Chapter 4
On Hecke Algebras

We will start this chapter by giving the definition and the classification of *complex reflection groups*. We will also define the *braid group* and the *pure braid group* associated to a complex reflection group. We will then introduce the *generic Hecke algebra* of a complex reflection group, which is a quotient of the group algebra of the associated braid group defined over a Laurent polynomial in a finite number of indeterminates. Under certain assumptions, which have been verified for all but a finite number of cases, we prove (Theorem 4.2.5) that the generic Hecke algebras of complex reflection groups are essential. Therefore, all results obtained in Chapter 3 apply to the case of the generic Hecke algebras.

A *cyclotomic Hecke algebra* is obtained from the generic Hecke algebra via a *cyclotomic specialization* (Definition 4.3.1). We prove (Theorem 4.3.3) that any cyclotomic specialization is essentially an adapted morphism. Thus, we can use Theorem 3.3.2 in order to obtain the *Rouquier blocks* of a cyclotomic Hecke algebra (*i.e.,* its blocks over the *Rouquier ring*, defined in Section 4.4), which are a substitute for the families of characters that can be applied to all complex reflection groups. We will see that the Rouquier blocks have the property of *semi-continuity*, thus depending only on some "essential" hyperplanes for the group, which are determined by the generic Hecke algebra.

The theory developed in this chapter will allow us to determine the Rouquier blocks of the cyclotomic Hecke algebras of all (irreducible) complex reflection groups in the next chapter.

4.1 Complex Reflection Groups and Associated Braid Groups

Let μ_∞ be the group of all the roots of unity in $\mathbb{C}$ and K a number field contained in $\mathbb{Q}(\mu_\infty)$. We denote by $\mu(K)$ the group of all the roots of unity of K. For every integer $d > 1$, we set $\zeta_d := \exp(2\pi i/d)$ and denote by μ_d the group of all the d-th roots of unity. Let V be a K-vector space of finite dimension r.

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4.1.1 Complex Reflection Groups

Definition 4.1.1. A *pseudo-reflection* is a non-trivial element s of $\mathrm{GL}(V)$ which acts trivially on a hyperplane, called the *reflecting hyperplane* of s.

If W is a finite subgroup of $\mathrm{GL}(V)$ generated by pseudo-reflections, then (V, W) is called a K-reflection group of rank r.

We have the following classification of complex reflection groups, also known as the "Shephard-Todd classification". For more details about the classification, one may refer to [61].

Theorem 4.1.2. *Let (V, W) be an irreducible complex reflection group (i.e., W acts irreducibly on V). Then one of the following assertions is true:*

- *There exist non-zero integers d, e, r such that $(V, W) \cong G(de, e, r)$, where $G(de, e, r)$ is the group of all $r \times r$ monomial matrices with non-zero entries in μ_{de} such that the product of all non-zero entries lies in μ_d.*
- *(V, W) is isomorphic to one of the 34 exceptional groups $G_n (n = 4, \ldots, 37)$.*

Remark. Among the irreducible complex reflection groups, we encounter the irreducible real reflection groups. In particular, we have:

- *$G(1, 1, r) \cong A_{r-1}$ for $r \geq 2$,*
- *$G(2, 1, r) \cong B_r$ (or C_r) for $r \geq 2$,*
- *$G(2, 2, r) \cong D_r$ for $r \geq 4$,*
- *$G(e, e, 2) \cong I_2(e)$, where $I_2(e)$ denotes the dihedral group of order $2e$,*
- *$G_{23} = H_3$, $G_{28} = F_4$, $G_{30} = H_4$, $G_{35} = E_6$, $G_{36} = E_7$, $G_{37} = E_8$.*

The following theorem has been proved (using a case by case analysis) by Benard [5] and Bessis [7] and generalizes a well known result for Weyl groups.

Theorem-Definition 4.1.3 *Let (V, W) be a reflection group. Let K be the field generated by the traces on V of all the elements of W. Then all irreducible KW-representations are absolutely irreducible, i.e., K is a splitting field for W. The field K is called the* field of definition *of the group W.*

- If $K \subseteq \mathbb{R}$, then W is a (finite) Coxeter group.
- If $K = \mathbb{Q}$, then W is a Weyl group.

4.1.2 Braid Groups Associated to Complex Reflection Groups

For all definitions and results about braid groups we follow [22]. Note that for a given topological space X and a point $x_0 \in X$, we denote by $\Pi_1(X, x_0)$ the fundamental group with base point x_0.

Let V be a K-vector space of finite dimension r. Let W be a finite subgroup of $\mathrm{GL}(V)$ generated by pseudo-reflections and acting irreducibly on V. We denote by $\mathcal{A}$ the set of its reflecting hyperplanes. We define the *regular variety* $V^{\mathrm{reg}} := \mathbb{C}\otimes V - \bigcup_{H\in\mathcal{A}} \mathbb{C}\otimes H$. For $x_0 \in V^{\mathrm{reg}}$, we define $P := \Pi_1(V^{\mathrm{reg}}, x_0)$ the *pure braid group* (at x_0) associated with W. If $p : V^{\mathrm{reg}} \to V^{\mathrm{reg}}/W$ denotes the canonical surjection, we define $B := \Pi_1(V^{\mathrm{reg}}/W, p(x_0))$ the *braid group* (at x_0) associated with W.

The projection p induces a surjective map $B \twoheadrightarrow W, \sigma \mapsto \bar{\sigma}$ as follows: Let $\tilde{\sigma} : [0,1] \to V^{\mathrm{reg}}$ be a path in V^{reg} such that $\tilde{\sigma}(0) = x_0$, which lifts σ. Then $\bar{\sigma}$ is defined by the equality $\bar{\sigma}(x_0) = \tilde{\sigma}(1)$. Note that the map $\sigma \mapsto \bar{\sigma}$ is an anti-morphism.

Denoting by W^{op} the group opposite to W, we have the following short exact sequence

$$1 \to P \to B \to W^{\mathrm{op}} \to 1,$$

where the map $B \to W^{\mathrm{op}}$ is defined by $\sigma \mapsto \bar{\sigma}$.

Now, for every hyperplane $H \in \mathcal{A}$, we set e_H the order of the group W_H, where W_H is the subgroup of W formed by id_V and all the reflections fixing the hyperplane H. The group W_H is cyclic: if s_H denotes an element of W_H with determinant $\zeta_H := \zeta_{e_H}$, then $W_H = \langle s_H \rangle$ and s_H is called a *distinguished reflection* in W.

Let $L_H := \mathrm{Im}(s - \mathrm{id}_V)$. Then, for all $x \in V$, we have $x = \mathrm{pr}_H(x) + \mathrm{pr}_{L_H}(x)$ with $\mathrm{pr}_H(x) \in H$ and $\mathrm{pr}_{L_H}(x) \in L_H$. Thus, $s_H(x) = \mathrm{pr}_H(x) + \zeta_H \mathrm{pr}_{L_H}(x)$.

If $t \in \mathbb{R}$, we set $\zeta_H^t := \exp(2\pi i t/e_H)$ and we denote by s_H^t the element of $\mathrm{GL}(V)$ (a pseudo-reflection if $t \neq 0$) defined by

$$s_H^t(x) := \mathrm{pr}_H(x) + \zeta_H^t \mathrm{pr}_{L_H}(x).$$

For $x \in V$, we denote by $\sigma_{H,x}$ the path in V from x to $s_H(x)$ defined by

$$\sigma_{H,x} : [0,1] \to V,\ t \mapsto s_H^t(x).$$

Let γ be a path in V^{reg} with initial point x_0 and terminal point x_H. Then γ^{-1} is the path in V^{reg} with initial point x_H and terminal point x_0 such that

$$\gamma^{-1}(t) = \gamma(1-t) \text{ for all } t \in [0,1].$$

Thus, we can define the path $s_H(\gamma^{-1}) : t \mapsto s_H(\gamma^{-1}(t))$, which goes from $s_H(x_H)$ to $s_H(x_0)$ and lies also in V^{reg}, since for all $x \in V^{\mathrm{reg}}$, $s_H(x) \in V^{\mathrm{reg}}$ (If $s_H(x) \notin V^{\mathrm{reg}}$, then $s_H(x)$ must belong to a hyperplane H'. If $s_{H'}$ is a distinguished pseudo-reflection with reflecting hyperplane H', then $s_{H'}(s_H(x)) = s_H(x)$ and ${s_H}^{-1}(s_{H'}(s_H(x))) = x$. However, ${s_H}^{-1} s_{H'} s_H$ is a reflection and x belongs to its reflecting hyperplane, $s_H^{-1}(H')$. This contradicts the fact that x belongs to V^{reg}.). Now we define a path from x_0 to $s_H(x_0)$ as follows:

$$\sigma_{H,\gamma} := s_H(\gamma^{-1}(t)) \cdot \sigma_{H,x_H} \cdot \gamma.$$

If x_H is chosen "close to H and far from the other reflecting hyperplanes", the path $\sigma_{H,\gamma}$ lies in V^{reg} and its homotopy class does not depend on the choice of x_H. The element it induces in the braid group B, $\mathbf{s}_{H,\gamma}$, is a distinguished braid reflection around the image of H in V^{reg}/W.

