, then obviously $D(S) = M$.

For a proof of the following proposition, see [1, Chap. 14, Thm. 38].

Proposition 3. *For any achronal subset $A \subset M$ the interior* $\mathrm{int}(D(A))$ *of the Cauchy development is globally hyperbolic (if nonempty).*

From this we conclude that a spacetime is globally hyperbolic if it possesses a Cauchy hypersurface. In view of Examples 17 and 18, this shows that Robertson–

Fig. 2.26 Cauchy development



Walker spacetimes, Schwarzschild exterior spacetime, and Schwarzschild black hole are all globally hyperbolic.

The following theorem is very powerful and describes the structure of globally hyperbolic manifolds explicitly: they are foliated by *smooth spacelike* Cauchy hypersurfaces.

Theorem 1. *Let M be a connected time-oriented Lorentzian manifold. Then the following are equivalent:*

(1) M is globally hyperbolic.
(2) There exists a Cauchy hypersurface in M.
(3) M is isometric to $\mathbb{R} \times S$ with metric $-\beta dt^2 + g_t$ where β is a smooth positive function, g_t is a Riemannian metric on S depending smoothly on $t \in \mathbb{R}$ and each $\{t\} \times S$ is a smooth spacelike Cauchy hypersurface in M.

Proof. The crucial point in this theorem is that (1) implies (3). This has been shown by A. Bernal and M. Sánchez in [6, Theorem 1.1] using work of R. Geroch [7, Theorem 11]. See also [8, Preposition 6.6.8] and [2, p. 209] for earlier mentionings of this fact. That (3) implies (2) is trivial, and Proposition 3 provides the implication (2)⇒(1). □

Corollary 2. *On every globally hyperbolic Lorentzian manifold M there exists a smooth function $h : M \to \mathbb{R}$ whose gradient is past-directed timelike at every point and all of whose level sets are spacelike Cauchy hypersurfaces.*

Proof. Define h to be the composition $t \circ \Phi$ where $\Phi : M \to \mathbb{R} \times S$ is the isometry given in Theorem 1 and $t : \mathbb{R} \times S \to \mathbb{R}$ is the projection onto the first factor. □

Such a function h on a globally hyperbolic Lorentzian manifold is called a *Cauchy time function*. Note that a Cauchy time function is strictly monotonically increasing along any future-directed causal curve.

We conclude with an enhanced form of Theorem 1, due to A. Bernal and M. Sánchez (see [9, Theorem 1.2]).

Theorem 2. *Let M be a globally hyperbolic manifold and S be a spacelike smooth Cauchy hypersurface in M. Then there exists a Cauchy time function $h : M \to \mathbb{R}$ such that $S = h^{-1}(\{0\})$.* $\square$

Any given smooth spacelike Cauchy hypersurface in a (necessarily globally hyperbolic) spacetime is therefore the leaf of a foliation by smooth spacelike Cauchy hypersurfaces.

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