Chapter 2
The Time-Dependent Schrödinger Equation Revisited: Quantum Optical and Classical Maxwell Routes to Schrödinger's Wave Equation[1]

Marlan O. Scully

2.1 Introduction

In a previous paper [1, 2] we presented quantum field theoretical and classical (Hamilton–Jacobi) routes to the time-dependent Schrödinger equation (TDSE) in which the time t and position $\mathbf{r}$ are regarded as parameters, not operators. From this perspective, the time in quantum mechanics is argued as being the same as the time in Newtonian mechanics. We here provide a parallel argument, based on the photon wave function, showing that the time in quantum mechanics is the same as the time in Maxwell equations.

The next section is devoted to a review of the photon wave function which is based on the premise that a photon is what a photodetector detects. In particular, we show that the time-dependent Maxwell equations for the photon are to be viewed in the same way we look at the time-dependent Dirac–Schrödinger equation for the (massive) π meson particle or (massless) neutrino.

In Sect. 2.3 we then recall previous work which casts the classical Maxwell equations into a form which is very similar to the Dirac equation for the neutrino. Thus, we are following de Broglie more closely than did Schrödinger, who followed a Hamilton–Jacobi approach to the quantum mechanical wave equation. In this way, with nearly a century of hindsight, we arrive naturally at the time-dependent Schrödinger equation without operator baggage. Figures 2.1 and 2.2 summarize the physics of the present chapter.

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[1] It is a pleasure to dedicate this chapter to David Woodling who has enriched our lives through his engineering and mechanical gifts and his insightful and gentle ways.

Scully, M.O.: *The Time-Dependent Schrödinger Equation Revisited: Quantum Optical and Classical Maxwell Routes to Schrödinger's Wave Equation*. Lect. Notes Phys. **789**, 15–24 (2010)
DOI 10.1007/978-3-642-03174-8_2 © Springer-Verlag Berlin Heidelberg 2010

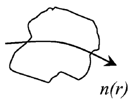
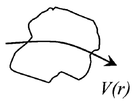
	Photon (spin 1)	Neutrino (spin ½)	π Meson (spin 0)
Quantum Field Theory	$\lvert\dot{\Psi}_\gamma\rangle = -\frac{1}{\hbar}H_\gamma\lvert\Psi_\gamma\rangle$	$\lvert\dot{\Psi}_\nu\rangle = -\frac{1}{\hbar}H_\nu\lvert\Psi_\nu\rangle$	$\lvert\dot{\Psi}_\pi\rangle = -\frac{1}{\hbar}H_\pi\lvert\Psi_\pi\rangle$
"Wave Mechanics"	$\Psi_\gamma = \begin{bmatrix}\varphi_\gamma \\ \chi_\gamma\end{bmatrix}$ $\dot{\Psi}_\gamma = -\frac{i}{\hbar}\begin{bmatrix}0 & -cs\cdot p \\ cs\cdot p & 0\end{bmatrix}\Psi_\gamma$	$\Psi_\nu = \begin{bmatrix}\varphi_\nu \\ \chi_\nu\end{bmatrix}$ $\dot{\Psi}_\nu = -\frac{i}{\hbar}\begin{bmatrix}0 & -c\sigma\cdot p \\ c\sigma\cdot p & 0\end{bmatrix}\Psi_\nu$	$\dot{\Psi}_\pi = -\frac{i}{\hbar}\sqrt{p^2c^2 + m_0^2c^4}\,\Psi_\pi$ $\Rightarrow \frac{i\hbar}{2m_0}\nabla^2\Psi_\pi$
Classical limit: Eikonal physics	 Ray optics $\delta\int n\,ds = 0$ Fermat's principle	 Classical Mechanics $\delta\int L\,dt = 0$ Hamilton's principle	