Proposition 4.1.4.

(1) The braid group B is generated by the distinguished braid reflections around the images of the hyperplanes $H \in \mathcal{A}$ in V^{reg}/W.
(2) The image of $\mathbf{s}_{H,\gamma}$ in W is s_H.
(3) Whenever γ' is a path in V^{reg} from x_0 to x_H, if λ denotes the loop in V^{reg} defined by $\lambda := \gamma'^{-1}\gamma$, then

$$\sigma_{H,\gamma'} = s_H(\lambda) \cdot \sigma_{H,\gamma} \cdot \lambda^{-1}.$$

In particular, $\mathbf{s}_{H,\gamma}$ and $\mathbf{s}_{H,\gamma}$ are conjugate in P.
(4) The path $\prod_{j=e_H-1}^{j=0} \sigma_{H,s_H^j(\gamma)}$, a loop in V^{reg}, induces the element $\mathbf{s}_{H,\gamma}^{e_H}$ in the braid group B and belongs to the pure braid group P. It is a distinguished braid reflection around H in P.

Definition 4.1.5. Let s be a distinguished pseudo-reflection in W with reflecting hyperplane H. An *s-distinguished braid reflection* or *monodromy generator* is a distinguished braid reflection $\boldsymbol{s}$ around the image of H in V^{reg}/W such that $\bar{\boldsymbol{s}} = s$.

Definition 4.1.6. Let $x_0 \in V^{\text{reg}}$ as before. We denote by $\boldsymbol{\tau}$ the element of P defined by the loop $t \mapsto x_0 \exp(2\pi i t)$.

Lemma 4.1.7. *We have $\boldsymbol{\tau} \in ZP$.*

Theorem-Definition 4.1.8 *Given $\mathcal{C} \in \mathcal{A}/W$, there exists a unique length function $l_{\mathcal{C}} : B \to \mathbb{Z}$ defined as follows: if $b = \mathbf{s}_1^{n_1} \cdot \mathbf{s}_2^{n_2} \cdot \cdots \cdot \mathbf{s}_m^{n_m}$ where (for all j) $n_j \in \mathbb{Z}$ and $\mathbf{s}_j$ is a distinguished braid reflection around an element of $\mathcal{C}_j$, then*

$$l_{\mathcal{C}}(b) = \sum_{\{j \,|\, \mathcal{C}_j = \mathcal{C}\}} n_j.$$

The length function $l : B \to \mathbb{Z}$ *is defined, for all $b \in B$, as*

$$l(b) = \sum_{\mathcal{C} \in \mathcal{A}/W} l_{\mathcal{C}}(b).$$

We say that B has an *Artin-like* presentation (cf. [57], 5.2), if it has a presentation of the form

$$\langle\, \mathbf{s} \in \mathbf{S} \,|\, \{\mathbf{v}_i = \mathbf{w}_i\}_{i \in I} \,\rangle\,,$$

where $\mathbf{S}$ is a finite set of distinguished braid reflections and I is a finite set of relations which are multi-homogeneous, *i.e.,* such that, for each i, $\mathbf{v}_i$ and $\mathbf{w}_i$ are positive words in elements of $\mathbf{S}$ (and hence, for each $\mathcal{C} \in \mathcal{A}/W$, we have $l_{\mathcal{C}}(\mathbf{v}_i) = l_{\mathcal{C}}(\mathbf{w}_i)$).

The following result by Bessis ([8], Theorem 0.1) shows that any braid group has an Artin-like presentation.

Theorem 4.1.9. *Let W be a complex reflection group with associated braid group B. Then there exists a subset $\mathbf{S} = \{\mathbf{s}_1, \ldots, \mathbf{s}_n\}$ of B such that*

(1) The elements $\mathbf{s}_1, \ldots, \mathbf{s}_n$ are distinguished braid reflection and therefore, their images $s_1, \ldots, s_n$ in W are distinguished reflections.
(2) The set $\mathbf{S}$ generates B and therefore, $S := \{s_1, \ldots, s_n\}$ generates W.
(3) There exists a set $\mathcal{R}$ of relations of the form $\mathbf{w}_1 = \mathbf{w}_2$, where $\mathbf{w}_1$ and $\mathbf{w}_2$ are positive words of equal length in the elements of $\mathbf{S}$, such that $\langle\, \mathbf{S} \mid \mathcal{R} \,\rangle$ is a presentation of B.
(4) Viewing now $\mathcal{R}$ as a set of relations in S, the group W is presented by

$$\langle\, S \mid \mathcal{R}; (\forall s \in S)(s^{e_s} = 1) \,\rangle ,$$

where e_s denotes the order of s in W.

4.2 Generic Hecke Algebras

Let $K, V, W, \mathcal{A}, P, B$ be defined as in the previous section. For every orbit $\mathcal{C}$ of W on $\mathcal{A}$, we set $e_{\mathcal{C}}$ the common order of the subgroups W_H, where H is any element of $\mathcal{C}$ and W_H the subgroup formed by id_V and all the reflections fixing the hyperplane H.

We choose a set of indeterminates $\mathbf{u} = (u_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\le j\le e_{\mathcal{C}}-1)}$ and we denote by $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ the Laurent polynomial ring in all the indeterminates $\mathbf{u}$. We define the *generic Hecke algebra* $\mathcal{H}$ of W to be the quotient of the group algebra $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]B$ by the ideal generated by the elements of the form

$$(\mathbf{s} - u_{\mathcal{C},0})(\mathbf{s} - u_{\mathcal{C},1}) \cdots (\mathbf{s} - u_{\mathcal{C},e_{\mathcal{C}}-1}),$$

where $\mathcal{C}$ runs over the set $\mathcal{A}/W$ and $\mathbf{s}$ runs over the set of monodromy generators around the images in V^{reg}/W of the elements of the hyperplane orbit $\mathcal{C}$.

Example 4.2.1. Let $W := G_2 = \left\langle s, t \mid ststst = tststs, s^2 = t^2 = 1 \right\rangle$ be the dihedral group of order 12. The generic Hecke algebra of G_2 is defined over the Laurent polynomial ring in four indeterminates $\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, w_0, w_0^{-1}, w_1, w_1^{-1}]$ and can be presented as follows:

$$\mathcal{H}(G_2) = \left\langle S, T \;\middle|\; STSTST = TSTSTS,\ \begin{array}{l}(S-u_0)(S-u_1)=0\\(T-w_0)(T-w_1)=0\end{array} \right\rangle .$$

Example 4.2.2. Let $W := G_4 =< s, t \,|\, sts = tst, s^3 = t^3 = 1 >$. Then s and t are conjugate in W and their reflecting hyperplanes belong to the same orbit of W on $\mathcal{A}$. The generic Hecke algebra of G_4 is defined over the Laurent polynomial ring in three indeterminates $\mathbb{Z}[u_0, u_0^{-1}, u_1, u_1^{-1}, u_2, u_2^{-1}]$ and can be presented as follows:

$$\mathcal{H}(G_4) = \left\langle S, T \;\middle|\; TST = TST, \begin{array}{c} (S-u_0)(S-u_1)(S-u_2) = 0 \\ (T-u_0)(T-u_1)(T-u_2) = 0 \end{array} \right\rangle.$$

We make some assumptions for the generic Hecke algebra $\mathcal{H}$. Note that they have been verified for all but a finite number of irreducible complex reflection groups ([21], remarks before 1.17, § 2; [36]).

