Chapter 2
General Motion of a Rigid Body

Abstract The tools that allow the description of the motion of the rigid body are recalled. We used both Euler's angles and Euler's parameters (normalized quaternions) to describe the orientations of the body. Precession of the rigid body and air resistance and the dissipation of the energy at successive collisions are discussed.

2.1 Basic Equations of Motion of a Rigid Body

In the motion analysis of a coin, a die, and roulette ball free (unconstrained) body motion and constraint body motion are analyzed. Free body motion is described by dynamics equations of a rigid body [1–7]. At the instant of the body (coin, die, ball) contact with surroundings (base, table, wheel) some additional equations – constraint equations – are necessary.

2.1.1 Dynamics Equations in General Form

The general form of body dynamics equations can be derived from body linear momentum and angular momentum laws

$$\frac{d\mathbf{Q}}{dt} = \mathbf{F} \quad \text{and} \quad \frac{d\mathbf{K}}{dt} = \mathbf{M}, \tag{2.1}$$

where $\mathbf{F}$, $\mathbf{M}$ are vectors of body external force and moments whereas $\mathbf{Q}$, $\mathbf{K}$ are vectors of linear momentum and angular momentum of body defined by

$$\mathbf{Q} = \int_V \mathbf{v}\,\rho dV \quad \text{and} \quad \mathbf{K} = \int_V \mathbf{r}\times\mathbf{v}\,\rho dV\ . \tag{2.2}$$

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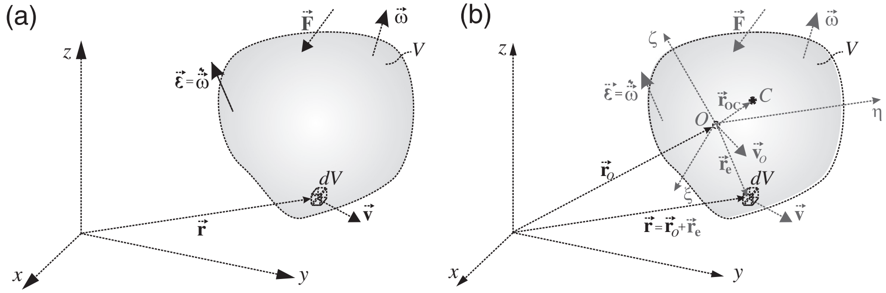

Fig. 2.1 Body point position (**r**) and velocity (**v**) vectors: **(a)** in the inertial frame xyz, and **(b)** in the body embedded frame $O\xi\eta\zeta$

In (2.2) V denotes a body volume (Fig. 2.1), **r** position vector[1] of a body point (volume dV), **v** velocity vector of the point described in inertial frame xyz, and ρ body material density.

Using the velocity vector $\mathbf{v}_o$ represented by the components in the body embedded frame ($\xi\eta\zeta$) to describe body point velocity vector ($\mathbf{v} = \mathbf{v}_o + \boldsymbol{\omega} \times \mathbf{r}_e$) on the basis of (2.1), (2.2) we obtain the general form of body dynamics equations:

$$m\dot{\mathsf{v}}_o + m\mathsf{R}_{oc}^{\mathsf{T}}\dot{\tilde{\omega}} + m\tilde{\boldsymbol{\Omega}}\mathsf{v}_o + m\tilde{\boldsymbol{\Omega}}^{\mathsf{T}}\tilde{\boldsymbol{\Omega}}\mathsf{R}_{oc} = \mathsf{F} \ , \tag{2.3}$$

$$m\mathsf{R}_{oc}\dot{\mathsf{v}}_o + \mathsf{J}_o\dot{\tilde{\omega}} + m\mathsf{R}_{oc}\tilde{\boldsymbol{\Omega}}\mathsf{v}_o + \tilde{\boldsymbol{\Omega}}\mathsf{J}_o\tilde{\boldsymbol{\omega}} = \mathsf{M}_o \ . \tag{2.4}$$

In the mentioned equations m is the mass of body, $\dot{\mathsf{v}}_o$ and $\tilde{\boldsymbol{\omega}}$ are column matrices of vectors[2] $\dot{\mathbf{v}}_o$, $\boldsymbol{\omega}$ components, R_{oc} and $\tilde{\boldsymbol{\Omega}}$ are antisymmetric matrices representing the vectors $\mathbf{r}_{oc}$ and $\boldsymbol{\omega}$, J_o is inertia matrix of the body including mass moments of inertia of the body with respect to $O\xi\eta\zeta$ frame. F and M_o are column matrices of vectors $\mathbf{F}$, $\mathbf{M}_o$ components in $O\xi\eta\zeta$ frame.