Fig. 2.1 Comparison of the quantum field, wave mechanical, and classical descriptions of the spin 1 photon, spin $\frac{1}{2}$ neutrino, and spin 0 meson; adapted from Scully and Zubairy "Quantum Optics" [3]

$$\boxed{\langle 0|\hat{E}(r)|\Psi(t)\rangle}$$
$$\Downarrow$$
$$\boxed{i\hbar\frac{\partial\Psi(r,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\Psi(r,t)}$$
$$\Uparrow$$
$$\boxed{\Psi_m(r,t) = E(r,t) + iH(r,t)}$$

Fig. 2.2 *Top Down*: The time-dependent Schrödinger wave equation follows from the quantum optical "a photon is what a photodetector detects" definition. This is in accord with the usual wave function definition $\Psi(\mathbf{r},t) = \langle\mathbf{r}|\Psi(t)\rangle$ since $|\mathbf{r}\rangle = \hat{\Psi}^{+}(r)|0\rangle$. *Bottom Up*: The time-dependent Schrödinger wave follows nicely from the classical Maxwell equations by, for example, working with a combination of electric and magnetic fields

2.2 The Quantum Optical Route to the Time-Dependent Schrödinger Equation

Quantum optics is an offshoot of quantum field theory in which we are often interested in intense light beams such as provided by the laser. However the issue of the photon concept, and how we should think of the "photon," is a topic of current and reoccurring discussion.

Perhaps the most logical, at least the most operational, approach is to say that the photon is what a photodetector detects. In this spirit we consider the excitation of a single atom at point $\mathbf{r}$ at time t to be our photodetector and, following [3], write the probability of exciting the atom as

$$P_{\Psi}(\mathbf{r},t) = \eta\langle\Psi\left|\hat{E}^{\dagger}(\mathbf{r},t)\hat{E}(\mathbf{r},t)\right|\Psi\rangle \; . \tag{2.1}$$

Several points should be made:

1. We consider the state $|\Psi\rangle$ to be a single photon state. For example, the state generated by the emission of a single photon (see [3], Eq. 6.3.18)

$$\left|\psi_{\gamma}\right\rangle = \sum_{\mathbf{k}} c_{\mathbf{k}}\left|\mathbf{k}\right\rangle \;\; , \tag{2.2}$$

where the state $|\mathbf{k}\rangle$ is expressed in terms of the radiation creation operator $\hat{a}^{\dagger}_{\mathbf{k}}$ as $|\mathbf{k}\rangle = \hat{a}^{\dagger}_{\mathbf{k}}\,|0\rangle$ and in the simple case of a scalar photon, we find

$$c_{\mathbf{k}} = g_{\mathbf{k}}\frac{\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{r}_0}}{(\nu_k-\omega_0)+\mathrm{i}\Gamma/2} \; , \tag{2.3}$$

where $g_{\mathbf{k}}$ is the atom-field coupling constant, $\mathbf{r}_0$ is the atomic position vector, ν_k and ω_0 are the photon and atomic frequencies, and Γ is the atomic decay rate.
2. The uninteresting photodetection efficiency constant η will be ignored in the following.
3. $\hat{E}^{\dagger}(\mathbf{r},t)$ and $\hat{E}(\mathbf{r},t)$ are the creation and annihilation operators defined by

$$\hat{E}^{\dagger}(\mathbf{r},t) = \sum_{\mathbf{k},\lambda} \boldsymbol{\varepsilon}^{(\lambda)}_{\mathbf{k}}\mathcal{E}_k\hat{a}^{\dagger}_{\mathbf{k},\lambda}\mathrm{e}^{-\mathrm{i}\nu_k t+\mathrm{i}\mathbf{k}\cdot\mathbf{r}} \; , \tag{2.4}$$

where $\boldsymbol{\varepsilon}^{\lambda}_{\mathbf{k}}$ is the unit vector for light having polarization λ and wave vector $\mathbf{k}$, $\nu_k = ck = c|\mathbf{k}|$ and the electric field "per photon" $\mathcal{E}_k = \sqrt{\hbar\nu_k/2\varepsilon_0 V}$, where we use MKS units so that $\varepsilon_0\mu_0 = 1/c^2$ and V is the quantization volume.