Assumptions 4.2.3 *The algebra $\mathcal{H}$ is a free $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$-module of rank $|W|$. Moreover, there exists a linear form $t : \mathcal{H} \to \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ with the following properties:*

(1) t is a symmetrizing form on $\mathcal{H}$, i.e., $t(hh') = t(h'h)$ for all $h, h' \in \mathcal{H}$ and the map

$$\begin{aligned} \hat{t} : \mathcal{H} &\to \mathrm{Hom}(\mathcal{H}, \mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]) \\ h &\mapsto \quad (h' \mapsto t(hh')) \end{aligned}$$

is an isomorphism.

(2) Via the specialization $u_{\mathcal{C},j} \mapsto \zeta_{e_\mathcal{C}}^j$, the form t becomes the canonical symmetrizing form on the group algebra $\mathbb{Z}_K[W]$.

(3) If we denote by $\alpha \mapsto \alpha^*$ the automorphism of $\mathbb{Z}[\mathbf{u}, \mathbf{u}^{-1}]$ consisting of the simultaneous inversion of the indeterminates, then for all $b \in B$, we have

$$t(b^{-1})^* = \frac{t(b\boldsymbol{\tau})}{t(\boldsymbol{\tau})},$$

where $\boldsymbol{\tau}$ is the (central) element of P defined by the loop $t \mapsto x_0\exp(2\pi it)$.

We know that the form t is unique ([21], 2.1). From now on, we suppose that the assumptions 4.2.3 are satisfied. Then we have the following result by G. Malle ([51], 5.2).

Theorem 4.2.4. *Let $\mathbf{v} = (v_{\mathcal{C},j})_{(\mathcal{C}\in\mathcal{A}/W)(0\leq j\leq e_\mathcal{C}-1)}$ be a set of $\sum_{\mathcal{C}\in\mathcal{A}/W} e_\mathcal{C}$ indeterminates such that, for every $\mathcal{C}, j$, we have $v_{\mathcal{C},j}^{|\mu(K)|} = \zeta_{e_\mathcal{C}}^{-j} u_{\mathcal{C},j}$. Then the $K(\mathbf{v})$-algebra $K(\mathbf{v})\mathcal{H}$ is split semisimple.*

By Tits' deformation theorem (Theorem 2.4.9), it follows that the specialization $v_{\mathcal{C},j} \mapsto 1$ induces a bijection $\chi_\mathbf{v} \mapsto \chi$ from the set $\mathrm{Irr}(K(\mathbf{v})\mathcal{H})$ of absolutely irreducible characters of $K(\mathbf{v})\mathcal{H}$ to the set $\mathrm{Irr}(W)$ of absolutely irreducible characters of W, such that the following diagram is commutative

$$\begin{array}{rcl} \chi_{\mathbf{v}}: \quad \mathcal{H} & \to & \mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}] \\ \downarrow & & \downarrow \\ \chi:\mathbb{Z}_K[W] & \to & \mathbb{Z}_K. \end{array}$$

Since the assumptions 4.2.3 are satisfied and the algebra $K(\mathbf{v})\mathcal{H}$ is split semisimple, we can define the Schur element $s_\chi(\mathbf{v})$ for every irreducible character $\chi_{\mathbf{v}}$ of $K(\mathbf{v})\mathcal{H}$ with respect to the symmetrizing form t. The following result describes the form of the Schur elements associated to the irreducible characters of $K(\mathbf{v})\mathcal{H}$.

Theorem 4.2.5. *The Schur element $s_\chi(\mathbf{v})$ associated to the irreducible character $\chi_{\mathbf{v}}$ of $K(\mathbf{v})\mathcal{H}$ is an element of $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ of the form*

$$s_\chi(\mathbf{v}) = \xi_\chi N_\chi \prod_{i \in I_\chi} \Psi_{\chi,i}(M_{\chi,i})^{n_{\chi,i}},$$

where

(a) ξ_χ is an element of $\mathbb{Z}_K$,
(b) $N_\chi = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{b_{\mathcal{C},j}}$ is a monomial in $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ with $\sum_{j=0}^{e_\mathcal{C}-1} b_{\mathcal{C},j} = 0$ for all $\mathcal{C} \in \mathcal{A}/W$,
(c) I_χ is an index set,
(d) $(\Psi_{\chi,i})_{i\in I_\chi}$ is a family of K-cyclotomic polynomials in one variable (i.e., minimal polynomials of the roots of unity over K),
(e) $(M_{\chi,i})_{i\in I_\chi}$ is a family of monomials in $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$ such that if $M_{\chi,i} = \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$, then $\gcd(a_{\mathcal{C},j}) = 1$ and $\sum_{j=0}^{e_\mathcal{C}-1} a_{\mathcal{C},j} = 0$ for all $\mathcal{C} \in \mathcal{A}/W$,
(f) $(n_{\chi,i})_{i\in I_\chi}$ is a family of positive integers.

Proof. By Proposition 2.2.10, we have that $s_\chi(\mathbf{v}) \in \mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$. The rest is a case by case analysis: Let us first consider the group $G(d,1,r)$. The Schur elements of $\mathcal{H}(G(d,1,r))$ have been calculated independently by Geck, Iancu and Malle [36] and by Mathas [54]. Following Theorem A.7.2, they are obviously of the desired form. Moreover, in the Appendix we give the generic Schur elements for the groups $G(2d,2,2)$, G_7, G_{11}, G_{19}, G_{26}, G_{32} (calculated by Malle in [49] and [50]) and F_4 (calculated by Lusztig in [47]) and show that they are of the form described above. In the Appendix, we also give the specializations of the parameters which make:

- $\mathcal{H}(G(de,1,r))$ the twisted symmetric algebra of the cyclic group C_e over $\mathcal{H}(G(de,e,r))$ in the case where $r>2$ or $r=2$ and e is odd.
- $\mathcal{H}(G(de,2,2))$ the twisted symmetric algebra of the cyclic group $C_{e/2}$ over $\mathcal{H}(G(de,e,2))$ in the case where e is even.
- $\mathcal{H}(G_7)$ the twisted symmetric algebra of some finite cyclic group over $\mathcal{H}(G_4)$, $\mathcal{H}(G_5)$ and $\mathcal{H}(G_6)$.
- $\mathcal{H}(G_{11})$ the twisted symmetric algebra of some finite cyclic group over $\mathcal{H}(G_8)$, $\mathcal{H}(G_9)$, $\mathcal{H}(G_{10})$, $\mathcal{H}(G_{12})$, $\mathcal{H}(G_{13})$, $\mathcal{H}(G_{14})$ and $\mathcal{H}(G_{15})$.

- $\mathcal{H}(G_{19})$ the twisted symmetric algebra of some finite cyclic group over $\mathcal{H}(G_{16})$, $\mathcal{H}(G_{17})$, $\mathcal{H}(G_{18})$, $\mathcal{H}(G_{20})$, $\mathcal{H}(G_{21})$ and $\mathcal{H}(G_{22})$.
- $\mathcal{H}(G_{26})$ the twisted symmetric algebra of the cyclic group C_2 over $\mathcal{H}(G_{25})$.

In all these cases, Proposition 2.3.15 implies that the Schur elements of the twisted symmetric algebra are scalar multiples of the Schur elements of the subalgebra. Due to the nature of the specializations (each indeterminate is sent to an indeterminate or a root of unity or a product of the two), the Schur elements of the subalgebra are also of the desired form.

Finally, if W is one of the remaining exceptional irreducible complex reflection groups, then W has one hyperplane orbit $\mathcal{C}$ with $e_{\mathcal{C}} = 2$. The generic Hecke algebra of W is defined over a Laurent polynomial ring in two indeterminates $v_{\mathcal{C},0}$ and $v_{\mathcal{C},1}$. Its Schur elements should be products of K-cyclotomic polynomials in one variable $v := v_{\mathcal{C},0}v_{\mathcal{C},1}^{-1}$. The generic Schur elements have been calculated

- for E_6 and E_7 by Surowski [62],
- for E_8 by Benson [6],
- for H_3 by Lusztig [44],
- for H_4 by Alvis and Lusztig [1],
- for G_{24}, G_{27}, G_{29}, G_{31}, G_{33} and G_{34} by Malle [50],

and they are indeed products of K-cyclotomic polynomials in "one" variable.

