Chapter 3
The Method of Small-Volume Expansions for Medical Imaging

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3.1 Introduction

Inverse problems in medical imaging are in their most general form ill-posed [47]. They literally have no solution. If, however, in advance we have additional structural information or supply missing information, then we may be able to determine specific features about what we wish to image with a satisfactory resolution and accuracy. One such type of information can be that the imaging problem is to find unknown small anomalies with significantly different parameters from those of the surrounding medium. These anomalies may represent potential tumors at early stage.

Over the last few years, the method of small-volume expansions has been developed for the imaging of such anomalies. The aim of this chapter is to provide a synthetic exposition of the method, a technique that has proven useful in dealing with many medical imaging problems. The method relies on deriving asymptotics. Such asymptotics have been investigated in the case of the conduction equation, the elasticity equation, the Helmholtz equation, the Maxwell system, the wave equation, the heat equation, and the Stokes system. A remarkable feature of this method is that it allows a stable and accurate reconstruction of the location and of some geometric features of the anomalies, even with moderately noisy data.

In this chapter we first provide asymptotic expansions for internal and boundary perturbations due to the presence of small anomalies. We then apply the asymptotic formulas for the purpose of identifying the location and certain properties of the shape of the anomalies. We shall restrict ourselves to conductivity and elasticity imaging and single out simple fundamental

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algorithms. We should emphasize that, since biological tissues are nearly incompressible, the model problem in elasticity imaging we shall deal with is the Stokes system rather than the Lamé system. The method of small-volume expansions also applies to the optical tomography and microwave imaging. However, these techniques are not discussed here. We refer the interested reader to, for instance, [2].

Applications of the method of small-volume expansions in medical imaging are described in this chapter. In particular, the use of the method of small-volume expansions to improve a multitude of emerging imaging techniques is highlighted. These imaging modalities include electrical impedance tomography (EIT), magnetic resonance elastography (MRE), impediography, magneto-acoustic imaging, infrared thermography, and acoustic radiation force imaging.

EIT uses low-frequency electrical current to probe a body; the method is sensitive to changes in electrical conductivity. By injecting known amounts of current and measuring the resulting electrical potential field at points on the boundary of the body, it is possible to "invert" such data to determine the conductivity or resistivity of the region of the body probed by the currents. This method can also be used in principle to image changes in dielectric constant at higher frequencies, which is why the method is often called "impedance" tomography rather than "conductivity" or "resistivity" tomography. However, the aspect of the method that is most fully developed to date is the imaging of conductivity/resistivity. Potential applications of electrical impedance tomography include determination of cardiac output, monitoring for pulmonary edema, and in particular screening for breast cancer. See, for instance, [35–37, 41, 44–46].

Recently, a commercial system called TransScan TS2000 (TransScan Medical, Ltd, Migdal Ha'Emek, Israel) has been released for adjunctive clinical uses with X-ray mammography in the diagnostic of breast cancer. The mathematical model of the TransScan can be viewed as a realistic or practical version of the general electrical impedance system. In the TransScan, a patient holds a metallic cylindrical reference electrode, through which a constant voltage of 1–2.5 V, with frequencies spanning 100 Hz–100 KHz, is applied. A scanning probe with a planar array of electrodes, kept at ground potential, is placed on the breast. The voltage difference between the hand and the probe induces a current flow through the breast, from which information about the impedance distribution in the breast can be extracted. See [25]. The method of small-volume expansions provides a rigorous mathematical framework for the TransScan. See Chap. 1 for a detailed study of this EIT system.

Since all the present EIT technologies are only practically applicable in feature extraction of anomalies, improving EIT calls for innovative measurement techniques that incorporate structural information. A very promising direction of research is the recent magnetic resonance imaging technique, called current density imaging, which measures the internal current density distribution. See the breakthrough work by Seo and his group described in Chap. 1. See also [52, 53]. However, this technique has a number of disadvantages, among which the lack of portability and a potentially long imaging time. Moreover, it uses an expensive magnetic resonance imaging scanner.

Impediography is another mathematical direction for future EIT research in view of biomedical applications. It keeps the most important merits of EIT (real time imaging, low cost, portability). It is based on the simultaneous measurement of an electric current and of acoustic vibrations induced by ultrasound waves. Its intrinsic resolution depends on the size of the focal spot of the acoustic perturbation, and thus it may provide high resolution images.

In magneto-acoustic imaging, an acoustic wave is applied to a biological tissue placed in a magnetic field. The probe signal produces by the Lorentz force an electric current that is a function of the local electrical conductivity of the biological tissue [59]. We provide the mathematical basis for this magneto-acoustic imaging approach and propose a new algorithm for solving the inverse problem which is quite similar to the one we design for impediography.

Extensive work has been carried out in the past decade to image, by inducing motion, the elastic properties of human soft tissues. This wide application field, called elasticity imaging or elastography, is based on the initial idea that shear elasticity can be correlated with the pathological state of tissues. Several imaging modalities can be used to estimate the resulting tissue displacements.

Magnetic resonance elastography is a recently developed technique that can directly visualize and quantitatively measure the displacement field in tissues subject to harmonic mechanical excitation at low-frequencies. A phase-contrast magnetic resonance imaging technique is used to spatially map and measure the complete three-dimensional displacement patterns. From this data, local quantitative values of shear modulus can be calculated and images that depict tissue elasticity or stiffness can be generated. The inverse problem for magnetic resonance elastography is to determine the shape and the elastic parameters of an elastic anomaly from internal measurements of the displacement field. In most cases the most significant elastic parameter is the stiffness coefficient. See, for instance, [42, 58, 60, 64, 65].

Another interesting approach to assessing elasticity is to use the acoustic radiation force of an ultrasonic focused beam to remotely generate mechanical vibrations in organs. The acoustic force is due to the momentum transfer from the acoustic wave to the medium. This technique is particularly suited for in vivo applications as it allows in depth vibrations of tissues exactly at the desired location. The radiation force acts as a dipolar source at the pushing ultrasonic beam focus. A spatio-temporal sequence of the propagation of the induced transient wave can be acquired, leading to a quantitative estimation of the viscoelastic parameters of the studied medium in a source-free region.

Infrared thermal imaging is becoming a common screening modality in the area of breast cancer. By carefully examining aspects of temperature and blood vessels of the breasts in thermal images, signs of possible cancer or pre-cancerous cell growth may be detected up to 10 years prior to being discovered using any other procedure. This provides the earliest detection of cancer possible. Because of thermal imaging's extreme sensitivity, these temperature variations and vascular changes may be among the earliest signs of breast cancer and/or a pre-cancerous state of the breast. An abnormal infrared image of the breast is an important marker of high risk for developing breast cancer.

3.2 Conductivity Problem

In this section we provide an asymptotic expansion of the voltage potentials in the presence of a diametrically small anomaly with conductivity different from the background conductivity.

Let Ω be a smooth bounded domain in $\mathbb{R}^d, d \geq 2$ and let ν_x denote the outward unit normal to $\partial\Omega$ at x. Define $N(x,z)$ to be the Neumann function for $-\Delta$ in Ω corresponding to a Dirac mass at z. That is, N is the solution to

$$\begin{cases} -\Delta_x N(x,z) = \delta_z \quad \text{in } \Omega, \\ \displaystyle \frac{\partial N}{\partial \nu_x}\Big|_{\partial\Omega} = -\frac{1}{|\partial\Omega|}, \int_{\partial\Omega} N(x,z)\, d\sigma(x) = 0 \quad \text{for } z \in \Omega. \end{cases} \tag{3.1}$$

Note that the Neumann function $N(x,z)$ is defined as a function of $x \in \overline{\Omega}$ for each fixed $z \in \Omega$.

Let B be a smooth bounded domain in $\mathbb{R}^d$, $0 < k \neq 1 < +\infty$, and let $\hat{v} = \hat{v}(B,k)$ be the solution to

$$\begin{cases} \Delta \hat{v} = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B}, \\ \Delta \hat{v} = 0 \quad \text{in } B, \\ \hat{v}|_- - \hat{v}|_+ = 0 \quad \text{on } \partial B, \\ \displaystyle k \frac{\partial \hat{v}}{\partial \nu}\Big|_- - \frac{\partial \hat{v}}{\partial \nu}\Big|_+ = 0 \quad \text{on } \partial B, \\ \hat{v}(\xi) - \xi \to 0 \quad \text{as } |\xi| \to +\infty. \end{cases} \tag{3.2}$$

Here we denote

$$v|_\pm(\xi) := \lim_{t \to 0^+} v(\xi \pm t\nu_\xi), \quad \xi \in \partial B,$$

and

$$\frac{\partial v}{\partial \nu_\xi}\Big|_\pm (\xi) := \lim_{t \to 0^+} \langle \nabla v(\xi \pm t\nu_\xi), \nu_\xi \rangle \ , \quad \xi \in \partial B,$$

if the limits exist, where ν_ξ is the outward unit normal to ∂B at ξ, and $\langle , \rangle$ is the scalar product in $\mathbb{R}^d$. For ease of notation we will sometimes use the dot for the scalar product in $\mathbb{R}^d$.

Let D denote a smooth anomaly inside Ω with conductivity $0 < k \neq 1 < +\infty$. The voltage potential in the presence of the set D of conductivity anomalies is denoted by u. It is the solution to the conductivity problem

$$\begin{cases} \nabla \cdot \Big(\chi(\Omega \setminus \overline{D}) + k\chi(D) \Big) \nabla u = 0 \quad \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu}\Big|_{\partial \Omega} = g \quad \Big(g \in L^2(\partial\Omega), \displaystyle\int_{\partial\Omega} g \, d\sigma = 0 \Big), \\ \displaystyle\int_{\partial\Omega} u \, d\sigma = 0 \, , \end{cases} \tag{3.3}$$

where $\chi(D)$ is the characteristic function of D.