Next we insert a sum over a complete set of states, $\sum_n |n\rangle\langle n| = 1$ in Eq. (2.1) and note that since there is only one photon in ψ_{γ} (and $\hat{E}(\mathbf{r},t)$ annihilates it), only the vacuum term $|0\rangle\langle 0|$ will contribute. Hence we have

$$P_{\psi_{\gamma}}(\mathbf{r},t) = \langle\psi_{\gamma}|\hat{E}^{\dagger}(\mathbf{r},t)|0\rangle\langle 0|\hat{E}(\mathbf{r},t)|\psi_{\gamma}\rangle \, , \tag{2.5}$$

and we are therefore led to define the single photon detection amplitude as

$$\Psi_{\mathcal{E}}(\mathbf{r},t) = \langle 0|\hat{E}(\mathbf{r},t)|\psi_{\gamma}\rangle \, . \tag{2.6}$$

As shown in detail in Sect. 2.4, the one photon state $|\psi_{\gamma}\rangle$ yields

$$\Psi_{\mathcal{E}}(\mathbf{r},t) = \frac{\mathcal{E}}{\Delta r}\Theta\left(t-\frac{\Delta r}{c}\right)\mathrm{e}^{-\mathrm{i}(t-\Delta r/c)(\omega-\mathrm{i}\Gamma/2)} \; , \tag{2.7}$$

where $\mathcal{E}$ is a constant, Δr is the distance from the atom to the detector, and $\Theta(x)$ is the usual step function. More generally we have

$$\Psi_{\mathcal{E}}(\mathbf{r},t) = \langle 0|\hat{E}(\mathbf{r},t)|\psi_\gamma\rangle = \left\langle 0\left|\sum_{\mathbf{k},\lambda}\hat{\varepsilon}_{\mathbf{k}}^{\lambda}\sqrt{\frac{\hbar\nu_k}{2\varepsilon_0 V}}\hat{a}_{\mathbf{k},\lambda}\mathrm{e}^{-\mathrm{i}\nu_k t+\mathrm{i}\mathbf{k}\cdot\mathbf{r}}\right|\psi_\gamma\right\rangle . \tag{2.8}$$

The field is sharply peaked about the frequency ω so that we may replace the frequency ν_k as it appears in the square root factor by ω and write

$$\Psi_{\mathcal{E}}(\mathbf{r},t) = \sqrt{\frac{\hbar\omega}{2\varepsilon_0}}\varphi_\gamma(\mathbf{r},t) , \tag{2.9}$$

where

$$\varphi_\gamma(\mathbf{r},t) = \sum_{\mathbf{k},\lambda}\hat{\varepsilon}_{\mathbf{k}}^{(\lambda)}\left\langle 0\left|\hat{a}_{\mathbf{k},\lambda}\frac{\mathrm{e}^{-\mathrm{i}\nu_k t+\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}}\right|\psi_\gamma\right\rangle . \tag{2.10}$$

The complete “photon wave function” also involves the magnetic analog of the proceeding. To that end we write

$$\Psi_{\mathcal{H}}(\mathbf{r},t) = \langle 0|\hat{H}(\mathbf{r},t)|\psi_\gamma\rangle , \tag{2.11}$$

where $\hat{H}(\mathbf{r},t)$ is the annihilation operator for the magnetic field which is given by

$$\hat{H}(\mathbf{r},t) = \sum_{\mathbf{r},\lambda}\frac{\mathbf{k}}{k}\times\hat{\varepsilon}_{\mathbf{k}}^{(\lambda)}\sqrt{\frac{\hbar\nu_k}{2\mu_0}}\hat{a}_{\mathbf{k},\lambda}\frac{\mathrm{e}^{-\mathrm{i}\nu_k t+\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}} , \tag{2.12}$$