Note that in order to write the Schur elements in the desired form, we have used the GAP Package CHEVIE (some mistakes in the articles cited above have been spotted and corrected). ■

Remark. It is a consequence of [59], *Theorem* 3.5, *that the irreducible factors of the generic Schur elements over* $\mathbb{C}[\mathbf{v},\mathbf{v}^{-1}]$ *are divisors of Laurent polynomials of the form* $M(\mathbf{v})^n - 1$, *where*

- $M(\mathbf{v})$ *is a monomial in* $\mathbb{C}[\mathbf{v},\mathbf{v}^{-1}]$,
- n *is a positive integer.*

We have seen that the specialization $v_{\mathcal{C},j} \mapsto 1$ induces a bijection $\chi_{\mathbf{v}} \mapsto \chi$ from $\mathrm{Irr}(K(\mathbf{v})\mathcal{H})$ to $\mathrm{Irr}(W)$. Due to the assumptions 4.2.3, it maps $s_\chi(\mathbf{v})$ to $|W|/\chi(1)$, which is the Schur element of χ with respect to the canonical symmetrizing form. Therefore, the first cyclotomic polynomial does not appear in the factorization of $s_\chi(\mathbf{v})$ (otherwise the specialization $v_{\mathcal{C},j} \mapsto 1$ would map $s_\chi(\mathbf{v})$ to 0).

The following result is an immediate application of Definition 3.1.1.

Theorem 4.2.6. *The algebra* $\mathcal{H}$, *defined over the ring* $\mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}]$, *is an essential algebra.*

Thanks to Theorem 4.2.6, all the results of Chapter 3 can be applied to the generic Hecke algebra of an irreducible complex reflection group.

Definition 4.2.7. Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. We say that a (primitive) monomial M in $\mathbb{Z}_K[\boldsymbol{v},\boldsymbol{v}^{-1}]$ is $\mathfrak{p}$-*essential for* W, if M is $\mathfrak{p}$-essential for $\mathcal{H}$.

Example 4.2.8. Let $W := G_2$. The group G_2 is a Weyl group. We have seen that

$$\mathcal{H}(G_2) = \left\langle S,T \;\middle|\; STSTST = TSTSTS,\ \begin{array}{l}(S-u_0)(S-u_1)=0\\(T-w_0)(T-w_1)=0\end{array}\right\rangle.$$

Set $x_0^2 := u_0$, $x_1^2 := -u_1$, $y_0^2 := w_0$, $y_1^2 := -w_1$. By Theorem 4.2.4, the algebra $\mathbb{Q}(x_0,x_1,y_0,y_1)\mathcal{H}(G_2)$ is split semisimple and hence, there exists a bijection between its irreducible characters and the irreducible characters of G_2. The group G_2 has 4 irreducible characters of degree 1 and 2 irreducible characters of degree 2. Set

$$s_1(x_0,x_1,y_0,y_1) := \Phi_4(x_0x_1^{-1})\cdot\Phi_4(y_0y_1^{-1})\cdot\Phi_3(x_0x_1^{-1}y_0y_1^{-1})\cdot\Phi_6(x_0x_1^{-1}y_0y_1^{-1}),$$

$$s_2(x_0,x_1,y_0,y_1) := 2x_0^{-2}x_1^2\cdot\Phi_3(x_0x_1^{-1}y_0y_1^{-1})\cdot\Phi_6(x_0x_1^{-1}y_0^{-1}y_1),$$

where $\Phi_3(x) = x^2+x+1$, $\Phi_4(x) = x^2+1$, $\Phi_6(x) = x^2-x+1$.

The Schur elements of $\mathcal{H}(G_2)$ are

$$s_1(x_0,x_1,y_0,y_1), s_1(x_0,x_1,y_1,y_0), s_1(x_1,x_0,y_0,y_1), s_1(x_1,x_0,y_1,y_0),$$

$$s_2(x_0,x_1,y_0,y_1), s_2(x_0,x_1,y_1,y_0).$$

Since $\Phi_3(1)=3$, $\Phi_4(1)=2$ and $\Phi_6(1)=1$, we obtain that

- the (2)-essential monomials for G_2 are $x_0x_1^{-1}$ and $y_0y_1^{-1}$ (and their inverses),
- the (3)-essential monomials for G_2 are $x_0x_1^{-1}y_0y_1^{-1}$ and $x_0x_1^{-1}y_0^{-1}y_1$ (and their inverses).

Example 4.2.9. Let $W := G_4$. The field of definition of G_4 is $\mathbb{Q}(\zeta_3)$. We have seen that

$$\mathcal{H}(G_4) = \left\langle S,T \;\middle|\; TST = TST,\ \begin{array}{l}(S-u_0)(S-u_1)(S-u_2)=0\\(T-u_0)(T-u_1)(T-u_2)=0\end{array}\right\rangle.$$

Set $v_0^6 := u_0$, $v_1^6 := \zeta_3^2u_1$, $v_2^6 := \zeta_3u_2$. By Theorem 4.2.4, the algebra $\mathbb{Q}(\zeta_3)(v_0,v_1,v_2)\mathcal{H}(G_4)$ is split semisimple and hence, there exists a bijection between its irreducible characters and the irreducible characters of G_4. The group G_4 has 3 irreducible characters of degree 1, 3 irreducible characters of degree 2 and 1 irreducible character of degree 3. Set

$$\begin{aligned} s_1(v_0,v_1,v_2) := {} & \Phi''_9(v_0v_1^{-1})\cdot\Phi'_{18}(v_0v_1^{-1})\cdot\Phi_4(v_0v_1^{-1})\cdot\Phi'_{12}(v_0v_1^{-1})\cdot\Phi''_{12}(v_0v_1^{-1})\\ & \cdot\Phi'_{36}(v_0v_1^{-1})\cdot\Phi'_9(v_0v_2^{-1})\cdot\Phi''_{18}(v_0v_2^{-1})\cdot\Phi_4(v_0v_2^{-1})\cdot\Phi'_{12}(v_0v_2^{-1})\\ & \cdot\Phi''_{12}(v_0v_2^{-1})\cdot\Phi''_{36}(v_0v_2^{-1})\cdot\Phi_4(v_0^2v_1^{-1}v_2^{-1})\\ & \cdot\Phi'_{12}(v_0^2v_1^{-1}v_2^{-1})\cdot\Phi''_{12}(v_0^2v_1^{-1}v_2^{-1}), \end{aligned}$$

$$\begin{aligned} s_2(v_0,v_1,v_2) := {} & -\zeta_3^2v_2^6v_1^{-6}\cdot\Phi'_9(v_1v_0^{-1})\cdot\Phi''_{18}(v_1v_0^{-1})\cdot\Phi''_9(v_2v_0^{-1})\cdot\Phi'_{18}(v_2v_0^{-1})\\ & \cdot\Phi_4(v_1v_2^{-1})\cdot\Phi'_{12}(v_1v_2^{-1})\cdot\Phi''_{12}(v_1v_2^{-1})\cdot\Phi'_{36}(v_1v_2^{-1})\\ & \cdot\Phi_4(v_0^{-2}v_1v_2)\cdot\ \Phi'_{12}(v_0^{-2}v_1v_2)\cdot\Phi''_{12}(v_0^{-2}v_1v_2), \end{aligned}$$

$$\begin{aligned} s_3(v_0,v_1,v_2) := {} & \Phi_4(v_0^2v_1^{-1}v_2^{-1})\cdot\Phi'_{12}(v_0^2v_1^{-1}v_2^{-1})\cdot\Phi''_{12}(v_0^2v_1^{-1}v_2^{-1})\\ & \cdot\Phi_4(v_1^2v_2^{-1}v_0^{-1})\cdot\Phi'_{12}(v_1^2v_2^{-1}v_0^{-1})\cdot\Phi''_{12}(v_1^2v_2^{-1}v_0^{-1})\\ & \cdot\Phi_4(v_2^2v_0^{-1}v_1^{-1})\cdot\Phi'_{12}(v_2^2v_0^{-1}v_1^{-1})\cdot\Phi''_{12}(v_2^2v_0^{-1}v_1^{-1}), \end{aligned}$$

where $\Phi_4(x) = x^2+1$, $\Phi'_9(x) = x^3-\zeta_3$, $\Phi''_9(x) = x^3-\zeta_3^2$, $\Phi''_{12}(x) = x^2+\zeta_3$, $\Phi'_{12}(x) = x^2+\zeta_3^2$, $\Phi''_{18}(x) = x^3+\zeta_3$, $\Phi'_{18}(x) = x^3+\zeta_3^2$, $\Phi''_{36}(x) = x^6+\zeta_3$, $\Phi'_{36}(x) = x^6+\zeta_3^2$.

