Chapter 2
Modal Logics

2.1 Introduction

This chapter introduces modal and continuous time stochastic logics. We discuss these logics here in some detail since we want to define the classical notions of bisimilarity, logical and behavioral equivalence for logics the interpretation of which is well understood. This will enable us later on to draw from this source knowledge as well as experience when defining these terms for coalgebraic logics, and when investigating them. We indicate the results and give sketches for the more interesting proofs, which we will need in later chapters as well. These considerations make substantial use of the techniques that have been developed in the context of stochastic relations, about which we reported in Chapter 1, in particular in Sections 1.6 and 1.7. We will return to the topics dealt with in the present chapter mainly in Chapter 4, when coalgebraic logics are investigated. It will turn out that nearly all results are special cases of a coalgebraic scenario. This might advise against dealing with, e.g., modal logics and continuous time stochastic logics on this level of detail. On the other hand it turns out that the approaches for the general, coalgebraic case are essentially motivated by these specific logics, and that these logics serve as excellent examples for the somewhat abstract treatment in Chapter 4. For instance, we will study residence times in some detail in the present chapter, and this will help in our appreciating the properties of some important functors when dealing with stochastic right coalgebras in Section 4.4.

We first deal with modal logics which are introduced fairly generally through modal similarity types. Kripke models are introduced in both their non-deterministic and their stochastic versions, and we give examples from well-known logics illustrating these models and highlighting their different approaches. Before jumping into this discussion in Section 2.3, however, we define in Section 2.2 bisimilar stochastic relations in terms of spans of morphisms and give a criterion for them to be bisimilar; this is based on simu-

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lation equivalent congruences. Because each modal logic defines congruences on the state space of a Kripke model, we put this criterion to work when looking into the relationship of bisimilarity, behavioral equivalence — which is defined in terms of a cospan of morphisms — and logical equivalence in Section 2.3.2. Investigating congruences like that is also at the heart of looking into the same relationship when discussing continuous time stochastic logic in Section 2.4. This is a logic akin to the well-known logic **CTL** used for model checking which takes time explicitly into consideration. We augment the logic by adding a fixed-point operator, partly for indicating that the techniques we present here are so flexible that adding a substantially different operator does not affect them, but also to prepare for things to come when discussing coalgebraic stochastic logics which contain infinitesimal operators as abstractions for fixed point operators. The discussion of the continuous time logics is technically a bit more involved than the one for the conceptually simpler modal logic because we need to argue on two levels, viz., on the level of states — here we will define state formulas — and on the level of infinite paths, which will be captured through path formulas; see Section 2.4.5. This foreshadows the development for some coalgebraic generalization as well; see, e.g., Section 4.3.

The models are defined —nearly by default — over analytic spaces, and sometimes even over Polish ones. We show in an appendix that weaker results can be obtained already in Kripke models that are defined over general measurable spaces. It will be shown that two such Kripke models are logically equivalent iff they are behaviorally equivalent; this will be established for a very simple negation-free Hennessy-Milner logic.

2.2 Bisimulations

Bisimulations are introduced as spans of morphisms such that common events exist. They relate two systems in terms of their elements, and hence in terms of nondeterministic relations of their state spaces.

In fact, assume that $(S, (\rightarrow_a)_{a \in A})$ and $(T, (\rightarrow'_a)_{a \in A})$ are two labeled transition systems; then a relation $R \subseteq S \times T$ is called a bisimulation iff

- Whenever $\langle s, t \rangle \in R$ and $s \rightarrow_a s_1$, then there exists t_1 with $t \rightarrow'_a t_1$ and $\langle s_1, t_1 \rangle \in R$.
- Whenever $\langle s, t \rangle \in R$ and $t \rightarrow'_a t_1$, then there exists s_1 with $s \rightarrow_a s_1$ and $\langle s_1, t_1 \rangle \in R$.

We interpret a labeled transition system as a coalgebra (S, α) for the functor $\mathfrak{F} := \mathcal{P}(A \times -)$ which sends the set S to $\mathcal{P}(A \times S)$ and the map $f : S \rightarrow T$ to $\mathcal{P}(A \times S) \ni Q \mapsto \{\langle a, f(s) \rangle \mid \langle a, s \rangle \in Q\} \in \mathcal{P}(A \times T)$. It is an easy exercise to show that R is a bisimulation iff there exists a coalgebraic structure xi on R such that this diagram commutes:

$$\begin{tikzcd}
S \arrow[d, "\alpha"'] & R \arrow[l, "\pi_S"'] \arrow[r, "\pi_T"] \arrow[d, "xi"] & T \arrow[d, "\beta"] \\
\mathfrak{F}(S) & \mathfrak{F}(R) \arrow[l, "\mathfrak{F}(\pi_S)"] \arrow[r, "\mathfrak{F}(\pi_T)"'] & \mathfrak{F}(T)
\end{tikzcd}$$

Definition 2.2.1. *The stochastic relations* $\mathsf{K} = (X, Y, K)$ *and* $\mathsf{L} = (V, W, L)$ *are called* bisimilar *iff there exist a stochastic relation* $\mathsf{M} = (A, B, M)$ *and morphisms* $\mathsf{f} = (\phi, \psi) : \mathsf{M} \to \mathsf{L}, \mathsf{g} = (\gamma, \delta) : \mathsf{M} \to \mathsf{L}$ *such that*

a. the diagram

$$\begin{tikzcd}
X \arrow[d, "K"'] & A \arrow[l, "\phi"'] \arrow[r, "\gamma"] \arrow[d, "M"] & V \arrow[d, "L"] \\
\mathfrak{S}(Y) & \mathfrak{S}(B) \arrow[l, "\mathfrak{S}(\psi)"] \arrow[r, "\mathfrak{S}(\delta)"'] & \mathfrak{S}(W)
\end{tikzcd}$$

is commutative,

b. the σ*-algebra* $\psi^{-1}[\mathcal{B}(Y)] \cap \delta^{-1}[\mathcal{B}(W)]$ *is nontrivial, i.e., contains not only* $\emptyset$ *and* B*.*

The relation M *is called* mediating.

The first condition on bisimilarity is in accordance with the general definition of bisimilarity of coalgebras in Section 1.2.1; it requests that f and g form a span of morphisms

$$\mathsf{K} \xleftarrow{\mathsf{f}} \mathsf{M} \xrightarrow{\mathsf{g}} \mathsf{L},$$

so that we have for each $a \in A, D \in \mathcal{B}(Y), E \in \mathcal{B}(W)$ the equalities

$$K\big(\phi(a)\big)(D) = \big(\mathfrak{S}(\psi) \circ M\big)(a)(D) = M(a)\big(\psi^{-1}[D]\big)$$

and

$$L\big(\gamma(a)\big)(E) = \big(\mathfrak{S}(\delta) \circ M\big)(a)(E) = M(a)\big(\delta^{-1}[E]\big).$$

The second condition, however, states that we can find an event $C^* \in \mathcal{B}(B)$ which is common to both K and L in the sense that

$$\psi^{-1}[D] = C^* = \delta^{-1}[E]$$

for some $D \in \mathcal{B}(Y)$ and $E \in \mathcal{B}(W)$ such that both $C^* \neq \emptyset$ and $C^* \neq B$ hold (note that for $C^* = \emptyset$ or $C^* = W$ we can always take the empty and the full set, respectively). Given such a C^* with D and E from above we get for each $a \in A$

$$\begin{aligned} K(\phi(a))(D) &= M(a)(\psi^{-1}[D]) \\ &= M(a)(C^*) \\ &= M(a)(\delta^{-1}[E]) \\ &= L(\gamma(a))(E); \end{aligned}$$

thus the event C^* ties K and L together. Loosely speaking, $\psi^{-1}[\mathcal{B}(Y)] \cap \delta^{-1}[\mathcal{B}(W)]$ can be described as the σ-algebra of common events, which is required to be nontrivial. Note that without the second condition two relations K and L which are strictly probabilistic (i.e., for which the entire space is always assigned probability 1) would always be bisimilar: Put $A := X \times V, B := Y \times W$ and set for $\langle x, v\rangle \in A$ as the mediating relation $M(x, v) := K(x) \otimes L(v)$; then the projections will make the diagram commutative. It is also clear that this argument does not work for the subprobabilistic case. The second condition in Definition 2.2.1 serves to prevent this somewhat anomalous behavior; it is technically not too restrictive, as we will see below.

A criterion for stochastic relations to be bisimilar is derived from simulation-equivalent congruences. They will be introduced now, and the relation to bisimilarity is indicated as well.

Simulation Equivalence

Simulation-equivalent relations behave on their classes in exactly the same fashion. This requires the equivalence classes, in particular the Borel structure on the respective factor spaces, to be related in a suitable way: knowing one factor space and its Borel structure entails detailed knowledge about the Borel structure of the other one, in particular about its generators. This is captured through the idea that one equivalence relation *spawns* the other one — if it is known how to generate the Borel structure on one factor space, then this knowledge is carried over to the other one.

Definition 2.2.2. *Let α and β be smooth equivalence relations on the analytic spaces X resp. Y, and assume that $\Upsilon : X/\alpha \to Y/\beta$ is a map between the equivalence classes. We say that α* spawns *β* via *$(\Upsilon, \mathcal{A}_0)$ iff $\mathcal{A}_0$ is a countable generator of $\mathbf{\Sigma}\big(\mathcal{B}(X), \alpha\big)$ such that*

a. $\mathcal{A}_0$ is closed under finite intersections,
b. $\{\Upsilon_A \mid A \in \mathcal{A}_0\}$ is a generator of $\mathbf{\Sigma}\big(\mathcal{B}(Y), \beta\big)$, where $\Upsilon_A := \bigcup\{\Upsilon([x]_\alpha) \mid x \in A\}$.

Thus if α spawns β, then the measurable structure induced by α on X is all we need for constructing the measurable structure induced by β on Y: the map Υ can be made to carry over the generator $\mathcal{A}_0$ from $\mathbf{\Sigma}\big(\mathcal{B}(X), \alpha\big)$ to $\mathbf{\Sigma}\big(\mathcal{B}(Y), \beta\big)$ and to transport the atoms from one σ-algebra to the other. This is of particular interest since the atoms are just the equivalence classes by Lemma 1.7.11. Hence α together with Υ and the generator $\mathcal{A}_0$ is all we may care to know or to learn about β.

Spawning is used to model congruences that behave in the same fashion; this requires that at least the Borel structures of the underlying factor spaces be comparable. The definition of simulation-equivalent congruences then reads as follows.

Definition 2.2.3. *Let* $\mathsf{K} = (X, Y, K)$ *and* $\mathsf{K}' = (X', Y', K')$ *be stochastic relations over Standard Borel spaces with congruences* $\mathsf{c} = (\alpha, \beta)$ *and* $\mathsf{c}' = (\alpha', \beta')$*, respectively.*

a. Congruence c simulates c' *(symbolically* $\mathsf{c} \propto \mathsf{c}'$*) iff* α *spawns* α' *via* $(\Upsilon, \mathcal{A}_0)$ *and* β *spawns* β' *via* $(\Theta, \mathcal{B}_0)$ *such that*

$$\forall x \in X \forall x' \in \Upsilon([x]_\alpha) \forall B \in \mathcal{B}_0 : K(x)(B) = K'(x')(\Theta_B).$$

b. Call these congruences simulation-equivalent *iff both* $\mathsf{c} \propto \mathsf{c}'$ *and* $\mathsf{c}' \propto \mathsf{c}$ *hold.*

Simulation-equivalent congruences behave in exactly the same way. The same behavior is exhibited on each equivalence class, as far as the input is concerned, and on the respective invariant output sets. It becomes plain at this point that a characterization of equivalent behavior through congruences exhibits the double face of congruences: it is certainly necessary to use the equivalence relation on the input spaces; but since the behavior on the output spaces is modelled through probabilities, we need also the invariant Borel sets for a characterization.

Simulation-equivalent congruences on stochastic relations give rise to a factor object built on their sum, as we have seen in Section 1.7.3. This construction will be used for investigating the bisimilarity of stochastic relations, and later on for a closer discussion of the bisimilarity of Kripke models for a multitude of logics, ranging from general modal logics (Section 2.3.2) to continuous time logics (Section 2.4.5) and to coalgebraic logics (Section 4.3.3).

Assume that c and c' are simulation-equivalent congruences on the Polish objects $\mathsf{K} = (X, Y, K)$ and $\mathsf{K}' = (X', Y', K')$, respectively. Assume that α spawns α' via $(\Upsilon, \{(C_n)_{n\in\mathbb{N}}\})$, and that β spawns β' via $(\Theta, \{(D_n)_{n\in\mathbb{N}}\})$. Construct for K and K' the direct sum

$$\mathsf{K} \oplus \mathsf{K}' := \big(X + X', Y + Y', K \oplus K'\big);$$

where the only non-obvious construction is $K \oplus K'$: put for the Borel set $E \subseteq Y + Y'$

$$(K \oplus K')(z)(E) := \begin{cases} K(z)(E \cap Y), & \text{if } z \in X \\ K'(z)(E \cap Y'), & \text{if } z \in X', \end{cases}$$

then clearly $K \oplus K' : X + X' \rightsquigarrow Y + Y'$ (we omit the injections). Define respectively on $X + X'$ and $Y + Y'$ the σ-algebras

$$\mathcal{G} := \{C + \Upsilon_{C'} \mid C \in \mathbf{\Sigma}(\mathcal{B}(X), \alpha), C' \in \mathbf{\Sigma}(\mathcal{B}(X'), \alpha')\}$$
$$\mathcal{H} := \{D + \Theta_{D'} \mid D \in \mathbf{\Sigma}(\mathcal{B}(Y), \beta), D' \in \mathbf{\Sigma}(\mathcal{B}(Y'), \beta')\}.$$

Then $\mathcal{G}$ and $\mathcal{H}$ are countably generated sub-σ-algebras of the respective Borel sets. Because the σ-algebras in question are countably generated, so is their sum. Both $\mathcal{G}$ and $\mathcal{H}$ respectively define smooth equivalence relations on $X+X'$ and $Y + Y'$ by Corollary 1.7.14. By simulation-equivalence it follows that these equivalences are just the amalgamation of the participating relations (the amalgamation is defined on page 52).

Because the congruences are simulation-equivalent, we see that $z\ (\alpha \diamond \alpha')\ z'$ implies $(K \oplus K')(z)(F) = (K \oplus K')(z')(F)$ for all $F \in \mathcal{H}$. Hence,

$$\mathcal{G} = \mathbf{\Sigma}(\mathcal{B}(X + X'), \alpha \diamond \alpha')$$
$$\mathcal{H} = \mathbf{\Sigma}(\mathcal{B}(Y + Y'), \beta \diamond \beta'),$$

and $\mathsf{c} \diamond \mathsf{c}' := (\alpha \diamond \alpha', \beta \diamond \beta')$ is a congruence on $\mathsf{K} \oplus \mathsf{K}'$.

Bisimilarity

The factor object $(\mathsf{K} \oplus \mathsf{K}')/(\mathsf{c} \diamond \mathsf{c}')$ constructed in this way will be of interest when helping us establish the bisimilarity of K and K', provided they have simulation-equivalent nontrivial congruences.

Proposition 2.2.4. *If there exists nontrivial congruences c_i on the Polish objects K_i for $i = 1, 2$ that are simulation-equivalent, then*

a. there exist morphisms $\mathsf{f}_1 : \mathsf{K}_1 \to (\mathsf{K}_1 \oplus \mathsf{K}_2)/(\mathsf{c}_1 \diamond \mathsf{c}_2)$ and $\mathsf{f}_2 : \mathsf{K}_2 \to (\mathsf{K}_1 \oplus \mathsf{K}_2)/(\mathsf{c}_1 \diamond \mathsf{c}_2)$.
b. K_1 and K_2 are bisimilar.

Proof (Sketch) 1. Assume $\mathsf{K}_i = (X_i, Y_i, K_i)$ and $\mathsf{c}_i = (\alpha_i, \beta_i)$ for $i = 1, 2$. Construct the sum $\mathsf{K}_1 \oplus \mathsf{K}_2$ as above, and let (κ_i, λ_i) be the corresponding injections, which are, however, no morphisms. Let

$$(\eta_{\alpha_1 \diamond \alpha_2}, \eta_{\beta_1 \diamond \beta_2}) : \mathsf{K}_1 \oplus \mathsf{K}_2 \to (\mathsf{K}_1 \oplus \mathsf{K}_2)/(\mathsf{c}_1 \diamond \mathsf{c}_2)$$

be the factor map. Then $(\eta_{\alpha_1 \diamond \alpha_2} \circ \kappa_i, \eta_{\beta_1 \diamond \beta_2} \circ \lambda_i)$ constitutes a morphism $\mathsf{K}_i \to (\mathsf{K}_1 \oplus \mathsf{K}_2)/(\mathsf{c}_1 \diamond \mathsf{c}_2)$, as will be shown now. Surjectivity has to be established, and we have to show that the σ-algebra of common events is nontrivial.

2. Each equivalence class $a \in (X_1 + X_2)/(\alpha_1 \diamond \alpha_2)$ can be represented as $a = [x_1]_{\alpha_1} + [x_2]_{\alpha_2}$ for some suitably chosen $x_1 \in X_1, x_2 \in X_2$. Similarly, each equivalence class $b \in (Y_1 + Y_2)/(\beta_1 \diamond \beta_2)$ can be written as $b = [y_1]_{\beta_1} + [y_2]_{\beta_2}$ for some $y_1 \in Y_1, y_2 \in Y_2$. Conversely, the sum of classes is a class again.

3. Now we have this diagram:

$$\begin{tikzcd} & \mathsf{K}_1 \arrow[d, "\left(\eta_{\alpha_1 \diamond \alpha_2} \circ \kappa_1, \eta_{\beta_1 \diamond \beta_2} \circ \lambda_1\right)"] \\ \mathsf{K}_2 \arrow[r, "\left(\eta_{\alpha_1 \diamond \alpha_2} \circ \kappa_2, \eta_{\beta_1 \diamond \beta_2} \circ \lambda_2\right)"'] & (\mathsf{K}_1 \oplus \mathsf{K}_2)/(\mathsf{c}_1 \diamond \mathsf{c}_2) \end{tikzcd}$$

This yields part a.