and we introduce the notation

$$\Psi_{\mathcal{H}}(\mathbf{r},t) = \sqrt{\frac{\hbar\omega}{2\mu_0}}\chi_\gamma(\mathbf{r},t) , \tag{2.13}$$

where

$$\chi_\gamma(\mathbf{r},t) = \left\langle 0\left|\sum_{\mathbf{k},\lambda}\frac{\mathbf{k}}{k}\times\hat{\varepsilon}_{\mathbf{k}}^{(\lambda)}a_{\mathbf{k},\lambda}\frac{\mathrm{e}^{-\mathrm{i}\nu_k t+\mathrm{i}\mathbf{k}\cdot\mathbf{r}}}{\sqrt{V}}\right|\psi_\gamma\right\rangle . \tag{2.14}$$

Finally, we write $\varphi_\gamma(\mathbf{r},t)$ and $\chi_\gamma(\mathbf{r},t)$ in matrix form as

$$\varphi_\gamma = \begin{bmatrix}\varphi_x\\ \varphi_y\\ \varphi_z\end{bmatrix} , \quad \chi_\gamma = \begin{bmatrix}\chi_x\\ \chi_y\\ \chi_z\end{bmatrix} , \tag{2.15}$$

in terms of which Maxwell equations may be written as

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix}\varphi_\gamma\\ \chi_\gamma\end{bmatrix}=\begin{bmatrix}0 & -c\mathbf{s}\cdot\mathbf{p}\\ c\mathbf{s}\cdot\mathbf{p} & 0\end{bmatrix}\begin{bmatrix}\varphi_\gamma\\ \chi_\gamma\end{bmatrix}, \tag{2.16}$$

where $\mathbf{p}=\frac{\hbar}{i}\nabla$ and

$$s_x=\begin{bmatrix}0&0&0\\0&0&-1\\0&1&0\end{bmatrix},\quad s_y=\begin{bmatrix}0&0&1\\0&0&0\\-1&0&0\end{bmatrix},\quad s_z=\begin{bmatrix}0&-1&0\\1&0&0\\0&0&0\end{bmatrix} \tag{2.17}$$

are the 3×3 matrices for the (spin 1) photon.

Finally, we note the close correspondence with the two-component (spin $\frac{1}{2}$) neutrino,

$$\begin{bmatrix}\varphi_{\text{photon}}\\ \chi_{\text{photon}}\end{bmatrix}\longleftrightarrow\begin{bmatrix}\varphi_{\text{neutrino}}\\ \chi_{\text{neutirno}}\end{bmatrix}, \tag{2.18}$$

and the Dirac equation for the neutrino

$$i\hbar\frac{\partial}{\partial t}\begin{bmatrix}\varphi_\nu\\ \chi_\nu\end{bmatrix}=\begin{bmatrix}0 & -c\sigma\cdot\mathbf{p}\\ c\sigma\cdot\mathbf{p} & 0\end{bmatrix}\begin{bmatrix}\varphi_\nu\\ \chi_\nu\end{bmatrix}, \tag{2.19}$$

where σ is given in terms of the 2×2 Pauli matrices and $\mathbf{p}=\frac{\hbar}{i}\nabla$.

We conclude by noting that, just as in the quantum field theory [4, 5] route to the Schrödinger equation, the appearance of $\frac{\partial}{\partial t}$ and ∇ in Eq. (2.16) has not arisen from operator arguments. In the next sections, we follow a de Broglie wave–particle duality path to the Schrödinger equation.

2.3 The Classical Maxwell Route to the Schrödinger Equation

In the previous section, we followed a top-down quantum field route to the Schrödinger equation, see Fig. 2.2. In particular, we saw that the quantum optical analysis of the single photon wave equation provided an interesting connection between the Schrödinger (Dirac) equations for photons and neutrinos.

In the present section, we start with the classical Maxwell equations and obtain a Schrödinger equation for the combination $\mathbf{E}+i\mathbf{H}$ which previous workers [6, 7] call the photon wave function. It is then natural to follow de Broglie and associate a wave function with matter waves. This provides another (operator-free) route to the Schrödinger equation.

