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Properties of $3jm$- and $3nj$-Symbols

2.1 Properties of $3jm$-Symbols

Basic elements in the theory of angular momentum are the Clebsch–Gordan coefficients for coupling two states characterized by j_1, m_1 and j_2, m_2 into a new state with quantum numbers J, M. The numbers j and their projections, or magnetic quantum numbers[1], m have integer or half-odd integer values. Hence, in an exponent to (-1) the quantum numbers must combine to integers as in $j \pm m$ or $j_1 + j_2 + J$, for example. The notation used for the corresponding Clebsch–Gordan coefficient is $(j_1 m_1 j_2 m_2 | JM)$ and it can be non-zero only if j_1, j_2, J fulfil the triangle condition $\triangle\{j_1, j_2, J\}$ equivalent to the condition $j_1 + j_2 \geq J \geq |j_1 - j_2|$ and, furthermore, provided that $m_1 + m_2 = M$. The Clebsch–Gordan coefficients are chosen to be purely real and constitute a unitary transformation.

The coupling coefficients introduced in the previous paragraph are closely related to the so-called Wigner coefficients, $3j$-symbols or, more precisely, $3jm$-symbols:

$$\begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} = \frac{(-1)^{j_1 - j_2 + M}}{\sqrt{(2J+1)}} (j_1 m_1 j_2 m_2 | JM) \ .$$

$3jm$-symbols are also purely real and this convention is adopted for all $3nj$-symbols encountered later. Note that, the exponent $j_1 - j_2 + M$ is integer. Although the $3jm$-symbols, or Wigner symbols, are usually referred to as just $3j$-symbols in the literature, we have chosen the term $3jm$-symbol here in order to distinguish these from the $3j$-, $6j$- and $9j$-symbol etc. in the theory. The Clebsch–Gordan coefficients correspond to the notation and phase used in Condon and Shortley [2], although, unlike them, we do not repeat j_1, j_2 in the right-hand part of the coefficient. The definition above can be inverted to give

[1] The terms ‘projection’ and ‘magnetic quantum number’ are used interchangeably in the text, with the first term favoured overall.

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$$(j_1 m_1 j_2 m_2 | JM) = (-1)^{j_2 - j_1 - M} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} . \tag{2.1}$$

From this definition follows the transformation from plain product states $|j_1 m_1\rangle |j_2 m_2\rangle$ to the coupled state $|j_1 j_2 JM\rangle$ and the inverse transformation, with the Clebsch–Gordan coefficients replaced by $3jm$-symbols as in (2.1).

$$|j_1 j_2 JM\rangle = \sum_{m_1 m_2} (-1)^{j_2 - j_1 - M} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} |j_1 m_1\rangle |j_2 m_2\rangle ,$$

$$|j_1 m_1\rangle |j_2 m_2\rangle = \sum_{JM} (-1)^{j_2 - j_1 - M} \sqrt{2J+1} \begin{pmatrix} j_1 & j_2 & J \\ m_1 & m_2 & -M \end{pmatrix} |j_1 j_2 JM\rangle .$$

As mentioned before, the sum of the arguments $j_1 + j_2 + J$ is integer and the triangle condition $\triangle\{j_1, j_2, J\}$ must be fulfilled.

An advantage of the $3jm$-symbols over the coupling coefficients lies in their neat symmetry properties. E.g., even permutation of the columns leaves a $3jm$-symbol invariant,

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = \begin{pmatrix} j_2 & j_3 & j_1 \\ m_2 & m_3 & m_1 \end{pmatrix} = \begin{pmatrix} j_3 & j_1 & j_2 \\ m_3 & m_1 & m_2 \end{pmatrix} , \tag{2.2}$$

while an odd permutation introduces a phase factor $(-1)^{j_1+j_2+j_3}$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_2 & j_1 & j_3 \\ m_2 & m_1 & m_3 \end{pmatrix} = \text{etc.} \tag{2.3}$$

The same phase factor is required for a sign change of m_1, m_2 and m_3

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1+j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ -m_1 & -m_2 & -m_3 \end{pmatrix} . \tag{2.4}$$

The $3jm$-symbol can be non-zero only if $m_1 + m_2 + m_3 = 0$ and the triangle condition $\triangle\{j_1, j_3, j_3\}$ for j_1, j_2 and j_3 is fulfilled. At least one of the three angular momenta in a $3jm$-symbol has to be integer and the sum of all three must also be an integer. From this we see that, two of the angular momenta in a $3jm$-symbol may be half-integer and this has the following consequences for the simplification of a general phase factor: every phase $(-1)^{4j_i} \equiv 1$ or $(-1)^{2j_i \pm 2m_i} \equiv 1$ or, for the j_i of a $3jm$-symbol, $(-1)^{2j_1+2j_2+2j_3} \equiv 1$ can thus be removed or added because the exponents are even integers in all these cases.