4. The semi-pullback of the pair of morphisms with a joint target constructed in the first step exists by Proposition 1.6.25. It is a Polish object (A, B, M), where

$$A := \{\langle x_1, x_2 \rangle \in X_1 \times X_2 \mid [x_1]_{\alpha_1 \diamond \alpha_2} = [x_2]_{\alpha_1 \diamond \alpha_2}\},$$
$$B := \{\langle y_1, y_2 \rangle \in Y_1 \times Y_2 \mid [y_1]_{\beta_1 \diamond \beta_2} = [y_2]_{\beta_1 \diamond \beta_2}\}.$$

We finally are required to establish that there are indeed nontrivial common events. Since c is nontrivial, we can find an invariant Borel set $D \in \mathbf{\Sigma}\big(\mathcal{B}(Y_1), \beta_1\big)$ with $\emptyset \neq D \neq Y_1$. Assume that β_1 spawns β_2 via $(\Theta, \{D_n \mid n \in \mathbb{N}\})$, then $\emptyset \neq \Theta_D \neq Y_2$ also holds. Because D is β_1-invariant,

$$\pi_{1,Y_1}^{-1}[D] = \{\langle y_1, y_2 \rangle \mid y_1 \in D\} = \{\langle y_1, y_2 \rangle \mid y_2 \in \Theta_D\} = \pi_{2,Y_2}^{-1}[\Theta_D];$$

thus

$$\pi_{1,Y_1}^{-1}[D] \in \pi_{1,Y_1}^{-1}[\mathcal{B}(Y_1)] \cap \pi_{2,Y_2}^{-1}[\mathcal{B}(Y_2)],$$

and we are done once it is shown that $\pi_{1,Y_1}^{-1}[D] \neq B$. Since $D \neq Y_1$ is invariant, there exists y_1 with

$$[y_1]_{\beta_1 \diamond \beta_2} \cap D = [y_1]_{\beta_1} \cap D = \emptyset.$$

Let $[y_2]_{\beta_2} := \Theta([\psi_1]_{\beta_1})$, then $[y_2]_{\beta_1 \diamond \beta_2} \cap \Theta_D = [y_2]_{\beta_2} \cap \Theta_D = \emptyset$. Consequently, $\langle y_1, y_2 \rangle \in B \setminus \pi_{1,Y_1}^{-1}[D]$. This shows that $\pi_{1,Y_1}^{-1}[\mathcal{B}(Y_1)] \cap \pi_{2,Y_2}^{-1}[\mathcal{B}(Y_2)]$ is nontrivial. $\square$

We note the following for later use.

Corollary 2.2.5. *Under the conditions of Proposition 2.2.4, the stochastic relation* $(\mathsf{K}_1 \oplus \mathsf{K}_2)/(\mathsf{c}_1 \diamond \mathsf{c}_2)$ *is isomorphic both to* $\mathsf{K}_1/\mathsf{c}_1$ *and to* $\mathsf{K}_2/\mathsf{c}_2$. $\square$

The strategy of the proof to Proposition 2.2.4 has been to make sure that the classes associated with the congruences are distributed evenly among the summands in the sense that each class in the sum is the sum of appropriate classes. This then implies that we can construct surjective maps, and from them morphisms through some general mechanisms. The idea works in particular with isomorphic factor spaces.

Proposition 2.2.6. *Let* K *and* K' *be analytic objects such that* K/c *is isomorphic to* K'/c' *for some nontrivial congruences* c *and* c'*. Then*

a. c *and* c' *are simulation-equivalent,*
b. K *and* K' *are bisimilar.*

Proof (Sketch) 0. Let $\mathsf{K} = (X, Y, K)$ with $\mathsf{c} = (\alpha, \beta)$; similarly for K' and c'. Assume that $\mathsf{f} = (\Phi, \Psi)$ is the isomorphism $\mathsf{K}/\mathsf{c} \to \mathsf{K}'/\mathsf{c}'$ which is composed of the Borel isomorphisms $\Phi : X/\alpha \to X'/\alpha'$ and $\Psi : \ Y/\beta \to Y'/\beta'$. Let moreover $\mathcal{A}$ and $\mathcal{B}$ be countable generators of $\mathbf{\Sigma}\big(\mathcal{B}(X), \alpha\big)$ and $\mathbf{\Sigma}\big(\mathcal{B}(Y), \beta\big)$ which are closed under finite intersections. Then α spawns α' via $(\Phi, \mathcal{A})$, and β spawns β' via $(\Psi, \mathcal{B})$. Hence we have to establish for each $x \in X, x' \in \Phi([x]_\alpha)$ and for each β-invariant Borel subset $B \subseteq Y$ that $K(x)(B) = K'(x')(\Psi_B)$ holds. This will imply that c simulates c'; interchanging the rôles of c and c' then will yield simulation-equivalence.

1. Given $B \in \mathbf{\Sigma}\big(\mathcal{B}(Y), \beta\big)$ we know from Lemma 1.7.10 that we can find a Borel set $B_1 \in \mathcal{B}(Y/\beta)$ such that $B = \eta_\beta^{-1}\left[B_1\right]$. Since Ψ is a Borel isomorphism, we find $B_2 \in \mathcal{B}(Y'/\beta')$ with $B_1 = \Psi^{-1}\left[B_2\right]$. A routine calculation shows that $\Psi_B = \eta_{\beta'}^{-1}\left[B_2\right]$. Now assume that $x \in X, x' \in \Phi([x]_\alpha)$; then the following chain of equations is obtained from the argumentation above, and from the assumption that f is an isomorphism

$$\begin{aligned} K(x)(B) &= K(x)(\eta_\beta^{-1}\left[\Psi^{-1}\left[B_2\right]\right]) \\ &= K_{\alpha,\beta}([x]_\alpha)(\Psi^{-1}\left[B_2\right]) \\ &= K'_{\alpha',\beta'}(\Phi([x]_\alpha)(B_2) \\ &= K'(x')(\eta_{\beta'}^{-1}\left[B_2\right]) \\ &= K'(x')(\Psi_B). \end{aligned}$$

This establishes the desired relation $\mathsf{c} \propto \mathsf{c}'$ and completes the proof for the first part.

2. Bisimilarity now follows through Proposition 2.2.4. $\square$

This will be a helpful tool and construction for the investigations to follow.

2.3 Modal Logics: Syntax and Semantics

We did establish a criterion for bisimilarity through simulation-equivalent congruences and discussed bisimilarity in terms of isomorphic factor spaces. We will now apply this to modal logics.

This section defines modal logic, and Kripke models are defined in their usual nondeterministic and their stochastic versions, together with their satisfaction relation. Some examples are given in order to exhibit probabilistic models for specific logics. Then we discuss bisimilarity and the related notions of logical and behavioral equivalence for these Kripke models.

Let P be a countable set of propositional letters which is fixed throughout; $O \neq \emptyset$ is a set of modal operators. $\tau = (O, \mathfrak{ar})$ is called a *modal similarity* *type* iff $O \neq \emptyset$, and if $\mathfrak{ar} : O \to \mathbb{N}$ is a map, assigning each modal operator $\triangle$ its arity $\mathfrak{ar}(\triangle) \geq 1$. We will not deal with modal operators of arity 0, since they do not have to be dealt with as modal constants in an interpretation. The similarity type τ will be fixed.

We define three modal languages based on τ and P. The formulas of the *basic modal language* $\mathfrak{M}_{\mathfrak{b}}(\tau, P)$ are given by the syntax

$$\phi ::= p \mid \top \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \triangle(\phi_1, \ldots \phi_{\mathfrak{ar}(\triangle)}),$$

where $p \in P$ is a propositional letter. If we have $O = \{\Diamond\}$ with $\mathfrak{ar}(\Diamond) = 1$, we obtain the formulas of the well-known *basic modal language* with negation. Omitting negation in $\mathfrak{M}_{\mathfrak{b}}(\tau, P)$ defines the formulas in the *negation-free basic modal language* $\mathfrak{M}_{\mathfrak{1}}(\tau, P)$. Finally the *extended modal language* $\mathfrak{M}_{\mathfrak{s}}(\tau, P)$ is defined through the syntax

$$\phi ::= p \mid \top \mid \phi_1 \wedge \phi_2 \mid \neg\phi \mid \triangle_q(\phi_1, \ldots \phi_{\mathfrak{ar}(\triangle)}),$$

where $q \in \mathbb{Q} \cap [0, 1]$ is a rational number, and $p \in P$ is a propositional letter. Again, if we deal with $O = \{\Diamond\}$ as the similarity type, then we get an entire line of new formulas through $(\Diamond_q)_{q \in \mathbb{Q} \cap [0,1]}$.

Nondeterministic Kripke Models

A *nondeterministic τ-Kripke model* $\mathcal{R} = (S, R_\tau, V)$ consists of a state space S, a family $R_\tau = (R_\triangle)_{\triangle \in O}$ of set-valued maps $R_\triangle : S \to \mathcal{P}\left(S^{\mathfrak{ar}(\triangle)}\right)$, and a set-valued map $V : P \to \mathcal{P}(S)$.

The satisfaction relation $\models$ for a nondeterministic τ-Kripke model $\mathcal{R}$ is defined as usual for $\mathfrak{M}_{\mathfrak{b}}(\tau, P)$:

$$\begin{aligned}
\mathcal{R}, s \models p &\Leftrightarrow s \in V(p) \\
\mathcal{R}, s \models \neg\phi &\Leftrightarrow \mathcal{R}, s \not\models \phi \\
\mathcal{R}, s \models \phi_1 \wedge \phi_2 &\Leftrightarrow \mathcal{R}, s \models \phi_1 \text{ and } \mathcal{R}, s \models \phi_2 \\
\mathcal{R}, s \models \triangle(\phi_1, \ldots, \phi_{\mathfrak{ar}(\triangle)}) &\Leftrightarrow \exists \langle s_1, \ldots, s_{\mathfrak{ar}(\triangle)} \rangle \in R_\triangle(s) : \mathcal{R}, s_i \models \phi_i \\
&\qquad \text{for } 1 \leq i \leq \mathfrak{ar}(\triangle).
\end{aligned}$$

Denote by

$$[\![\phi]\!]_{\mathcal{R}} := \{s \in S \mid \mathcal{R}, s \models \phi\}$$

the set of states for which formula ϕ is valid (the *extension of formula* ϕ), and by

$$Th_{\mathcal{R}}(s) := \{\phi \in \mathfrak{M}_{\mathfrak{b}}(\tau, P) \mid \mathcal{R}, s \models \phi\}$$

the *theory of state* s in $\mathcal{R}$.

Stochastic Kripke Models

In analogy, a *stochastic τ-Kripke model* $\mathcal{K} = (S, K_\tau, V)$ has a state space S which is endowed with a σ-algebra $\mathcal{A}$, a family $K_\tau = (K_\triangle)_{\triangle \in O}$ of stochastic relations $K_\triangle : S \rightsquigarrow S^{\mathfrak{ar}(\triangle)}$, and a set-valued map $V : P \to \mathcal{A}$. The stochastic relation $K_\triangle : S \rightsquigarrow S^{\mathfrak{ar}(\triangle)}$ is denoted by $F_\triangle(\mathcal{K})$. We will usually assume that S is a Polish space, and that the σ-algebra are the Borel sets.

The interpretation of formulas in $\mathfrak{M}_\mathfrak{s}(\tau, P)$ for a stochastic τ-Kripke model $\mathcal{K}$ is fairly straightforward, the interesting case arising when a modal operator is involved:

$$\mathcal{K}, s \models \triangle_q(\phi_1, \ldots, \phi_{\mathfrak{ar}(\triangle)})$$

holds iff there exists measurable subsets $A_1, \ldots, A_{\mathfrak{ar}(\triangle)}$ of S such that $\mathcal{K}, s_i \models \phi_i$ holds for all $s_i \in A_i$ for $1 \leq i \leq \mathfrak{ar}(\triangle)$, and

$$K_\triangle(s)(A_1 \times \ldots \times A_{\mathfrak{ar}(\triangle)}) \geq q.$$

Arguing with state transition systems in mind, this interpretation of validity reflects that upon the move indicated by $\triangle_q$, a state s satisfies $\triangle_q(\phi_1, \ldots, \phi_{\mathfrak{ar}(\triangle)})$ iff we can find states s_i satisfying ϕ_i with a $K_\triangle$-probability not smaller than q. Note that the usual operators $\triangle$ and ∇ are replaced by a whole spectrum of operators $\triangle_q$ which permit a finer and probabilistically more adequate notion of satisfaction.

Again, let $[\![\phi]\!]_\mathcal{K}$ be the set of all states for which $\phi \in \mathfrak{M}_\mathfrak{s}(\tau, P)$ is satisfied under $\mathcal{K}$, and

$$Th_\mathcal{K}(s) := \{\phi \in \mathfrak{M}_\mathfrak{s}(\tau, P) \mid \mathcal{K}, s \models \phi\}$$

the theory for state $s \in S$.

An easy inductive argument shows that the sets $[\![\phi]\!]_\mathcal{K}$ are measurable, so that they may be used as arguments for the stochastic relations we are working with:

Lemma 2.3.1. $[\![\phi]\!]_\mathcal{K}$ *is a measurable subset of* S *for each* $\phi \in \mathfrak{M}_\mathfrak{s}(\tau, P)$. $\square$

As in the case of stochastic relations we need to exclude trivial cases.

Definition 2.3.2. *A τ-Kripke model $\mathcal{K}$ with state space S is called* degenerate *iff* $[\![\phi]\!]_\mathcal{K} = S$ *or* $[\![\phi]\!]_\mathcal{K} = \emptyset$ *holds for each formula* $\phi \in \mathfrak{M}_\mathfrak{s}(\tau, P)$.

A degenerate model does not usually carry useful information. The restriction is quite similar to not permitting the universal relation as a part of a congruence, and of requesting the existence of nontrivial common events for bisimulations. We will see that these constraints are closely related.

2.3.1 Examples

We show how some well-known logics may be interpreted through Kripke models, indicating that specific logics require specific probabilistic arguments. We introduce first the logic associated with labeled transition systems. This example is of historic significance [58]. It is shown also that the basic temporal language may be interpreted stochastically by reversing a relation. Arrow logic as a popular logic modelling simple programming constructs is interpreted through a simple transformation of a distribution. In presenting these examples we follow essentially the representation of the respective logics in [8].

Example 2.3.3. Suppose that the set Act of labels is a countable set; it is thought of as an alphabet of actions. Each action $a \in \mathsf{Act}$ is associated with a unary modal operator $\langle a \rangle$; so put $\tau := (O, \mathfrak{ar})$ with $O := \{\langle a \rangle \mid a \in \mathsf{Act}\}$ and $\mathfrak{ar}(\langle a \rangle) := 1$.

1. A nondeterministic τ-Kripke model is based on a labeled transition system $(S, (\rightarrow_a)_{a \in \mathsf{Act}})$ which associates a binary relation $\rightarrow_a \subseteq S \times S$ with each action a. Thus

$$s \models \langle a \rangle \phi \Leftrightarrow \exists s' : s \rightarrow_a s' \wedge s' \models \phi.$$

2. A stochastic τ-Kripke model is based on a *labeled Markov transition system*, say $(S, (k_a)_{a \in \mathsf{Act}})$, which associates with each action a a stochastic relation $k_a : S \rightsquigarrow S$. Thus

$$s \models \langle a \rangle_q \phi \Leftrightarrow k_a(s)([\![\phi]\!]) \geq q;$$

   hence a transition is replaced by the probability with which it can occur.

Variants of the logic $\mathfrak{M}_\mathfrak{s}(\tau, P)$ with $P = \emptyset$ were investigated in the literature by Larsen and Skou, and by Desharnais, Edalat and Panangaden with a reference to the logic investigated by Hennessy and Milner [44]; we refer to them also as *Hennessy-Milner logic* $\mathfrak{L}(\mathsf{Act}, \mathbb{Q} \cap [0,1])$. Consequently, this logic's formulas are given through

$$\phi ::= \top \mid \phi_1 \wedge \phi_2 \mid \langle a \rangle_r \phi.$$

Here $a \in \mathsf{Act}$ is an action, and the threshold r is a rational number from the unit interval. $\dashv$

Example 2.3.4. The basic temporal language has two unary modal operators $\mathbf{F}$ (forward) and $\mathbf{B}$ (backward), so that $O = \{\mathbf{F}, \mathbf{B}\}$.

1. A nondeterministic τ-Kripke model interprets the forward operator $\mathbf{F}$ through a relation $R \subseteq S \times S$ and the backward operator $\mathbf{B}$ through the converse $R^{\smile}$ of relation R; thus $R^{\smile} := \{\langle s', s \rangle \mid \langle s, s' \rangle \in R\}$. Consequently, we have

$$s \models \mathbf{B}\phi \Leftrightarrow \exists t \in S : \langle t, s \rangle \in R \wedge t \models \phi.$$

2. A probabilistic interpretation interprets $\mathbf{F}$ through a stochastic relation $K : S \rightsquigarrow S$, so that

$$s \models \mathbf{F}_q\phi \Leftrightarrow K(s)([\![\phi]\!]) \geq q.$$

The backward operator $\mathbf{B}$ is interpreted through the converse $K_\mu^\smile : S \rightsquigarrow S$, provided the state space S is Standard Borel and an initial probability μ is given. The converse $K_\mu^\smile$ of a stochastic relation K given an initial probability μ is a stochastic relation $L : S \rightsquigarrow S$ such that

$$\int_S K(s)(B_s)\ \mu(ds) = \int_S L(t)(B^t)\ \mu(dt)$$

holds for each Borel set $B \subseteq S \times S$. It is known that the converse relation exists whenever the state space S is a Polish space [23]; this carries over obviously to Standard Borel spaces. Thus

$$s \models \mathbf{B}_q\phi \Leftrightarrow K_\mu^\smile(s)([\![\phi]\!]) \geq q.$$

An easy calculation shows that

$$\begin{aligned} s \models \mathbf{B}_1\mathbf{F}_1\phi &\Leftrightarrow K_\mu^\smile(s)\left(\{s' \mid K(s')([\![\phi]\!]) = 1\}\right) = 1 \\ &\Leftrightarrow \int_S K(s')([\![\phi]\!])\ K_\mu^\smile(s)(ds') = 1. \end{aligned}$$

Note that the definition of the converse requires an initial probability (this is intuitively clear: if the probability for a backward running process is described, one has to say where to start). It is also noteworthy that a topological assumption has been made; if the state space is not a Polish space, then the technical arguments permitting the definition of the converse are not available. $\dashv$

Example 2.3.5. Arrow logic has three modal operators modelling reversal, composition, and skip respectively. Thus $O = \{\mathbf{1}, \otimes, \circ\}$ with respective arities $\mathfrak{ar}(\mathbf{1}) = 0, \mathfrak{ar}(\otimes) = 1,$ and $\mathfrak{ar}(\circ) = 2$.