Thus, we define the "classical" photon wave function as

$$\Psi_{\mathrm{m}}(\mathbf{r},t)=\mathbf{E}(\mathbf{r},t)+\mathrm{i}\mathbf{H}(\mathbf{r},t)=\begin{bmatrix}\psi_x(\mathbf{r},t)\\ \psi_y(\mathbf{r},t)\\ \psi_z(\mathbf{r},t)\end{bmatrix}, \tag{2.20}$$

where the subscript m stands for Maxwell. Along the lines of the discussion in Sect. 2.2, we may write the Maxwell equations as

$$i\hbar\dot{\Psi}_{\mathrm{m}}(\mathbf{r},t)=-c\mathbf{s}\cdot\mathbf{p}\Psi_{\mathrm{m}}(\mathbf{r},t)\,, \tag{2.21}$$

where $\mathbf{p}=\frac{\hbar}{i}\nabla$, as before, but now

$$s_x=\begin{bmatrix}0&0&0\\0&0&-i\\0&i&0\end{bmatrix},\quad s_y=\begin{bmatrix}0&0&i\\0&0&0\\-i&0&0\end{bmatrix},\quad s_z=\begin{bmatrix}0&-i&0\\i&0&0\\0&0&0\end{bmatrix}. \tag{2.22}$$

The present $\mathbf{s}$ matrix is related to the $\mathbf{s}$ of Sect. 2.2 by the factor i. It also should be noted that the present photon wave function ψ_{m} is a 1×3 matrix whereas that of 2.2 is a 1×6 matrix. That is, the quantum optical analysis involves a two-component wave function in Ψ_{ε} and $\Psi_{\mathcal{H}}$; in the present analysis we find it convenient to combine the electric and magnetic contributions at the outset.

Since the energy per photon is $\hbar\omega=\hbar ck=cp$, we write

$$i\hbar\dot{\Psi}_{\mathrm{m}}(\mathbf{r},t)=H\Psi_{\mathrm{m}}(\mathbf{r},t)\,, \tag{2.23}$$

where the Hamiltonian is given by

$$H=-c\mathbf{s}\cdot\mathbf{p}\,. \tag{2.24}$$

The natural extension of this Schrödinger equation for the spin one massless photon to the case of a spin zero particle of mass m is clear. That is, since $E=\sqrt{m_0^2c^4+p^2c^2}$ is the finite mass extension of $E=pc$, we follow the lead of de Broglie and write

$$\mathrm{i}\hbar\dot{\Psi}(\mathbf{r},t)=\sqrt{m_0^2c^4+p^2c^2}\,\Psi(\mathbf{r},t)\,, \tag{2.25}$$

where $\mathbf{p}=\frac{\hbar}{\mathrm{i}}\nabla$, just as it is for the photon.

Hence when $m_0c^2\gg pc$ we may write $\sqrt{m_0^2c^4+p^2c^2}\cong\frac{p^2}{2m}+m_0c^2$, and we have

$$\mathrm{i}\hbar\dot{\Psi}(\mathbf{r},t)=\frac{-\hbar^2}{2m_0}\nabla^2\Psi(\mathbf{r},t)\,, \tag{2.26}$$

which is the non-relativistic wave equation, again obtained without introducing operator-valued time or momentum.

2.4 The Single Photon and Two Photon Wave Functions

The photon wave function concept really comes into its own when solving problems involving photon–photon correlations. Then, as is explained in [8], the two photon wave function

$$\psi^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) \equiv \langle 0|\hat{E}(\mathbf{r}_2, t_2)\hat{E}(\mathbf{r}_1, t_1)|\Psi\rangle \tag{2.27}$$

is the subject of interest. Under some conditions this may be written in terms of single photon wave functions, as in the case of two photon cascade discussed below. Some of the calculational details will be given since the physics (and the devil) is in the details.