The following identities will be very useful in subsequent derivations. From the properties of the Clebsch–Gordan coefficients follow the orthogonality properties of the $3jm$-symbols

$$\sum_{m_1, m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{\delta_{j_3, j_3'} \delta_{m_3, m_3'}}{(2j_3 + 1)} , \tag{2.5}$$

and

$$\sum_{j_3,m_3} (2j_3+1) \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m'_1 & m'_2 & m_3 \end{pmatrix} = \delta_{m_1,m'_1}\delta_{m_2,m'_2} \; . \tag{2.6}$$

These identities are so important, they merit an entire section in our subsequent presentation. Because the sum of the m_i in a $3jm$-symbol is zero by definition, some of the summations over magnetic quantum numbers may be purely formal and this can be seen for a special case of (2.5):

$$\sum_{m_1,m_2,m_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = 1 \; . \tag{2.7}$$

The following equality uses the cyclic permutation symmetry, (2.2), and shows the value of this special $3jm$-symbol with one j and its projection m set zero:

$$\begin{pmatrix} j & 0 & j \\ -m & 0 & m \end{pmatrix} = \begin{pmatrix} j & j & 0 \\ m & -m & 0 \end{pmatrix} = (-1)^{j-m} \frac{1}{\sqrt{2j+1}} \; . \tag{2.8}$$

This result will be frequently used in the more general form

$$\begin{aligned} &(-1)^{j_1-m_1} \begin{pmatrix} j_1 & 0 & j_2 \\ -m_1 & 0 & m_2 \end{pmatrix} \\ &= (-1)^{j_1-m_1} \begin{pmatrix} j_2 & j_1 & 0 \\ m_2 & -m_1 & 0 \end{pmatrix} = \frac{\delta_{j_1,j_2}\delta_{m_1,m_2}}{\sqrt{2j_1+1}} \; , \end{aligned} \tag{2.9}$$

or

$$\begin{pmatrix} j_1 & 0 & j_2 \\ -m_1 & 0 & m_2 \end{pmatrix} = \begin{pmatrix} j_2 & j_1 & 0 \\ m_2 & -m_1 & 0 \end{pmatrix} = (-1)^{j_1-m_1} \frac{\delta_{j_1,j_2}\delta_{m_1,m_2}}{\sqrt{2j_1+1}} \; . \tag{2.10}$$

where the Kronecker-δ take care of the triangle condition $\triangle\{j_1, j_2, 0\}$ and the condition $m_2 - m_1 = 0$ for this special case.

The algebraic formula used by Rotenberg et al. [9] for the calculation of $3jm$-symbols is

$$\begin{aligned} &\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3} \\ &\times \left[\frac{(j_1+j_2-j_3)!(j_1-j_2+j_3)!(-j_1+j_2+j_3)!}{(j_1+j_2+j_3+1)!} \right]^{1/2} \\ &\times \left[(j_1+m_1)!(j_1-m_1)!(j_2+m_2)!(j_2-m_2)!(j_3+m_3)!(j_3-m_3)! \right]^{1/2} \\ &\times \left[\sum_k \frac{(-1)^k}{k!(j_1+j_2-j_3-k)!(j_1-m_1-k)!(j_2+m_2-k)!} \right. \\ &\qquad \left. \times \frac{1}{(j_3-j_2+m_1+k)!(j_3-j_1-m_2+k)!} \right] \; . \end{aligned}$$

Tables of $3jm$-symbols have been given by Edmonds [4], Rotenberg et al. [9] and Varshalovich et al. [12], to cite just a few. Of course, nowadays computer programs will provide values for $3jm$-symbols, in some cases even in algebraic form.

2.2 Properties of $3nj$-Symbols

If the theory of angular momentum is extended to consider more than two angular momenta one encounters several possible coupling schemes. For three angular momenta j_1, j_2 and j_3 the coupling into a state of total angular momentum J could proceed via two routes. As a first step one could couple j_1 and j_2 to j_{12} and then j_{12} with j_3 to obtain the total J. Alternatively, one could first couple j_2 and j_3 to j_{23} and then j_1 with j_{23} for J. The states belonging to the two coupling schemes can be transformed into each other [7] and the elements of the transformation matrix are proportional to a $6j$-symbol. A symbolic relation for these couplings is,

$$\langle [(j_1, j_2)j_{12}, j_3]J | [j_1, (j_2, j_3)j_{23}]J \rangle \propto \begin{Bmatrix} j_3 & J & j_{12} \\ j_1 & j_2 & j_{23} \end{Bmatrix} .$$

The symmetry properties of $6j$-symbols will be discussed later in section 5.2.

The next natural extension considers the coupling schemes in which four angular momenta are involved. These two coupling schemes are well known in atomic physics and referred to as SL and jj-coupling. In this case one finds [7] the transformation matrix is proportional to a $9j$-symbol, and symbolically,

$$\langle [(j_1, j_2)j_{12}, (j_3, j_4)j_{34}]J | [(j_1, j_3)j_{13}, (j_2, j_4)j_{24}]J \rangle \propto \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{Bmatrix} .$$

Again, the discussion of the symmetry properties of the $9j$-symbol will be postponed.

In the following sections we will present a graphical technique equivalent to the algebraic treatment of complex expressions built from $3jm$-symbols, $3nj$-symbols like $6j$-, $9j$-symbols and symbols of higher order.

2.3 Comment on Notation

To simplify the notation, the subscripted variables $j_1, j_2, j_3, \ldots$ will be replaced by $a, b, c, \ldots$ and the magnetic quantum numbers, or projections, $m_1, m_2, m_3, \ldots$ will be represented by corresponding Greek letters $\alpha, \beta, \gamma, \ldots$ but we will continue to refer to these as j- and m-values, respectively.

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