1. The usual nondeterministic interpretation of arrow logic is done over a world of pairs; so the base state space is $S \times S$ for some S, with associated relations

$$\begin{aligned} R_\mathbf{1} &= \Delta_S = \{\langle s, s \rangle \mid s \in S\}, \\ R_\otimes &= \{\langle\langle s_0, s_1\rangle, \langle s_1, s_0\rangle\rangle \mid s_0, s_1 \in S\}, \\ R_\circ &= \{\langle\langle s_0, s_1\rangle, \langle s_0, s\rangle, \langle s, s_1\rangle\rangle \mid s, s_0, s_1 \in S\}. \end{aligned}$$

Thus, e.g.,

$$\langle s, s' \rangle \models \phi \circ \psi \Leftrightarrow \exists s_0 : \langle s, s_0 \rangle \models \phi \wedge \langle s_0, s' \rangle \models \psi$$

and

$$\langle s, s' \rangle \models \otimes\phi \Leftrightarrow \langle s', s \rangle \models \phi.$$

2. Now assume again that S is a Polish space, and let $\mu \in \mathfrak{P}(S)$ be a probability. Put for $A \in \mathcal{B}(S \times S)$

$$\hat{\mu}(A) := \mu(\{s \in S \mid \langle s, s \rangle \in A\});$$

thus $\hat{\mu}$ transports a Borel set in S to a Borel set in the diagonal of $S \times S$. Interpret the composition operator $\circ_q$ through the stochastic relation

$$K_\circ(s, s') := \delta_s \otimes \hat{\mu} \otimes \delta_{s'}.$$

Note that the operator $\otimes$ is somewhat overloaded: it denotes the modal operator for reversal, and the product operator for measures. The context should make it clear which version is meant.
We obtain then

$$\begin{aligned} K_\circ(s, s')([\![\phi]\!] \times [\![\psi]\!]) &= (\delta_s \otimes \hat{\mu} \otimes \delta_{s'})\,([\![\phi]\!] \times [\![\psi]\!]) \\ &= \hat{\mu}(\{\langle s_1, s_2 \rangle \mid \langle s, s_1 \rangle \in [\![\phi]\!], \langle s_2, s' \rangle \in [\![\psi]\!]\}) \\ &= \mu(\{s_1 \mid \langle s, s_1 \rangle \in [\![\phi]\!], \langle s_1, s' \rangle \in [\![\psi]\!]\}). \end{aligned}$$

Consequently,

$$\langle s, s' \rangle \models \phi \circ_1 \psi \Leftrightarrow \langle s, s_1 \rangle \models \phi \wedge \langle s_1, s' \rangle \models \psi \text{ for } \mu\text{-almost all } s_1$$

(here *μ-almost all* s_1 means as usual that the set of all s_1 for which the property does not hold has μ-measure 0). More generally, $\langle s, s' \rangle \models \phi \circ_q \psi$ iff $\langle s, s_1 \rangle \models \phi \wedge \langle s_1, s' \rangle \models \psi$ for all s_1 from a Borel set S_0 with $\mu(S_0) \geq q$. Finally, put $K_\otimes(s, s') := \delta_{\langle s', s \rangle}$; then $\langle s, s' \rangle \models \otimes_q \phi \Leftrightarrow \langle s', s \rangle \models \phi$, for all rational q with $0 \leq q \leq 1$ (which is evidently independent of q), and let

$$K_{\mathbf{1}}(s, s') := \begin{cases} 0, & s \neq s' \\ \delta_{\langle s, s \rangle}, & s = s' \end{cases}$$

(here 0 is the null measure); then

$$\langle s, s' \rangle \models \mathbf{1} \Leftrightarrow s = s'.$$

Note that in general we did exclude modal constants, i.e., modal operators of arity 0, when defining modal similarity types. The example shows that it is possible to include them nevertheless without much ado. $\dashv$

2.3.2 Bisimulations for Kripke Models

This section investigates morphisms for stochastic τ-Kripke models. Bisimilarity and logical equivalence are related to each other.

Definition 2.3.6. *The stochastic τ-Kripke models $\mathcal{K}$ and $\mathcal{K}'$ are said to be* logical equivalent *iff $\{Th_{\mathcal{K}}(s) \mid s \in S\} = \{Th_{\mathcal{K}'}(s') \mid s' \in S'\}$.*

Thus $\mathcal{K}$ and $\mathcal{K}'$ are logically equivalent iff given $s \in S$ there exists $s' \in S'$ such that $Th_{\mathcal{K}}(s) = Th_{\mathcal{K}'}(s')$, and vice versa.

Morphisms for stochastic Kripke models should be based on morphisms for the underlying stochastic relations, and they should take the propositional constants into account.

Definition 2.3.7. *Let $\mathcal{K}$ and $\mathcal{K}'$ be stochastic τ-Kripke models with $\mathcal{K} = \big(S, (K_\triangle)_{\triangle \in O}, V\big)$ and $\mathcal{K}' = \big(S', (K'_\triangle)_{\triangle \in O}, V'\big)$. A* strong morphism *$\Phi : \mathcal{K} \to \mathcal{K}'$ is determined through a measurable and surjective map $\Phi : S \to S'$ so that these conditions are satisfied:*

a. $V(p) = \Phi^{-1}\left[V'(p)\right]$ holds for all $p \in P$,
b. $K'_\triangle \circ \Phi = \mathfrak{S}\left(\Phi^{\mathfrak{ar}(\triangle)}\right) \circ K_\triangle$ holds for each modal operator $\triangle$.

Here $\Phi^{\mathfrak{ar}(\triangle)} : \langle x_1, \ldots, x_{\mathfrak{ar}(\triangle)}\rangle \mapsto \langle \Phi(x_1), \ldots, \Phi(x_{\mathfrak{ar}(\triangle)})\rangle$ distributes the map Φ into the components. Consequently, if $\Phi : \mathcal{K} \to \mathcal{K}'$ is a strong morphism, then

$$(\Phi, \Phi^{\mathfrak{ar}(\triangle)}) : F_\triangle(\mathcal{K}) \to F_\triangle(\mathcal{K}')$$

is a morphism between the corresponding stochastic relations for each modal operator $\triangle \in O$. In addition we know for each propositional letter p that $\mathcal{K}, s \models p$ iff $\mathcal{K}', \Phi(s) \models p$.

Bisimulations are defined again as spans of — strong — morphisms. Similarly, we define behavioral equivalence through a cospan of morphisms, essentially mimicking the corresponding definition for stochastic relations.

Definition 2.3.8. *Let $\mathcal{K}_1$ and $\mathcal{K}_2$ be stochastic τ-Kripke models.*

a. $\mathcal{K}_1$ and $\mathcal{K}_2$ are called strongly bisimilar *iff there exists a stochastic τ-Kripke model $\mathcal{M}$ and strong morphisms*

$$\mathcal{K}_1 \xleftarrow{\Phi_1} \mathcal{M} \xrightarrow{\Phi_2} \mathcal{K}_2,$$

such that the σ-algebra of common events $\Phi_1^{-1}\left[\mathcal{B}(S_1)\right] \cap \Phi_2^{-1}\left[\mathcal{B}(S_2)\right]$ is nontrivial (here S_i is the state space of $\mathcal{K}_i, i = 1, 2$).

b. $\mathcal{K}_1$ and $\mathcal{K}_2$ are called behaviorally equivalent *iff there exists a stochastic τ-Kripke model $\mathcal{L}$ and strong morphisms*

$$\mathcal{K}_1 \xrightarrow{\Psi_1} \mathcal{L} \xleftarrow{\Psi_2} \mathcal{K}_2.$$

We relate logical equivalence, strong bisimilarity and behavioral equivalence of Kripke models $\mathcal{K}$ and $\mathcal{K}'$, provided the models are based on Polish spaces. Fix the stochastic τ-Kripke models $\mathcal{K} := \big(S, (K_\triangle)_{\triangle \in O}), V\big)$ and $\mathcal{K}' := \big(S', (K'_\triangle)_{\triangle \in O}, V'\big)$.

It is well known that morphisms preserve theories for logics of the Hennessy-Milner type. This is also true for stochastic relations:

Lemma 2.3.9. *If $\Phi : \mathcal{K} \to \mathcal{K}'$ is a strong morphism, then $Th_{\mathcal{K}}(s) = Th_{\mathcal{K}'}(\Phi(s))$ holds for all states $s \in S$.* $\square$

Define the equivalence relation α on state space S through

$$s_1 \ \alpha \ s_2 \Leftrightarrow Th_{\mathcal{K}}(s_1) = Th_{\mathcal{K}}(s_2);$$

thus two states are α-equivalent iff they satisfy exactly the same formulas in $\mathfrak{M}_{\mathfrak{s}}(\tau, P)$. Thus they are equivalent iff they cannot be separated through a formula. In a similar way α' is defined on S'. Because we have at most countably many formulas, α and α' are smooth equivalence relations. Define the equivalence relation $\beta_\triangle$ on $S^{\mathfrak{ar}(\triangle)}$ through

$$\langle s_1, \ldots, s_{\mathfrak{ar}(\triangle)} \rangle \ \beta_\triangle \ \langle t_1, \ldots, t_{\mathfrak{ar}(\triangle)} \rangle \Leftrightarrow s_1 \ \alpha \ t_1 \wedge \cdots \wedge s_{\mathfrak{ar}(\triangle)} \ \alpha \ t_{\mathfrak{ar}(\triangle)};$$

then $\beta_\triangle = \times_{i=1}^{\mathfrak{ar}(\triangle)} \beta$ is smooth, and we know that the σ-algebra of β-invariant sets can be written in terms of the α-invariant sets, viz.,

$$\mathbf{\Sigma}\big(\mathcal{B}(S^{\mathfrak{ar}(\triangle)}), \beta_\triangle\big) = \bigotimes_{i=1}^{\mathfrak{ar}(\triangle)} \mathbf{\Sigma}\big(\mathcal{B}(S), \alpha\big)$$

(see Lemma 1.7.19). The relation $\beta'_\triangle$ is defined in the same way for α'.

The equivalence of $\mathcal{K}$ and $\mathcal{K}'$ makes these relations into simulation-equivalent congruences.

Lemma 2.3.10. *If the nondegenerate Kripke models $\mathcal{K}$ and $\mathcal{K}'$ are logically equivalent, then $(\alpha, \beta_\triangle)$ and $(\alpha', \beta'_\triangle)$ are simulation-equivalent and nontrivial congruences for the stochastic relations $F_\triangle(\mathcal{K})$ and $F_\triangle(\mathcal{K}')$.* $\square$

Accordingly, we know from Proposition 2.2.4 that for logical equivalent Kripke models $\mathcal{K}$ and $\mathcal{K}'$ and for each modal operator $\triangle$ the stochastic relations $F_\triangle(\mathcal{K})$ and $F_\triangle(\mathcal{K}')$ are bisimilar. All the mediating relations can be collected to form a mediating Kripke model. This yields the following.

Theorem 2.3.11. *Assume that $\mathcal{K}$ and $\mathcal{K}'$ are nondegenerate stochastic τ-Kripke models over analytic spaces. Then the following statements are equivalent:*

a. $\mathcal{K}$ and $\mathcal{K}'$ are strongly bisimilar,
b. $\mathcal{K}$ and $\mathcal{K}'$ are logically equivalent,

c. $\mathcal{K}$ *and* $\mathcal{K}'$ *are behaviorally equivalent.* □

Commenting on the development, it is noted that Theorem 2.3.11 is derived from Proposition 2.2.4, and hence from a condition that arose from the consideration of stochastic relations alone. This is in marked contrast to the proof carried out in [27] which starts from the logic and develop the properties of simulation-equivalent congruences implicitly. Analyzing the proof, it becomes clear that the model constructed there will usually not be defined over a Standard Borel space. This is so since factoring destroys the property of being a Polish space, rendering the factor space analytic instead, see Example 1.7.6.

Logical equivalence appears here as some sort of catalyst which permits proving that bisimilarity and behavioral equivalence describe the same phenomenon, a link that is missing in the general development of stochastic relations; see Section 2.2. There we have simulation-equivalent congruences at our disposal, which are always tied to a relation, while the logic serves here as an arbitrator which is completely independent of the Kripke model interpreting it.

2.4 Temporal Logics: μCSL

We will define as a further illustration *continuous time stochastic logic with fixed points operators*, abbreviated as $\mu\mathbf{CSL}$. The logic will be introduced formally first; then models and their morphisms are introduced and the interpretation of $\mu\mathbf{CSL}$ is given. Some standard properties like Borel measurability are established, and logical equivalence is defined. We deal with properties on states and on paths; the equivalence relations defined by the logic on these sets are related to each other. The relations obtained from these constructions are modified so that they fit into the mold of the models for the logic. The main result is that logical equivalence and bisimilarity are equivalent, and that this holds also for behavioral equivalence, provided the factor space induced by the theory of states is a Standard Borel space again (this is so since the projective limit construction, on which interpretations are based, does not seem to work for general analytic spaces, but only for their Standard Borel brethren, see Proposition 1.6.35).

2.4.1 The Logic μCSL

State formulas and path formulas for $\mu\mathbf{CSL} = \mu\mathbf{CSL}(\mathsf{AP}, \mathsf{SV}, \mathsf{PV})$ are given through this syntax (with mutually disjoint and countable sets AP, SV, and PV of atomic proposition, state variables, and path variables, respectively): The intuitive idea is that the logic works over infinite paths which have as a component alternating a state and a time; the time is interpreted as residence time for the state.

The formal definition reads as follows.

- *State formulas* are defined through the syntax

$$\phi ::= \top \mid a \mid Z \mid \neg\phi \mid \phi \wedge \phi' \mid \mathcal{S}_{\ltimes p}(\phi) \mid \mathcal{P}_{\ltimes p}(\psi).$$

  Here $a \in \mathsf{AP}$ is an atomic proposition, $Z \in \mathsf{SV}$ is a state variable, ψ is a path formula, $\ltimes$ is one of the relational operators $<$, $\leq$, $>$, $\geq$, and $p \in [0,1]$ is a rational number.
- *Path formulas* are defined through

$$\psi ::= \tilde{\top} \mid P \mid \neg\psi \mid \psi \wedge \psi' \mid \mathcal{X}^I\, \psi \mid \phi\, \mathcal{U}^I\, \phi' \mid \mu P.\psi$$

  with $P \in \mathsf{PV}$ as a path variable, ϕ, ϕ' as state formulas, $I \subseteq \mathbb{R}_+$ a closed interval of the real numbers with rational bounds (including $I = \mathbb{R}_+$); these intervals will be called *rational intervals*. The operator μ describes the smallest fixed point; it binds variables in the usual sense. We assume that the variable bound by it is in the range of an even number of negations.

The informal interpretation of the operators is as follows.

1. The operator $\mathcal{S}_{\ltimes p}(\phi)$ gives the *steady state probability* for ϕ to hold with the boundary condition $\ltimes p$, where ϕ is a state formula. This is a state formula again.
2. The *path quantifier* formula $\mathcal{P}_{\ltimes p}(\psi)$ holds for a state iff the probability of all paths starting in this state and satisfying path formula ψ is specified by $\ltimes p$. Thus, e.g., ψ holds on almost all paths starting from that state iff it satisfies $\mathcal{P}_{\geq 1}(\psi)$. Of course, $\mathcal{P}_{\ltimes p}(\psi)$ is a state formula.
3. The *next operator* $\mathcal{X}^I\, \phi$ is assumed to hold on an infinite path of states and residence times iff the residence time for the first state is an element of interval I, and if the second state satisfies ϕ.
4. The *until-operator* $\phi_1\, \mathcal{U}^I\, \phi_2$ holds on path σ iff we can find a point in time $t \in I$ such that the state $\sigma@t$ which the infinite path σ denotes at time t satisfies ϕ_2, and for all times t' before that, $\sigma@t'$ satisfies ϕ_1 (the notation $\sigma@t$ will be defined formally on page 89).

2.4.2 Defining Models and Their Morphisms

We are ready for the definition of models for μ**CSL** and their morphisms. We will work with projective limits (see Section 1.6.4) for interpreting path formulas; models will be based on Polish spaces rather than more generally on analytic spaces.

Definition 2.4.1. $\mathcal{M} = (S, M, \mathcal{I}, V)$ *is called a* model for μ**CSL** *iff*

1. S is a Polish space, the state space *of $\mathcal{M}$,*
2. $M : S \rightsquigarrow \mathbb{R}_+ \times S$ is a stochastic relation with $M(s)(\mathbb{R}_+ \times S) = 1$ for all $s \in S$; the stochastic relation M is the law of change *of $\mathcal{M}$,*
3. $\mathcal{I} = (\Sigma, \Pi)$ interprets the variables,

a. $\Sigma : \mathsf{SV} \to \mathcal{B}(S)$ assigns each state variable a Borel set in S,
b. $\Pi : \mathsf{PV} \to \mathcal{B}((S \times \mathbb{R}_+)^\infty)$ assigns each path variable a Borel set of paths,

4. $V : \mathsf{AP} \to \mathcal{B}(S)$ maps each atomic proposition to a Borel set of states.

Thus a model says how residence times and state changes are to be handled: if $s \in S$ is the present state, then $M(s)(I \times B)$ gives the probability that after $t \in I$ time units a state change will happen, and that the new state will be a member of Borel set $B \subseteq S$. Each model says how the variables are to be interpreted; this is written down through the maps Σ and Π, and we say what sets the atomic propositions are taken from. Note that we assume in each case that the sets under consideration are Borel. Otherwise we could not assign them any probability directly or indirectly; hence this assumption is made for keeping the model within the realm of our probabilistic reasoning. We postulate that the law of change assigns probability 1 to $\mathbb{R}_+ \times S$ for each state. Without this assumption, mass along infinite paths might vanish too fast; from a technical point of view, this assumption is necessary because we will construct from this law a projective limit which requires its components to assign probability 1 to the base space (see Section 1.6.4). Consequently, we will use the probability functor $\mathfrak{P}$ rather than its cousin $\mathfrak{S}$.

Morphisms

We define a morphism $\Phi : \mathcal{M} \to \mathcal{N}$ for the models $\mathcal{M}$ and $\mathcal{N}$. It is based on a map $\Phi : S \to S'$ between state spaces, which is extended to a map $\Phi_\infty : (S \times \mathbb{R}_+)^\infty \to (S' \times \mathbb{R}_+)^\infty$ upon setting

$$\Phi_\infty(\langle s_0, t_0, s_1, t_1, \ldots \rangle) := \langle \Phi(s_0), t_0, \Phi(s_1), t_1, \ldots \rangle;$$

thus we transform the states according to Φ but leave the residence times alone; define additionally $id_{\mathbb{R}_+} \times \Phi : \langle t, s \rangle \mapsto \langle t, \Phi(s) \rangle$, and similarly, $\Phi \times id_{\mathbb{R}_+}$.