The Schur elements of $\mathcal{H}(G_4)$ are

$$s_1(v_0,v_1,v_2), s_1(v_1,v_2,v_0), s_1(v_2,v_0,v_1),$$

$$s_2(v_0,v_1,v_2), s_2(v_1,v_2,v_0), s_2(v_2,v_0,v_1), s_3(v_0,v_1,v_2).$$

We deduce that the (2)-essential monomials for G_4 are

$$v_0v_1^{-1}, v_0v_2^{-1}, v_1v_2^{-1}, v_0^2v_1^{-1}v_2^{-1}, v_1^2v_2^{-1}v_0^{-1}, v_2^2v_0^{-1}v_1^{-1}.$$

The first three are also the $(1-\zeta_3)$-essential monomials for G_4.

4.3 Cyclotomic Hecke Algebras

Let y be an indeterminate. We set $x := y^{|\mu(K)|}$.

Definition 4.3.1. A *cyclotomic specialization* of $\mathcal{H}$ is a $\mathbb{Z}_K$-algebra morphism $\phi : \mathbb{Z}_K[\mathbf{v},\mathbf{v}^{-1}] \to \mathbb{Z}_K[y,y^{-1}]$ with the following properties:

- $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ where $n_{\mathcal{C},j} \in \mathbb{Z}$ for all $\mathcal{C}$ and j.
- For all $\mathcal{C} \in \mathcal{A}/W$, if z is another indeterminate, the element of $\mathbb{Z}_K[y,y^{-1},z]$ defined by
$$\Gamma_{\mathcal{C}}(y,z) := \prod_{j=0}^{e_{\mathcal{C}}-1}(z-\zeta_{e_{\mathcal{C}}}^j y^{n_{\mathcal{C},j}})$$
is invariant by the action of $\mathrm{Gal}(K(y)/K(x))$.

If ϕ is a cyclotomic specialization of $\mathcal{H}$, the corresponding *cyclotomic Hecke algebra* is the $\mathbb{Z}_K[y, y^{-1}]$-algebra, denoted by $\mathcal{H}_\phi$, which is obtained as the specialization of the $\mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$-algebra $\mathcal{H}$ via the morphism ϕ. It also has a symmetrizing form t_ϕ defined as the specialization of the canonical form t.

Remark. Sometimes we describe the morphism ϕ by the formula

$$u_{\mathcal{C},j} \mapsto \zeta_{e_\mathcal{C}}^j x^{n_{\mathcal{C},j}}.$$

*If now we set $q := \zeta x$ for some root of unity $\zeta \in \mu(K)$, then the cyclotomic specialization ϕ becomes a ζ-*cyclotomic specialization *and $\mathcal{H}_\phi$ can be also considered over $\mathbb{Z}_K[q, q^{-1}]$.*

Example 4.3.2. The "spetsial" cyclotomic Hecke algebra $\mathcal{H}_q^s(W)$ is the 1-cyclotomic algebra obtained by the specialization

$$u_{\mathcal{C},0} \mapsto q,\ u_{\mathcal{C},j} \mapsto \zeta_{e_\mathcal{C}}^j \text{ for } 1 \leq j \leq e_\mathcal{C} - 1, \text{ for all } \mathcal{C} \in \mathcal{A}/W.$$

For example,

$$\mathcal{H}_q^s(G_2) = \langle S, T \mid STSTST = TSTSTS, (S-q)(S+1) = (T-q)(T+1) = 0 \rangle .$$

and

$$\mathcal{H}_q^s(G_4) = \left\langle S, T \mid STS = TST, (S-q)(S^2+S+1) = (T-q)(T^2+T+1) = 0 \right\rangle .$$

Set $A := \mathbb{Z}_K[\mathbf{v}, \mathbf{v}^{-1}]$ and $\Omega := \mathbb{Z}_K[y, y^{-1}]$. Let $\phi : A \to \Omega$ be a cyclotomic specialization such that $\phi(v_{\mathcal{C},j}) = y^{n_{\mathcal{C},j}}$ for all $\mathcal{C}, j$. Recall that, for $\alpha \in \mathbb{Z}\setminus\{0\}$, we denote by $I^\alpha : \Omega \to \Omega$ the monomorphism $y \mapsto y^\alpha$.

Theorem 4.3.3. *Let $\phi : A \to \Omega$ be a cyclotomic specialization like above. Then there exist an adapted $\mathbb{Z}_K$-algebra morphism $\varphi : A \to \Omega$ and $\alpha \in \mathbb{Z}\setminus\{0\}$ such that*

$$\phi = I^\alpha \circ \varphi.$$

Proof. We set $d := \gcd(n_{\mathcal{C},j})$ and consider the cyclotomic specialization $\varphi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}/d}$. We have $\phi = I^d \circ \varphi$. Since $\gcd(n_{\mathcal{C},j}/d) = 1$, there exist $a_{\mathcal{C},j} \in \mathbb{Z}$ such that

$$\sum_{\mathcal{C},j} a_{\mathcal{C},j}(n_{\mathcal{C},j}/d) = 1.$$

We have $y = \varphi(\prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}})$ an hence, φ is surjective. Then, by Proposition 1.4.12, φ is adapted. ∎

Let φ be defined as in Theorem 4.3.3 and $\mathcal{H}_\varphi$ the corresponding cyclotomic Hecke algebra. Proposition 3.2.1 implies that the algebra $K(y)\mathcal{H}_\varphi$ is split semisimple. Due to Corollary 2.4.11 and the theorem above, we deduce that

Proposition 4.3.4. *The algebra $K(y)\mathcal{H}_\phi$ is split semisimple.*

For $y = 1$, the algebra $K(y)\mathcal{H}_\phi$ specializes to the group algebra KW (the form t_ϕ becoming the canonical form on the group algebra). Thus, by Tits' deformation theorem, the specialization $v_{\mathcal{C},j} \mapsto 1$ defines the following bijections

$$\begin{array}{ccccc} \mathrm{Irr}(K(\mathbf{v})\mathcal{H}) & \leftrightarrow & \mathrm{Irr}(K(y)\mathcal{H}_\phi) & \leftrightarrow & \mathrm{Irr}(W) \\ \chi_{\mathbf{v}} & \mapsto & \chi_\phi & \mapsto & \chi. \end{array}$$

The following result is an immediate consequence of Theorem 4.2.5.

Proposition 4.3.5. *The Schur element $s_{\chi_\phi}(y)$ associated to the irreducible character χ_ϕ of $K(y)\mathcal{H}_\phi$ is a Laurent polynomial in y of the form*

$$s_{\chi_\phi}(y) = \psi_{\chi,\phi} y^{a_{\chi,\phi}} \prod_{\Phi \in C_K} \Phi(y)^{n_{\chi,\phi,\Phi}}$$

where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi,\Phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

4.3.1 Essential Hyperplanes

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization of $\mathcal{H}$ and let φ be an adapted morphism as in Theorem 4.3.3. By Corollary 3.4.2, the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$ coincide with the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\varphi$ and the latter can be calculated with the use of Theorem 3.3.2. Therefore, we need to know which $\mathfrak{p}$-essential monomials are sent to 1 by φ.

Let $M := \prod_{\mathcal{C},j} v_{\mathcal{C},j}^{a_{\mathcal{C},j}}$ be a $\mathfrak{p}$-essential monomial for W. Then

$$\varphi(M) = 1 \Leftrightarrow \phi(M) = 1 \Leftrightarrow \sum_{\mathcal{C},j} a_{\mathcal{C},j} n_{\mathcal{C},j} = 0.$$

Set $m := \sum_{\mathcal{C} \in \mathcal{A}/W} e_{\mathcal{C}}$. The hyperplane defined in $\mathbb{C}^m$ by the relation

$$\sum_{\mathcal{C},j} a_{\mathcal{C},j} t_{\mathcal{C},j} = 0,$$

where $(t_{\mathcal{C},j})_{\mathcal{C},j}$ is a set of m indeterminates, is called $\mathfrak{p}$-*essential hyperplane* for W. A hyperplane in $\mathbb{C}^m$ is called *essential* for W, if it is $\mathfrak{p}$-essential for some prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$.