We get a special form of body dynamics equations introducing the body embedded frame $C\xi\eta\zeta$, where C is the body mass center (Fig. 2.2), and using the mass center velocity vector ($\mathsf{v}_c = [\,v_{c\xi} \;\; v_{c\eta} \;\; v_{c\zeta}\,]^T$) instead of v_o:

$$m\dot{\mathsf{v}}_c + m\tilde{\boldsymbol{\Omega}}\mathsf{v}_c = \mathsf{F} \ , \tag{2.5}$$

$$\mathsf{J}_c\dot{\tilde{\omega}} + \tilde{\boldsymbol{\Omega}}\mathsf{J}_c\tilde{\boldsymbol{\omega}} = \mathsf{M}_c \ . \tag{2.6}$$

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[1] In most of figures the vector quantities are denoted with an arrow above the symbol.

[2] The symbol $\boldsymbol{\omega}$ is used for the angular velocity vector, and $\tilde{\boldsymbol{\omega}}$ for its matrix representation, i.e., the column matrix of the angular velocity components. For the square matrix representing the angular velocity vector ($\boldsymbol{\omega}$) the symbol $\tilde{\boldsymbol{\Omega}}$ is used.

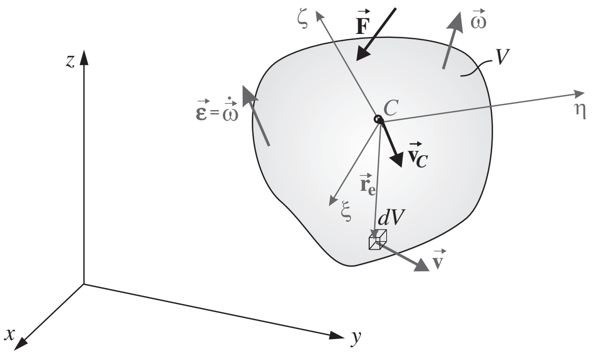


Fig. 2.2 Body mass center velocity vector ($\mathbf{v}_C$) and external force $\mathbf{F}$ in the inertial frame xyz, angular velocity and acceleration vectors in the body embedded frame ($C\xi\eta\zeta$)

2.1.2 Newton–Euler Equations

The simplest equations describing dynamics of a rigid body are the well-known form of the Newton–Euler equations based on body linear momentum and angular momentum laws

$$\frac{d\mathbf{Q}}{dt} = \mathbf{F} \quad \text{and} \quad \frac{d\mathbf{K}_C}{dt} = \mathbf{M}_C \, , \tag{2.7}$$

where $\mathbf{K}_C$ and $\mathbf{M}_C$ are vectors of angular momentum of a body and body force moments with respect to the body mass center (C). Newton–Euler equations have the form:

$$m\dot{\mathsf{v}}_C = \mathsf{F} \, , \tag{2.8}$$

$$\mathsf{J}_C\dot{\tilde{\boldsymbol{\omega}}} + \tilde{\boldsymbol{\Omega}}\,\mathsf{J}_C\dot{\tilde{\boldsymbol{\omega}}} = \mathsf{M}_C, \tag{2.9}$$

where the mass center velocity and external force are expressed by vector components with respect to the inertial frame (xyz), i.e., $\mathsf{v}_C = [\,\dot{x} \quad \dot{y} \quad \dot{z}]^T$, $\mathsf{F} = [\,f_x \quad f_y \quad f_z]^T$, whereas J_C, $\tilde{\boldsymbol{\omega}}$, $\tilde{\boldsymbol{\Omega}}$, and M_C are described in the frame $C\xi\eta\zeta$.

On the other hand, it is more convenient to describe the rotations of a body (2.4) by their components with respect to the body embedded frame ($C\xi\eta\zeta$).