Definition 2.4.2. *Let $\mathcal{M} = (S, M, \mathcal{I}, V)$ and $\mathcal{N} = (S', N, \mathcal{I}', V')$ be models for μ**CSL**. Then $\Phi : \mathcal{M} \to \mathcal{N}$ is called a* morphism *from $\mathcal{M}$ to $\mathcal{N}$ iff*

a. $\Phi : S \to S'$ is a surjective and Borel measurable map between the state spaces,
b. $(\Phi, id_{\mathbb{R}_+} \times \Phi) : M \to N$ is a morphism for the associated stochastic relations M and N,
c. $\Phi^{-1}\left[\Sigma'(Z)\right] = \Sigma(Z)$ for each state variable Z,
d. $\Phi_\infty^{-1}\left[\Pi'(P)\right] = \Pi(P)$ for each path variable P,

e. $\Phi^{-1}[V'(a)] = V(a)$ *for each atomic proposition* a.

We require the map underlying a morphism to be onto since we want to be able to trace each state in S' back to a state in S, inheriting the corresponding property from the basic stochastic relations. Condition b says that this diagram is commutative:

$$\begin{tikzcd} S \arrow[r, "\Phi"] \arrow[d, "M"'] & S' \arrow[d, "N"] \\ \mathfrak{P}(\mathbb{R}_+ \times S) \arrow[r, "\mathfrak{P}\left(id_{\mathbb{R}_+} \times \Phi\right)"'] & \mathfrak{P}(\mathbb{R}_+ \times S') \end{tikzcd}$$

Thus we have in particular

$$N(\Phi(s))(I \times B) = M(s)\left(I \times \Phi^{-1}[B]\right)$$

for every state $s \in S$, every rational interval I, and every Borel set $B \in \mathcal{B}(S)$. Conditions c to e relate the interpretations of variables and atomic propositions. For example, condition c says that for a state s and a state variable Z we have $s \in \Sigma(Z)$ iff $\Phi(s) \in \Sigma'(Z)$.

Projective Limits

We model the one-step behavior of a model through its transition law, but we are not yet able to say how this relates to the behavior along paths. This is done iteratively for paths of finite length, and by passing to a limit (in this case to the projective limit) for infinite paths.

The construction works like this.

Let $M : S \rightsquigarrow \mathbb{R}_+ \times S$ be the stochastic relation underlying a model. Fix a state $s \in S$, and proceed inductively along finite paths: Put

$$M_1(s) := M(s),$$

and set in the inductive step for the Borel set $D \subseteq (\mathbb{R}_+ \times S)^{n+1}$

$$\begin{aligned} M_{n+1}(s)(D) := \\ &\int_{(\mathbb{R}_+ \times S)^n} M(s_n)\left(\{\langle t, s\rangle \mid \langle t_0, s_1, \dots, t_{n-1}, s_n, t, s\rangle \in D\}\right) \times \\ &\qquad \times M_n(s)(d\langle t_0, s_1, \dots, t_{n-1}, s_n\rangle) = \\ &\qquad\qquad \int_{(\mathbb{R}_+ \times S)^n} M(\hbar_S(\mathbf{w}))(D_{\mathbf{w}})\; M_n(s)(d\mathbf{w}), \end{aligned}$$

where we have set $\hbar_S(t_0, s_1, \ldots, t_{n-1}, s_n) := s_n$ for simplifying the notation. Thus the argument to $M(s_n) = M(\hbar_S(\mathbf{w}))$ is the set of all times and states $\langle t, s\rangle$ such that $\langle \mathbf{w}, t, s\rangle = \langle t_0, s_1, \ldots, t_{n-1}, s_n, t, s\rangle$ is a member of D. Analyzing the expression further, we see that at step $n+1$ the probability for the pair that consists of timing a transition and changing a state is an element of $\{\langle t, s\rangle \mid \langle t_0, s_1, \ldots, t_{n-1}, s_n, t, s\rangle \in D\}$ equals

$$M(\hbar_S(\mathbf{w}))(D_{\mathbf{w}}) = M(s_n)\left(\{\langle t, s\rangle \mid \langle t_0, s_1, \ldots, t_{n-1}, s_n, t, s\rangle \in D\}\right),$$

provided the corresponding times and states that have been run through during steps $1, \ldots, n$ are given by $\mathbf{w} = \langle t_0, s_1, \ldots, t_{n-1}, s_n\rangle$ which in turn is captured through $M_n(s)(d\mathbf{w})$.

Standard arguments show that $M_n : S \rightsquigarrow (\mathbb{R}_+ \times S)^n$ is a stochastic relation. For each state $s \in S$ the sequence $(M_n(s))_{n\in\mathbb{N}}$ forms a projective system (Definition 1.6.32), provided $M(s)(\mathbb{R}_+ \times S) = 1$ holds for each $s \in S$: for each Borel set $B \subseteq (\mathbb{R}_+ \times S)^n$ the equality

$$M_{n+1}(B \times (\mathbb{R}_+ \times S)) = M_n(s)(B)$$

holds. Consistency of this family has as a consequence the fact that the measures can be extended to Borel sets of infinite sequences. We obtain from Proposition 1.6.35 the existence of the projective limit.

Proposition 2.4.3. *Given a stochastic relation $M : S \rightsquigarrow \mathbb{R}_+ \times S$ such that $M(s)(\mathbb{R}_+ \times S) = 1$ for all $s \in S$, there exists a unique stochastic relation $M_\infty : S \rightsquigarrow (\mathbb{R}_+ \times S)^\infty$ such that*

$$M_\infty(s)\big(B \times \prod_{j>n}(\mathbb{R}_+ \times S)\big) = M_n(s)(B)$$

for each Borel set $B \in \mathcal{B}((\mathbb{R}_+ \times S)^n)$ and each state $s \in S$. M_∞ is the projective limit of $(M_n)_{n\in\mathbb{N}}$. $\square$

Intuitively, this equation means that the behavior of the infinite paths up to horizon n is uniquely determined by the transition law M_n. This construction entails that a morphism Φ between models may be interpreted as a morphism (Φ, Φ_∞) between these projective limits. To be specific:

Proposition 2.4.4. *Let $\mathcal{M}$ and $\mathcal{N}$ be models, $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism from $\mathcal{M}$ to $\mathcal{N}$. Then $(\Phi, \Phi_\infty) : M_\infty \to N_\infty$ is a morphism between the stochastic relations M_∞ and N_∞.* $\square$

2.4.3 Interpreting μCSL

We are now ready for an interpretation of μ**CSL**. Fix a model $\mathcal{M} = (S, M, \mathcal{I}, V)$ over the Standard Borel space S and let $M_\infty : S \rightsquigarrow \mathbb{R}_+ \times (S \times$

$\mathbb{R}_+)^\infty = (\mathbb{R}_+ \times S)^i nfty$ be the associated stochastic relation that relates (initial) states to paths.

The Semantics

The semantics of μ**CSL** is then described recursively through relation $\models$ between states respectively paths, and formulas as described below. We will need to describe in what state the model is at a given time t. Assume the behavior is given through a path $\sigma = \langle s_0, t_0, s_1, t_1, \ldots \rangle$ which may more graphically be written through $s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} s_2 \xrightarrow{t_2} s_3 \ldots$. Put $\sigma[1\ldots] := s_1 \xrightarrow{t_1} s_2 \xrightarrow{t_2} s_3 \ldots$ as the path after step 1, and define $\delta(\sigma, 0) := t_0$ as the first residence time. Given $t \in \mathbb{R}_+$, assume that there is a smallest index k such that $t < \sum_{i=0}^{k} t_i$; then we would intuitively say that σ is at time k in state s_k; define accordingly $\sigma@t := s_k$. It can be shown that the set $\{\langle \sigma, t \rangle \in (S \times \mathbb{R}_+)^\infty \mid \sigma@t \text{ is defined}\}$ is a Borel subset of $(S \times \mathbb{R}_+)^\infty$, and that $\langle \sigma, t \rangle \mapsto \sigma@t$ constitutes a measurable map from the latter set to S.

Let $\mathcal{M} = (S, M, (\Sigma, \Pi), V)$ be a model. Given a state variable Z and a Borel set $Q \in \mathcal{B}(S)$, denote by $\mathcal{M}[Z\backslash Q]$ the model $(S, M, (\Sigma', \Pi), V)$ with $\Sigma'(Z) := Q$; otherwise Σ' coincides with Σ. Similarly, the model $\mathcal{M}[P\backslash U]$ is defined for the path variable P and the Borel set $U \in \mathcal{B}((S \times \mathbb{R}_+)^\infty)$. Substituting values in this way may be iterated.

We again denote for model $\mathcal{M}$ the extensions of formulas through

$$[\![\phi]\!]_{\mathcal{M}} := \{s \in S \mid \mathcal{M}, s \models \phi\}$$

and

$$[\![\psi]\!]_{\mathcal{M}} := \{\sigma \in (S \times \mathbb{R}_+)^\infty \mid \mathcal{M}.\sigma \models \psi\}$$

Hence these sets denote all states or paths for which the respective formula holds. Relation $\models$ is defined inductively; let $s \in S$ be a state, and $\sigma \in (S \times \mathbb{R}_+)^\infty$ be an infinite path alternating between states and residence times.

$\mathcal{M}, s \models \top$ is true for all $s \in S$.
$\mathcal{M}, s \models a \Leftrightarrow s \in V(a)$.
$\mathcal{M}, s \models Z \Leftrightarrow s \in \Sigma(Z)$ for $Z \in \mathsf{SV}$.
$\mathcal{M}, s \models \phi_1 \wedge \phi_2 \Leftrightarrow \mathcal{M}, s \models \phi_1$ and $\mathcal{M}, s \models \phi_2$.
$\mathcal{M}, s \models \neg\phi \Leftrightarrow \mathcal{M}, s \models \phi$ is false.
$\mathcal{M}, s \models \mathcal{S}_{\ltimes p}(\phi) \Leftrightarrow \lambda := \lim_{t\to\infty} M_\infty(s)(\{\tau \mid \langle s, \tau\rangle@t \models \phi\})$ exists, and $\lambda \ltimes p$.
$\mathcal{M}, s \models \mathcal{P}_{\ltimes p}(\psi) \Leftrightarrow M_\infty(s)(\{\tau \mid \langle s, \tau\rangle \models \psi\}) \ltimes p$.

$\mathcal{M}, \sigma \models \tilde{\top}$ is true for all $\sigma \in (S \times \mathbb{R}_+)^\infty$
$\mathcal{M}, \sigma \models P \Leftrightarrow \sigma \in \Pi(P)$ for $P \in \mathsf{PV}$.
$\mathcal{M}, \sigma \models \psi_1 \wedge \psi_2 \Leftrightarrow \mathcal{M}, \sigma \models \psi_1$ and $\mathcal{M}, \sigma \models \psi_2$.
$\mathcal{M}, \sigma \models \neg\psi \Leftrightarrow \mathcal{M}, \sigma \models \psi$ is false.
$\mathcal{M}, \sigma \models \mathcal{X}^I\, \psi \Leftrightarrow \mathcal{M}, \sigma[1\ldots] \models \psi$ and $\delta(\sigma, 0) \in I$.
$\mathcal{M}, \sigma \models \phi_1\, \mathcal{U}^I\, \phi_2 \Leftrightarrow \exists t \in I : \mathcal{M}, \sigma@t \models \phi_2$ and $\forall t' \in [0, t[: \mathcal{M}, \sigma@t' \models \phi_1$.
$\mathcal{M}, \sigma \models \mu P.\psi \Leftrightarrow \sigma \in \bigcup_{i\geq 0} R_i$ with $R_0 := [\![\psi]\!]_{\mathcal{M}[P\backslash\emptyset]}, R_{i+1} := [\![\psi]\!]_{\mathcal{M}[P\backslash R_i]}$.

Define the theory $Th_{\mathcal{M}}(s)$ of a state s as above as the formulas which hold in s:

$$Th_{\mathcal{M}}(s) := \{\phi \mid \phi \text{ is a state formula}, \mathcal{M}, s \models \phi\}.$$

Similarly, the theory $Th_{\mathcal{M}}(\sigma)$ of a path σ is defined:

$$Th_{\mathcal{M}}(\sigma) := \{\psi \mid \psi \text{ is a path formula}, \mathcal{M}, \sigma \models \psi\}.$$

The extensions of the formulas are Borel-measurable. This is established through induction on the structure of a formula. Those formulas that contain the μ-operator need special consideration.

Proposition 2.4.5. *$[\![\phi]\!]_{\mathcal{M}} \in \mathcal{B}(S)$ for all state formulas ϕ, and $[\![\psi]\!]_{\mathcal{M}} \in \mathcal{B}((S \times \mathbb{R}_+)^\infty)$ for all path formulas ψ.* $\square$

Of course it is important to know that the sets under consideration are Borel, for otherwise the corresponding sets are not in the range of the corresponding probability, and one cannot compute probabilities like $M_\infty(s)(\{\tau \mid \langle s, \tau\rangle \models \psi\})$.

We note that the μ-operator plays a special rôle: intuitively, it models the smallest fixed point. This is noted just for the sake of completeness.

Proposition 2.4.6. *$[\![\mu P.\psi]\!]_{\mathcal{M}}$ is the smallest fixed point of $R \mapsto [\![\psi]\!]_{\mathcal{M}[P\backslash R]}$.* $\square$

Theories are invariant under model morphisms; to be specific:

Proposition 2.4.7. *Let $\mathcal{M}$ and $\mathcal{M}'$ be models for μ**CSL**, and assume that $\Phi : \mathcal{M} \to \mathcal{M}'$ is a morphism. Then*

a. $[\![\phi]\!]_{\mathcal{M}} = \Phi^{-1}\left[[\![\phi]\!]_{\mathcal{M}'}\right]$ *for all state formulas* ϕ.
b. $[\![\psi]\!]_{\mathcal{M}} = \Phi_{\infty}^{-1}\left[[\![\psi]\!]_{\mathcal{M}'}\right]$ *for all state formulas* ψ. $\square$

2.4.4 Congruences

We will define two equivalence relations on states respectively on paths. These relations are fundamental for discussing bisimilarity and behavioral as well as logical equivalence later on.

Fix the model $\mathcal{M} = (S, M, \mathcal{I}, V)$ and define

$$s \;\zeta_{\mathcal{M}}\; s' \Leftrightarrow Th_{\mathcal{M}}(s) = Th_{\mathcal{M}}(s'),$$
$$\sigma \;\omega_{\mathcal{M}}\; \sigma' \Leftrightarrow Th_{\mathcal{M}}(\sigma) = Th_{\mathcal{M}}(\sigma').$$

Then both $\zeta_{\mathcal{M}}$ and $\omega_{\mathcal{M}}$ are smooth equivalence relations on S resp. on $(S \times \mathbb{R}_+)^{\infty}$. This is so since there are only countably many formulas, and because we have

$$\begin{aligned} s \;\zeta_{\mathcal{M}}\; s' &\Leftrightarrow \big[\mathcal{M}, s \models \phi \Leftrightarrow \mathcal{M}, s' \models \phi\big] \text{ for all state formulas } \phi,\\ &\Leftrightarrow \big[s \in [\![\phi]\!]_{\mathcal{M}} \Leftrightarrow s' \in [\![\phi]\!]_{\mathcal{M}}\big] \text{ for all state formulas } \phi. \end{aligned}$$

From this it is clear that the countable set $\big\{[\![\phi]\!]_{\mathcal{M}} \mid \phi \text{ is a state formula}\big\}$ determines the relation $\zeta_{\mathcal{M}}$, and that

$$\mathbf{\Sigma}\big(\mathcal{B}(S), \zeta_{\mathcal{M}}\big) = \sigma\big(\{[\![\phi]\!]_{\mathcal{M}} \mid \phi \text{ is a state formula}\}\big).$$

In a similar way we see that $\omega_{\mathcal{M}}$ is smooth, and that

$$\mathbf{\Sigma}\big(\mathcal{B}((S \times \mathbb{R}_+)^{\infty}), \omega_{\mathcal{M}}\big) = \sigma\big(\{[\![\psi]\!]_{\mathcal{M}} \mid \psi \text{ is a path formula}\}\big)$$

holds as well. These two relations will be studied now in some detail, and it will turn out that the relationship of $\zeta_{\mathcal{M}}$ and $\omega_{\mathcal{M}}$ is closer than meets the eye.

In a first step it is established that $\zeta_{\mathcal{M}}$ and $\omega_{\mathcal{M}}$ form essentially a congruence for M_{∞}.

Proposition 2.4.8. *The pair $(\zeta_{\mathcal{M}}, \Delta_{\mathbb{R}_+} \times \omega_{\mathcal{M}})$ of smooth equivalence relations is a congruence for $M_{\infty} : S \rightsquigarrow \mathbb{R}_+ \times (S \times \mathbb{R}_+)^{\infty}$.* $\square$

Relating the Relations

Two infinite paths are $\omega_{\mathcal{M}}$-equivalent iff their state components are $\zeta_{\mathcal{M}}$-equivalent (and the timing information is identical). This will support the investigation of logical equivalence later on, mainly since the information available for states is easier to handle than that for infinite paths. We obtain through a detailed analysis.

Proposition 2.4.9. $\omega_{\mathcal{M}} = (\zeta_{\mathcal{M}} \times \varDelta_{\mathbb{R}_+})^{\infty}$. □

This has two interesting consequences:

Corollary 2.4.10. *We have*

$$\mathbf{\Sigma}\big(\mathcal{B}((S \times \mathbb{R}_+)^{\infty}), (\zeta_{\mathcal{M}} \times \varDelta_{\mathbb{R}_+})^{\infty}\big) = \mathbf{\Sigma}\big(\mathcal{B}((S \times \mathbb{R}_+)^{\infty}), \omega_{\mathcal{M}}\big)$$

and

$$M_{\infty} : (S, \mathbf{\Sigma}\big(\mathcal{B}(S), \zeta_{\mathcal{M}}\big)) \rightsquigarrow \Big(\mathbb{R}_+ \times (S \times \mathbb{R}_+)^{\infty}, \mathbf{\Sigma}\big(\mathcal{B}(\mathbb{R}_+ \times (S \times \mathbb{R}_+)^{\infty}), \big(\varDelta_{\mathbb{R}_+} \times \zeta_{\mathcal{M}}\big)^{\infty}\big)\Big)$$

is a stochastic relation. □

The consequence of the equality in Proposition 2.4.9 is that we may check the equivalence of paths locally, i.e., through the equivalence of states. This represents a considerable reduction in complexity, because the equivalence relation $\omega_{\mathcal{M}}$ that operates on infinite paths is uniquely determined through the relation $\zeta_{\mathcal{M}}$ which in turn operates on states. It will be reflected in the representation of the equivalence classes, as we will see in Corollary 2.4.11. The reduction makes checking some properties of course much easier, and it has also technical advantages when it comes to checking the semi-pullback of two models, as we will in the next section.