Example 4.3.6. Let $W := G_2$. Following Example 4.2.8, let

$$\phi : x_0 \mapsto y^{n_0}, x_1 \mapsto y^{n_1}, y_0 \mapsto y^{m_0}, y_1 \mapsto y^{m_1}$$

be a cyclotomic specialization. Then

- the (2)-essential hyperplanes for G_2 are $N_0 - N_1 = 0$ and $M_0 - M_1 = 0$,
- the (3)-essential hyperplanes for G_2 are $N_0 - N_1 + M_0 - M_1 = 0$ and $N_0 - N_1 - M_0 + M_1 = 0$.

Example 4.3.7. Let $W := G_4$. Following Example 4.2.9, let $\phi : v_i \mapsto y^{n_i}$ for $i = 0, 1, 2$ be a cyclotomic specialization. Then the hyperplanes

- $N_0 - N_1 = 0$, $N_0 - N_2 = 0$ *and* $N_1 - N_2 = 0$ *are* (2)*-essential and* $(1 - \zeta_3)$*-essential for* G_4,
- $2N_0 - N_1 - N_2 = 0$, $2N_1 - N_2 - N_0 = 0$ *and* $2N_2 - N_0 - N_1 = 0$ *are just* (2)*-essential for* G_4.

In order to calculate the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$, we check to which $\mathfrak{p}$-essential hyperplanes the $n_{\mathcal{C},j}$ belong and we apply Theorem 3.3.2:

- If the $n_{\mathcal{C},j}$ belong to no $\mathfrak{p}$-essential hyperplane, then the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$ coincide with the blocks of $A_{\mathfrak{p}A}\mathcal{H}$. We call these blocks $\mathfrak{p}$*-blocks associated with no essential hyperplane.*
- If the $n_{\mathcal{C},j}$ belong to exactly one $\mathfrak{p}$-essential hyperplane H_M, corresponding to the $\mathfrak{p}$-essential monomial M, then the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$ coincide with the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$, where $\mathfrak{q}_M := \mathfrak{p}A + (M-1)A$. We call these blocks $\mathfrak{p}$*-blocks associated with the essential hyperplane* H_M.
- If the $n_{\mathcal{C},j}$ belong to more than one $\mathfrak{p}$-essential hyperplane, then, following Theorem 3.3.2, the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$ are unions of the $\mathfrak{p}$-blocks associated with the $\mathfrak{p}$-essential hyperplanes to which the $n_{\mathcal{C},j}$ belong and they are minimal with respect to that property.

This last property of the $\mathfrak{p}$-blocks is called "property of *semi-continuity*" (the name is due to C. Bonnafé). The property of semi-continuity also appears in works on Kazhdan-Lusztig cells (cf. [9, 10, 40]) and on Cherednik algebras (cf. [38]). In the next section, we will see that the Rouquier blocks of the cyclotomic Hecke algebras also have this property.

4.3.2 Group Algebra

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$ lying over a prime number p and let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization of $\mathcal{H}$. If $n_{\mathcal{C},j} = n \in \mathbb{Z}$ for all $\mathcal{C}$ and j, then the $n_{\mathcal{C},j}$ belong to all $\mathfrak{p}$-essential hyperplanes for W and we have $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi \cong \Omega_{\mathfrak{p}\Omega}W$. Note that, since the ring $\Omega_{\mathfrak{p}\Omega}$ is a discrete valuation ring (by Theorem 1.2.24), the blocks of $\Omega_{\mathfrak{p}\Omega}W$ are the p-blocks of W as determined by Brauer theory. Due to Theorem 3.3.2, we obtain the following result which relates the $\mathfrak{p}$-blocks of any cyclotomic Hecke algebra to the p-blocks of W.

Proposition 4.3.8. *Let* $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ *be a cyclotomic specialization of* $\mathcal{H}$. *If two irreducible characters* $\chi, \psi \in \mathrm{Irr}(W)$ *are in the same block of* $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$, *then they are in the same* p*-block of* W.

Proof. The blocks of $\Omega_{\mathfrak{p}\Omega}H_\phi$ are unions of the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ for all $\mathfrak{p}$-essential monomials M such that $\phi(M)=1$, whereas the p-blocks of W are unions of the blocks of $A_{\mathfrak{q}_M}\mathcal{H}$ for all $\mathfrak{p}$-essential monomials M. ■

However, we know from Brauer theory that if the order of the group W is prime to p, then every character of W is a p-block by itself (see, for example, [60], 15.5, Proposition 43). The following result is an immediate consequence of Proposition 4.3.8.

Corollary 4.3.9. *If $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$ lying over a prime number p which does not divide the order of the group W, then the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$ are singletons.*

4.4 Rouquier Blocks of the Cyclotomic Hecke Algebras

Definition 4.4.1. We call *Rouquier ring* of K and denote by $\mathcal{R}_K(y)$ the $\mathbb{Z}_K$-subalgebra of $K(y)$

$$\mathcal{R}_K(y) := \mathbb{Z}_K[y, y^{-1}, (y^n - 1)^{-1}_{n\geq 1}].$$

Let $\phi : v_{\mathcal{C},j} \mapsto y^{n_{\mathcal{C},j}}$ be a cyclotomic specialization and $\mathcal{H}_\phi$ the corresponding cyclotomic Hecke algebra. The *Rouquier blocks* of $\mathcal{H}_\phi$ are the blocks of the algebra $\mathcal{R}_K(y)\mathcal{H}_\phi$.

It has been shown by Rouquier (cf. [58]), that if W is a Weyl group and $\mathcal{H}_\phi$ is obtained via the "spetsial" cyclotomic specialization (see Example 4.3.2), then the Rouquier blocks of $\mathcal{H}_\phi$ coincide with the families of characters defined by Lusztig. Thus, the Rouquier blocks generalize the notion of "families of characters" to all complex reflection groups.

*Remark. We have seen that if we set $q := \zeta y^{|\mu(K)|}$ for some root of unity $\zeta \in \mu(K)$, then the cyclotomic Hecke algebra $\mathcal{H}_\phi$ can be also considered over the ring $\mathbb{Z}_K[q, q^{-1}]$. We define the Rouquier blocks of $\mathcal{H}_\phi$ to be the blocks of $\mathcal{R}_K(y)\mathcal{H}_\phi$. However, in other texts (*e.g., *in* [18]*), the Rouquier blocks are defined to be the blocks of $\mathcal{R}_K(q)\mathcal{H}_\phi$. Since $\mathcal{R}_K(y)$ is the integral closure of $\mathcal{R}_K(q)$ in the splitting field $K(y)$ for $\mathcal{H}_\phi$, Proposition 2.1.9 establishes the connection between the blocks of $\mathcal{R}_K(y)\mathcal{H}_\phi$ and the blocks of $\mathcal{R}_K(q)\mathcal{H}_\phi$.*

The Rouquier ring $\mathcal{R}_K(y)$ has many interesting properties. The next result describes some of them.

Proposition 4.4.2.

(1) The group of units $\mathcal{R}_K(y)^\times$ of the Rouquier ring $\mathcal{R}_K(y)$ consists of the elements of the form

$$uy^n \prod_{\Phi \in \mathrm{Cycl}(K)} \Phi(y)^{n_\Phi},$$

where $u \in \mathbb{Z}_K^\times$, $n, n_\Phi \in \mathbb{Z}$, $\mathrm{Cycl}(K)$ *is the set of* K*-cyclotomic polynomials and* $n_\Phi = 0$ *for all but a finite number of* Φ.

(2) The prime ideals of $\mathcal{R}_K(y)$ *are*

- *the zero ideal* $\{0\}$,
- *the ideals of the form* $\mathfrak{p}\mathcal{R}_K(y)$, *where* $\mathfrak{p}$ *is a prime ideal of* $\mathbb{Z}_K$,
- *the ideals of the form* $P(y)\mathcal{R}_K(y)$, *where* $P(y)$ *is an irreducible element of* $\mathbb{Z}_K[y]$ *of degree at least* 1, *prime to* y *and to* $\Phi(y)$ *for all* $\Phi \in \mathrm{Cycl}(K)$.

(3) The Rouquier ring $\mathcal{R}_K(y)$ *is a Dedekind ring.*

Proof. (1) This part is immediate from the definition of K-cyclotomic polynomials.

