# Preface

## Motivation

Modal logics are usually interpreted through Kripke models, branching logics find their interpretation through models which deal with infinite paths. These seemingly structurally different interpretations can be unified by considering coalgebras which model the underlying worlds suitably; the predicates through which the formulas are represented in their interpretation are modelled using natural transformations between functors, to which the functor that underlies the coalgebra contributes. The basic functor is usually based on the power set functor. Adopting this general approach, we see that a fairly general and uniform way of interpreting modal logics and their step twins arises through coalgebras and the generalization of predicates into suitable natural transformations. We will show in this treatise that coalgebras based on the subprobability functor are amenable to these ideas as well.

Having arrived at this general approach of interpreting a rather broad family of logics, the question of comparing different models relative to a given logic presents itself. So we ask (and answer) the question about the conditions under which the well investigated relationship of logical equivalence, bisimilarity, and behavioral equivalence holds in this generalized, uniform scenario.

This is one of two driving topics in this book. The other one notes that stochastic interpretations — such as the ones indicated above — rest on stochastic relations, which in turn are the Kleisli morphisms for the Giry monad. The morphisms considered so far are based on measurable maps which are the morphisms of the base category. But at least bisimilarity and behavioral equivalence are formulated through morphisms, viz., through the existence of a span or a cospan, respectively. These formulations apply verbatim to Kleisli morphisms as well. Generalizing logical equivalence to distributional equivalence lifts the entire stage to the level of the Kleisli category, and, again, the problem of the relationship of the various behavioral descriptions presents itself. Since morphisms and congruences are very closely related, a study of these issues needs to be accompanied by a careful investigation of congruences on the Kleisli category.

This is what you will find in this book.

## Overview

A brief survey of the contents of the individual chapters is in order.

### *Chapter 1*

Stochastic relations form the mathematical basis for a probabilistic interpretation of coalgebraic logics. They provide also a foundation for Markov transition systems. These relations in turn are based on transition probabilities, and because we do not confine ourselves to the probabilistic case but rather accept models in which mass vanishes, we base the theory on transition subprobabilities. Capturing the properties of these relations and applying them to the logics under consideration requires a substantial amount of Measure Theory when models are required that are based on spaces containing more than a finite or a countable number of elements. Thus we study measures on general measurable spaces, and, since these spaces are usually too general, we make a topological assumption and consider Polish spaces as well as spaces that arise from their images under measurable maps, i.e., analytic spaces. Analytic spaces find usually the right balance of generality (for representing the application) and manageability (for the model). They provide the stage on which our play is going to be performed.

Chapter 1 is devoted to giving an overview of Borel sets and measures on analytic spaces, as far as this theory is needed; the purpose of this chapter is to provide a self-contained introduction to these topics. In contrast to Chapter 1 in the present author's book on stochastic relation [20], however, not all proofs are given, and whenever necessary we provide a pointer to that book's *Gentle Tutorial to All Things Considered*, where an attempt has been made to collect useful topics from the somewhat scattered literature. Nevertheless, the reader finds all necessary information on the topological and measure-theoretic base of what is going to be discussed in the latter chapters.

Because we will deal extensively with equivalence relations that are countably generated, and with the lifting of these relations to the space of all subprobabilities over the base space, we devote two sections to their study and consider this an investment which pays off largely in the later chapters.

Oh, yes: we assume a very basic familiarity with the theory of categories; this is assembled as well in this chapter (in fact, the reader is welcomed with a bunch of definitions from categories). Coalgebras together with the fundamental notions of behavioral equivalence and bisimilarity are discussed here as well; although in most applications to follow a detailed definition is given, the coalgebraic definitions provided here give a frame of reference. More advanced concepts of categories like monads are introduced whenever we need them; in this case, monads are not discussed before Section 3.2.