We give a first consequence of this equality in terms of a representation of the equivalence classes.

Corollary 2.4.11. *Given* $\sigma \in (S \times \mathbb{R}_+)^{\infty}$, *the* $\omega_{\mathcal{M}}$*-class of* $\sigma = s_0 \xrightarrow{t_0} s_1 \xrightarrow{t_1} \ldots$ *can be represented as*

$$[\sigma]_{\omega_{\mathcal{M}}} = \prod_{j \geq 0} \Big([s_j]_{\zeta_{\mathcal{M}}} \times \{t_j\}\Big).$$

Moreover, we have Borel isomorphisms between these analytic spaces:

$$(S \times \mathbb{R}_+)^{\infty}/\omega_{\mathcal{M}} \cong \big((S \times \mathbb{R}_+)/(\zeta_{\mathcal{M}} \times \varDelta_{\mathbb{R}_+})\big)^{\infty} \cong \big((S/\zeta_{\mathcal{M}}) \times \mathbb{R}_+\big)^{\infty}. \ \square$$

We are now in a position to define the logical equivalence of models, and to relate it to spans of morphisms.

2.4.5 Logical Equivalence and Bisimilarity

Logical equivalence between two models says roughly that, given a state in one model, there exists a state in the other model so that in both exactly the same formulas are valid; similarly for paths. This equivalence is modelled after the corresponding equivalence that has been investigated in modal logics; see Definition 2.3.6. We have seen there that it is closely tied to the notion of bisimulation through the Hennessy-Milner Theorem. The relationship of this equivalence to bisimulations will be discussed now.

Let $\mathcal{M} = (S, M, \mathcal{I}, V)$ and $\mathcal{N} = (S', N, \mathcal{J}, W)$ be models for μ**CSL**. We assume that $\mathcal{M}$ is *nondegenerate*, i.e., that there exists a state formula ϕ with $\emptyset \neq [\![\phi]\!]_{\mathcal{M}} \neq S$. Being nondegenerate implies that the factor space $S/\zeta_{\mathcal{M}}$ is not trivial. Corollary 2.4.11 entails the existence of a path formula ψ such that $\emptyset \neq [\![\psi]\!]_{\mathcal{M}} \neq (S \times \mathbb{R}_+)^\infty$.

Basic Definitions

Define the models $\mathcal{M}$ and $\mathcal{N}$ as logically equivalent iff they accept exactly the same formulas. This is similar to logical equivalence for Kripke models, as the discussion in Section 2.3.2 indicates. In addition and in contrast, however, it has to take two levels into account, since we are dealing here with state formulas together with path formulas, so that formulas may hold in states or on paths — this situation is familiar from model checking where one has this dichotomy as well.

Definition 2.4.12. *The models $\mathcal{M}$ and $\mathcal{N}$ are called* logically equivalent *iff both*

$$\{Th_{\mathcal{M}}(s) \mid s \in S\} = \{Th_{\mathcal{N}}(s') \mid s' \in S'\}$$

and

$$\{Th_{\mathcal{M}}(\sigma) \mid \sigma \in (S \times \mathbb{R}_+)^\infty\} = \{Th_{\mathcal{N}}(\sigma') \mid \sigma' \in (S' \times \mathbb{R}_+)^\infty\}$$

hold.

Note that we take both states and infinite paths into consideration. Thus the models are logically equivalent iff these conditions are satisfied:

1. Given a state $s \in S$, there exists a state $s' \in S'$ such that $\big[\mathcal{M}, s \models \phi \Leftrightarrow \mathcal{N}, s' \models \phi\big]$ holds for all state formulas ϕ, and vice versa.
2. Given a path $\sigma \in (S \times \mathbb{R}_+)^\infty$, there exists a path $\sigma' \in (S' \times \mathbb{R}_+)^\infty$ such that $\big[\mathcal{M}, \sigma \models \psi \Leftrightarrow \mathcal{N}, \sigma' \models \psi\big]$ holds for all path formulas ψ, and vice versa.

As usual, the existence of a morphism between models entails their logical equivalence.

Proposition 2.4.13. *Let $\Phi : \mathcal{M} \to \mathcal{N}$ be a morphism. Then $\mathcal{M}$ and $\mathcal{N}$ are logically equivalent.* □

We will show that logically equivalent models are bisimilar. Bisimilarity is again introduced as a span of morphisms, and behavioral equivalence through a cospan.

Definition 2.4.14. *Let $\mathcal{M}$ and $\mathcal{N}$ be nondegenerate models for μ**CSL**.*

a. $\mathcal{M}$ and $\mathcal{N}$ are said to be bisimilar *iff there exists a model $\mathcal{Q}$ for μ**CSL** and morphisms*

$$\mathcal{M} \xleftarrow{\Phi} \mathcal{Q} \xrightarrow{\Psi} \mathcal{N}.$$

b. $\mathcal{M}$ and $\mathcal{N}$ are said to be behaviorally equivalent *iff there exists a model $\mathcal{Q}$ for μ**CSL** and morphisms*

$$\mathcal{M} \xrightarrow{\Phi} \mathcal{Q} \xleftarrow{\Psi} \mathcal{N}.$$

It is clear that bisimilar models are logically equivalent, because this notion of equivalence is transitive; see Proposition 2.4.7. Now suppose model $\mathcal{M} = (S, M, \mathcal{I}, V)$ is bisimilar to model $\mathcal{N} = (S', N, \mathcal{J}, W)$ with mediating model $\mathcal{Q}$ over the state space S'' and the morphisms according to Definition 2.4.14. Then the condition on bisimilarity implies in the present scenario that

1. $M(\Phi(s''))(I \times B) = N(\Psi(s''))(I \times B')$ for every state $s'' \in S''$, every rational interval I, and all common events $B \in \mathcal{B}(S), B' \in \mathcal{B}(S')$ (thus every pair of events B, B' such that $\Phi^{-1}[B] = \Psi^{-1}[B']$, as the discussion following Definition 2.2.1 indicates). Consequently, the probability for $\mathcal{M}$ changing the state during interval I and entering a state in B from state $\Phi(s'')$ equals the probability of $\mathcal{N}$ changing the state during time interval I and entering a state in B' from state $\Psi(s'')$. This illustrates again the mediating work done through model $\mathcal{Q}$.
2. For a state $s'' \in S''$, $\Phi(s'')$ is a member of the valuation for a state variable Z in model $\mathcal{M}$ iff $\Psi(s'')$ is a member for this variable in model $\mathcal{N}$, and similarly for path variables, and for atomic propositions.

The situation is a bit different with behavioral equivalence. By implication, the model which constitutes the range of the cospan is based on a Polish space. Otherwise we could not always conclude that behaviorally equivalent models are logically equivalent as well. This is so since computing the set of all states in which a given formula is valid requires the knowledge of a projective limit, and we did establish the existence of such a limit only for the case of Polish spaces, not for general analytic ones. On the other hand, these cospans are constructed usually through factoring (see, e.g., Propositions 2.2.4 and 2.4.22 below), and the factor space of a Polish space is not always a Polish one. So we need to exercise some care. Specifically, the behavioral equivalence of the models means that, if $\Phi(s) = \Psi(s')$ for states $s \in S$, $s' \in S'$ then

1. $M(s)(I \times \Phi^{-1}[B'']) = N(s')(I \times \Psi^{-1}[B''])$ whenever $B'' \in \mathcal{B}(S'')$ is a Borel set in S'', and I is a rational interval. Consequently, the probability of $\mathcal{M}$ changing the state during interval I and entering a state $s^{\bullet} \in S$ with $\Phi(s^{\bullet}) \in B''$ from state s equals the probability of $\mathcal{N}$ changing the state during time interval I and entering a state $s^{\star} \in S'$ with $\Psi(s^{\star}) \in B''$ from state s'.
2. s is a member for the valuation of a state variable Z in model $\mathcal{M}$ iff s' is a member for the valuation of a state variable Z in model $\mathcal{N}$; similarly for path variables, and for atomic propositions.

Returning to the general discussion, fix the models $\mathcal{M} = (S, M, \mathcal{I}, V)$ and $\mathcal{N} = (S', N, \mathcal{J}, W)$ such that $\mathcal{M}$ and $\mathcal{N}$ are logically equivalent; hence both models are logically equivalent. Each model has the equivalence relations $\zeta_{\mathcal{M}}$ and $\omega_{\mathcal{M}}$ or $\zeta_{\mathcal{N}}$ and $\omega_{\mathcal{N}}$ associated with it, as defined in Section 2.4.4.

The stochastic relations M_{∞} and N_{∞} will be investigated with respect to bisimilarity first, and it will be shown first that they are bisimilar *as stochastic relations on Polish spaces.*

Lemma 2.4.15. *$\zeta_{\mathcal{M}}$ and $\zeta_{\mathcal{N}}$ spawn each other; so do $\omega_{\mathcal{M}}$ and $\omega_{\mathcal{N}}$.*

Proof (Sketch) 0. We will show only that $\zeta_{\mathcal{M}}$ spawns $\zeta_{\mathcal{N}}$; interchanging the rôles of $\mathcal{M}$ and $\mathcal{N}$ will show that $\zeta_{\mathcal{N}}$ spawns $\zeta_{\mathcal{M}}$. The argumentation for $\omega_{\mathcal{M}}$ and $\omega_{\mathcal{N}}$ is nearly verbatim the same, so the reader is invited to fill in the details.

1. Define for the state $s \in S$ the map $\Upsilon([s]_{\zeta_{\mathcal{M}}}) := [s']_{\zeta_{\mathcal{N}}}$, whenever $Th_{\mathcal{M}}(s) = Th_{\mathcal{N}}(s')$. Because $s_1 \; \zeta_{\mathcal{M}} \; s_2$ iff $Th_{\mathcal{M}}(s_1) = Th_{\mathcal{M}}(s_2)$, and similarly for $\mathcal{N}$, the map is well defined. For the state formula ϕ its class $[\![\phi]\!]_{\mathcal{M}}$ can be represented as

$$\bigcup\{[s]_{\zeta_{\mathcal{M}}} \mid \mathcal{M}, s \models \phi\};$$

thus it is readily verified that $\Upsilon_{[\![\phi]\!]_{\mathcal{M}}} = [\![\phi]\!]_{\mathcal{N}}$. Consequently,

$$\{\Upsilon_{[\![\phi]\!]_{\mathcal{M}}} \mid \phi \text{ is a state formula}\}$$

is a generator of $\mathbf{\Sigma}\big(\mathcal{B}(S'), \zeta_{\mathcal{N}}\big)$. This generator is closed under intersections, since the conjunction of two state formulas is again one. $\square$

We know that both $(\zeta_{\mathcal{M}}, \Delta_{\mathbb{R}_+} \times \omega_{\mathcal{M}})$ and $(\zeta_{\mathcal{N}}, \Delta_{\mathbb{R}_+} \times \omega_{\mathcal{N}})$ are congruences for the stochastic relations M_{∞} resp. N_{∞}. We will show now that they are simulation equivalent, so that the situation is here very similar to that prevailing for logically equivalent modal logics in Section 2.3.2.

Proposition 2.4.16. *Let $\mathcal{M}$ and $\mathcal{N}$ be logically equivalent models. Then the congruences $\mathsf{c}_{\mathcal{M}} := (\zeta_{\mathcal{M}}, \Delta_{\mathbb{R}_+} \times \omega_{\mathcal{M}})$ and $\mathsf{c}_{\mathcal{N}} := (\zeta_{\mathcal{N}}, \Delta_{\mathbb{R}_+} \times \omega_{\mathcal{N}})$ are simulation equivalent.*

Proof (Sketch) 1. We know that $\zeta_{\mathcal{M}}$ and $\zeta_{\mathcal{N}}$ are in a mutually spawning relationship; so are $\omega_{\mathcal{M}}$ and $\omega_{\mathcal{N}}$. Consequently, $\Delta_{\mathbb{R}_+} \times \omega_{\mathcal{M}}$ and $\Delta_{\mathbb{R}_+} \times \omega_{\mathcal{N}}$ are related through spawning as well, where

$$\{I \times [\![\psi]\!]_{\mathcal{M}} \mid I \text{ is a rational interval, } \psi \text{ is a path formula}\}$$

and

$$\{I \times [\![\psi]\!]_{\mathcal{N}} \mid I \text{ is a rational interval, } \psi \text{ is a path formula}\}$$

are the generators that relate to each other.

2. Using the map $\Upsilon : S/\zeta_{\mathcal{M}} \to S'/\zeta_{\mathcal{N}}$ defined in the proof of Lemma 2.4.15, we show that

$$M_\infty(s)(I \times [\![\psi]\!]_{\mathcal{M}}) = N_\infty(s')(I \times [\![\psi]\!]_{\mathcal{N}})$$

for each $s \in S$, $s' \in \Upsilon([s]_{\zeta_{\mathcal{M}}})$, and for each rational interval I and each path formula ψ. Because $s' \in \Upsilon([s]_{\zeta_{\mathcal{M}}})$ means $Th_{\mathcal{M}}(s) = Th_{\mathcal{N}}(s')$, we obtain for an arbitrary rational number p

$$\begin{aligned} M_\infty(s)(I \times [\![\psi]\!]_{\mathcal{M}}) \leq p &\Leftrightarrow \mathcal{M}, s \models \mathcal{P}_{\leq p}(\mathcal{X}^I\, \psi) \\ &\Leftrightarrow \mathcal{N}, s' \models \mathcal{P}_{\leq p}(\mathcal{X}^I\, \psi) \\ &\Leftrightarrow N_\infty(s')(I \times [\![\psi]\!]_{\mathcal{M}}) \leq p; \end{aligned}$$

consequently, both probabilities are identical. This implies that $(\zeta_{\mathcal{M}}, \Delta_{\mathbb{R}_+} \times \omega_{\mathcal{M}})$ simulates $(\zeta_{\mathcal{N}}, \Delta_{\mathbb{R}_+} \times \omega_{\mathcal{N}})$. Interchanging the rôles of $\mathcal{M}$ and $\mathcal{N}$ gives the result now. $\square$

This yields the properties we are interested in for the associated stochastic relations.

Proposition 2.4.17. *Let $\mathcal{M}$ and $\mathcal{N}$ be logically equivalent models. Then the associated stochastic relations $M_\infty : S \rightsquigarrow (\mathbb{R}_+ \times S)^\infty$ and $N_\infty : S' \rightsquigarrow (\mathbb{R}_+ \times S')^\infty$ are bisimilar and behaviorally equivalent.* $\square$

Tuning the Mediator

This result is quite welcome when being looked at from the point of view of stochastic relations: Given two models for μ**CSL** that are logically equivalent, we can show that the associated stochastic relations are bisimilar. It does not give us, however, in this present and preliminary form a *model* that mediates between $\mathcal{M}$ and $\mathcal{N}$ (a similar situation has been encountered already with stochastic Kripke models in Section 2.3.2). An analysis of the construction leading to the mediating relation will again provide information for the construction of a model $\mathcal{L}$ and the desired morphisms $\mathcal{L} \to \mathcal{M}$ and $\mathcal{L} \to \mathcal{N}$. The construction leading to Proposition 2.4.17 is again based on a semi-pullback construction.

Lemma 2.4.18. *Let $\mathcal{M}$ and $\mathcal{N}$ be logically equivalent models for μ**CSL**. Define*

$$\begin{aligned}
A &:= \{\langle s, s'\rangle \in S \times S' \mid Th_{\mathcal{M}}(s) = Th_{\mathcal{N}}(s')\},\\
B &:= \big\{\langle\langle t, \sigma\rangle, \langle t, \sigma'\rangle\rangle \in (\mathbb{R}_+ \times (S \times \mathbb{R}_+)^\infty) \times (\mathbb{R}_+ \times (S' \times \mathbb{R}_+)^\infty) \mid \\
&\qquad\qquad\qquad\qquad\qquad\qquad Th_{\mathcal{M}}(\sigma) = Th_{\mathcal{N}}(\sigma')\big\}.
\end{aligned}$$

Then A and B are Standard Borel, and there exists a stochastic relation $L_0 : A \rightsquigarrow B$ that mediates between M_∞ and N_∞. The morphisms are composed from the corresponding projections. $\square$

We know from Proposition 2.4.9 that $\omega_{\mathcal{M}} = (\zeta_{\mathcal{M}} \times \varDelta_{\mathbb{R}_+})^\infty$; similarly for $\zeta_{\mathcal{N}}$ and $\omega_{\mathcal{N}}$. Thus B is essentially the set of all paths over A, extended by timing information.

Corollary 2.4.19. *Define A and B according to Lemma 2.4.18. There exists a bijection $\varLambda : B \to (A \times \mathbb{R}_+)^\infty$ that is also a Borel isomorphism.* $\square$

Define for this bijection $\varLambda$ the map $L' := \mathfrak{P}(\varLambda) \circ L_0$; then this is a stochastic relation $L' : A \rightsquigarrow \mathbb{R}_+ \times (A \times \mathbb{R}_+)^\infty$ that mediates between M_∞ and N_∞. But, still, this is not enough, because we cannot ascertain that L' is actually generated from a model, since we do not know whether or not L' is actually a projective limit of some sort. However, the semi-pullback is a rather flexible construction, and we will show now that we may construct from L' a mediator L_0 with the desired shape, viz., $L_0 = L_\infty$ for some stochastic relation $L : A \rightsquigarrow \mathbb{R}_+ \times A$.

In fact, put for $\langle s, s'\rangle \in A$ and for $E \in \mathcal{B}(\mathbb{R}_+ \times A)$

$$L(s, s')(E) := L'(s, s')\big(E \times \prod_{j>1} (\mathbb{R}_+ \times A)\big).$$

Thus the semi-pullback is restricted to its first component, yielding a stochastic relation $L : A \rightsquigarrow \mathbb{R}_+ \times A$, for which the projective limit can be constructed. This is what we will have a closer look at now.

Define for $n \in \mathbb{N}$ the map $\ell_n : (\mathbb{R}_+ \times A)^n \to (\mathbb{R}_+ \times S)^n$ through

$$\ell_n(t_1, s_1, s_1', \ldots, t_n, s_n, s_n') := \langle t_1, s_1, \ldots, t_n, s_n\rangle;$$

the map $r_n : (\mathbb{R}_+ \times A)^n \to (\mathbb{R}_+ \times S')^n$ is defined analogously.