The background voltage potential U satisfies

$$\begin{cases} \Delta U = 0 \quad \text{in } \Omega \, , \\ \dfrac{\partial U}{\partial \nu}\Big|_{\partial \Omega} = g \, , \\ \displaystyle\int_{\partial\Omega} U \, d\sigma = 0 \, . \end{cases} \tag{3.4}$$

The following theorem gives asymptotic formulas for both boundary and internal perturbations of the voltage potential that are due to the presence of a conductivity anomaly.

Theorem 3.1 (Voltage perturbations). *Suppose that $D = \delta B + z$, δ being the characteristic size of D, and let u be the solution of (3.3), where $0 < k \neq 1 < +\infty$.*

(i) *The following asymptotic expansion of the voltage potential on $\partial\Omega$ holds for $d = 2, 3$:*

$$u(x) \approx U(x) - \delta^d \nabla U(z) M(k, B) \partial_z N(x, z). \tag{3.5}$$

Here U is the background solution, that is, the solution to (3.4), $N(x, z)$ is the Neumann function, that is, the solution to (3.1), and $M(k, B) = (m_{pq})_{p,q=1}^d$ is the polarization tensor (PT) given by

$$M(k, B) := (k - 1) \int_B \nabla \hat{v}(\xi) \, d\xi, \tag{3.6}$$

where $\hat{v}$ is the solution to (3.3).

(ii) *Let w be a smooth harmonic function in Ω. The weighted boundary measurements I_w satisfies*

$$I_w := \int_{\partial\Omega} (u - U)(x) \frac{\partial w}{\partial \nu}(x) \, d\sigma(x) \approx -\delta^d \nabla U(z) \cdot M(k, B) \nabla w(z). \tag{3.7}$$

(iii) *The following inner asymptotic formula holds:*

$$u(x) \approx U(z) + \delta\hat{v}(\frac{x-z}{\delta}) \cdot \nabla U(z) \quad \text{for } x \text{ near } z\,. \tag{3.8}$$

The inner asymptotic expansion (3.8) uniquely characterizes the shape and the conductivity of the anomaly. In fact, suppose for two Lipschitz domains B and B' and two conductivities k and k' that $\hat{v}(B,k) = \hat{v}(B',k')$ in a domain englobing B and B' then using the jump conditions satisfied by $\hat{v}(B,k)$ and $\hat{v}(B',k')$ we can easily prove that $B = B'$ and $k = k'$.

The asymptotic expansion (3.5) expresses the fact that the conductivity anomaly can be modeled by a dipole far away from z. It does not hold uniformly in Ω. It shows that, from an imaging point of view, the location z and the polarization tensor M of the anomaly are the only quantities that can be determined from boundary measurements of the voltage potential, assuming that the noise level is of order δ^{d+1}. It is then important to precisely characterize the polarization tensor and derive some of its properties, such as symmetry, positivity, and isoperimetric inequalities satisfied by its elements, in order to develop efficient algorithms for reconstructing conductivity anomalies of small volume.

We list in the next theorem important properties of the PT.

Theorem 3.2 (Properties of the polarization tensor). *For $0 < k \neq 1 < +\infty$, let $M(k,B) = (m_{pq})_{p,q=1}^d$ be the PT associated with the bounded domain B in $\mathbb{R}^d$ and the conductivity k. Then*

(i) *M is symmetric.*
(ii) *If $k > 1$, then M is positive definite, and it is negative definite if $0 < k < 1$.*
(iii) *The following isoperimetric inequalities for the PT*

$$\begin{cases} \dfrac{1}{k-1}\,\mathrm{trace}(M) \leq (d-1+\dfrac{1}{k})|B|, \\ (k-1)\,\mathrm{trace}(M^{-1}) \leq \dfrac{d-1+k}{|B|}, \end{cases} \tag{3.9}$$

hold, where trace *denotes the trace of a matrix.*

The polarization tensor M can be explicitly computed for disks and ellipses in the plane and balls and ellipsoids in three-dimensional space. See [18, pp. 81–89]. The formula of the PT for ellipses will be useful here. Let B be an ellipse whose semi-axes are on the x_1- and x_2-axes and of length a and b, respectively. Then, we recall that $M(k,B)$ takes the form

$$M(k,B) = (k-1)|B| \begin{pmatrix} \dfrac{a+b}{a+kb} & 0 \\ 0 & \dfrac{a+b}{b+ka} \end{pmatrix}, \tag{3.10}$$

where $|B|$ denotes the volume of B.

Formula (3.5) shows that from boundary measurements we can always represent and visualize an arbitrary shaped anomaly by means of an equivalent ellipse of center z with the same polarization tensor. Further, it is impossible to extract the conductivity from the polarization tensor. The information contained in the polarization tensor is a mixture of the conductivity and the volume. A small anomaly with high conductivity and larger anomaly with lower conductivity can have the same polarization tensor.

The bounds (3.9) are known as the Hashin–Shtrikman bounds. By making use of these bounds, a size estimation of B can be obtained.

3.3 Wave Equation

With the notation of Sect. 3.2, consider the initial boundary value problem for the (scalar) wave equation

$$\begin{cases} \partial_t^2 u - \nabla \cdot \Big(\chi(\Omega \setminus \overline{D}) + k\chi(D) \Big) \nabla u = 0 \quad \text{in} \quad \Omega_T, \\ u(x,0) = u_0(x), \quad \partial_t u(x,0) = u_1(x) \quad \text{for} \quad x \in \Omega\,, \\ \dfrac{\partial u}{\partial \nu} = g \quad \text{on} \quad \partial\Omega_T\,, \end{cases} \tag{3.11}$$

where $T < +\infty$ is a final observation time, $\Omega_T = \Omega \times]0,T[$, $\partial\Omega_T = \partial\Omega \times]0,T[$. The initial data $u_0, u_1 \in \mathcal{C}^\infty(\overline{\Omega})$, and the Neumann boundary data $g \in \mathcal{C}^\infty(0,T;\mathcal{C}^\infty(\partial\Omega))$ are subject to compatibility conditions.

Define the background solution U to be the solution of the wave equation in the absence of any anomalies. Thus U satisfies

$$\begin{cases} \partial_t^2 U - \Delta U = 0 \quad \text{in} \quad \Omega_T\,, \\ U(x,0) = u_0(x), \quad \partial_t U(x,0) = u_1(x) \quad \text{for} \quad x \in \Omega\,, \\ \dfrac{\partial U}{\partial \nu} = g \quad \text{on} \quad \partial\Omega_T. \end{cases}$$

The following asymptotic expansion holds as $\delta \to 0$.

Theorem 3.3 (Perturbations of weighted boundary measurements). *Set $\Omega_T = \Omega \times]0,T[$ and $\partial\Omega_T = \partial\Omega \times]0,T[$. Let $w \in C^\infty(\overline{\Omega}_T)$ satisfy $(\partial_t^2 - \Delta)w(x,t) = 0$ in Ω_T with $\partial_t w(x,T) = w(x,T) = 0$ for $x \in \Omega$. Define the weighted boundary measurements*

$$I_w(T) := \int_{\partial\Omega_T} (u-U)(x,t) \frac{\partial w}{\partial \nu}(x,t)\, d\sigma(x)\, dt\,.$$

Then, for any fixed $T > diam(\Omega)$, the following asymptotic expansion for $I_w(T)$ holds as $\delta \to 0$:

$$I_w(T) \approx \delta^d \int_0^T \nabla U(z,t) M(k,B) \nabla w(z,t)\, dt, \tag{3.12}$$

where $M(k,B)$ is defined by (3.6).

Expansion (3.12) is a weighted expansion. Pointwise expansions similar to those in Theorem 3.1 which is for the steady-state model can also be obtained.

Let $y \in \mathbb{R}^3$ be such that $|y-z| \gg \delta$. Choose

$$U(x,t) := U_y(x,t) := \frac{\delta_{t=|x-y|}}{4\pi|x-y|} \quad \text{for } x \neq y\,. \tag{3.13}$$

It is easy to check that U_y is the outgoing Green function to the wave equation:

$$(\partial_t^2 - \Delta) U_y(x,t) = \delta_{x=y}\delta_{t=0} \quad \text{in } \mathbb{R}^3 \times]0, +\infty[\,.$$

Moreover, U_y satisfies the initial conditions: $U_y(x,0) = \partial_t U_y(x,0) = 0$ for $x \neq y$. Consider now for the sake of simplicity the wave equation in the whole three-dimensional space with appropriate initial conditions:

$$\begin{cases} \partial_t^2 u - \nabla \cdot \Big(\chi(\mathbb{R}^3 \setminus \overline{D}) + k\chi(D) \Big) \nabla u = \delta_{x=y}\delta_{t=0} & \text{in} \quad \mathbb{R}^3 \times]0, +\infty[, \\ u(x,0) = 0, \quad \partial_t u(x,0) = 0 \quad \text{for} \quad x \in \mathbb{R}^3, x \neq y\,. \end{cases} \tag{3.14}$$

For $\rho > 0$, define the operator P_ρ on tempered distributions by

$$P_\rho[\psi](t) = \int_{|\omega| \leq \rho} e^{-\sqrt{-1}\omega t} \hat{\psi}(\omega)\, d\omega,$$

where $\hat{\psi}$ denotes the Fourier transform of ψ. Clearly, the operator P_ρ truncates the high-frequency component of ψ.

Theorem 3.4 (Pointwise perturbations). *Let u be the solution to (3.14). Set U_y to be the background solution. Suppose that $\rho = O(\delta^{-\alpha})$ for some $\alpha < \frac{1}{2}$.*

(i) *The following outer expansion holds*

$$P_\rho[u - U_y](x,t) \approx -\delta^3 \int_{\mathbb{R}} \nabla P_\rho[U_z](x, t-\tau) \cdot M(k,B) \nabla P_\rho[U_y](z,\tau)\, d\tau, \tag{3.15}$$

for x away from z, where $M(k,B)$ is defined by (3.6) and U_y and U_z by (3.13).