(2) Since $\mathcal{R}_K(y)$ is an integral domain, the zero ideal is prime. Now, the ring $\mathbb{Z}_K$ is a Dedekind ring and thus a Krull ring, by Proposition 1.2.26. Proposition 1.2.25 implies that the ring $\mathbb{Z}_K[y]$ is also a Krull ring whose prime ideals of height 1 are of the form $\mathfrak{p}\mathbb{Z}_K[y]$ ($\mathfrak{p}$ prime in $\mathbb{Z}_K$) and $P(y)\mathbb{Z}_K[y]$ ($P(y)$ irreducible in $\mathbb{Z}_K[y]$ of degree at least 1). Moreover, $\mathbb{Z}_K$ has an infinite number of non-zero prime ideals whose intersection is the zero ideal. Since all non-zero prime ideals of $\mathbb{Z}_K$ are maximal, we obtain that every prime ideal of $\mathbb{Z}_K$ is the intersection of maximal ideals. Thus $\mathbb{Z}_K$ is, by definition, a Jacobson ring (cf. [31], §4.5). The general form of the Nullstellensatz ([31], Theorem 4.19) implies that for every maximal ideal $\mathfrak{m}$ of $\mathbb{Z}_K[y]$, the ideal $\mathfrak{m} \cap \mathbb{Z}_K$ is a maximal ideal of $\mathbb{Z}_K$. We deduce that the maximal ideals of $\mathbb{Z}_K[y]$ are of the form $\mathfrak{p}\mathbb{Z}_K[y] + P(y)\mathbb{Z}_K[y]$ ($\mathfrak{p}$ prime in $\mathbb{Z}_K$ and $P(y)$ of degree at least 1 irreducible modulo $\mathfrak{p}$). Since $\mathbb{Z}_K[y]$ has Krull dimension 2, we have now described all its prime ideals.

The Rouquier ring $\mathcal{R}_K(y)$ is a localization of $\mathbb{Z}_K[y]$. Therefore, in order to prove that the non-zero prime ideals of $\mathcal{R}_K(y)$ are the ones described above, it is enough to show that $\mathfrak{m}\mathcal{R}_K(y) = \mathcal{R}_K(y)$ for all maximal ideals $\mathfrak{m}$ of $\mathbb{Z}_K[y]$. For this, it suffices to show that $\mathfrak{p}\mathcal{R}_K(y)$ is a maximal ideal of $\mathcal{R}_K(y)$ for all prime ideals $\mathfrak{p}$ of $\mathbb{Z}_K$.

Let $\mathfrak{p}$ be a prime ideal of $\mathbb{Z}_K$. Then

$$\mathcal{R}_K(y)/\mathfrak{p}\mathcal{R}_K(y) \cong \mathbb{F}_\mathfrak{p}[y, y^{-1}, (y^n - 1)^{-1}_{n \geq 1}],$$

where $\mathbb{F}_\mathfrak{p}$ denotes the finite field $\mathbb{Z}_K/\mathfrak{p}$. Since $\mathbb{F}_\mathfrak{p}$ is finite, every non-zero polynomial in $\mathbb{F}_\mathfrak{p}[y]$ is a product of elements which divide y or $y^n - 1$ for some $n \in \mathbb{N}$. Thus every non-zero element of $\mathbb{F}_\mathfrak{p}[y]$ is invertible in $\mathcal{R}_K(y)/\mathfrak{p}\mathcal{R}_K(y)$. Consequently, we obtain that

$$\mathcal{R}_K(y)/\mathfrak{p}\mathcal{R}_K(y) \cong \mathbb{F}_\mathfrak{p}(y),$$

whence $\mathfrak{p}$ generates a maximal ideal in $\mathcal{R}_K(y)$.

(3) The ring $\mathcal{R}_K(y)$ is the localization of a Noetherian integrally closed ring and thus Noetherian and integrally closed itself. Moreover, following the description of its prime ideals in part 2, it has Krull dimension 1. ■

Remark. If $P(y)$ is an irreducible element of $\mathbb{Z}_K[y]$ of degree at least 1, prime to y and to $\Phi(y)$ for all $\Phi \in \mathrm{Cycl}(K)$, then the field $\mathcal{R}_K(y)/P(y)\mathcal{R}_K(y)$ is isomorphic to the field of fractions of the ring $\mathbb{Z}_K[y]/P(y)\mathbb{Z}_K[y]$.

Now let us recall the form of the Schur elements of the cyclotomic Hecke algebra $\mathcal{H}_\phi$ given in Proposition 4.3.5. If χ_ϕ is an irreducible character of $K(y)\mathcal{H}_\phi$, then its Schur element $s_{\chi_\phi}(y)$ is of the form

$$s_{\chi_\phi}(y) = \psi_{\chi,\phi} y^{a_{\chi,\phi}} \prod_{\Phi \in C_K} \Phi(y)^{n_{\chi,\phi,\Phi}},$$

where $\psi_{\chi,\phi} \in \mathbb{Z}_K$, $a_{\chi,\phi} \in \mathbb{Z}$, $n_{\chi,\phi,\Phi} \in \mathbb{N}$ and C_K is a set of K-cyclotomic polynomials.

Definition 4.4.3. A prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ lying over a prime number p is *ϕ-bad* for W, if there exists $\chi_\phi \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$ with $\psi_{\chi,\phi} \in \mathfrak{p}$. If $\mathfrak{p}$ is ϕ-bad for W, we say that p is a *ϕ-bad prime number* for W.

Remark. If W is a Weyl group and ϕ is the "spetsial" cyclotomic specialization, then the ϕ-bad prime ideals are the ideals generated by the bad prime numbers (in the "usual" sense) for W (see [35], 5.2*).*

Note that if p is a ϕ-bad prime number for W, then p must divide the order of the group (since $s_{\chi_\phi}(1) = |W|/\chi(1)$).

Let us denote by $\mathcal{O}$ the Rouquier ring. By Proposition 2.1.10, the Rouquier blocks of $\mathcal{H}_\phi$ are unions of the blocks of $\mathcal{O}_\mathcal{P}\mathcal{H}_\phi$, where $\mathcal{P}$ runs over the set of prime ideals of $\mathcal{O}$. However, in all of the following cases, due to the form of the Schur elements, the blocks of $\mathcal{O}_\mathcal{P}\mathcal{H}_\phi$ are singletons (*i.e.*, $e_{\chi_\phi} = \chi_\phi^\vee / s_{\chi_\phi} \in \mathcal{O}_\mathcal{P}\mathcal{H}_\phi$ for all $\chi_\phi \in \mathrm{Irr}(K(y)\mathcal{H}_\phi)$):

- $\mathcal{P}$ is the zero ideal $\{0\}$.
- $\mathcal{P}$ is of the form $P(y)\mathcal{O}$, where $P(y)$ is an irreducible element of $\mathbb{Z}_K[y]$ of degree at least 1, prime to y and to $\Phi(y)$ for all $\Phi \in \mathrm{Cycl}(K)$.
- $\mathcal{P}$ is of the form $\mathfrak{p}\mathcal{O}$, where $\mathfrak{p}$ is a prime ideal of $\mathbb{Z}_K$ which is not ϕ-bad for W.

Therefore, the blocks of $\mathcal{O}\mathcal{H}_\phi$ are, simply, unions of the blocks of $\mathcal{O}_{\mathfrak{p}\mathcal{O}}\mathcal{H}_\phi$, where $\mathfrak{p}$ runs over the set of ϕ-bad prime ideals $\mathfrak{p}$ of $\mathbb{Z}_K$. More precisely, we have the following:

Proposition 4.4.4. *Let $\chi, \psi \in \mathrm{Irr}(W)$. The characters χ_ϕ and ψ_ϕ are in the same Rouquier block of $\mathcal{H}_\phi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of ϕ-bad prime ideals for W such that*

- $(\chi_0)_\phi = \chi_\phi$ *and* $(\chi_n)_\phi = \psi_\phi$,
- *for all* j $(1 \leq j \leq n)$, $(\chi_{j-1})_\phi$ *and* $(\chi_j)_\phi$ *are in the same block of* $\mathcal{O}_{\mathfrak{p}_j\mathcal{O}}\mathcal{H}_\phi$.