Equations (2.8) and (2.9) are general equations of motion and are suitable for the dynamics analysis of any body, i.e., a nonsymmetric and nonhomogeneous body. In the case in which the forces F are independent of the angular velocity of the body and moments M_C are not the functions of body mass center accelerations, (2.8) and (2.9) are uncoupled. This happens when the air resistance of a body is neglected.

The scalar form of a body dynamics equations obtained from the general equation (2.8) and (2.9) can be written in the well-known form of Newton–Euler equations:

$$m\,\ddot{x} = f_x \, , \quad m\,\ddot{y} = f_y \, , \quad m\,\ddot{z} = f_z \, , \tag{2.10}$$

$$
\begin{aligned}
&J_\xi \dot{\omega}_\xi + \left(J_\zeta - J_\eta\right) \omega_\eta \omega_\zeta - J_{\xi\zeta} \dot{\omega}_\zeta - J_{\xi\eta} \dot{\omega}_\eta + J_{\eta\zeta} \left({\omega_\zeta}^2 - {\omega_\eta}^2\right) + \\
&\quad + \left(J_{\xi\eta}\omega_\zeta - J_{\xi\zeta}\omega_\eta\right) \omega_\xi = M_{C\xi} \ , \\
&J_\eta \dot{\omega}_\eta + \left(J_\xi - J_\zeta\right) \omega_\xi \omega_\zeta - J_{\xi\eta} \dot{\omega}_\xi - J_{\eta\zeta} \dot{\omega}_\zeta + J_{\xi\zeta} \left({\omega_\xi}^2 - {\omega_\zeta}^2\right) + \\
&\quad + \left(J_{\eta\zeta}\omega_\xi - J_{\xi\eta}\omega_\zeta\right) \omega_\eta = M_{C\eta} \ , \\
&J_\zeta \dot{\omega}_\zeta + \left(J_\eta - J_\xi\right) \omega_\xi \omega_\eta - J_{\eta\zeta} \dot{\omega}_\eta - J_{\xi\zeta} \dot{\omega}_\xi + J_{\xi\eta} \left({\omega_\eta}^2 - {\omega_\xi}^2\right) + \\
&\quad + \left(J_{\xi\zeta}\omega_\eta - J_{\eta\zeta}\omega_\xi\right) \omega_\zeta = M_{C\zeta} \ .
\end{aligned}
\tag{2.11}
$$

The moments of inertia in this case are determined with respect to the body embedded frame $C\xi\eta\zeta$ with the origin in the body mass center and in the general case, for a nonsymmetric or nonhomogeneous body, the inertia products are nonzero terms, because the axes $C\xi\eta\zeta$ will not be principal axes.

In the case of symmetric body the axes $C\xi\eta\zeta$ are principal axes ($J_{\xi\eta} = 0$, $J_{\eta\zeta} = 0$, $J_{\zeta\xi} = 0$) and from (2.11) we obtain

$$
\begin{aligned}
J_\xi \frac{d\omega_\xi}{dt} - (J_\eta - J_\zeta)\, \omega_\eta \omega_\zeta &= M_{C\xi} \ , \\
J_\eta \frac{d\omega_\eta}{dt} - (J_\zeta - J_\xi)\, \omega_\zeta \omega_\xi &= M_{C\eta} \ , \\
J_\zeta \frac{d\omega_\zeta}{dt} - (J_\xi - J_\eta)\, \omega_\xi \omega_\eta &= M_{C\zeta} \ .
\end{aligned}
\tag{2.12}
$$

Expressing $\omega_\xi, \omega_\eta, \omega_\zeta$ in terms of Euler angles (φ, ϑ, ψ) or in normalized quaternions (e_0, e_1, e_2, e_3) one can obtain the motion equations in the chosen coordinates.

2.2 Precession of a Body

The term precession is used to indicate that the body angular velocity vector rotates during the motion.