(ii) *The following inner approximation holds:*

$$P_\rho[u - U_y](x,t) \approx \delta \hat{v}\left(\frac{x-z}{\delta}\right) \cdot \nabla P_\rho[U_y](x,t) \quad \textit{for } x \textit{ near } z\,, \quad (3.16)$$

where $\hat{v}$ *is given by (3.2) and* U_y *by (3.13).*

Formula (3.15) shows that the perturbation due to the anomaly is in the time-domain a wavefront emitted by a dipolar source located at the point z.

Taking the Fourier transform of (3.15) in the time variable gives an expansion of the perturbations resulting from the presence of a small anomaly for solutions to the Helmholtz equation at low frequencies (at wavelengths large compared to the size of the anomaly).

3.4 Heat Equation

Suppose that the background Ω is homogeneous with thermal conductivity 1 and that the anomaly $D = \delta B + z$ has thermal conductivity $0 < k \neq 1 < +\infty$. In this section we consider the following transmission problem for the heat equation:

$$\begin{cases} \partial_t u - \nabla \cdot \Big(\chi(\Omega \setminus \overline{D}) + k\chi(D)\Big)\nabla u = 0 \quad \text{in} \quad \Omega_T, \\ u(x,0) = u_0(x) \quad \text{for} \quad x \in \Omega\,, \\ \dfrac{\partial u}{\partial \nu} = g \quad \text{on} \quad \partial\Omega_T\,, \end{cases} \quad (3.17)$$

where the Neumann boundary data g and the initial data u_0 are subject to a compatibility condition. Let U be the background solution defined as the solution of

$$\begin{cases} \partial_t U - \Delta U = 0 \quad \text{in} \quad \Omega_T\,, \\ U(x,0) = u_0(x) \quad \text{for} \quad x \in \Omega\,, \\ \dfrac{\partial U}{\partial \nu} = g \quad \text{on} \quad \partial\Omega_T. \end{cases}$$

The following asymptotic expansion holds as $\delta \to 0$.

Theorem 3.5 (Perturbations of weighted boundary measurements). *Let* $w \in C^\infty(\overline{\Omega_T})$ *be a solution to the adjoint problem, namely, satisfy* $(\partial_t + \Delta)w(x,t) = 0$ *in* Ω_T *with* $w(x,T) = 0$ *for* $x \in \Omega$. *Define the weighted boundary measurements*

$$I_w(T) := \int_{\partial\Omega_T} (u - U)(x,t) \frac{\partial w}{\partial \nu}(x,t)\, d\sigma(x)\, dt\,.$$

Then, for any fixed $T > 0$, the following asymptotic expansion for $I_w(T)$ holds as $\delta \to 0$:

$$I_w(T) \approx -\delta^d \int_0^T \nabla U(z,t) \cdot M(k,B) \nabla w(z,t)\, dt, \tag{3.18}$$

where $M(k,B)$ is defined by (3.6).

Note that (3.18) holds for any fixed positive final time T while (3.12) holds only for $T > \operatorname{diam}(\Omega)$. This difference comes from the finite speed propagation property for the wave equation compared to the infinite one for the heat equation.

Consider now the background solution to be the Green function of the heat equation at y:

$$U(x,t) := U_y(x,t) := \begin{cases} \dfrac{e^{-\frac{|x-y|^2}{4t}}}{(4\pi t)^{d/2}} & \text{for } t > 0, \\ 0 \quad \text{for } t < 0 \,. \end{cases} \tag{3.19}$$

Let u be the solution to the following heat equation with an appropriate initial condition:

$$\begin{cases} \partial_t u - \nabla \cdot \Big(\chi(\mathbb{R}^d \setminus \overline{D}) + k\chi(D) \Big) \nabla u = 0 \quad \text{in} \quad \mathbb{R}^d \times]0,+\infty[, \\ u(x,0) = U_y(x,0) \quad \text{for} \quad x \in \mathbb{R}^d. \end{cases} \tag{3.20}$$

Proceeding as in the derivation of (3.15), we can prove that $\delta u(x,t) := u - U$ is approximated by

$$-(k-1) \int_0^t \frac{1}{(4\pi(t-\tau))^{d/2}} \int_{\partial D} e^{-\frac{|x-x'|^2}{4(t-\tau)}} \frac{\partial \hat{v}}{\partial \nu}\Big|_- (\frac{x'-z}{\delta}) \cdot \nabla U_y(x',\tau)\, d\sigma(x')\, d\tau, \tag{3.21}$$

for x near z. Therefore, analogously to Theorem 3.4, the following pointwise expansion follows from the approximation (3.21).

Theorem 3.6 (Pointwise perturbations). *Let $y \in \mathbb{R}^3$ be such that $|y - z| \gg \delta$. Let u be the solution to (3.20). The following expansion holds*

$$(u-U)(x,t) \approx -\delta^d \int_0^t \nabla U_z(x,t-\tau) M(k,B) \nabla U_y(z,\tau)\, d\tau \text{ for } |x-z| \gg O(\delta)\ , \tag{3.22}$$

where $M(k,B)$ is defined by (3.6) and U_y and U_z by (3.19).

When comparing (3.22) and (3.15), we shall point out that for the heat equation the perturbation due to the anomaly is accumulated over time.

An asymptotic formalism for the realistic half space model for thermal imaging well suited for the design of anomaly reconstruction algorithms has been developed in [22].

3.5 Modified Stokes System

Consider the modified Stokes system, i.e., the problem of determining $\mathbf{v}$ and q in a domain Ω from the conditions:

$$\begin{cases} (\Delta + \kappa^2)\mathbf{v} - \nabla q = 0, \\ \nabla \cdot \mathbf{v} = 0, \\ \mathbf{v}|_{\partial \Omega} = \mathbf{g} \,. \end{cases} \tag{3.23}$$

The problem (3.23) governs elastic wave propagation in nearly-incompressible media. In biological media, the compression modulus is 4–6 orders higher than the shear modulus. We can prove that the Lamé system converges to (3.23) as the compression modulus goes to $+\infty$.

Let $(G_{il})_{i,l=1}^d$ be the Dirichlet Green's function for the operator in (3.23), i.e., for $y \in \Omega$,

$$\begin{cases} (\Delta_x + \kappa^2) G_{il}(x,y) - \dfrac{\partial F_i(x-y)}{\partial x_l} = \delta_{il}\delta_y(x) \quad \text{in } \Omega \,, \\ \displaystyle\sum_{l=1}^d \frac{\partial}{\partial x_l} G_{il}(x,y) = 0 \quad \text{in } \Omega, \\ G_{il}(x,y) = 0 \quad \text{on } \partial\Omega \,. \end{cases} \tag{3.24}$$

Denote by $(\mathbf{e}_1, \ldots, \mathbf{e}_d)$ an orthonormal basis of $\mathbb{R}^d$. Let $d(\xi) := (1/d) \sum_k \xi_k \mathbf{e}_k$ and $\hat{\mathbf{v}}_{pq}$, for $p, q = 1, \ldots, d$, be the solution to

$$\begin{cases} \mu \Delta \hat{\mathbf{v}}_{pq} + \nabla \hat{p} = 0 \quad \text{in } \mathbb{R}^d \setminus \overline{B}, \\ \widetilde{\mu} \Delta \hat{\mathbf{v}}_{pq} + \nabla \hat{p} = 0 \quad \text{in } B, \\ \hat{\mathbf{v}}_{pq}|_- - \hat{\mathbf{v}}_{pq}|_+ = 0 \quad \text{on } \partial B, \\ (\hat{p}\mathbf{N} + \widetilde{\mu} \dfrac{\partial \hat{\mathbf{v}}_{pq}}{\partial \mathbf{N}})|_- - (\hat{p}\mathbf{N} + \mu \dfrac{\partial \hat{\mathbf{v}}_{pq}}{\partial \mathbf{N}})|_+ = 0 \quad \text{on } \partial B, \\ \nabla \cdot \hat{\mathbf{v}}_{pq} = 0 \quad \text{in } \mathbb{R}^d, \\ \hat{\mathbf{v}}_{pq}(\xi) \to \xi_p \mathbf{e}_q - \delta_{pq} d(\xi) \quad \text{as } |\xi| \to \infty, \\ \hat{p}(\xi) \to 0 \quad \text{as } |\xi| \to +\infty \,. \end{cases} \tag{3.25}$$

Here $\partial \mathbf{v}/\partial \mathbf{N} = (\nabla \mathbf{v} + (\nabla \mathbf{v})^T) \cdot \mathbf{N}$ and T denotes the transpose.

Define the viscous moment tensor (VMT) $(V_{ijpq})_{i,j,p,q=1,\ldots,d}$ by

$$V_{ijpq} := (\widetilde{\mu} - \mu) \int_B \nabla \hat{\mathbf{v}}_{pq} \cdot (\nabla(\xi_i \mathbf{e}_j) + \nabla(\xi_i \mathbf{e}_j)^T) \, d\xi \, . \tag{3.26}$$

Consider an elastic anomaly D inside a nearly-compressible medium Ω. The anomaly D has a shear modulus $\widetilde{\mu}$ different from that of Ω, μ. The displacement field $\mathbf{u}$ solves the following transmission problem for the modified Stokes problem:

$$\begin{cases} (\mu\Delta + \omega^2)\mathbf{u} + \nabla p = 0 \quad \text{in } \Omega \setminus \overline{D}, \\ (\widetilde{\mu}\Delta + \omega^2)\mathbf{u} + \nabla p = 0 \quad \text{in } D, \\ \mathbf{u}\big|_- = \mathbf{u}\big|_+ \quad \text{on } \partial D \, , \\ (p|_+ - p|_-)\mathbf{N} + \mu \dfrac{\partial \mathbf{u}}{\partial \mathbf{N}}\Big|_+ - \widetilde{\mu} \dfrac{\partial \mathbf{u}}{\partial \mathbf{N}}\Big|_- = 0 \quad \text{on } \partial D, \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \, , \\ \mathbf{u} = \mathbf{g} \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega p = 0 \, , \end{cases} \tag{3.27}$$

where $\mathbf{g} \in L^2(\partial\Omega)$ satisfies the compatibility condition $\displaystyle\int_{\partial\Omega} \mathbf{g} \cdot \mathbf{N} = 0$.