Lemma 2.4.20. *Define $L_n : A \rightsquigarrow (\mathbb{R}_+ \times A)^n$ inductively from L in the same way as M_n is defined from M for the statement of Proposition 2.4.3, and let π_i be the i^{th} projection. Then the diagram*

$$\begin{tikzcd}
S & A \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "L_n"'] & S' \arrow[d, "N_n"] \\
\mathfrak{P}((\mathbb{R}_+ \times S)^n) & \mathfrak{P}((\mathbb{R}_+ \times A)^n) \arrow[l, "\mathfrak{P}(\ell_n)"] \arrow[r, "\mathfrak{P}(r_n)"'] & \mathfrak{P}((\mathbb{R}_+ \times S')^n)
\arrow[from=1-1, to=2-1, "M_n"']
\end{tikzcd}$$

commutes for every $n \in \mathbb{N}$.

Proof (Sketch) 1. The proof proceeds by induction on n. For $n = 1$ there is not much to show: By construction, L' mediates between M_∞ and N_∞, and the latter relations are projective limits, so that for $\langle s, s'\rangle \in A$ and $E \in \mathcal{B}(\mathbb{R}_+ \times S)$

$$\begin{aligned} M_1(s)(E) &= M_\infty(s)\big(E \times \prod_{j>1}(\mathbb{R}_+ \times S)\big) \\ &= L'(s,s')\big(\ell_1^{-1}[E] \times \prod_{j>1}(\mathbb{R}_+ \times A)\big) \\ &= L_1(s,s')(\ell_1^{-1}[E]). \end{aligned}$$

Similarly, the right hand side of the diagram above is shown to commute for $n = 1$.

2. Now assume the assertion is established for n; then we get from the induction hypothesis together with the Change of Variables formula (see Lemma 1.6.20) for $g : (\mathbb{R}_+ \times S)^n \to \mathbb{R}$ measurable and bounded, and for $\langle s, s'\rangle \in A$ the equality

$$\int_{(\mathbb{R}_+\times S)^n} g(\mathbf{v})\ M_n(s)(d\mathbf{v}) = \int_{(\mathbb{R}_+\times A)^n} (g \circ \ell_n)(\mathbf{w})\ L_n(s,s')(d\mathbf{w}).$$

This is shown first for $g = \chi_D$ for $D \in \mathcal{B}((\mathbb{R}_+ \times S)^n)$, whence it is equivalent to the induction hypothesis; then it is shown for step functions by the linearity of the integral, subsequently for nonnegative measurable and bounded g by the Monotone Convergence Theorem (Proposition 1.6.1), and finally for general g by decomposing the map into a positive and a negative part.

3. But now we can perform the induction step: Let $\langle s_0, s_0'\rangle \in A$ and $F \in \mathcal{B}((\mathbb{R}_+ \times S)^{n+1})$ be a Borel set; then

$$\begin{aligned} &M_{n+1}(s_0)(F) = \\ &\quad\int_{(\mathbb{R}_+\times S)^n} M(\hbar_S(\mathbf{v}))(\{\langle t,s\rangle \mid \langle \mathbf{v},t,s\rangle \in F\})\ M_n(s_0)(d\mathbf{v}) = \\ &\int_{(\mathbb{R}_+\times A)^n} M(\pi_1(\hbar_A(\mathbf{w})))(\{\langle t,s\rangle \mid \langle \mathbf{w},t,s,s'\rangle \in \ell_{n+1}^{-1}[F]\})L_n(s_0,s_0')(d\mathbf{w}) = \\ &\int_{(\mathbb{R}_+\times A)^n} L(\hbar_A(\mathbf{w}))(\{\langle t,s,s'\rangle \mid \langle \mathbf{w},t,s,s'\rangle \in \ell_{n+1}^{-1}[F]\})L_n(s_0,s_0')(d\mathbf{w}) = \\ &\qquad\qquad L_{n+1}(s_0,s_0')(\ell_{n+1}^{-1}[F]). \end{aligned}$$

□

Now extend ℓ_n and r_n to the corresponding infinite products, yielding maps ℓ_∞ and r_∞.

Proposition 2.4.21. *Assume that $\mathcal{M}$ and $\mathcal{N}$ are Hennessy-Milner equivalent; construct the Standard Borel space A and the stochastic relation $L : A \rightsquigarrow \mathbb{R}_+ \times A$ as above. Then*

i. $(\pi_1, \ell_\infty) : L_\infty \to M_\infty$ *and* $(\pi_2, r_\infty) : L_\infty \to N_\infty$ *are morphisms.*
ii. M_∞ *and* N_∞ *are bisimilar with* L_∞ *as a mediator.* $\square$

We are nearly ready for the main result, which will be stated after dealing with behavioral equivalence.

Proposition 2.4.22. *Let $\mathcal{M}$ and $\mathcal{N}$ be logically equivalent models for μ***CSL**. *If $S/\zeta_{\mathcal{M}}$ is a Standard Borel space, then $\mathcal{M}$ and $\mathcal{N}$ are behaviorally equivalent.* $\square$

This, now, is the main result:

Theorem 2.4.23. *Let $\mathcal{M}$ and $\mathcal{N}$ be nontrivial models for μ***CSL**. *Consider these statements:*

a. $\mathcal{M}$ and $\mathcal{N}$ are behaviorally equivalent.
b. $\mathcal{M}$ and $\mathcal{N}$ are logically equivalent.
c. $\mathcal{M}$ and $\mathcal{N}$ are bisimilar.

Then $a \Rightarrow b \Leftrightarrow c$, and if $S/\zeta_{\mathcal{M}}$ is a Standard Borel space, then all three statements are equivalent.

Proof (Sketch) 1. The implications $c \Rightarrow b$ and $a \Rightarrow b$ both follow from Proposition 2.4.13. If $S/\zeta_{\mathcal{M}}$ is Standard Borel, then $b \Rightarrow a$ follows from Proposition 2.4.22, so that we have to take care of $b \Rightarrow c$.

2. Construct the Standard Borel space A and the stochastic relation $L : A \rightsquigarrow \mathbb{R}_+ \times (A \times \mathbb{R}_+)^\infty$ together with the maps ℓ_∞ and r_∞ as in Proposition 2.4.21. Assume that the interpretation $\mathcal{J}$ for model $\mathcal{N}$ is $\mathcal{J} = (\Sigma', \Pi')$, and define

$$\mathcal{L} := (L, A, (\Sigma^*, \Pi^*), V^*)$$

with

1. $V^* := (V(a) \times W(a)) \cap A$ for the atomic propositions $a \in \mathsf{AP}$,
2. $\Sigma^*(Z) := (\Sigma(Z) \times \Sigma'(Z)) \cap A$ for the state variable $Z \in \mathsf{SV}$,
3. $\Pi^*(P) := \{\rho \in (A \times \mathbb{R}_+)^\infty \mid \ell_\infty(\rho) \in \Pi(P), r_\infty(\rho) \in \Pi'(P)\}$ for the path variable $P \in \mathsf{PV}$.

Then both $\ell_\infty : \mathcal{L} \to \mathcal{M}$ and $r_\infty : \mathcal{L} \to \mathcal{N}$ are morphisms. $\square$

2.5 Bibliographic Notes

The exposition follows essentially the discussion in [20], in particular, proofs which have been omitted in the present discussion can be found there. Nevertheless, some notes are in order.

Modal Logics

We follow essentially the exposition in the monograph [8], with an occasional look at Rutten's overview of coalgebras [72]. It may be interesting for the reader to look also at the massive collection [9], and in particular at the chapters on proof theory in [35]. An early collection of mathematical questions pertaining to modal logic can be found in [40]. The treatment of nondeterministic Kripke models is fairly standard, stochastic Kripke models and their morphisms were inspired by the work [58] of Larsen and Skou on testing. The paper [22] proposes stochastic Kripke models for general modal logics; the results on bisimilarity are from there.

Continuous Time Stochastic Logics

The paper [3] introduces and studies a logic called **CSL**, *continuous time stochastic logics*, with applications to model checking; after all, **CSL** is fashioned after the popular logics **CTL** studied extensively in model checking [16]. Some mathematical questions for **CSL** were discussed in [20], in particular the use of projective limits. There the fixed-point operator is introduced into this scenario, evolving **CSL** into μ**CSL**. The investigation of the relation of logical equivalence, bisimilarity, and behavioral equivalence found there seems to originate the investigation of continuous stochastic models for logics outside the direct realm of modal logics. It is also shown how previous *ad hoc* approaches to probabilistic modelling fit into the general model.

2.6 Appendix: Behavioral and Logical Equivalence Reconsidered

We have shown that logical equivalence, bisimilarity, and behavioral equivalence are the same for Kripke models that are based on analytic spaces. This appendix is intended to address the question of behavioral and logical equivalence without topological assumptions; thus we will work in general measurable spaces, and we will show that both notions are equivalent as well. We did not include bisimilarity in this discussion. If we want to show that two behavioral equivalent models are bisimilar, we are requested to construct a mediating model, and it is currently not clear how this can be done without constructing a semi-pullback which in turn requires at least analytic base spaces. For simplicity, the constructions will be carried out for the negation free Hennessy-Milner logic $\mathfrak{L} = \mathfrak{L}(\mathsf{Act}, [0,1])$, the formulas of which are given through the grammar

$$\top \mid \phi_1 \wedge \phi_2 \mid \langle a \rangle_r \phi.$$

Here $a \in \mathsf{Act}$ is an action, and the threshold r is a real number from the unit interval (see Example 2.3.3). This logic plays an important rôle in other places in this book as well, but we do not require here the set Act of possible actions to be countable, and we do not restrict ourselves to rationals for the values of thresholds. Because the logic is so simple we can keep the interpreting Kripke models simple, too.

2.6.1 Discussing the Strategy

Let us briefly reconsider the strategy for the analytic case, where we have a countable number of actions, and where the thresholds are taken from the rational numbers. Given a Kripke model over an analytical space, the logic defines an equivalence relation $\mathsf{r}_{\mathcal{K}}$ which is smooth; this is so since we only have a countable number of formulas at our disposal (having all real numbers in the unit interval rather than only the rationals is not essential, since the rationals are dense; see Section 3.4.1, in particular Lemma 3.4.1, for a discussion). This equivalence relation is used for factoring, and since it is smooth, we obtain an analytic space again, when adopting the final σ-algebra as the σ-algebra on the factor space. This σ-algebra has a fairly rich structure: it constitutes the Borel sets of an analytic space, thus making in particular Souslin's Theorem available; its generators can be computed directly through the logic, and it is in direct correspondence with the $\mathsf{r}_{\mathcal{K}}$-invariant Borel sets. All constructions remain within the realm of analytic sets, so that in particular the amalgamated sum leads to an analytic space which in turn can be made the state space of a Kripke model through standard constructions. The logic influences these discussions only through the corresponding equivalence relations, witnessed by the observation that the general criterion for bisimilarity enters the discussion, this criterion being formulated in terms of general smooth equivalence relations.

We show in this appendix that it is also possible to construct a cospan of Kripke models without having to use the machinery of Polish and analytic spaces. So we start from general measurable spaces, investigating the equivalence relation which is induced by the logic on the state space. Since analycity is not available, we will not be able to observe such a convenient interplay of the measurable structures induced by the logic on the state space and on the factor space; specifically we are no longer able to observe that the $\mathsf{r}_{\mathcal{K}}$-invariant measurable sets are exactly the inverse images of the elements of the final σ-algebra with respect to the factor map $\eta_{\mathsf{r}_{\mathcal{K}}}$. Thus we need to construct explicitly an σ-algebra on the factor space which is closely adapted to the logic, and to derive a Kripke model from it which plays the rôle of the factor model. Similarly, the amalgamation of the equivalences on the individual models needs to be investigated more closely, since the interesting properties are no longer being made available through analycity and smoothness. A technical observation notes that the sum of two Kripke models is no longer helpful, so that another avenue has to be considered. We solve this obstacle by first constructing a σ-algebra on the factor space for the amalgamation, and then construct a relation on this space; finding the σ-algebra with the right properties appears as the key point. The leading idea is based on the observation that the equivalence classes induced by the logic for logically equivalent Kripke models are in a one-to-one correspondence. This basically sketches the strategy for this excursion.

2.6.2 The Equivalence Relation Induced by the Logic $\mathfrak{L}$

Fix a Kripke model $\mathcal{K} = \big((S, \mathcal{S}), (k_a)_{a\in\mathsf{Act}}\big)$ with a measurable state space $(S, \mathcal{S})$; thus $k_a : (S, \mathcal{S}) \rightsquigarrow (S, \mathcal{S})$ is a stochastic relation for each action a. A *morfism* $f : \mathcal{K} \to \big((T, \mathcal{T}), (\ell_a)_{a\in\mathsf{Act}}\big)$ between Kripke models is an $\mathcal{S}$-$\mathcal{T}$-measurable map $f : S \to T$ such that

$$\forall a \in \mathsf{Act} : \ell_a \circ f = \mathfrak{S}\,(f) \circ k_a$$

holds. Note that we do not require f to be onto which otherwise is assumed nearly everywhere in this treatise (for emphasizing this, we have dubbed these maps *mor**f**isms* rather than *mor**ph**isms*). Just for the record:

Lemma 2.6.1. *Let $f : \mathcal{K} \to \mathcal{L}$ be a morfism and ϕ a formula in $\mathfrak{L}$, then*

a. $\mathcal{K}, s \models \phi \Leftrightarrow \mathcal{L}, f(s) \models \phi$ holds for each state s of K,
b. $f^{-1}\left[[\![\phi]\!]_{\mathcal{L}}\right] = [\![\phi]\!]_{\mathcal{K}}$. $\square$

The equivalence relation $\mathsf{r}_{\mathcal{K}}$ induced by $\mathfrak{L}$ on S is defined as above through

$$s\ \mathsf{r}_{\mathcal{K}}\ s' \text{ iff } \forall \phi : \mathcal{K}, s \models \phi \Leftrightarrow \mathcal{K}, s' \models \phi.$$

Define the set $\mathcal{E}_{\mathcal{K}}$ of all extensions for formulas through

$$\mathcal{E}_{\mathcal{K}} := \{[\![\phi]\!]_{\mathcal{K}} \mid \phi \text{ is a formula}\},$$

and define as σ-algebra $\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$ on the factor space $S/\mathsf{r}_{\mathcal{K}}$ as the smallest σ-algebra which contains all the sets the inverse image of which lies in $\mathcal{E}_{\mathcal{K}}$:

$$\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}} := \sigma\big(\{A \subseteq S/\mathsf{r}_{\mathcal{K}} \mid \eta_{\mathsf{r}_{\mathcal{K}}}^{-1}[A] \in \mathcal{E}_{\mathcal{K}}\}\big).$$

We analyze this construction, and then we enter a discussion of behavioral and logical equivalence.

Lemma 2.6.2. *The set $\mathcal{A} := \{\eta_{\mathsf{r}_{\mathcal{K}}}\left[[\![\phi]\!]_{\mathcal{K}}\right] \mid \phi$ is a formula$\}$ is a generator of $\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$, which is closed under finite intersections.*

Proof Because each extension $[\![\phi]\!]_{\mathcal{K}}$ is $\eta_{\mathsf{r}_{\mathcal{K}}}$-invariant, and because the logic $\mathfrak{L}$ is closed under conjunction, $\mathcal{A}$ is closed under finite intersections. The factor map is onto; thus we have $[\![\phi]\!]_{\mathcal{K}} = \eta_{\mathsf{r}_{\mathcal{K}}}^{-1}\left[\eta_{\mathsf{r}_{\mathcal{K}}}\left[[\![\phi]\!]_{\mathcal{K}}\right]\right]$. Consequently, if ϕ is a formula, then $\eta_{\mathsf{r}_{\mathcal{K}}}\left[[\![\phi]\!]_{\mathcal{K}}\right] \in \mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$, thus $\sigma(\mathcal{A}) \subseteq \mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$. On the other hand, if $\eta_{\mathsf{r}_{\mathcal{K}}}^{-1}[A] \in \mathcal{E}_{\mathcal{K}}$ for some $A \subseteq S/\mathsf{r}_{\mathcal{K}}$, then there exists a formula ϕ such that $[\![\phi]\!]_{\mathcal{K}} = \eta_{\mathsf{r}_{\mathcal{K}}}^{-1}[A]$; hence $A = \eta_{\mathsf{r}_{\mathcal{K}}}\left[[\![\phi]\!]_{\mathcal{K}}\right]$, since $\eta_{\mathsf{r}_{\mathcal{K}}}$ is onto. This yields

$$\{A \subseteq S/\mathsf{r}_{\mathcal{K}} \mid \eta_{\mathsf{r}_{\mathcal{K}}}^{-1}[A] \in \mathcal{E}_{\mathcal{K}}\} \subseteq \mathcal{A},$$

establishing the equality. □

This has the following as an immediate consequence.

Corollary 2.6.3. *The factor map $\eta_{\mathsf{r}_{\mathcal{K}}} : S \to S/\mathsf{r}_{\mathcal{K}}$ is $\mathcal{S}$-$\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$-measurable.*

Proof Consider
$$\mathcal{D} := \{A \in \mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}} \mid \eta_{\mathsf{r}_{\mathcal{K}}}^{-1}[A] \in \mathcal{S}\}.$$
Because $\mathcal{S} \ni [\![\phi]\!]_{\mathcal{K}} = \eta_{\mathsf{r}_{\mathcal{K}}}^{-1}\left[\eta_{\mathsf{r}_{\mathcal{K}}}\left[[\![\phi]\!]_{\mathcal{K}}\right]\right]$, we may conclude that $\mathcal{A} \subseteq \mathcal{D}$, where the generator $\mathcal{A}$ is defined as in Lemma 2.6.2. Because $\mathcal{D}$ is a σ-algebra, we obtain $\sigma(\mathcal{A}) \subseteq \mathcal{D}$. Thus the assertion follows from Lemma 2.6.2. □

Because the final σ-algebra with respect to a map is the largest σ-algebra on the codomain rendering this map measurable, we obtain the following as an immediate consequence.

Corollary 2.6.4. *Let $\mathcal{S}/\mathsf{r}_{\mathcal{K}}$ be the final σ-algebra with respect to $\eta_{\mathsf{r}_{\mathcal{K}}} : S \to S/\mathsf{r}_{\mathcal{K}}$ and $\mathcal{S}$. Then $\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}} \subseteq \mathcal{S}/\mathsf{r}_{\mathcal{K}}$.* □

A closer analysis reveals that we can say even more, viz., that $\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$ coincides with the factor algebra, provided the $\eta_{\mathsf{r}_{\mathcal{K}}}$-invariant sets are generated from the formulas. Recall that $\mathbf{\Sigma}(\mathcal{S}, \mathsf{r}_{\mathcal{K}})$ is σ-algebra of $\eta_{\mathsf{r}_{\mathcal{K}}}$-invariant measurable sets.