Consider first the single photon wave function. From Eqs. (2.3) and (2.4) and ignoring polarization, we find

$$\langle 0|\hat{E}(\mathbf{r}, t)|\psi_\gamma\rangle = \sqrt{\frac{\hbar}{2\varepsilon_0 V}} \sum_{\mathbf{k}} (\nu_k)^{1/2} g_{\mathbf{k}} e^{-i\nu_k t} e^{i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}_0)} \frac{1}{(\nu_k - \omega) + i\Gamma/2} \,. \tag{2.28}$$

We now evaluate this function by converting the sum into an integral. The ϕ- and θ-integrations can be carried out by choosing a coordinate system in which the vector $\mathbf{r}-\mathbf{r}_0$ points along the z-axis. We then carry out the integration over $|\mathbf{k}|$ by evaluating the density of states and matrix elements at resonance. We are left with the integral

$$\int_{-\infty}^{\infty} d\nu_k \frac{e^{-i\nu_k t + i\nu_k \Delta r/c}}{(\nu_k - \omega) + i\Gamma/2},$$

which is evaluated via contour methods and where $\Delta r = |\mathbf{r} - \mathbf{r}_0|$ is the distance from the atom located at position $\mathbf{r}_0$ to the detector. For $t < \Delta r/c$, the contour lies in the upper half-plane and if $t > \Delta r/c$, in the lower half-plane. On performing the integration, we find

$$\langle 0|\hat{E}(\mathbf{r}, t)|\psi_\gamma\rangle = \frac{\mathcal{E}}{\Delta r} \Theta\left(t - \frac{\Delta r}{c}\right) e^{-i(t-\frac{\Delta r}{c})(\omega - i\Gamma/2)} \,, \tag{2.29}$$

where Θ is a unit step function and $\mathcal{E}$ is an overall constant with the units of electric field.

Next we consider the problem of "interrupted" emission, see Fig. 2.3. The first photon, associated with the $a \leftrightarrow b$ transition, is described in the long time limit by our "old friend"

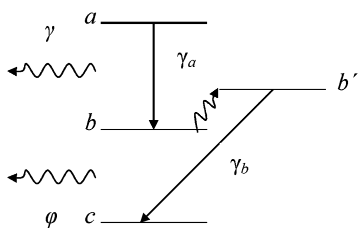


Fig. 2.3 Figure illustrating decay of atom excited to state a at a rate γ_a to non-decaying level b. Upon detection of $a \to b$ photon, population in level b is transferred to b' by means of an external field indicated by *wavy line*. Level b' decays to c at rate γ_b

$$|\gamma\rangle = \sum_{\mathbf{k}} \frac{g_{a,\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}}}{(\omega_{ab} - c|\mathbf{k}|) - i\gamma_a} |1_{\mathbf{k}}\rangle \ . \tag{2.30}$$

Likewise the second photon, associated with the $b' \to c$ transition, is given in the long time limit by

$$|\phi\rangle = \sum_{\mathbf{q}} \frac{g_{b,\mathbf{q}} e^{-i(\mathbf{q}\cdot\mathbf{r} - cqt_0)}}{(\omega_{ac} - c|\mathbf{q}|) - i\gamma_b} |1_{\mathbf{q}}\rangle \ , \tag{2.31}$$

where t_0 is the time of detection of γ photon and the transfer from $b \to b'$.