Let $(\mathbf{U}, q)$ denote the background solution to the modified Stokes system in the absence of any anomalies, that is, the solution to

$$\begin{cases} (\mu\Delta + \omega^2)\mathbf{U} + \nabla q = 0 \quad \text{in } \Omega, \\ \nabla \cdot \mathbf{U} = 0 \quad \text{in } \Omega \, , \\ \mathbf{U} = \mathbf{g} \quad \text{on } \partial\Omega, \\ \displaystyle\int_\Omega q = 0 \, . \end{cases} \tag{3.28}$$

The following asymptotic expansions hold.

Theorem 3.7 (Expansions of the displacement field). *Suppose that* $D = \delta B + z$, *and let* u *be the solution of* (3.27), *where* $0 < \widetilde{\mu} \neq \mu < +\infty$.

(i) *The following inner expansion holds:*

$$\mathbf{u}(x) \approx \mathbf{U}(z) + \delta \sum_{p,q=1}^{d} \partial_q \mathbf{U}(z)_p \hat{\mathbf{v}}_{pq}\left(\frac{x-z}{\delta}\right) \quad \textit{for } x \textit{ near } z \, , \tag{3.29}$$

where $\hat{\mathbf{v}}_{pq}$ *is defined by (3.25)*

(ii) *Let* (V_{ijpq}) *be the VMT defined by (3.26). The following outer expansion holds uniformly for* $x \in \partial\Omega$*:*

$$(\mathbf{u}-\mathbf{U})(x) \approx \delta^d \Bigg[\sum_{i,j,p,q,\ell=1}^{d} \mathbf{e}_\ell \partial_j G_{\ell i}(x,z) \partial_q \mathbf{U}(z)_p V_{ijpq} \Bigg] , \qquad (3.30)$$

where V_{ijpq} *is given by (3.26), and the Green function* $(G_{il})_{i,l=1}^d$ *is defined by (3.24) with* $\kappa^2 = \omega^2/\mu$*,* μ *being the shear modulus of the background medium.*

The notion of a viscous moment tensor extends the notion of a polarization tensor to quasi-incompressible elasticity. The VMT V characterizes all the information about the elastic anomaly that can be learned from the leading-order term of the outer expansion (3.30). It can be explicitly computed for disks and ellipses in the plane and balls and ellipsoids in three-dimensional space. If B is a two dimensional disk, then

$$V = 4\,|B|\,\mu \frac{(\tilde{\mu}-\mu)}{\tilde{\mu}+\mu} P \ ,$$

where $P = (P_{ijpq})$ is the orthogonal projection from the space of symmetric matrices onto the space of symmetric matrices of trace zero, i.e.,

$$P_{ijpq} = \frac{1}{2}(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) - \frac{1}{d}\delta_{ij}\delta_{pq} \ .$$

If B is an ellipse of the form

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1, \quad a \geq b > 0 \ , \qquad (3.31)$$

then the VMT for B is given by

$$\begin{cases} V_{1111} = V_{2222} = -V_{1122} = -V_{2211} = |B| \dfrac{2\mu(\tilde{\mu}-\mu)}{\mu+\tilde{\mu}-(\tilde{\mu}-\mu)m^2}, \\ V_{1212} = V_{2112} = V_{1221} = V_{2121} = |B| \dfrac{2\mu(\tilde{\mu}-\mu)}{\mu+\tilde{\mu}+(\tilde{\mu}-\mu)m^2}, \\ \text{the remaining terms are zero,} \end{cases} \qquad (3.32)$$

where $m = (a-b)/(a+b)$.

If B is a ball in three dimensions, the VMT associated with B and an arbitrary $\widetilde{\mu}$ is given by

$$\begin{cases} V_{iiii} = \dfrac{20\mu|B|}{3}\dfrac{\widetilde{\mu}-\mu}{2\widetilde{\mu}+3\mu}\,, \quad V_{iijj} = -\dfrac{10\mu|B|}{3}\dfrac{\widetilde{\mu}-\mu}{2\widetilde{\mu}+3\mu}\ (i \neq j)\,, \\ V_{ijij} = V_{ijji} = 5\mu|B|\dfrac{\widetilde{\mu}-\mu}{2\widetilde{\mu}+3\mu},\ (i \neq j)\,, \\ \text{the remaining terms are zero.} \end{cases} \tag{3.33}$$

Theorem 3.8 (Properties of the viscous moment tensor). *For $0 < \widetilde{\mu} \neq \mu < +\infty$, let $V = (V_{ijpq})_{i,p,q=1}^d$ be the VMT associated with the bounded domain B in $\mathbb{R}^d$ and the pair of shear modulus $(\widetilde{\mu}, \mu)$. Then*

(i) *For $i, j, p, q = 1, \ldots, d$,*

$$V_{ijpq} = V_{jipq}, \quad V_{ijpq} = V_{ijqp}, \quad V_{ijpq} = V_{pqij}\ . \tag{3.34}$$

(ii) *We have*

$$\sum_p V_{ijpp} = 0 \quad \textit{for all } i,j \quad \textit{and} \quad \sum_i V_{iipq} = 0 \quad \textit{for all } p,q\,,$$

or equivalently, $V = PVP$.

(iii) *The tensor V is positive (negative, resp.) definite on the space of symmetric matrices of trace zero if $\widetilde{\mu} > \mu$ ($\widetilde{\mu} < \mu$, resp.).*

(iv) *The tensor $(1/(2\mu))\,V$ satisfies the following bounds*

$$\mathrm{Tr}\Big(\frac{1}{2\mu}V\Big) \leq |B|\Big(\frac{\tilde{\mu}}{\mu}-1\Big)\Big((d-1)\frac{\mu}{\widetilde{\mu}} + \frac{d(d-1)}{2}\Big)\,, \tag{3.35}$$

$$\mathrm{Tr}\Big(\frac{1}{2\mu}V\Big)^{-1} \leq \frac{1}{|B|(\frac{\tilde{\mu}}{\mu}-1)}\Big((d-1)\frac{\widetilde{\mu}}{\mu} + \frac{d(d-1)}{2}\Big)\,, \tag{3.36}$$

where for $C = (C_{ijpq})$, $\mathrm{Tr}(C) := \sum_{i,j=1}^d C_{ijij}$.

Note that the VMT V is a 4-tensor and can be regarded, because of its symmetry, as a linear transformation on the space of symmetric matrices. Note also that, in view of Theorem 3.2, the right-hand sides of (3.35) and (3.36) are exactly in the two-dimensional case ($d = 2$) the Hashin–Shtrikman bounds (3.9) for the PT associated with the same domain B and the conductivity contrast $k = \widetilde{\mu}/\mu$.

3.6 Electrical Impedance Imaging

In this section we apply the asymptotic formula (3.5) for the purpose of identifying the location and certain properties of the shape of the conductivity anomalies. We single out two simple fundamental algorithms that take advantage of the smallness of the anomalies: projection-type algorithms and multiple signal classification (MUSIC)-type algorithms. These algorithms are fast, stable, and efficient.

We refer to Chap. 1 for basic mathematical and physical concepts of electrical impedance tomography.

3.6.1 Detection of a Single Anomaly: A Projection-Type Algorithm

We briefly discuss a simple algorithm for detecting a single anomaly. We refer to Chap. 1 for further details. The projection-type location search algorithm makes use of constant current sources. We want to apply a special type of current that makes ∂U constant in D. Injection current $g = a \cdot \nu$ for a fixed unit vector $a \in \mathbb{R}^d$ yields $\nabla U = a$ in Ω.

Assume for the sake of simplicity that $d = 2$ and D is a disk. Set

$$w(y) = -(1/2\pi) \log |x - y| \quad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}, y \in \Omega \, .$$

Since w is harmonic in Ω, then from (3.10) and (3.7), it follows that

$$I_w[a] \approx \frac{(k-1)|D|}{\pi(k+1)} \frac{(x-z) \cdot a}{|x-z|^2} \, , \quad x \in \mathbb{R}^2 \setminus \overline{\Omega} \, . \tag{3.37}$$

The first step for the reconstruction procedure is to locate the anomaly. The location search algorithm is as follows. Take two observation lines Σ_1 and Σ_2 contained in $\mathbb{R}^2 \setminus \overline{\Omega}$ given by

$$\Sigma_1 := \text{ a line parallel to } a,$$

$$\Sigma_2 := \text{ a line normal to } a \, .$$

Find two points $P_i \in \Sigma_i, i = 1, 2$, so that

$$I_w[a](P_1) = 0, \quad I_w[a](P_2) = \max_{x \in \Sigma_2} |I_w[a](x)| \, .$$

From (3.37), we can see that the intersecting point P of the two lines

$$\Pi_1(P_1) := \{x \mid a \cdot (x - P_1) = 0\}, \tag{3.38}$$

$$\Pi_2(P_2) := \{x \mid (x - P_2) \text{ is parallel to } a\} \tag{3.39}$$

is close to the center z of the anomaly D: $|P - z| = O(\delta^2)$.

Once we locate the anomaly, the factor $|D|(k-1)/(k+1)$ can be estimated. As we said before, this information is a mixture of the conductivity and the volume.