Corollary 2.6.5. *These statements are equivalent:*

a. $\sigma\big(\{[\![\phi]\!]_{\mathcal{K}} \mid \phi \textit{ is a formula}\}\big) = \mathbf{\Sigma}\big(\mathcal{S}, \mathsf{r}_{\mathcal{K}}\big)$.
b. $\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}} = \mathcal{S}/\mathsf{r}_{\mathcal{K}}$.

Proof 1. For establishing $a \Rightarrow b$ we show first that

$$\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}} = \{A \subseteq \mathcal{S}/\mathsf{r}_{\mathcal{K}} \mid \eta_{\mathsf{r}_{\mathcal{K}}}^{-1}[A] \in \sigma(\mathcal{E}_{\mathcal{K}})\}.$$

The construction of $\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$ implies that it is contained in the latter σ-algebra. For establishing the reverse inclusion, we argue as follows. Because each element of $\sigma(\mathcal{E}_{\mathcal{K}})$ is $\eta_{\mathsf{r}_{\mathcal{K}}}$-invariant, the π-λ-Theorem 1.3.1 is used to show that $\eta_{\mathsf{r}_{\mathcal{K}}}[B] \in \mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$ for each $B \in \sigma(\mathcal{E}_{\mathcal{K}})$. Because $\eta_{\mathsf{r}_{\mathcal{K}}}$ is onto, we know $\eta_{\mathsf{r}_{\mathcal{K}}}\left[\eta_{\mathsf{r}_{\mathcal{K}}}^{-1}[A]\right] = A$ for $A \subseteq \mathcal{S}/\mathsf{r}_{\mathcal{K}}$; hence the inclusion follows.

The assertion follows now from the observation that the factor σ-algebra $\mathcal{S}/\mathsf{r}_{\mathcal{K}}$ can be written as $\mathcal{S}/\mathsf{r}_{\mathcal{K}} = \sigma(\{\eta_{\mathsf{r}_{\mathcal{K}}}[A] \mid A \in \mathcal{S} \text{ is } \eta_{\mathsf{r}_{\mathcal{K}}}\text{-invariant}\})$.

2. In order to prove $b \Rightarrow a$ we note first that the sets $[\![\phi]\!]_{\mathcal{K}}$ are $\mathsf{r}_{\mathcal{K}}$-invariant, so that $\sigma\big(\{[\![\phi]\!]_{\mathcal{K}} \mid \phi \text{ is a formula}\}\big) \subseteq \mathbf{\Sigma}\big(\mathcal{S}, \mathsf{r}_{\mathcal{K}}\big)$. On the other hand, Lemma 2.6.2 yields

$$\eta_{\mathsf{r}_{\mathcal{K}}}^{-1}\left[\mathcal{S}_{\mathsf{r}_{\mathcal{K}}}^{\ddagger}\right] = \sigma(\{[\![\phi]\!]_{\mathcal{K}} \mid \phi \text{ is a formula}\}).$$

Now take $A \in \mathbf{\Sigma}\big(\mathcal{S}, \mathsf{r}_{\mathcal{K}}\big)$. Then $A = \eta_{\mathsf{r}_{\mathcal{K}}}^{-1}\left[\eta_{\mathsf{r}_{\mathcal{K}}}\left[A\right]\right]$ with $\eta_{\mathsf{r}_{\mathcal{K}}}\left[A\right] \in \mathcal{S}/\mathsf{r}_{\mathcal{K}}$. Consequently the assumption implies $A \in \eta_{\mathsf{r}_{\mathcal{K}}}^{-1}\left[\mathcal{S}_{\mathsf{r}_{\mathcal{K}}}^{\ddagger}\right]$, from which the reverse inclusion follows. □

Comparing the construction for the general case with the one for analytic spaces, it follows that we can determine the crucial σ-algebra $\mathcal{S}_{\mathsf{r}_{\mathcal{K}}}^{\ddagger}$ through the factor map. This is actually a straightforward consequence of Corollary 1.7.13 and Corollary 2.6.5.

Corollary 2.6.6. *Assume that S is an analytic space with $\mathcal{S} = \mathcal{B}(S)$, and that* Act *is countable. Then $\mathcal{S}_{\mathsf{r}_{\mathcal{K}}}^{\ddagger} = \mathcal{B}(S/\mathsf{r}_{\mathcal{K}})$.* □

If $\mathcal{S}_{\mathsf{r}_{\mathcal{K}}}^{\ddagger}$ is a proper sub-σ-algebra of $\mathcal{S}/\mathsf{r}_{\mathcal{K}}$, then we conclude with Proposition 1.7.21 that the equivalence relation induced by the logic constitutes no longer a congruence for the Kripke model. This is a fairly peculiar situation which indicates that analytic state spaces play a somewhat special rôle (and invites further investigations for the general case).

Example 2.6.7. Put $S := \{x, y, z\}$ as the state space, $\mathcal{S} := \big\{\emptyset, S, \{x\}, \{y, z\}\big\}$, as the σ-algebra over S, and fix $\mu \in \mathfrak{S}\left(S, \mathcal{S}\right)$. Let $\mathsf{Act} := \{*\}$ be a singleton set of actions, and put $k_*(s) := \mu$ for all $s \in S$. Then we know for the Kripke model $\mathcal{K} := \big((S, \mathcal{S}), (k_a)_{a \in \mathsf{Act}}\big)$ that $\{x\}$ is $\mathsf{r}_{\mathcal{K}}$-invariant; hence

$$\mathbf{\Sigma}\big(\mathcal{S}, \mathsf{r}_{\mathcal{K}}\big) = \mathcal{S}.$$

Furthermore we establish by induction that $\mathcal{K}, s \models \phi \Leftrightarrow \mathcal{K}, s' \models \phi$ for any states $s, s' \in S$, so that $[\![\phi]\!]_{\mathcal{K}} \neq \emptyset$ implies $[\![\phi]\!]_{\mathcal{K}} = S$. Consequently,

$$\sigma\big(\{[\![\phi]\!]_{\mathcal{K}} \mid \phi \text{ is a formula}\}\big) = \{\emptyset, S\}.$$

This implies by Corollary 2.6.5 that $\mathcal{S}_{\mathsf{r}_{\mathcal{K}}}^{\ddagger} \neq \mathcal{S}/\mathsf{r}_{\mathcal{K}}$. Thus we cannot dispose of the assumption that the state space is analytic. ⊣

Now consider the Kripke model $\mathcal{K}$. Let $a \in \mathsf{Act}$ be an action; the factor relation $k_{a,\mathsf{r}_{\mathcal{K}}}$ is defined through

$$k_{a,\mathsf{r}_{\mathcal{K}}}\big([s]_{\mathsf{r}_{\mathcal{K}}}\big)(A) := k_a(s)(\eta_{\mathsf{r}_{\mathcal{K}}}^{-1}\left[A\right])$$

whenever $A \in \mathcal{S}_{\mathsf{r}_{\mathcal{K}}}^{\ddagger}$ (compare the definition of the factor relation in the general setting on page 53). This definition is possible since we know from Corollary 2.6.4 that $\eta_{\mathsf{r}_{\mathcal{K}}}^{-1}\left[A\right] \in \mathcal{S}$ for $A \in \mathcal{S}_{\mathsf{r}_{\mathcal{K}}}^{\ddagger}$; it determines in fact a stochastic relation.

Proposition 2.6.8. $k_{a,\mathsf{r}_\mathcal{K}} : (S/\mathsf{r}_\mathcal{K}, \mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}) \rightsquigarrow (S/\mathsf{r}_\mathcal{K}, \mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}})$ *is a stochastic relation.*

Proof 1. Because $k_{a,\mathsf{r}_\mathcal{K}}\big([s]_{\mathsf{r}_\mathcal{K}}\big) = \big(\mathfrak{S}\,(\eta_{\mathsf{r}_\mathcal{K}}) \circ k_a\big)(s)$, it follows immediately that $k_{a,\mathsf{r}_\mathcal{K}}\big([s]_{\mathsf{r}_\mathcal{K}}\big)$ is a subprobability on $\mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}$ for each $s \in S$.

2. Fix $A \in \mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}$; then the $\mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}$-measurability of $v \mapsto k_{a,\mathsf{r}_\mathcal{K}}(v)(A)$ has to be established. Let for this

$$\mathcal{D} := \{A \in \mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}} \mid v \mapsto k_{a,\mathsf{r}_\mathcal{K}}(v)(A) \text{ is } \mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}\text{-measurable}\}.$$

We observe these properties

(i) $\mathcal{D}$ is closed under complementation.
(ii) $\mathcal{D}$ is closed under disjoint countable unions.
(iii) $\mathcal{A} \subseteq \mathcal{D}$, where $\mathcal{A}$ is the generator defined in Lemma 2.6.2.

The first and the second property follow from the usual properties of measurable functions; so only the last property needs to be verified. Because

$$\mathcal{K}, s \models \langle a \rangle_r \phi \Leftrightarrow k_a(s)\big([\![\phi]\!]_\mathcal{K}\big) \geq r,$$

we infer

$$\begin{aligned}\{[s]_{\mathsf{r}_\mathcal{K}} \mid k_{a,\mathsf{r}_\mathcal{K}}\big([s]_{\mathsf{r}_\mathcal{K}}\big)\big(\eta_{\mathsf{r}_\mathcal{K}}\,[[\![\phi]\!]_\mathcal{K}]\big) \geq r\} &= \eta_{\mathsf{r}_\mathcal{K}}\,\big[\{s \in S \mid k_a(s)\big([\![\phi]\!]_\mathcal{K}\big) \geq r\}\big] \\ &= \eta_{\mathsf{r}_\mathcal{K}}\,[[\![\langle a \rangle_r \phi]\!]_\mathcal{K}]\,,\end{aligned}$$

and the latter set is a member of $\mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}$. Consequently, $\eta_{\mathsf{r}_\mathcal{K}}\,[[\![\phi]\!]_\mathcal{K}] \in \mathcal{D}$ for each formula ϕ. Using the π-λ-Theorem 1.3.1 and Lemma 2.6.2, we conclude now that $\mathcal{D} = \mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}$ holds. $\square$

The factor map defines a morfism between the stochastic relations k_a and $k_{a,\mathsf{r}_\mathcal{K}}$, as we will see now. In fact, define the Kripke model

$$\mathcal{K}_\mathfrak{L} := \big((S/\mathsf{r}_\mathcal{K}, \mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}), (k_{a,\mathsf{r}_\mathcal{K}})_{a \in \mathsf{Act}}\big);$$

then we make this observation, which will be useful for the investigations of behavioral and logical equivalence:

Corollary 2.6.9. $\eta_{\mathsf{r}_\mathcal{K}} : \mathcal{K} \to \mathcal{K}_\mathfrak{L}$ *is a morfism.*

Proof Because we have

$$k_{a,\mathsf{r}_\mathcal{K}}\big([s]_{\mathsf{r}_\mathcal{K}}\big)(A) = k_a(s)\big(\eta^{-1}_{\mathsf{r}_\mathcal{K}}\,[A]\big) = \big(\mathfrak{S}\,(\eta_{\mathsf{r}_\mathcal{K}}) \circ k_a\big)(s)(A)$$

for all actions $a \in \mathsf{Act}$, for all states $s \in S$, and for all sets $A \in \mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}$, the claim is easily established. $\square$

2.6.3 Logical Equivalence

Now let $\mathcal{L} = \big((T, \mathcal{T}), (\ell_a)_{a \in \mathsf{Act}}\big)$ be another Kripke model. Denote the equivalence relation defined by the logic $\mathfrak{L}$ on T by $\mathsf{r}_{\mathcal{L}}$. All constructions with the σ-algebra $\mathcal{S}$ and the equivalence relation $\mathsf{r}_{\mathcal{K}}$ are carried out with $\mathcal{T}$ and $\mathsf{r}_{\mathcal{L}}$, so we may construct a σ-algebra $\mathcal{T}^{\ddagger}_{\mathsf{r}_{\mathcal{L}}}$ on $T/\mathsf{r}_{\mathcal{L}}$, and we obtain a new Kripke model $\mathcal{L}_{\mathfrak{L}} = \big((T/\mathsf{r}_{\mathcal{L}}, \mathcal{T}^{\ddagger}_{\mathsf{r}_{\mathcal{L}}}), (\ell_{a,\mathsf{r}_{\mathcal{K}}})_{a \in \mathsf{Act}}\big)$ together with the morfism $\eta_{\mathsf{r}_{\mathcal{L}}} : \mathcal{L} \to \mathcal{L}_{\mathfrak{L}}$.

Define the relation

$$\Re := \{\langle s, t\rangle \in S \times T \mid Th_{\mathcal{K}}(s) = Th_{\mathcal{L}}(t)\}.$$

Consequently, $\langle s, t\rangle \in \Re$ iff s and t satisfy exactly the same formulas. In particular, we know that then $k_a(s)([\![\phi]\!]_{\mathcal{K}}) = \ell_a([\![\phi]\!]_{\mathcal{L}})$ holds for all formulas ϕ. This is so because $k_a(s)\big([\![\phi]\!]_{\mathcal{K}}\big) \geq r \Leftrightarrow \ell_a(t)\big([\![\phi]\!]_{\mathcal{L}}\big) \geq r$ for each r in the unit interval (it would suffice to restrict ourselves to rational numbers r).

In addition to relation $\Re$ a relation $\Re_0$ derived from it on the Cartesian product of the factor spaces is defined:

$$\Re_0 := \{\langle [s]_{\mathsf{r}_{\mathcal{K}}}, [t]_{\mathsf{r}_{\mathcal{L}}}\rangle \mid \langle s, t\rangle \in \Re\}.$$

Call models $\mathcal{K}$ and $\mathcal{L}$ *behaviorally equivalent* iff there exists a cospan

$$\mathcal{K} \xrightarrow{f} \mathcal{M} \xleftarrow{g} \mathcal{L}$$

of morfisms, as above. Logical equivalence may be defined through the relation $\Re$: $\mathcal{K}$ and $\mathcal{L}$ are said to be *logically equivalent* iff the relation $\Re$ is both right and left total. This is but a simple reformulation of the usual definition of logical equivalence; see Definition 2.3.6. Assume for the rest of this section that the Kripke models $\mathcal{K}$ and $\mathcal{L}$ are logically equivalent.

Lemma 2.6.10. *$\Re_0$ is the graph of a bijection $\tau : S/\mathsf{r}_{\mathcal{K}} \to T/\mathsf{r}_{\mathcal{L}}$; τ is $\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$-$\mathcal{T}^{\ddagger}_{\mathsf{r}_{\mathcal{L}}}$-measurable. Similarly, $\Re_0^{-1}$ is the graph of a bijection $\theta : S/\mathsf{r}_{\mathcal{K}} \to T/\mathsf{r}_{\mathcal{L}}$, which is $\mathcal{T}^{\ddagger}_{\mathsf{r}_{\mathcal{L}}}$-$\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$-measurable. τ and θ are inverse to each other.*

Proof 1. Define $\tau([s]_{\mathsf{r}_{\mathcal{K}}}) := [t]_{\mathsf{r}_{\mathcal{L}}}$ iff $\langle [s]_{\mathsf{r}_{\mathcal{K}}}, [t]_{\mathsf{r}_{\mathcal{L}}}\rangle \in \Re_0$. Then $\tau : S/\mathsf{r}_{\mathcal{K}} \to T/\mathsf{r}_{\mathcal{L}}$ is obviously well defined and injective. Because $\Re$ is a right total relation, τ is surjective as well.

2. Consider $\mathcal{D} := \{B \in \mathcal{T}^{\ddagger}_{\mathsf{r}_{\mathcal{L}}} \mid \tau^{-1}[B] \in \mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}\}$. Then $\mathcal{D}$ is closed under complementation and countable disjoint unions. Let ϕ be a formula; then $\tau^{-1}\left[\eta_{\mathsf{r}_{\mathcal{L}}}\left[[\![\phi]\!]_{\mathcal{L}}\right]\right] = \eta_{\mathsf{r}_{\mathcal{K}}}\left[[\![\phi]\!]_{\mathcal{K}}\right]$. Consequently, $\eta_{\mathsf{r}_{\mathcal{L}}}\left[[\![\phi]\!]_{\mathcal{L}}\right] \in \mathcal{D}$, so $\mathcal{D}$ contains a generator which is closed under finite intersections by Lemma 2.6.2. Thus $\mathcal{D} = \mathcal{T}^{\ddagger}_{\mathsf{r}_{\mathcal{L}}}$ by the π-λ-Theorem 1.3.1. Thus τ is $\mathcal{S}^{\ddagger}_{\mathsf{r}_{\mathcal{K}}}$-$\mathcal{T}^{\ddagger}_{\mathsf{r}_{\mathcal{L}}}$-measurable.

3. The argumentation for θ is verbatim the same, after interchanging the rôles of $\mathcal{K}$ and $\mathcal{L}$. It is also obvious that τ and θ are inverse to each other. $\square$

If $\Re$ would not be a left total relation, the map τ would only be partially defined; if $\Re$ would not be right total, τ would not be surjective. Consequently, this construction works only with logically equivalent Kripke models.

But τ and θ are even richer in structure.