In general case of a body motion we use (2.11) to describe the body rotation. There are no analytical solution of (2.11) in general case but we can solve these equations numerically for any data and obtain the body motion as well as the changes in angular velocity vector ($\boldsymbol{\omega}$) and angular momentum vector ($\mathbf{K}_C$). Some exemplary results of such solutions for torque-induced body motion are shown in Fig. 2.3 where vectors $\boldsymbol{\omega}$ and $\mathbf{K}_C$ are depicted for some time instant. We can see that both vectors change their directions and values during the body motion. From presented cases follow that these bodies undergo precession (i.e., the vector $\boldsymbol{\omega}$ rotates).

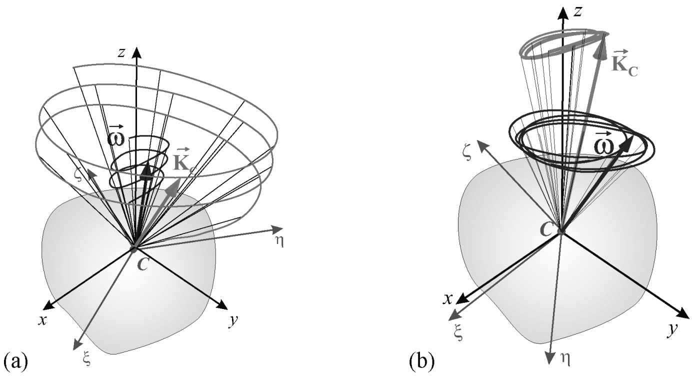


Fig. 2.3 The precessing nonsymmetric body: changes of angular momentum ($\mathbf{K}_C$) and angular velocity ($\boldsymbol{\omega}$) vectors direction due to nonzero moments of external forces: (**a**) for $M_{C\xi} \neq 0$, $M_{C\eta} \neq 0$, $M_{C\zeta} \neq 0$ and (**b**) for $M_{C\xi} \neq 0$, $M_{C\eta} \neq 0$, $M_{C\zeta} = 0$

Beyond the torque-induced precession torque-free precession is also possible. In the case of torque-free motion of a body the moment of external forces with respect to the body mass center is zero ($\mathbf{M}_C = \mathbf{0}$) and from (2.7) we have $d\mathbf{K}_C/dt = \mathbf{0}$. It means that the body angular momentum vector with respect to the mass center is constant vector and there are no changes in the value neither in the direction of $\mathbf{K}_C$ ($\mathbf{K}_C = \mathbf{const}$).

For some special cases – for symmetric bodies and zero moments of external forces ($\mathbf{M}_C = \mathbf{0}$) – an analytical solution of (2.11) can be obtained [4]. We present such solutions for symmetric top[3] (which is the model of ideal coin) and spherical top[4] (which is the model of ideal die and torque-free motion of a roulette ball).

2.2.1 Precession of Symmetric Top

Taking into account the symmetry of body (symmetric top) $J_\xi = J_\eta = J_1$, $J_\zeta = J_3$, $J_{\xi\eta} = J_{\xi\zeta} = J_{\eta\zeta} = 0$ and assuming zero value of forces moment acting on the body ($M_{C\xi} = M_{C\eta} = M_{C\zeta} = 0$), from (2.12) we get the following equations:

[3] A symmetric top is a body in which two moments of inertia are the same. There are two classes of symmetric tops, oblate symmetric top with $J_\xi = J_\eta < J_\zeta$ and prolate symmetric top (cigar shaped) with $J_\xi = J_\eta > J_\zeta$. (A body is an asymmetric top if all three moments of inertia are different.)

[4] A spherical top is a special case of a symmetric top (although it need not be spherical) with equal moment of inertia about all three axes ($J_\xi = J_\eta = J_\zeta$).

$$
\begin{aligned}
&J_\xi \dot{\omega}_\xi - (J_1 - J_3)\omega_\zeta \omega_\eta = 0 \ , \\
&J_1 \dot{\omega}_\eta - (J_3 - J_1)\omega_\zeta \omega_\xi = 0 \ , \\
&J_3 \dot{\omega}_\zeta = 0 \ .
\end{aligned}
\tag{2.13}
$$