3.6.2 Detection of Multiple Anomalies: A MUSIC-Type Algorithm

Consider m anomalies $D_s = \delta B_s + z_s$, $s = 1, \ldots, m$. Suppose for the sake of simplicity that all the domains B_s are disks. Let $y_l \in \mathbb{R}^2 \setminus \Omega$ for $l = 1, \ldots, n$ denote the source points. Set

$$U_{y_l} = w_{y_l} := -(1/2\pi) \log |x - y_l| \quad \text{for } x \in \Omega, \quad l = 1, \ldots, n \, .$$

The MUSIC-type location search algorithm for detecting multiple anomalies is as follows. For $n \in \mathbb{N}$ sufficiently large, define the matrix $A = [A_{ll'}]_{l,l'=1}^n$ by

$$A_{ll'} = I_{w_{y_l}}[y_{l'}] := \int_{\partial\Omega} (u - U_{y_{l'}})(x) \frac{\partial w_{y_l}}{\partial \nu}(x) \, d\sigma(x) \, .$$

Expansion (3.7) yields

$$A_{ll'} \approx -\delta^d \sum_{s=1}^m \frac{2(k_s - 1)|B_s|}{k_s + 1} \nabla U_{y_{l'}}(z_s) \nabla U_{y_l}(z_s) \, .$$

Introduce

$$g_z = \Big(U_{y_1}(z), \ldots, U_{y_n}(z) \Big)^* ,$$

where v^* denotes the transpose of the vector v.

Lemma 3.9. *Suppose that $n > m$. The following characterization of the location of the anomalies in terms of the range of the matrix A holds:*

$$g_z \in \text{Range}(A) \textit{ iff } z \in \{z_1, \ldots, z_m\} \, . \tag{3.40}$$

The MUSIC-type algorithm to determine the location of the anomalies is as follows. Let $P_{\text{noise}} = I - P$, where P is the orthogonal projection onto the range of A. Given any point $z \in \Omega$, form the vector g_z. The point z coincides with the location of an anomaly if and only if $P_{\text{noise}} g_z = 0$. Thus we can form an image of the anomalies by plotting, at each point z, the cost function $1/||P_{\text{noise}} g_z||$. The resulting plot will have large peaks at the locations of the anomalies.

Once we locate the anomalies, the factors $|D_s|(k_s - 1)/(k_s + 1)$ can be estimated from the significant singular values of A.

3.7 Impediography

The core idea of impediography is to couple electric measurements to localized elastic perturbations. A body (a domain $\Omega \subset \mathbb{R}^2$) is electrically probed: One or several currents are imposed on the surface and the induced potentials are measured on the boundary. At the same time, a circular region of a few millimeters in the interior of Ω is mechanically excited by ultrasonic waves, which dilate this region. The measurements are made as the focus of the ultrasounds scans the entire domain. Several sets of measurements can be obtained by varying amplitudes of the ultrasound waves and the applied currents.

Within each disk of (small) volume, the conductivity is assumed to be constant per volume unit. At a point $x \in \Omega$, within a disk B of volume V_B, the electric conductivity γ is defined in terms of a density ρ as $\gamma(x) = \rho(x) V_B$.

The ultrasonic waves induce a small elastic deformation of the disk B. If this deformation is isotropic, the material points of B occupy a volume V_B^p in the perturbed configuration, which at first order is equal to

$$V_B^p = V_B(1 + 2\frac{\Delta r}{r}) ,$$

where r is the radius of the disk B and Δr is the variation of the radius due to the elastic perturbation. As Δr is proportional to the amplitude of the ultrasonic wave, we obtain a proportional change of the deformation. Using two different ultrasonic waves with different amplitudes but with the same spot, it is therefore easy to compute the ratio V_B^p / V_B. As a consequence, the perturbed electrical conductivity γ^p satisfies

$$\forall\, x \in \Omega, \quad \gamma^p(x) = \rho(x) V_B^p \;=\; \gamma(x)\nu(x) ,$$

where $\nu(x) = V_B^p / V_B$ is a known function. We make the following realistic assumptions: (1) the ultrasonic wave expands the zone it impacts, and changes its conductivity: $\forall x \in \Omega$, $\nu(x) > 1$, and (2) the perturbation is not too small: $\nu(x) - 1 \gg V_B$.

3.7.1 A Mathematical Model

Let u be the voltage potential induced by a current g, in the absence of ultrasonic perturbations. It is given by

$$\begin{cases} \nabla_x \cdot (\gamma(x) \nabla_x u) = 0 & \text{in } \Omega, \\ \gamma(x) \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega , \end{cases} \tag{3.41}$$

with the convention that $\int_{\partial\Omega} u = 0$. We suppose that the conductivity γ of the region close to the boundary of the domain is known, so that ultrasonic probing is limited to interior points. We denote the region (open set) by Ω_1.

Let u_δ be the voltage potential induced by a current g, in the presence of ultrasonic perturbations localized in a disk-shaped domain $D := z + \delta B$ of volume $|D| = O(\delta^2)$. The voltage potential u_δ is a solution to

$$\begin{cases} \nabla_x \cdot (\gamma_\delta(x) \nabla_x u_\delta(x)) = 0 & \text{in } \Omega \, , \\ \gamma(x) \dfrac{\partial u_\delta}{\partial \nu} = g & \text{on } \partial\Omega \, , \end{cases} \tag{3.42}$$

with the notation

$$\gamma_\delta(x) = \gamma(x) \Big[1 + \chi(D)(x) \, (\nu(x) - 1) \Big] \, ,$$

where $\chi(D)$ is the characteristic function of the domain D.

As the zone deformed by the ultrasound wave is small, we can view it as a small volume perturbation of the background conductivity γ, and seek an asymptotic expansion of the boundary values of $u_\delta - u$. The method of small-volume expansions shows that comparing u_δ and u on $\partial\Omega$ provides information about the conductivity. Indeed, we can prove that

$$\begin{aligned} \int_{\partial\Omega} (u_\delta - u) g \, d\sigma &= \int_D \gamma(x) \frac{(\nu(x) - 1)^2}{\nu(x) + 1} \nabla u \cdot \nabla u \, dx + o(|D|) \\ &= |\nabla u(z)|^2 \int_D \gamma(x) \frac{(\nu(x) - 1)^2}{\nu(x) + 1} \, dx + o(|D|) \, . \end{aligned}$$

Therefore, we have

$$\gamma(z) \, |\nabla u(z)|^2 = \mathcal{E}(z) + o(1) \, , \tag{3.43}$$

where the function $\mathcal{E}(z)$ is defined by

$$\mathcal{E}(z) = \left(\int_D \frac{(\nu(x) - 1)^2}{\nu(x) + 1} dx \right)^{-1} \int_{\partial\Omega} (u_\delta - u) g \, d\sigma \, . \tag{3.44}$$

By scanning the interior of the body with ultrasound waves, given an applied current g, we then obtain data from which we can compute

$$\mathcal{E}(z) := \gamma(z) |\nabla u(z)|^2$$

in an interior sub-region of Ω. The new inverse problem is now to reconstruct γ knowing $\mathcal{E}$.

3.7.2 A Substitution Algorithm

The use of $\mathcal{E}$ leads us to transform (3.41), having two unknowns γ and u with highly nonlinear dependency on γ, into the following nonlinear PDE (the 0-Laplacian)

$$\begin{cases} \nabla_x \cdot \left(\dfrac{\mathcal{E}}{|\nabla u|^2} \nabla u \right) = 0 & \text{in } \Omega \,, \\ \dfrac{\mathcal{E}}{|\nabla u|^2} \dfrac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega \,. \end{cases} \tag{3.45}$$

We emphasize that $\mathcal{E}$ is a known function, constructed from the measured data (3.44). Consequently, all the parameters entering in (3.45) are known. So, the ill-posed inverse problem of EIT model is converted into less complicated direct problem (3.45).

The E-substitution algorithm, which will be explained below, uses two currents g_1 and g_2. We choose this pair of current patterns to have $\nabla u_1 \times \nabla u_2 \neq 0$ for all $x \in \Omega$, where $u_i, i = 1, 2$, is the solution to (3.41). We refer to Chap. 1 and the references therein for an evidence of the possibility of such a choice. The E-substitution algorithm is based on an approximation of a linearized version of problem (3.45).

Suppose that γ is a small perturbation of conductivity profile γ_0: $\gamma = \gamma_0 + \delta\gamma$. Let u_0 and $u = u_0 + \delta u$ denote the potentials corresponding to γ_0 and γ with the same Neumann boundary data g. It is easily seen that δu satisfies $\nabla \cdot (\gamma \nabla \delta u) = -\nabla \cdot (\delta\gamma \nabla u_0)$ in Ω with the homogeneous Dirichlet boundary condition. Moreover, from

$$\mathcal{E} = (\gamma_0 + \delta\gamma)|\nabla(u_0 + \delta u)|^2 \approx \gamma_0 |\nabla u_0|^2 + \delta\gamma |\nabla u_0|^2 + 2\gamma_0 \nabla u_0 \cdot \nabla \delta u \,,$$

after neglecting the terms $\delta\gamma \nabla u_0 \cdot \nabla \delta u$ and $\delta\gamma |\nabla \delta u|^2$, it follows that

$$\delta\gamma \approx \frac{\mathcal{E}}{|\nabla u_0|^2} - \gamma_0 - 2\gamma_0 \frac{\nabla \delta u \cdot \nabla u_0}{|\nabla u_0|^2} \,.$$

The E-substitution algorithm is as follows. We start from an initial guess for the conductivity γ, and solve the corresponding Dirichlet conductivity problem

$$\begin{cases} \nabla \cdot (\gamma \nabla u_0) = 0 \text{ in } \Omega, \\ u_0 = \psi \text{ on } \partial\Omega \,. \end{cases}$$

The data ψ is the Dirichlet data measured as a response to the current g (say $g = g_1$) in absence of elastic deformation. The discrepancy between the data and our guessed solution is

$$\epsilon_0 := \frac{\mathcal{E}}{|\nabla u_0|^2} - \gamma \,. \tag{3.46}$$

We then introduce a corrector, δu, computed as the solution to

$$\begin{cases} \nabla \cdot (\gamma \nabla \delta u) = -\nabla \cdot (\varepsilon_0 \nabla u_0) & \text{in } \Omega, \\ \delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

and update the conductivity

$$\gamma := \frac{\mathcal{E} - 2\gamma \nabla \delta u \cdot \nabla u_0}{|\nabla u_0|^2} \, .$$

We iteratively update the conductivity, alternating directions of currents (i.e., with $g = g_2$).