By Proposition 1.1.5(4), we obtain that $\mathcal{O}_{\mathfrak{p}\mathcal{O}} \cong \Omega_{\mathfrak{p}\Omega}$, where $\Omega := \mathbb{Z}_K[y, y^{-1}]$. In the previous section we saw how we can use Theorem 3.3.2 to calculate the blocks of $\Omega_{\mathfrak{p}\Omega}\mathcal{H}_\phi$ and thus obtain the Rouquier blocks of $\mathcal{H}_\phi$. We deduce that the Rouquier blocks of the cyclotomic Hecke algebras also have the property of *semi-continuity*:

- If the $n_{\mathcal{C},j}$ belong to no essential hyperplane for W, then the Rouquier blocks of $\mathcal{H}_\phi$ are the *Rouquier blocks associated with no essential hyperplane*.
- If the $n_{\mathcal{C},j}$ belong to exactly one essential hyperplane H for W, then the Rouquier blocks of $\mathcal{H}_\phi$ are the *Rouquier blocks associated with the essential hyperplane H*.
- If the $n_{\mathcal{C},j}$ belong to more than one essential hyperplane, then the Rouquier blocks of $\mathcal{H}_\phi$ are unions of the Rouquier blocks associated with the essential hyperplanes to which the $n_{\mathcal{C},j}$ belong and they are minimal with respect to that property.

4.4.1 Rouquier Blocks and Central Morphisms

The following description of the Rouquier blocks results from Proposition 2.1.15 and the description of ϕ-bad prime ideals for W.

Proposition 4.4.5. *Let $\chi, \psi \in \mathrm{Irr}(W)$. The characters χ_ϕ and ψ_ϕ are in the same Rouquier block of $\mathcal{H}_\phi$ if and only if there exist a finite sequence $\chi_0, \chi_1, \ldots, \chi_n \in \mathrm{Irr}(W)$ and a finite sequence $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of ϕ-bad prime ideals for W such that*

- $(\chi_0)_\phi = \chi_\phi$ *and* $(\chi_n)_\phi = \psi_\phi$,
- *for all* j $(1 \leq j \leq n)$, $\omega_{(\chi_{j-1})_\phi} \equiv \omega_{(\chi_j)_\phi} \bmod \mathfrak{p}_j\mathcal{O}_{\mathfrak{p}_j\mathcal{O}}$.

4.4.2 Rouquier Blocks and Functions a and A

Following the notation of [21], 6B, for every element $P(y) \in \mathbb{C}(y)$, we call

- *valuation of $P(y)$ at y* and denote by $\mathrm{val}_y(P)$ the order of $P(y)$ at 0 (we have $\mathrm{val}_y(P) < 0$ if 0 is a pole of $P(y)$ and $\mathrm{val}_y(P) > 0$ if 0 is a zero of $P(y)$),
- *degree of $P(y)$ at y* and denote by $\deg_y(P)$ the opposite of the valuation of $P(1/y)$.

Moreover, if $x := y^{|\mu(K)|}$, then

$$\mathrm{val}_x(P(y)) := \frac{\mathrm{val}_y(P)}{|\mu(K)|} \quad \text{and} \quad \deg_x(P(y)) := \frac{\deg_y(P)}{|\mu(K)|}.$$

For $\chi \in \mathrm{Irr}(W)$, we define

$$a_{\chi_\phi} := \mathrm{val}_x(s_{\chi_\phi}(y)) \quad \text{and} \quad A_{\chi_\phi} := \deg_x(s_{\chi_\phi}(y)).$$

The following result is proven in [18], Proposition 2.9.

Proposition 4.4.6.

(1) For all $\chi \in \mathrm{Irr}(W)$, we have

$$\omega_{\chi_\phi}(\boldsymbol{\tau}) = t_\phi(\boldsymbol{\tau}) x^{a_{\chi_\phi}+A_{\chi_\phi}},$$

where $\boldsymbol{\tau}$ is the central element of the pure braid group of Definition 4.1.6.

(2) Let $\chi, \psi \in \mathrm{Irr}(W)$. If χ_ϕ and ψ_ϕ belong to the same Rouquier block, then

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

Proof. (1) If $P(y) \in \mathbb{C}[y, y^{-1}]$, we denote by $P(y)^*$ the polynomial whose coefficients are the complex conjugates of those of $P(y)$. By [21], 2.8, we know that the Schur element $s_{\chi_\phi}(y)$ is semi-palindromic and satisfies

$$s_{\chi_\phi}(y^{-1})^* = \frac{t_\phi(\boldsymbol{\tau})}{\omega_{\chi_\phi}(\boldsymbol{\tau})} s_{\chi_\phi}(y).$$

We deduce ([21], 6.5, 6.6) that

$$\frac{t_\phi(\boldsymbol{\tau})}{\omega_{\chi_\phi}(\boldsymbol{\tau})} = \xi x^{-(a_{\chi_\phi}+A_{\chi_\phi})},$$

for some $\xi \in \mathbb{C}$. For $y = x = 1$, the first equation gives $t_\phi(\boldsymbol{\tau}) = \omega_{\chi_\phi}(\boldsymbol{\tau})$ and the second one $\xi = 1$. Thus we obtain

$$\omega_{\chi_\phi}(\boldsymbol{\tau}) = t_\phi(\boldsymbol{\tau}) x^{a_{\chi_\phi}+A_{\chi_\phi}}.$$

(2) Suppose that χ_ϕ and ψ_ϕ belong to the same Rouquier block. Due to Proposition 4.4.5, it is enough to show that if there exists a ϕ-bad prime ideal $\mathfrak{p}$ of $\mathbb{Z}_K$ such that $\omega_{\chi_\phi} \equiv \omega_{\psi_\phi} \mod \mathfrak{p}\mathcal{O}_{\mathfrak{p}\mathcal{O}}$, then $a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}$. If $\omega_{\chi_\phi} \equiv \omega_{\psi_\phi} \mod \mathfrak{p}\mathcal{O}_{\mathfrak{p}\mathcal{O}}$, then, in particular, $\omega_{\chi_\phi}(\boldsymbol{\tau}) \equiv \omega_{\psi_\phi}(\boldsymbol{\tau}) \mod \mathfrak{p}\mathcal{O}_{\mathfrak{p}\mathcal{O}}$. Part 1 implies that

$$t_\phi(\boldsymbol{\tau}) x^{a_{\chi_\phi}+A_{\chi_\phi}} \equiv t_\phi(\boldsymbol{\tau}) x^{a_{\psi_\phi}+A_{\psi_\phi}} \mod \mathfrak{p}\mathcal{O}_{\mathfrak{p}\mathcal{O}}.$$

We know by [21], 2.1 that $t_\phi(\boldsymbol{\tau})$ is of the form ξx^M, where ξ is a root of unity and $M \in \mathbb{Z}$. Thus $t_\phi(\boldsymbol{\tau}) \notin \mathfrak{p}\mathcal{O}_{\mathfrak{p}\mathcal{O}}$ and the above congruence gives

$$x^{a_{\chi_\phi}+A_{\chi_\phi}} \equiv x^{a_{\psi_\phi}+A_{\psi_\phi}} \bmod \mathfrak{p}\mathcal{O}_{\mathfrak{p}\mathcal{O}},$$

whence

$$a_{\chi_\phi} + A_{\chi_\phi} = a_{\psi_\phi} + A_{\psi_\phi}.$$

■

Remark. *For all Coxeter groups, Lusztig has proved (cf., for example,* [46], 3.3 *and* 3.4*) that if* χ_ϕ *and* ψ_ϕ *belong to the same Rouquier block of the Iwahori-Hecke algebra, then* $a_{\chi_\phi} = a_{\psi_\phi}$ *and* $A_{\chi_\phi} = A_{\psi_\phi}$*. This assertion has also been proved*

- *for almost all cyclotomic Hecke algebras of the groups* $G(d,1,r)$ *and* $G(e,e,r)$ *in* [18],
- *for the "spetsial" cyclotomic Hecke algebra of the "spetsial" exceptional complex reflection groups in* [53].

Using the results of the next chapter, we have been able to obtain the same result for all cyclotomic Hecke algebras

- *of the groups* $G(d,1,r)$ *in* [25],
- *of the groups* $G(de,e,r)$ *in* [26], *and*
- *of all exceptional irreducible complex reflection groups in* [23],

thus completing its proof for all complex reflection groups.

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