Solving (2.13) together with initial conditions (i.e., the initial angular velocity components $\omega_\xi(0) = \omega_{\xi 0}$, $\omega_\eta(0) = \omega_{\eta 0}$, $\omega_\zeta(0) = \omega_{\zeta 0}$) we obtain the components of angular velocity of the body in the form

$$
\begin{aligned}
&\omega_\xi(t) = \omega_{\eta 0} \sin\left(\tfrac{J_1 - J_3}{J_1}\omega_{\zeta 0}\, t\right) + \omega_{\xi 0} \cos\left(\tfrac{J_1 - J_3}{J_1}\omega_{\zeta 0}\, t\right), \\
&\omega_\eta(t) = \omega_{\eta 0} \cos\left(\tfrac{J_1 - J_3}{J_1}\omega_{\zeta 0}\, t\right) - \omega_{\xi 0} \sin\left(\tfrac{J_1 - J_3}{J_1}\omega_{\zeta 0}\, t\right) \ , \\
&\omega_\zeta(t) = \omega_{\zeta 0} \ .
\end{aligned}
\tag{2.14}
$$

For symmetric top the vector $\boldsymbol{\omega}$ has constant value

$$
\omega = \sqrt{\omega_{\xi 0}^2 + \omega_{\eta 0}^2 + \omega_{\zeta 0}^2} \ . \tag{2.15}
$$

From (2.14) arise that $\omega_\zeta(t)$ is time independent and two other components can be transformed to the following form:

$$
\begin{aligned}
&\omega_\xi(t) = \omega_\tau \sin\left(\omega_p t + \varphi_\tau\right) \ , \\
&\omega_\eta(t) = \omega_\tau \cos\left(\omega_p t + \varphi_\tau\right) \ ,
\end{aligned}
\tag{2.16}
$$

with the constant coefficients

$$
\omega_p = \frac{J_1 - J_3}{J_1}\omega_{\zeta 0} \ , \quad \omega_\tau = \sqrt{\omega_{\xi 0}^2 + \omega_{\eta 0}^2} \ , \quad \varphi_\tau = \operatorname{arctg} \frac{\omega_{\xi 0}}{\omega_{\eta 0}} \ . \tag{2.17}
$$

From presented solutions (2.14), (2.15), (2.16), and (2.17) it follows that for the symmetric top its angular momentum vector ($\mathbf{K}_C$) has components obtained as

$$
\begin{aligned}
&K_\xi(t) = J_1 \omega_\tau \sin\left(\omega_p t + \varphi_\tau\right) \ , \\
&K_\eta(t) = J_1 \omega_\tau \cos\left(\omega_p t + \varphi_\tau\right) \ , \\
&K_\zeta(t) = J_3 \omega_{\zeta 0} \ ,
\end{aligned}
\tag{2.18}
$$

and the constant value as

$$
K_C = \sqrt{J_1^2(\omega_{\xi 0}^2 + \omega_{\eta 0}^2) + J_3^2 \omega_{\zeta 0}^2} \ . \tag{2.19}
$$

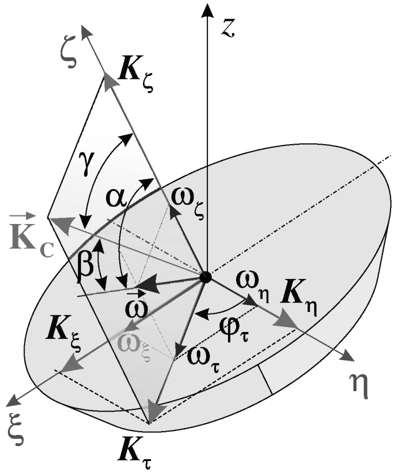


Fig. 2.4 Torque-free precession of symmetric top: vectors of angular velocity ($\boldsymbol{\omega}$) and angular momentum ($\mathbf{K}_C$)

We can also obtain the angle γ between the symmetry axis of the body (ζ) and the angular momentum vector ($\mathbf{K}_c$) and the angle α between ζ and the vector $\tilde{\boldsymbol{\omega}}$ (see Fig. 2.4) as

$$\gamma = \arccos \frac{J_3 \omega_{\zeta 0}}{K_c} = const \ , \quad \alpha = \arccos \frac{\omega_{\zeta 0}}{\omega} = const \ . \tag{2.20}$$