## *Chapter 2*

The probabilistic interpretation of some fairly well understood logics is discussed in this chapter. We first consider bisimilar stochastic relations and give a criterion for bisimilarity; this discussion is fairly basic also for the more coalgebraically oriented logics that follow later, so we discuss this concept carefully and with a view towards generalizations. It is applied to modal logics and to continuous time stochastic logics. These logics have been studied already [20]. We deal with them nevertheless in the present treatise in the context of general coalgebraic logic. The reason for this is that we feel that it is difficult to appreciate the step from modal logics to coalgebraic logics without an understanding of these approaches for well understood logics. It becomes only then visible where the interesting and critical points are. This applies in particular to the interplay of the functor and predicate liftings, the latter being the constructs replacing modal operators. The similarities of and the differences between interpretations of modal logics and continuous time stochastic logics are emphasized, and we encounter these topics again when discussing coalgebraic logics.

After all these interpretations the patient reader is given a vacation from analytic spaces: we show that Kripke models for a simple Hennessy-Milner logic are logical equivalent if and only if they are behavioral equivalent without imposing any topological assumptions. This is just an application of the techniques developed so far, but care has to be exercised, since working with just these topological assumptions is fairly tempting. A comparison of the general measurable setting against the analytic scenario shows that some phenomena which coincide in the analytic case display a fairly wide gap in the general measurable case, albeit the basic technical ideas are the same.

Given that Chapters 1 and 2 provide the fundamental tools, the discussion now splits into two branches, both of which are generalizations of interpretations for modal logics. The first one generalizes the state space, the second one generalizes the logic proper.

## *Chapter 3*

A Kripke model discusses validity on the basis of states: it defines the validity of formulas for individual states. But this is sometimes not adequate. Consider a large economy: here the behavior of an individual customer is only interesting insofar as it influences larger entities, the coalitions. Hence the question arises how to deal with these larger entities adequately, and we propose comparing the behavior of subprobability distributions rather than states proper — two distributions behave in the same way with respect to the logic iff they assign the same probability to all formulas' extensions. This gives rise to new ways of comparing the behavior of Kripke models, viz., on the distributional level rather than on the level of individual states.

The discussion in the previous chapter has shown that these questions are answered through an investigation of morphisms. Hence we need custom-tailored morphisms for this purpose. They are provided through the Kleisli category of the associated monad (the Giry monad), and we develop a theory of morphisms in this particular Kleisli category. Actually, we do a bit more than we need to, and discuss factoring for these morphisms as well, but this does not hurt (on the contrary, it provides some insight into the inner workings of these morphisms).

An investigation of the different ways for comparing the behavior of Kripke models is in order, and we show how the different ways relate to each other. An important tool in these discussions is provided by ergodic morphisms; we borrow the notion of ergodicity from the theory of dynamical systems for the purpose of modelling a very close relationship of Kleisli morphisms to the logic via the equivalence relations which they induce.

## *Chapter 4*

Modal logics are characterized by their modal operators. Take for example the logic that has the diamond as a modal operator in addition to the usual Boolean operations. Then an interpreting Kripke model $\mathcal{R} = (S, R)$ with state space $S$ and relation $R \subseteq S \times S$ has $\mathcal{R}, s \models \Diamond\phi$ iff we can find a successor $t$ to $s$ that satisfies $\phi$, hence iff $R(s) \cap [\![\phi]\!] \neq \emptyset$. Reformulating the latter condition says $s \in \bigl(R^{-1} \circ \lambda_S\bigr)([\![\phi]\!])$, where $\lambda_S(A) := \{B \subseteq S \mid B \cap A \neq \emptyset\}$. Then $\lambda$ is a natural transformation for a power set functor, and $\mathcal{R}$ is perceived as a coalgebra for this functor with relation $R$ as its dynamics.

Hence a modal operator is viewed as a natural transformation, and the transition structure in the Kripke model gives rise to a coalgebra. The required natural transformations will have some additional properties; they are called predicate liftings. This is basically the theme which will be developed for stochastic coalgebraic logics in this Chapter.