Lemma 2.6.11. *Define τ and θ as in Lemma 2.6.10. Then $\tau : \mathcal{K}_{\mathfrak{L}} \to \mathcal{L}_{\mathfrak{L}}$ and $\theta : \mathcal{L}_{\mathfrak{L}} \to \mathcal{K}_{\mathfrak{L}}$ are morfisms.*

Proof 1. Assume that $\tau\big([s]_{\mathsf{r}_{\mathcal{K}}}\big) = [t]_{\mathsf{r}_{\mathcal{L}}}$; then, since $\eta_{\mathsf{r}_{\mathcal{K}}}^{-1}\left[\eta_{\mathsf{r}_{\mathcal{K}}}\left[[\![\phi]\!]_{\mathcal{K}}\right]\right] = [\![\phi]\!]_{\mathcal{K}}$, and similarly for $[\![\phi]\!]_{\mathcal{L}}$,

$$\begin{aligned} k_{a,\mathsf{r}_{\mathcal{K}}}\big([s]_{\mathsf{r}_{\mathcal{K}}}\big)\big(\eta_{\mathsf{r}_{\mathcal{K}}}\left[[\![\phi]\!]_{\mathcal{K}}\right]\big) &= k_a(s)\big([\![\phi]\!]_{\mathcal{K}}\big) \\ &= \ell_a(t)\big([\![\phi]\!]_{\mathcal{L}}\big) \\ &= \ell_{a,\mathsf{r}_{\mathcal{L}}}\big([t]_{\mathsf{r}_{\mathcal{L}}}\big)\big(\eta_{\mathsf{r}_{\mathcal{L}}}\left[[\![\phi]\!]_{\mathcal{L}}\right]\big). \end{aligned}$$

2. We want to show that

$$\ell_{a,\mathsf{r}_{\mathcal{L}}}\big(\tau([s]_{\mathsf{r}_{\mathcal{K}}})\big)(A) = k_{a,\mathsf{r}_{\mathcal{K}}}([s]_{\mathsf{r}_{\mathcal{K}}})\big(\tau^{-1}[A]\big)$$

holds for each $s \in S$ and each measurable set $A \in \mathcal{T}^{\ddagger}_{\mathsf{r}_{\mathcal{L}}}$. Consider for fixed $a \in \mathsf{Act}$ and $s \in S$ the set

$$\mathcal{D} := \{A \in \mathcal{T}^{\ddagger}_{\mathsf{r}_{\mathcal{L}}} \mid \ell_{a,\mathsf{r}_{\mathcal{L}}}\big(\tau([s]_{\mathsf{r}_{\mathcal{K}}})\big)(A) = k_{a,\mathsf{r}_{\mathcal{K}}}([s]_{\mathsf{r}_{\mathcal{K}}})\big(\tau^{-1}[A]\big)\}.$$

Then $\mathcal{D}$ is closed under complementation and countable disjoint unions, and part 1 shows that $\eta_{\mathsf{r}_{\mathcal{L}}}\left[[\![\phi]\!]_{\mathcal{L}}\right] \in \mathcal{D}$ for each formula. Thus the π-λ-Theorem 1.3.1 together with Lemma 2.6.2 implies that $D = \mathcal{T}^{\ddagger}_{\mathsf{r}_{\mathcal{L}}}$. Consequently, τ constitutes a morfism; the same argumentation shows that θ is a morfism as well. $\square$

2.6.4 Logical vs. Behavioral Equivalence

Form the sum $S + T$ of the state space with injections $i_S : S \to S + T$ and $i_T : T \to S + T$, respectively. Let $\mathsf{r}_{\mathcal{K}} \diamond \mathsf{r}_{\mathcal{L}}$ be the amalgamation of $\mathsf{r}_{\mathcal{K}}$ and $\mathsf{r}_{\mathcal{L}}$, so that we have for $v, v' \in S + T$ in the present scenario

$$v \; \mathsf{r}_{\mathcal{K}} \diamond \mathsf{r}_{\mathcal{L}} \; v' \iff \begin{cases} s \; \mathsf{r}_{\mathcal{K}} \; s', & v = i_S(s), v' = i_S(s'), \\ t \; \mathsf{r}_{\mathcal{L}} \; t', & v = i_T(t), v' = i_T(t'), \\ \langle s, t\rangle \in \Re, & v = i_S(s), v' = i_T(t), \\ \langle t, s\rangle \in \Re^{-1}, & v = i_T(t), v' = i_S(s) \end{cases}$$

(see page 52). This construction entails also that

$$(*)\qquad \begin{cases} [i_S(s)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}} &= i_S\big[[s]_{\mathsf{r}_\mathcal{K}}\big] \cup i_T\big[\tau\big([s]_{\mathsf{r}_\mathcal{K}}\big)\big],\\ [i_T(t)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}} &= i_T\big[[t]_{\mathsf{r}_\mathcal{L}}\big] \cup i_S\big[\theta\big([t]_{\mathsf{r}_\mathcal{L}}\big)\big].\end{cases}$$

Thus the equivalence class of an element of $S + T$ has both a non-void component from S and from T, and these components are linked through τ and θ, respectively. Now define maps $I_S : S/\mathsf{r}_\mathcal{K} \to (S+T)/\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}$ and $I_T : T/\mathsf{r}_\mathcal{L} \to (S+T)/\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}$ through

$$\begin{aligned} I_S\big([s]_{\mathsf{r}_\mathcal{K}}\big) &:= [i_S(s)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}}\,,\\ I_S\big([t]_{\mathsf{r}_\mathcal{L}}\big) &:= [i_T(t)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}}\,.\end{aligned}$$

Hence we assign to each class in the participating spaces the class of its representative in the sum; it is clear from the characterization in $(*)$ that both maps are well defined.

Lemma 2.6.12. $I_S\left[\eta_{\mathsf{r}_\mathcal{K}}\left[[\![\phi]\!]_\mathcal{K}\right]\right] = I_T\left[\eta_{\mathsf{r}_\mathcal{L}}\left[[\![\phi]\!]_\mathcal{L}\right]\right]$ *holds for each formula* ϕ, *and the set* $\{I_S\left[\eta_{\mathsf{r}_\mathcal{K}}\left[[\![\phi]\!]_\mathcal{K}\right]\right] \mid \phi$ *is a formula*$\}$ *is closed under finite intersections.*

Proof The first claim is established by this direct computation:

$$\begin{aligned} I_S\left[\eta_{\mathsf{r}_\mathcal{K}}\left[[\![\phi]\!]_\mathcal{K}\right]\right] &= \{I_S(b) \mid b \in \eta_{\mathsf{r}_\mathcal{K}}\left[[\![\phi]\!]_\mathcal{K}\right]\}\\ &= \{[i_S(s)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}} \mid s \in [\![\phi]\!]_\mathcal{K}\}\\ &\overset{(\natural)}{=} \{[i_T(t)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}} \mid t \in [\![\phi]\!]_\mathcal{L}\}\\ &= \{I_T(c) \mid c \in \eta_{\mathsf{r}_\mathcal{L}}\left[[\![\phi]\!]_\mathcal{L}\right]\}\\ &= I_T\left[\eta_{\mathsf{r}_\mathcal{L}}\left[[\![\phi]\!]_\mathcal{L}\right]\right]\end{aligned}$$

The crucial equality $(\natural)$ follows immediately from $[i_S(s)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}} = [\tau(i_S(s))]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}}$ and $[i_T(t)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}} = [\theta(i_T(t))]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}}$ together with $\theta = \tau^{-1}$(Lemma 2.6.10). Because $\eta_{\mathsf{r}_\mathcal{K}}\left[[\![\phi]\!]_\mathcal{K}\right]$ is an I_S-invariant set, it follows from Lemma 1.4.30 that $\{I_S\left[\eta_{\mathsf{r}_\mathcal{K}}\left[[\![\phi]\!]_\mathcal{K}\right]\right] \mid \phi$ is a formula$\}$ is closed under finite intersections. □

Define the σ-algebra $\mathcal{W}$ on $(S+T)/\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}$ as

$$\mathcal{W} := \sigma\big(\{I_S\left[\eta_{\mathsf{r}_\mathcal{K}}\left[[\![\phi]\!]_\mathcal{K}\right]\right] \mid \phi \text{ is a formula}\}\big);$$

then $\mathcal{W} = \sigma\big(\{I_T\left[\eta_{\mathsf{r}_\mathcal{L}}\left[[\![\phi]\!]_\mathcal{L}\right]\right] \mid \phi \text{ is a formula}\}\big)$ follows from Lemma 2.6.12, and the maps I_S and I_T turn out to be measurable.

Lemma 2.6.13. $I_S : S \to (S+T)/\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}$ *is* $\mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}$-$\mathcal{W}$-*measurable, and* $I_T : T \to (S+T)/\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}$ *is* $\mathcal{T}^\ddagger_{\mathsf{r}_\mathcal{L}}$-$\mathcal{W}$-*measurable.*

Proof We focus on the map I_S and apply essentially the π-λ-Theorem 1.3.1 again. Let

$$\mathcal{D} := \{C \in \mathcal{W} \mid I_S^{-1}\left[C\right] \in \mathcal{S}^\ddagger_{\mathsf{r}_\mathcal{K}}\}.$$

Because $\eta_{\mathsf{r}_\mathcal{K}}[[\![\phi]\!]_\mathcal{K}]$ is an I_S-invariant set, Lemma 1.4.30 tells us for a formula ϕ that $I_S^{-1}[I_S[\eta_{\mathsf{r}_\mathcal{K}}[[\![\phi]\!]_\mathcal{K}]]] = \eta_{\mathsf{r}_\mathcal{K}}[[\![\phi]\!]_\mathcal{K}]$, and the latter is an element of $\mathcal{S}^{\ddagger}_{\mathsf{r}_\mathcal{K}}$. Consequently, $I_S[\eta_{\mathsf{r}_\mathcal{K}}[[\![\phi]\!]_\mathcal{K}]] \in \mathcal{D}$ for each formula ϕ. Now $\mathcal{D}$ is a σ-algebra, in particular closed under complementation and disjoint countable unions. The family $\{I_S[\eta_{\mathsf{r}_\mathcal{K}}[[\![\phi]\!]_\mathcal{K}]] \mid \phi \text{ is a formula}\}$ generates $\mathcal{W}$ and is closed under finite intersections by Lemma 2.6.12, so the π-λ-Theorem implies $\mathcal{D} = \mathcal{W}$. This establishes the assertion. □

This construction yields a measurable space $((S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}, \mathcal{W})$. It will serve as the state space for a Kripke model for which we will construct the transition law now. Before we do that, we need to make sure that the transition laws $k_{a,\mathsf{r}_\mathcal{K}}$ and $\ell_{a,\mathsf{r}_\mathcal{L}}$ coincide on certain crucial sets.

Lemma 2.6.14. *Assume that* $\langle s,t\rangle \in \mathfrak{R}$. *Then*

$$k_{a,\mathsf{r}_\mathcal{K}}([s]_{\mathsf{r}_\mathcal{K}})(I_S^{-1}[C]) = \ell_{a,\mathsf{r}_\mathcal{L}}([t]_{\mathsf{r}_\mathcal{L}})(I_T^{-1}[C])$$

for all $C \in \mathcal{W}$.

Proof It is by the π-λ-Theorem (Lemma 1.6.30) sufficient to take the set C from the generator $\{I_S[\eta_{\mathsf{r}_\mathcal{K}}[[\![\phi]\!]_\mathcal{K}]] \mid \phi \text{ is a formula}\}$ of $\mathcal{W}$. Thus, let ϕ be a formula; then

$$\begin{aligned}
k_{a,\mathsf{r}_\mathcal{K}}([s]_{\mathsf{r}_\mathcal{K}})\big(I_S^{-1}[I_S[\eta_{\mathsf{r}_\mathcal{K}}[[\![\phi]\!]_\mathcal{K}]]]\big) &= k_{a,\mathsf{r}_\mathcal{K}}([s]_{\mathsf{r}_\mathcal{K}})\big(\eta_{\mathsf{r}_\mathcal{K}}[[\![\phi]\!]_\mathcal{K}]\big) \\
&= k_a(s)(\eta_{\mathsf{r}_\mathcal{K}}^{-1}[\eta_{\mathsf{r}_\mathcal{K}}[[\![\phi]\!]_\mathcal{K}]]) \\
&= k_a(s)([\![\phi]\!]_\mathcal{K}) \\
&\overset{(\natural)}{=} \ell_a(t)([\![\phi]\!]_\mathcal{L}) \\
&= \ell_{a,\mathsf{r}_\mathcal{L}}([t]_{\mathsf{r}_\mathcal{L}})\big(\eta_{\mathsf{r}_\mathcal{L}}[[\![\phi]\!]_\mathcal{L}]\big) \\
&= \ell_{a,\mathsf{r}_\mathcal{L}}([t]_{\mathsf{r}_\mathcal{L}})\big(I_T^{-1}[I_T[\eta_{\mathsf{r}_\mathcal{L}}[[\![\phi]\!]_\mathcal{L}]]]\big).
\end{aligned}$$

The equation $(\natural)$ permits the comparisons between different Kripke models; it follows from the assumption that $\langle s,t\rangle \in \mathfrak{R}$. □

Now we are poised to define the transition law on the compound factor space. We put

$$m_a([i_S(s)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}})(C) := k_{a,\mathsf{r}_\mathcal{K}}([s]_{\mathsf{r}_\mathcal{K}})(I_S^{-1}[C])$$

for $C \in \mathcal{W}$. It follows from Lemma 2.6.14 that also

$$m_a([i_T(t)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}})(C) = \ell_{a,\mathsf{r}_\mathcal{L}}([t]_{\mathsf{r}_\mathcal{L}})(I_T^{-1}[C])$$

holds, provided $[i_S(s)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}} = [i_T(t)]_{\mathsf{r}_\mathcal{K}\diamond\mathsf{r}_\mathcal{L}}$. This is so because the latter condition is equivalent to $\langle [s]_{\mathsf{r}_\mathcal{K}}, [t]_{\mathsf{r}_\mathcal{L}}\rangle \in \mathfrak{R}_0$ which in turn is equivalent to $\langle s,t\rangle \in \mathfrak{R}$.

This defines for each action $a \in \mathsf{Act}$ a map

$$m_a : (S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L} \to \mathfrak{S}\left((S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}, \mathcal{W}\right),$$

and we have to make sure that it defines a Kripke model, i.e., that it is a stochastic relation. Thus we have to establish measurability.

Lemma 2.6.15. $m_a : \big((S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}, \mathcal{W}\big) \rightsquigarrow \big((S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}, \mathcal{W}\big)$ *is a stochastic relation for each action* $a \in \mathsf{Act}$.

Proof 1. We have to show that $w \mapsto m_a(w)(G)$ constitutes a $\mathcal{W}$-measurable map on $(S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}$ for each set $G \in \mathcal{W}$. This is established through the π-λ-Theorem 1.3.1 by investigating the set of all sets for which the assertion holds.

2. In fact, $\mathcal{D} := \{G \in \mathcal{W} \mid w \in m_a(w)(G) \text{ is } \mathcal{W}\text{-measurable}\}$ is closed under complementation and under countable disjoint unions. Now assume that $G = I_S\left[\eta_{\mathsf{r}_\mathcal{K}}\left[[\![\phi]\!]_\mathcal{K}\right]\right]$ is an element of the generators of $\mathcal{W}$, then

$$\begin{aligned} m_a\big([s]_{\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}}\big)(G) \geq r &\Leftrightarrow k_{a,\mathsf{r}_\mathcal{K}}\big([s]_{\mathsf{r}_\mathcal{K}}\big)\big(\eta_{\mathsf{r}_\mathcal{K}}\left[[\![\phi]\!]_\mathcal{K}\right]\big) \geq r \\ &\Leftrightarrow [s]_{\mathsf{r}_\mathcal{K}} \in [\![\langle a\rangle_r \phi]\!]_\mathcal{K}. \end{aligned}$$

Thus

$$\{w \in (S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L} \mid m_a(w)\big(I_S\left[\eta_{\mathsf{r}_\mathcal{K}}\left[[\![\phi]\!]_\mathcal{K}\right]\right]\big) \geq r\} = I_S\left[\eta_{\mathsf{r}_\mathcal{K}}\left[[\![\langle a\rangle_r \phi]\!]_\mathcal{K}\right]\right],$$

which is a generator itself.

Hence $\mathcal{D}$ contains the generator of $\mathcal{W}$ which is closed under finite intersections by Lemma 2.6.12, so that $\mathcal{D} = \mathcal{W}$, which establishes measurability. □

Thus we may use m_a as the transition law for a Kripke model.

Corollary 2.6.16. $\mathcal{K}_\mathfrak{L} \xrightarrow{I_S} \mathcal{M} \xleftarrow{I_T} \mathcal{L}_\mathfrak{L}$ *is a cospan of morfisms.* □

This is the main result.

Proposition 2.6.17. *Logical and behavioral equivalence are the same for Kripke models over general measurable spaces for the negation-free Hennessy-Milner logic* $\mathfrak{L}$.

Proof 1. Since morfisms preserve and reflect validity, behaviorally equivalent Kripke models are logically equivalent.

2. Let $\mathcal{K} = \big((S, \mathcal{S}), (k_a)_{a \in \mathsf{Act}}\big)$ and $\mathcal{L} = \big((T, \mathcal{T}), (\ell_a)_{a \in \mathsf{Act}}\big)$ be the Kripke models under consideration. Construct the factor space $(S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}$ for the amalgamation $\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}$ of the equivalence relations $\mathsf{r}_\mathcal{K}$ and $\mathsf{r}_\mathcal{L}$ which are constructed from logic $\mathfrak{L}$ over S respectively T together with the maps $I_S : S/\mathsf{r}_\mathcal{K} \to (S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}$ and $I_T : T/\mathsf{r}_\mathcal{L} \to (S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}$. Construct the σ-algebra $\mathcal{W}$ from these data, and define the stochastic relation $m_a : \big((S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}, \mathcal{W}\big) \rightsquigarrow \big((S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}, \mathcal{W}\big)$. Put

$$\mathcal{M} := \Big(\big((S+T)/\mathsf{r}_\mathcal{K} \diamond \mathsf{r}_\mathcal{L}, \mathcal{W}\big), (m_a)_{a \in \mathsf{Act}}\Big).$$

Then this diagram gives the desired cospan of morfisms.

$$\begin{tikzcd}
\mathcal{K} \arrow[d, "\eta_{\mathsf{r}_{\mathcal{K}}}"'] & & \mathcal{L} \arrow[d, "\eta_{\mathsf{r}_{\mathcal{L}}}"] \\
\mathcal{K}_{\mathfrak{L}} \arrow[dr, "I_S"'] & & \mathcal{L}_{\mathfrak{L}} \arrow[dl, "I_T"] \\
& \mathcal{M} &
\end{tikzcd}$$

The factor maps $\eta_{\mathsf{r}_{\mathcal{K}}}$ and $\eta_{\mathsf{r}_{\mathcal{L}}}$ are morfisms by Corollary 2.6.9; I_S and I_T are morfisms by Corollary 2.6.16. $\square$

One may wonder why we went through this slightly complicated construction process in order to obtain the σ-algebra on the target space. Taking the sum $S + T$ of the measurable spaces $(S, \mathcal{S})$ and $(T, \mathcal{T})$ with the sum-σ-algebra $\mathcal{S} + \mathcal{T}$, factoring this space through the amalgamated equivalence relation $\mathsf{r}_{\mathcal{K}} \diamond \mathsf{r}_{\mathcal{L}}$ and constructing the σ-algebra $(\mathcal{S} + \mathcal{T})^{\ddagger}_{\mathsf{r}_{\mathcal{K}} \diamond \mathsf{r}_{\mathcal{L}}}$ is not possible. Alas, this approach requires an underlying Kripke model which this process is just intended to construct.

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