The angles γ and α have constant value, but they are measured from axis ζ which moves (changes its direction). While the angular momentum vector $\mathbf{K}_c$ has constant value and direction in the spatial frame xyz we can use the angle $\beta = \alpha - \gamma$ to obtain the orientation of the vector $\tilde{\boldsymbol{\omega}}$ with respect to the fixed vector $\mathbf{K}_c$. From (2.20) we have

$$\beta = \alpha - \gamma = \arcsin \frac{(J_3 - J_1)\omega_{\zeta 0}\omega_\tau}{K_c \omega} = const \ . \tag{2.21}$$

The vectors of angular velocity and angular momentum for symmetric top are also presented in Fig. 2.5, where $\boldsymbol{\omega}$ and $\mathbf{K}_c$ are shown in the spatial frame xyz as well as in the body embedded frame $\xi\eta\zeta$. The angular velocity vector $\boldsymbol{\omega}$ in the spatial frame xyz rotates around the angular momentum vector (Fig. 2.5a). Both vectors rotate in the body embedded frame $\xi\eta\zeta$ (Fig. 2.5b).

2.2.2 Torque-Free Motion of Spherical Top

For torque-free motion of spherical top ($J_\xi = J_\eta = J_\zeta = J_1$), (2.13) simplify to the form

$$J_1 \dot{\omega}_\xi = 0 \ , \quad J_1 \dot{\omega}_\eta = 0 \ , \quad J_1 \dot{\omega}_\zeta = 0 \ , \tag{2.22}$$

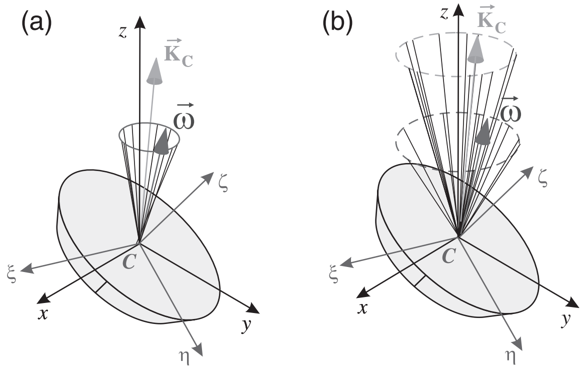


Fig. 2.5 The precessing symmetric top: angular momentum ($\mathbf{K}_C$) and angular velocity ($\boldsymbol{\omega}$) vectors in (**a**) spatial frame xyz and (**b**) body embedded frame $\xi\eta\zeta$

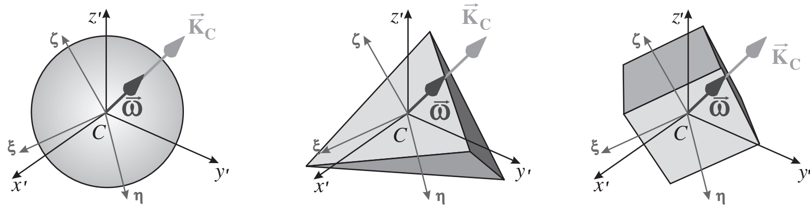


Fig. 2.6 Torque-free motion of spherical tops: constant angular momentum ($\mathbf{K}_C$) and angular velocity ($\boldsymbol{\omega}$) vectors' directions and values for a ball, tetrahedron die, and cubic die (no precession motion)

and we get the solution: $\omega_\xi = \omega_{\xi 0}$, $\omega_\eta = \omega_{\eta 0}$, $\omega_\zeta = \omega_{\zeta 0}$. It means that the angular momentum vector ($\mathbf{K}_c$) and angular velocity vector ($\boldsymbol{\omega}$) have constant values and directions during the body motion. We will call such a motion as no precession motion. Illustration of this case is shown in Fig. 2.6.

2.3 Orientation of a Rigid Body

The orientation of a rigid body (orientation of a body embedded frame $\xi\eta\zeta$) with respect to the local reference frame $x'y'z'$ is described by [3, 5, 8–10]

$$\mathsf{r} = \mathsf{R}\,\mathsf{r}' \,, \tag{2.23}$$

where

r – the column matrix representing the vector $\mathbf{r}$ of the position coordinates of an arbitrary point A (Fig. 2.7) of the body before its rotation (in the initial position),

r' – the column matrix of coordinates of the same point of the body after its rotation (in the final position),

R – the rotation matrix (representing the orientation of the local frame $\xi\eta\zeta$, with respect to the frame $x'y'z'$).