In the case of incomplete data, that is, if $\mathcal{E}$ is only known on a subset ω of the domain, we can follow an optimal control approach. See [29].

3.8 Magneto-Acoustic Imaging

Denote by $\gamma(x)$ the unknown conductivity and let the voltage potential v be the solution to the conductivity problem (3.41). Suppose that the γ is a known constant on a neighborhood of the boundary $\partial\Omega$ and let γ_* denote $\gamma|_{\partial\Omega}$.

In magneto-acoustic imaging, ultrasonic waves are focused on regions of small diameter inside a body placed on a static magnetic field. The oscillation of each small region results in frictional forces being applied to the ions, making them move. In the presence of a magnetic field, the ions experience Lorentz force. This gives rise to a localized current density within the medium. The current density is proportional to the local electrical conductivity [59]. In practice, the ultrasounds impact a spherical or ellipsoidal zone, of a few millimeters in diameter. The induced current density should thus be sensitive to conductivity variations at the millimeter scale, which is the precision required for breast cancer diagnostic. The feasibility of this conductivity imaging technique has been demonstrated in [43].

Let $z \in \Omega$ and D be a small impact zone around the point z. The created current by the Lorentz force density is given by

$$\mathbf{J}_z(x) = c\chi_D(x)\gamma(x)\mathbf{e}, \tag{3.47}$$

for some constant c and a constant unit vector $\mathbf{e}$ both of which are independent of z. Here, χ_D denotes the characteristic function of D. With the induced current $\mathbf{J}_z$ the new voltage potential, denoted by u_z, satisfies

$$\begin{cases} \nabla \cdot (\gamma \nabla u_z + \mathbf{J}_z) = 0 & \text{in } \Omega, \\ u_z = g & \text{on } \partial\Omega. \end{cases}$$

According to (3.47), the induced electrical potential $w_z := v - u_z$ satisfies the conductivity equation:

$$\begin{cases} \nabla \cdot \gamma \nabla w_z = c \nabla \cdot (\chi_D \gamma \mathbf{e}) & \text{for } x \in \Omega, \\ w_z(x) = 0 & \text{for } x \in \partial \Omega. \end{cases} \tag{3.48}$$

The inverse problem for the vibration potential tomography, which is a synonym of Magneto-Acoustic Imaging, is to reconstruct the conductivity profile γ from boundary measurements of $\frac{\partial u_z}{\partial \nu}|_{\partial \Omega}$ or equivalently $\frac{\partial w_z}{\partial \nu}|_{\partial \Omega}$ for $z \in \Omega$.

Since γ is assumed to be constant in D and $|D|$ is small, we obtain using Green's identity [5]

$$\int_{\partial \Omega} \gamma_* \frac{\partial w_z}{\partial \nu} g d\sigma \approx -c|D| \nabla(\gamma v)(z) \cdot \mathbf{e}. \tag{3.49}$$

The relation (3.49) shows that, by scanning the interior of the body with ultrasound waves, $c\nabla(\gamma v)(z) \cdot \mathbf{e}$ can be computed from the boundary measurements $\frac{\partial w_z}{\partial \nu}|_{\partial \Omega}$ in Ω. If we can rotate the subject, then $c\nabla(\gamma v)(z)$ for any z in Ω can be reconstructed. In practice, the constant c is not known. But, since γv and $\partial(\gamma v)/\partial \nu$ on the boundary of Ω are known, we can recover c and γv from $c\nabla(\gamma v)$ in a constructive way. See [5].

The new inverse problem is now to reconstruct the contrast profile γ knowing

$$\mathcal{E}(z) := \gamma(z) v(z) \tag{3.50}$$

for a given boundary potential g, where v is the solution to (3.41).

In view of (3.50), v satisfies

$$\begin{cases} \nabla \cdot \dfrac{\mathcal{E}}{v} \nabla v = 0 & \text{in } \Omega, \\ v = g & \text{on } \partial \Omega. \end{cases} \tag{3.51}$$

If we solve (3.51) for v, then (3.50) yields the conductivity contrast γ. Note that to be able to solve (3.51) we need to know the coefficient $\mathcal{E}(z)$ for all z, which amounts to scanning all the points $z \in \Omega$ by the ultrasonic beam.

Observe that solving (3.51) is quite easy mathematically: If we put $w = \ln v$, then w is the solution to

$$\begin{cases} \nabla \cdot \mathcal{E} \nabla w = 0 & \text{in } \Omega, \\ w = \ln g & \text{on } \partial \Omega, \end{cases} \tag{3.52}$$

as long as $g > 0$. Thus if we solve (3.52) for w, then $v := e^w$ is the solution to (3.51). However, taking an exponent may amplify the error which already exists in the computed data $\mathcal{E}$. In order to avoid this numerical instability, we solve (3.51) iteratively. To do so, we can adopt an iterative scheme similar to the one proposed in the previous section.

Start with γ_0 and let v_0 be the solution of

$$\begin{cases} \nabla \cdot \gamma_0 \nabla v_0 = 0 & \text{in } \Omega, \\ v_0 = g & \text{on } \partial\Omega. \end{cases} \tag{3.53}$$

According to (3.50), our updates, $\gamma_0 + \delta\gamma$ and $v_0 + \delta v$, should satisfy

$$\gamma_0 + \delta\gamma = \frac{\mathcal{E}}{v_0 + \delta v}, \tag{3.54}$$

where

$$\begin{cases} \nabla \cdot (\gamma_0 + \delta\gamma)\nabla(v_0 + \delta v) = 0 & \text{in } \Omega, \\ \delta v = 0 & \text{on } \partial\Omega, \end{cases}$$

or equivalently

$$\begin{cases} \nabla \cdot \gamma_0 \nabla \delta v + \nabla \cdot \delta\gamma \nabla v_0 = 0 & \text{in } \Omega, \\ \delta v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.55}$$

We then linearize (3.54) to have

$$\gamma_0 + \delta\gamma = \frac{\mathcal{E}}{v_0(1 + \delta v / v_0)} \approx \frac{\mathcal{E}}{v_0}\left(1 - \frac{\delta v}{v_0}\right). \tag{3.56}$$

Thus

$$\delta\gamma = -\frac{\mathcal{E}\delta v}{v_0^2} - \delta, \quad \delta = -\frac{\mathcal{E}}{v_0} + \gamma_0. \tag{3.57}$$

We then find δv by solving

$$\begin{cases} \nabla \cdot \gamma_0 \nabla \delta v - \nabla \cdot \left(\frac{\mathcal{E}\delta v}{v_0^2} + \delta\right) \nabla v_0 = 0 & \text{in } \Omega, \\ \delta v = 0 & \text{on } \partial\Omega, \end{cases}$$

or equivalently

$$\begin{cases} \nabla \cdot \gamma_0 \nabla \delta v - \nabla \cdot \left(\frac{\mathcal{E}\nabla v_0}{v_0^2}\delta v\right) = \nabla \cdot \delta \nabla v_0 & \text{in } \Omega, \\ \delta v = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.58}$$

In the case of incomplete data, that is, if $\mathcal{E}$ is only known on a subset ω of the domain, we can follow an optimal control approach. See [5].

3.9 Magnetic Resonance Elastography

Let $\mathbf{u}$ be the solution to the modified Stokes system (3.27). The inverse problem in the magnetic resonance elastography is to reconstruct the shape and the shear modulus of the anomaly D from internal measurements of $\mathbf{u}$.

Based on the inner asymptotic expansion (3.29) of $\delta\mathbf{u}$ $(:= \mathbf{u} - \mathbf{U})$ of the perturbations in the displacement field that are due to the presence of the anomaly, a reconstruction method of binary level set type can be designed.

The first step for the reconstruction procedure is to locate the anomaly. This can be done using the outer expansion of $\delta\mathbf{u}$, i.e., an expansion far away from the elastic anomaly.

Suppose that z is reconstructed. Since the representation $D = z + \delta B$ is not unique, we can fix δ. We use a binary level set representation f of the scaled domain B:

$$f(x) = \begin{cases} 1, & x \in B, \\ -1, & x \in \mathbb{R}^3 \setminus \overline{B}. \end{cases} \tag{3.59}$$

Let

$$2h(x) = \widetilde{\mu}\left(f(\frac{x-z}{\delta}) + 1\right) - \mu\left(f(\frac{x-z}{\delta}) - 1\right), \tag{3.60}$$

and let β be a regularization parameter. Then the second step is to fix a window W (for instance a sphere containing z) and solve the following constrained minimization problem

$$\begin{aligned} \min_{\widetilde{\mu}, f} L(f, \widetilde{\mu}) = \frac{1}{2}\left\| \delta\mathbf{u}(x) - \delta \sum_{p,q=1}^{d} \partial_q \mathbf{U}(z)_p \hat{\mathbf{v}}_{pq}(\frac{x-z}{\delta}) + \nabla\mathbf{U}(z)(x-z) \right\|^2_{L^2(W)} \\ + \beta \int_W |\nabla h(x)|\, dx\,, \end{aligned} \tag{3.61}$$

subject to (3.25). Here, $\int_W |\nabla h|\, dx$ is the total variation of the shear modulus and $|\nabla h|$ is understood as a measure:

$$\int_W |\nabla h| = \sup\left\{ \int_W h\nabla\cdot\mathbf{v}\, dx, \quad \mathbf{v} \in \mathcal{C}_0^1(W) \text{ and } |\mathbf{v}| \le 1 \text{ in } W \right\}.$$

This regularization indirectly controls both the length of the level curves and the jumps in the coefficients.