Using (2.30) and (2.31), it is easy to calculate the two photon wave function $\Psi^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2)$ as defined by (2.27). We find

$$\psi^{(2)}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \psi_\gamma(\mathbf{r}_1, t_1)\psi_\phi(\mathbf{r}_2, t_2) + \psi_\phi(\mathbf{r}_1, t_1)\psi_\gamma(\mathbf{r}_2, t_2) \ , \tag{2.32}$$

where

$$\psi_\gamma(\mathbf{r}_i, t_i) = \frac{\varepsilon_\gamma}{\Delta r_i} \Theta\left(t_i - \frac{\Delta r_i}{c}\right) e^{-\gamma_a (t_i - \frac{\Delta r_i}{c})} e^{-i\omega_{ab}(t_i - \frac{\Delta r_i}{c})}, \tag{2.33}$$

and

$$\psi_\phi(\mathbf{r}_i, t_i) = \frac{\varepsilon_\phi}{\Delta r_i} \Theta\left(t_i - t_0 - \frac{\Delta r_i}{c}\right) e^{-\gamma_a (t_i - t_0 - \frac{\Delta r_i}{c})} e^{-i\omega_{bc}(t_i - t_0 - \frac{\Delta r_i}{c})}, \tag{2.34}$$

where $i = 1, 2$ designates the detector positions.

2.5 Conclusions

One motive for this chapter is to show that the time appearing in the classical Maxwell equations is the same as the time parameter which appears in the TDSE. Thus, the times appearing in classical mechanics and electrodynamics and quantum mechanics are all the same.

Another motivation involves the definition of the photon wave function in terms of the electric and magnetic operators as

$$\Psi_{\mathcal{E}}(\mathbf{r}, t) = \langle 0|\hat{E}(\mathbf{r})|\Psi(t)\rangle, \tag{2.35}$$

and

$$\Psi_{\mathcal{H}}(\mathbf{r}, t) = \langle 0|\hat{H}(\mathbf{r})|\Psi(t)\rangle. \tag{2.36}$$

Equations (2.6) and (2.11) are the analog of the matter wave probability amplitudes

$$\Psi(\mathbf{r}, t) = \langle 0|\hat{\psi}(\mathbf{r})|\Psi(t)\rangle \tag{2.37}$$

discussed at length in Sect. 2.1.

As explained in [3], the discussion of the proceeding paragraph serves to put the nice question of Kramers [9] in perspective. Specifically, Kramers asks,

> When in 1924 De Broglie suggested that material particles should show wave phenomena ... such a comparison was of great heuristic importance. Now that wave mechanics has become a consistent formalism one could ask whether it is possible to consider the Maxwell equations to be a kind of Schrödinger equation of light particles ...?

Kramers answers his question in the negative, he says,

> Thus it is natural to ask what are the ϕ's for photons? Strictly speaking there are no such wave functions! One may not speak of particles in a radiation field in the same sense as in the elementary quantum mechanics of systems of particles as used in the last chapter. The reason is that the wave equation ... solutions of Schrödinger's time dependent wave function corresponding to an energy E_λ have a circular frequency $\omega_\lambda = +E_\lambda/\hbar$, while the monochromatic solutions of the wave equation have both $\pm\omega_\lambda$.

In other words, Kramers is saying that "the *real* electric wave has both $\exp(-\mathrm{i}\nu_k t)$ and $\exp(\mathrm{i}\nu_k t)$ parts while the matter wave has only $\exp(-\mathrm{i}\nu_p t)$ type terms."

However, from the quantum optical perspective, we see that the photon wave functions (2.35) and (2.36) and the matter wave function (2.37) are identical in spirit. An earlier discussion of the importance of the analytical (positive frequency) signal in this context was given by Sudarshan [10].

The present measurement theory, "a-photon-is-what-a-photodetector-detects" point-of-view is discussed further in [3]. We have also included in Sect. 2.4 a detailed photon–photon correlation analysis [8] for the convenience of the reader.

Acknowledgments I would like to thank R. Arnowitt, C. Summerfield, and S. Weinberg for useful and stimulating discussions. This work was supported by the Robert A. Welch Foundation grant number A-126 and the ONR award number N00014-07-1-1084.

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http://www.springer.com/978-3-642-03173-1

Time in Quantum Mechanics - Vol. 2
Muga, G.; Ruschhaupt, A.; del Campo, A. (Eds.)
2009, X, 423 p., Hardcover
ISBN: 978-3-642-03173-1