To this end we will look at two structurally different scenarios by developing the theory separately for left and for right coalgebras. A left coalgebra for functor $\mathfrak{F}$ is a coalgebra for the functor $\mathfrak{S} \circ \mathfrak{F}$. These coalgebras correspond to A. Sokolova's generative systems; in contrast, a right coalgebra for functor $\mathfrak{F}$ is a coalgebra for the functor $\mathfrak{F} \circ \mathfrak{S}$, corresponding to reactive systems in Sokolova's taxonomy. It is shown that modal logics can be modelled through left coalgebras (this applies to, e.g., continuous time stochastic logics), and that Markov transition systems are special cases of right coalgebras. We investigate the corresponding logics for both kinds of coalgebras.

For left coalgebras the subprobability functor is the dominating one, and it turns out that the techniques for dealing with modal logics developed so far from stochastic relations may be applied, albeit with a grain of salt. For right coalgebras, however, the dominant functor is $\mathfrak{F}$, and here some new techniques have to be found, in particular the selection of morphisms through suitable selection theorems from the theory of Borel sets becomes important.

Interestingly, the resulting characterization of logical equivalence, bisimilarity, and behavioral equivalence, which is formulated on a global level (i.e., for entire coalgebras) can be made to work for a local characterization of these properties (i.e., for individual states). This yields also a new characterization of bisimilar states in a Markov transition system.

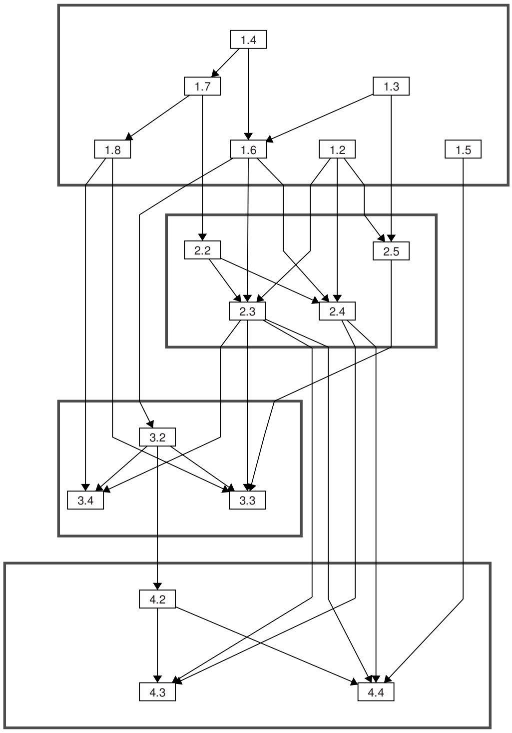
The graph below indicates the dependencies among the sections.

## *Acknowledgements*

Christoph Schubert convinced me to tackle the problems at hand with at least a small load of categories under the belt. I want to thank him for more than a few discussions and suggestions. The Deutsche Forschungsgemeinschaft (DFG) supported the investigations into the algebraic properties of stochastic relations for four years (*Algebraische Eigenschaften stochastischer Relationen*, grant DO-263/8-1/2). This is gratefully acknowledged.

The author is also grateful to Alberto Casagrande, Alexander Kurz, Carla Piazza, Chunlai Zhou, Dieter Pumplün, Eugenio Omodeo, Giovanna d'Agostino, Ingo Battenfeld, Johannes Köster, Klaus Keimel, Larry Moss, Lutz Schröder, Peter Buchholz, Peter Padawitz, Vincent Danos, and Yixiang Chen for discussions related in one way or other to the topics discussed here; the anonymous reviewer also made some insightful comments. Ingo Battenfeld and in particular Christoph Schubert offered numerous suggestions for improving the representation.

Special help was provided by Alla Stankjawitschene, Helmut Henning gave some hints when I needed to write my own indexing programs, and, again, Stefan Dissmann's organizational skills made many things easier for me. Petra




Mutzel provided the drawing of the dependency graph. The cooperation with Ronan Nugent from Springer-Verlag was a pleasure. The author is grateful for all this help.

Bochum and Dortmund, June 2009 Ernst-Erich Doberkat

Springer

http://www.springer.com/978-3-642-02994-3

Stochastic Coalgebraic Logic
Doberkat, E.-E.
2009, XV, 231 p. 81 illus., Hardcover
ISBN: 978-3-642-02994-3