The local character of the method is due to the decay of

$$\delta \sum_{p,q=1}^{d} \partial_q \mathbf{U}(z)_p \hat{\mathbf{v}}_{pq}(\frac{\cdot - z}{\delta}) - \nabla\mathbf{U}(z)(\cdot - z)$$

away from z. This is one of the main features of the method. In the presence of noise, because of a trade-off between accuracy and stability, we have to choose carefully the size of W. The size of W should not be so small to preserve some stability and not so big so that we can gain some accuracy. See [8].

The minimization problem (3.61) corresponds to a minimization with respect to $\widetilde{\mu}$ followed by a step of minimization with respect to f. The minimization steps are over the set of $\widetilde{\mu}$ and f, and can be performed using a gradient based method with a line search. Of importance to us are the optimal bounds satisfied by the viscous moment tensor V. We should check for each step whether the bounds (3.35) and (3.36) on V are satisfied or not. In the case they are not, we have to restate the value of $\widetilde{\mu}$. Another way to deal with (3.35) and (3.36) is to introduce them into the minimization problem (3.61) as a constraint. Set $\alpha = \mathrm{Tr}(V)$ and $\beta = \mathrm{Tr}(V^{-1})$ and suppose for simplicity that $\widetilde{\mu} > \mu$. Then, (3.35) and (3.36) can be rewritten (when $d = 3$) as follows

$$\begin{cases} \alpha \leq 2(\widetilde{\mu} - \mu)(3 + \dfrac{2\mu}{\widetilde{\mu}})|D|, \\ \dfrac{2\mu(\widetilde{\mu} - \mu)}{3\mu + 2\widetilde{\mu}}|D| \leq \beta^{-1} \, . \end{cases} \tag{3.62}$$

3.10 Imaging by the Acoustic Radiation Force

A model problem for the acoustic radiation force imaging is (3.14), where y is the location of the pushing ultrasonic beam. The transient wave $u(x, t)$ is the induced wave. The inverse problem is to reconstruct the shape and the conductivity of the small anomaly D from measurements of u on $\mathbb{R}^3 \times]0, +\infty[$. It is easy to detect $T = |y - z|$ and the location z of the anomaly from measurements of $u(x, t) - U_y(x, t)$.

Suppose that the wavefield in a window W containing the anomaly can be acquired. In view of Theorem 3.4, the shape and the conductivity of D can be approximately reconstructed, analogously to MRE, by minimizing over k and f the following functional:

$$\begin{aligned} L(f, k) = & \frac{1}{2\varDelta T} \int_{T - \frac{\varDelta T}{2}}^{T + \frac{\varDelta T}{2}} \left\| P_\rho[u - U_y] - \delta \hat{v} \left(\frac{x - z}{\delta} \right) \cdot \nabla P_\rho[U_y] \right\|_{L^2(W)}^2 dt \\ & + \beta \int_W |\nabla h(x)| \, dx \, , \end{aligned} \tag{3.63}$$

subject to (3.2). Here $\varDelta T = O(\delta/\sqrt{k})$ is small, $2h(x) = k(f(\frac{x-z}{\delta}) + 1) - (f(\frac{x-z}{\delta}) - 1)$, and f given by (3.59).

To detect the anomaly from measurements of the wavefield away from the anomaly one can use a time-reversal technique. As shown in Chap. 2, the main idea of time-reversal is to take advantage of the reversibility of the wave equation in a non-dissipative unknown medium in order to back-propagate signals to the sources that emitted them. In the context of anomaly detection, one measures the perturbation of the wave on a closed surface surrounding the anomaly, and retransmits it through the background medium in a time-reversed chronology. Then the perturbation will travel back to the location of the anomaly. We can show that the time-reversal perturbation focuses on the location z of the anomaly with a focal spot size limited to one-half the wavelength which is in agreement with the Rayleigh resolution limit.

In mathematical words, suppose that we are able to measure the perturbation $w := u - U_y$ and its normal derivative at any point x on a sphere S englobing the anomaly D. The time-reversal operation is described by the transform $t \mapsto t_0 - t$. Both the perturbation w and its normal derivative on S are time-reversed and emitted from S. Then a time-reversed perturbation, denoted by w_{tr}, propagates inside the volume Ω surrounded by S. Taking into account the definition of the outgoing fundamental solution (3.13) to the wave equation, spatial reciprocity and time reversal invariance of the wave equation, the time-reversed perturbation w_{tr} due to the anomaly D in Ω should be defined by

$$w_{\mathrm{tr}}(x,t) = \int_{\mathbb{R}} \int_S \left[U_x(x',t-s) \frac{\partial w}{\partial \nu}(x',t_0-s) - \frac{\partial U_x}{\partial \nu}(x',t-s) w(x',t_0-s) \right] d\sigma(x')\, ds ,$$

where

$$U_x(x',t-\tau) = \frac{\delta(t-\tau-|x-x'|)}{4\pi|x-x'|} .$$

However, with the high frequency component of w truncated as in Theorem 3.4, we take the following definition:

$$\begin{aligned} w_{\mathrm{tr}}(x,t) = \int_{\mathbb{R}} \int_S \Bigg[& U_x(x',t-s) \frac{\partial P_\rho[u-U_{\bar{y}}]}{\partial \nu}(x',t_0-s) \\ & - \frac{\partial U_x}{\partial \nu}(x',t-s) P_\rho[u-U_{\bar{y}}](x',t_0-s) \Bigg]\, d\sigma(x') . \end{aligned}$$

According to Theorem 3.4, we have

$$P_\rho[u-U_y](x,t) \approx -\delta^3 \int_{\mathbb{R}} \nabla P_\rho[U_z](x,t-\tau) \cdot p(z,\tau)\, d\tau ,$$

where

$$p(z,\tau) = M(k,B) \nabla P_\rho[U_y](z,\tau) .$$

Thus, since

$$\int_{\mathbb{R}} \int_S \Big[U_x(x', t-s) \frac{\partial P_\rho[U_z]}{\partial \nu}(x', t_0 - s - \tau) \\ - \frac{\partial U_x}{\partial \nu}(x', t-s) P_\rho[U_z](x', t_0 - s - \tau) \Big] \, d\sigma(x') \, ds \\ = P_\rho[U_z](x, t_0 - \tau - t) - P_\rho[U_z](x, t - t_0 + \tau) \, ,$$

we arrive at

$$w_{\mathrm{tr}}(x,t) \approx -\delta^3 \int_{\mathbb{R}} p(z,\tau) \cdot \nabla_z \big[P_\rho[U_z](x, t_0 - \tau - t) - P_\rho[U_z](x, t - t_0 + \tau) \big] \, d\tau \, .$$

Formula (3.64) can be interpreted as the superposition of incoming and outgoing waves, centered on the location z of the anomaly. Suppose that $p(z,\tau)$ is concentrated at the travel time $\tau = T$. Formula (3.64) takes therefore the form

$$w_{\mathrm{tr}}(x,t) \approx -\delta^3 p \cdot \nabla_z \big[P_\rho[U_z](x, t_0 - T - t) - P_\rho[U_z](x, t - t_0 + T) \big] \, ,$$

where $p = p(z,T)$. The wave w_{tr} is clearly sum of incoming and outgoing spherical waves.

Formula (3.64) has an important physical interpretation. By changing the origin of time, T can be set to 0 without loss of generality. By taking Fourier transform of (3.64) over the time variable t, we obtain that

$$\hat{w}_{\mathrm{tr}}(x,\omega) \propto \delta^3 p \cdot \nabla \left(\frac{\sin(\omega |x - z|)}{|x - z|} \right) \, ,$$

where $\hat{w}_{\mathrm{tr}}$ denotes the Fourier transform of w_{tr} and ω is the wavenumber and, which shows that the time-reversal perturbation w_{tr} focuses on the location z of the anomaly with a focal spot size limited to one-half the wavelength. An identity parallel to (3.64) can be rigorously derived in the frequency domain. It plays a key role in the resolution limit analysis. See [7].

3.11 Infrared Thermal Imaging

In this section we apply (3.18) (with an appropriate choice of test functions w and background solutions U) for the purpose of identifying the location of the anomaly D. The first algorithm makes use of constant heat flux and, not surprisingly, it is limited in its ability to effectively locate multiple anomalies.

Using many heat sources, we then describe an efficient method to locate multiple anomalies and illustrate its feasibility. For the sake of simplicity we consider only the two-dimensional case.

3.11.1 Detection of a Single Anomaly

For $y \in \mathbb{R}^2 \setminus \overline{\Omega}$, let

$$w(x,t) = w_y(x,t) := \frac{1}{4\pi(T-t)} e^{-\frac{|x-y|^2}{4(T-t)}} . \tag{3.64}$$

The function w satisfies $(\partial_t + \Delta)w = 0$ in Ω_T and the final condition $w|_{t=T} = 0$ in Ω.

Suppose that there is only one anomaly $D = z + \delta B$ with thermal conductivity k. For simplicity assume that B is a disk. Choose the background solution $U(x,t)$ to be a harmonic (time-independent) function in Ω_T. We compute

$$\nabla w_y(z,t) = \frac{y-z}{8\pi(T-t)^2} e^{-\frac{|z-y|^2}{4(T-t)}} ,$$

$$M(k,B)\nabla w_y(z,t) = \frac{(k-1)|B|}{k+1} \frac{y-z}{4\pi(T-t)^2} e^{-\frac{|z-y|^2}{4(T-t)}} ,$$

and

$$\int_0^T M(k,B)\nabla w_y(z,t)\,dt = \frac{(k-1)|B|}{k+1} \frac{y-z}{4\pi} \int_0^T \frac{e^{-\frac{|z-y|^2}{4(T-t)}}}{(T-t)^2}\,dt .$$

But

$$\frac{d}{dt} e^{-\frac{|z-y|^2}{4(T-t)}} = \frac{-|z-y|^2}{4} \frac{e^{-\frac{|z-y|^2}{4(T-t)}}}{(T-t)^2}$$

and therefore

$$\int_0^T M(k,B)\nabla w_y(z,t)\,dt = \frac{(k-1)|B|}{k+1} \frac{y-z}{\pi|z-y|^2} e^{-\frac{|z-y|^2}{4(T-t)}} .$$

Then the asymptotic expansion (3.18) yields

$$I_w(T)(y) \approx \delta^2 \frac{k-1}{k+1} |B| \frac{\nabla U(z)\cdot(y-z)}{\pi|y-z|^2} e^{-\frac{|y-z|^2}{4T}} . \tag{3.65}$$

Now we are in a position to present our projection-type location search algorithm for detecting a single anomaly. We prescribe the initial condition $u_0(x) = a \cdot x$ for some fixed unit constant vector a and choose $g = a \cdot \nu$ as an applied time-independent heat flux on $\partial\Omega_T$, where a is taken to be a coordinate unit vector. Take two observation lines Σ_1 and Σ_2 contained in $\mathbb{R}^2 \setminus \overline{\Omega}$ such that

$$\Sigma_1 := \text{a line parallel to } a, \qquad \Sigma_2 := \text{a line normal to } a\,.$$

Next we find two points $P_i \in \Sigma_i$ $(i = 1, 2)$ so that $I_w(T)(P_1) = 0$ and

$$I_w(T)(P_2) = \begin{cases} \min\limits_{x\in\Sigma_2} I_w(T)(x) & \text{if } k-1<0\,, \\ \max\limits_{x\in\Sigma_2} I_w(T)(x) & \text{if } k-1>0. \end{cases}$$

Finally, we draw the corresponding lines $\Pi_1(P_1)$ and $\Pi_2(P_2)$ given by (3.38). Then the intersecting point P of $\Pi_1(P_1) \cap \Pi_2(P_2)$ is close to the anomaly D: $|P - z| = O(\delta\,|\log\delta|)$ for δ small enough.

3.11.2 Detection of Multiple Anomalies: A MUSIC-Type Algorithm

Consider m anomalies $D_s = \delta B_s + z_s$, $s = 1, \ldots, m$, whose heat conductivity is k_s. Choose

$$U(x,t) = U_{y'}(x,t) := \frac{1}{4\pi t} e^{-\frac{|x-y'|^2}{4t}} \quad \text{for } y' \in \mathbb{R}^2 \setminus \overline{\Omega}$$

or, equivalently, g to be the heat flux corresponding to a heat source placed at the point source y' and the initial condition $u_0(x) = 0$ in Ω, to obtain that

$$\begin{aligned} I_w(T) \approx &-\delta^2 \sum_{s=1}^{m} \frac{(1-k_s)}{64\pi^2}(y'-z_s)M^{(s)}(y-z_s) \\ &\times \int_0^T \frac{1}{t^2(T-t)^2} \exp\left(-\frac{|y-z_s|^2}{4(T-t)} - \frac{|y'-z_s|^2}{4t}\right) dt, \end{aligned}$$

where w is given by (3.64) and $M^{(s)}$ is the polarization tensor of D_s.

Suppose for the sake of simplicity that all the domains B_s are disks. Then it follows from (3.10) that $M^{(s)} = m^{(s)} I_2$, where $m^{(s)} = 2(k_s-1)|B_s|/(k_s+1)$ and I_2 is the 2×2 identity matrix. Let $y_l \in \mathbb{R}^2 \setminus \overline{\Omega}$ for $l \in \mathbb{N}$ be the source points. We assume that the countable set $\{y_l\}_{l\in\mathbb{N}}$ has the property that any analytic function which vanishes in $\{y_l\}_{l\in\mathbb{N}}$ vanishes identically.

The MUSIC-type location search algorithm for detecting multiple anomalies is as follows. For $n \in \mathbb{N}$ sufficiently large, define the matrix $A = [A_{ll'}]_{l,l'=1}^{n}$ by

$$A_{ll'} := -\,\delta^2 \sum_{s=1}^{m} \frac{(1-k_s)}{64\pi^2} m^{(s)} (y_{l'} - z_s) \cdot (y_l - z_s) \\ \times \int_0^T \frac{1}{t^2(T-t)^2} \exp\left(-\frac{|y_l - z_s|^2}{4(T-t)} - \frac{|y_{l'} - z_s|^2}{4t} \right) dt \, .$$

For $z \in \Omega$, we decompose the symmetric real matrix C defined by

$$C := \left[\int_0^T \frac{1}{t^2(T-t)^2} \exp\left(-\frac{|y_l - z|^2}{4(T-t)} - \frac{|y_{l'} - z|^2}{4t} \right) dt \right]_{l,l'=1,\ldots,n}$$

as follows:

$$C = \sum_{l=1}^{p} v_l(z) v_l(z)^* \tag{3.66}$$

for some $p \leq n$, where $v_l \in \mathbb{R}^n$ and v_l^* denotes the transpose of v_l. Define the vector $g_z^{(l)} \in \mathbb{R}^{n \times 2}$ for $z \in \Omega$ by

$$g_z^{(l)} = \Big((y_1 - z) v_{l1}(z), \ldots, (y_n - z) v_{ln}(z) \Big)^*, \quad l = 1, \ldots, p. \tag{3.67}$$

Here $v_{l1}, \ldots, v_{ln}$ are the components of the vector v_l, $l = 1, \ldots, p$. Let $y_l = (y_{lx}, y_{ly})$ for $l = 1, \ldots, n$, $z = (z_x, z_y)$, and $z_s = (z_{sx}, z_{sy})$. We also introduce

$$g_{zx}^{(l)} = \Big((y_{1x} - z_x) v_{l1}(z), \ldots, (y_{nx} - z_x) v_{ln}(z) \Big)^*$$

and

$$g_{zy}^{(l)} = \Big((y_{1y} - z_y) v_{l1}(z), \ldots, (y_{ny} - z_y) v_{ln}(z) \Big)^* .$$

Lemma 3.10. *The following characterization of the location of the anomalies in terms of the range of the matrix A holds:*

$$g_{zx}^{(l)} \text{ and } g_{zy}^{(l)} \in \mathrm{Range}(A) \quad \forall\, l \in \{1, \ldots, p\} \quad \textit{iff} \quad z \in \{z_1, \ldots, z_m\} \, . \tag{3.68}$$

Note that the smallest number n which is sufficient to efficiently recover the anomalies depends on the (unknown) number m. This is the main reason to take n sufficiently large. As for the electrical impedance imaging, the MUSIC-type algorithm for the thermal imaging is as follows. Compute P_{noise}, the projection onto the noise space, by the singular value decomposition of the matrix A. Compute the vectors v_l by (3.66). Form an image of the locations,

$z_1, \ldots, z_m$, by plotting, at each point z, the quantity $\|g_z^{(l)} \cdot a\| / \|P_{\text{noise}}(g_z^{(l)} \cdot a)\|$ for $l = 1, \ldots, p$, where $g_z^{(l)}$ is given by (3.67) and a is a unit constant vector. The resulting plot will have large peaks at the locations of z_s, $s = 1, \ldots, m$.

The algorithms described for reconstructing thermal anomalies can be extended to the realistic half-space model. See [22].

3.12 Bibliography and Concluding Remarks

In this chapter, applications of the method of small-volume expansions in emerging medical imaging are outlined. This method leads to very effective and robust reconstruction algorithms in many imaging problems [15]. Of particular interest are emerging multi-physics or hybrid imaging approaches. These approaches allow to overcome the severe ill-posedness character of image reconstruction.

Part (i) in Theorem 3.1 was proven in [14, 33, 40] and in a more general form in [31]. The proof in [14] is based on a decomposition formula of the solution into a harmonic part and a refraction part first derived in [48]. In this connection, see [49–51, 54, 56]. Part (iii) is from [21]. The Hashin-Shtrikman bounds for the polarization tensor were proved in [32, 57]. Theorem 3.7 and the results on the viscous moment tensor in Theorem 3.8 are from [6]. The initial boundary-value problems for the wave equation in the presence of anomalies of small volume have been considered in [1, 17]. See [16] for the time-harmonic regime. Theorem 3.4 is from [7]. See also [19, 20, 30] for similar results in the case of compressible elasticity. In that paper, a time-reversal approach was designed for locating the anomaly from the outer expansion (3.15). We refer to Chap. 2 for basic physical principles of time reversal. See also [38, 39].

The projection algorithm was introduced in [23, 24, 55, 63]. The MUSIC-type algorithm for locating small electromagnetic anomalies from the response matrix was first developed in [28]. See also [9, 11–13, 34]. It is worth mentioning that the MUSIC-type algorithm is related to time reversal [61,62].

Impediography was proposed in [3] and the substitution algorithm proposed there. An optimal control approach for solving the inverse problem in impediography has been described in [29]. The inversion was considered as a minimization problem, and it was performed in two or three dimensions.

Magnetic resonance elastography was first proposed in [60]. The results provided on this technique are from [6]. For physical principles of radiation force imaging we refer to [26, 27]. Thermal imaging of small anomalies has been considered in [10]. See also [22] where a realistic half space model for thermal imaging was considered and accurate and robust reconstruction algorithms are designed.

To conclude this chapter, it is worth mentioning that the inner expansions derived for the heat equation can be used to improve reconstruction in ultrasonic temperature imaging. The idea behind ultrasonic temperature imaging hinges on measuring local temperature near anomalies. The aim is to reconstruct anomalies with higher spatial and contrast resolution as compared to those obtained from boundary measurements alone.

We would also like to mention that our approach for the magneto-acoustic tomography can be used in photo-acoustic imaging. The photo-acoustic effect is the physical basis for photo-acoustic imaging; it refers to the generation of acoustic waves by the absorption of optical energy. In [4], a new method for reconstructing absorbing regions inside a bounded domain from boundary measurements of the induced acoustic signal has been developed. There, the focusing property of the time-reversed acoustic signal has been shown.

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