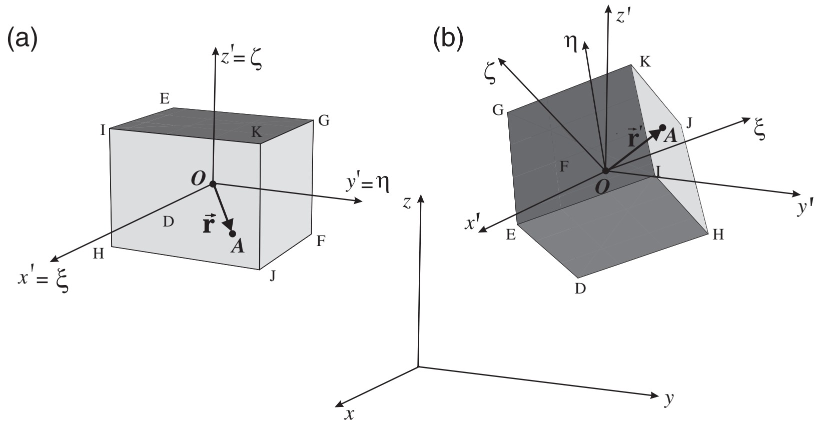


Fig. 2.7 Point A position vectors: (**a**) **r** – in the initial position and (**b**) **r**′ – in the final position

The inverse transformation is defined by

$$\mathsf{r}' = \mathsf{R}^{-1}\,\mathsf{r} = \mathsf{R}^{\mathsf{T}}\,\mathsf{r} \tag{2.24}$$

(R^{-1} is the inverse matrix of R, and $\mathsf{R}^{-1} = \mathsf{R}^{\mathsf{T}}$, whereas R^{T} is the transpose matrix of R).

The matrix R can be expressed in different ways. Its form depends on the coordinates that are chosen for the body orientation description. The elements of the rotation matrix (R) are cosines of the angles between the axes of body reference frames $x'y'z'$ and $\xi\eta\zeta$ (Fig. 2.7b)

$$\mathsf{R} = \begin{bmatrix} \cos\sphericalangle(x',\xi) & \cos\sphericalangle(x',\eta) & \cos\sphericalangle(x',\zeta) \\ \cos\sphericalangle(y',\xi) & \cos\sphericalangle(y',\eta) & \cos\sphericalangle(y',\zeta) \\ \cos\sphericalangle(z',\xi) & \cos\sphericalangle(z',\eta) & \cos\sphericalangle(z',\zeta) \end{bmatrix}. \tag{2.25}$$

On the basis of the rotation matrix (R) it is possible to define the matrix (or angular velocity tensor) that contains the components of the body angular velocity vector ($\tilde{\boldsymbol{\Omega}} = \dot{\mathsf{R}}\mathsf{R}^{\mathsf{T}}$), which is necessary in the dynamic analysis of the body. Matrix $\tilde{\boldsymbol{\Omega}}$ is obtained in the xyz reference frame. We use the symbol $\tilde{\boldsymbol{\Omega}}_{\xi}$ for the body angular velocity matrix defined by the components in the body embedded frame $\xi\eta\zeta$. This matrix is expressed as $\tilde{\boldsymbol{\Omega}}_{\xi} = \mathsf{R}^{\mathsf{T}}\tilde{\boldsymbol{\Omega}}\mathsf{R}$.

2.3.1 Euler Angles and Other Conventions

In what follows, we adapt the conventions of [1, 7, 9, 11] which are different from the ones used in [6]. Using Euler angles (or any other from 12 possible conventions of specifying the relative orientation of a body) the rotation matrix R is the composition of three consecutive rotations: φ_1, φ_2, φ_3, around axes $\xi\eta\zeta$ of the frame embedded and fixed in the body

$$\mathsf{R} = \mathsf{R}_1(\varphi_1)\mathsf{R}_2(\varphi_2)\mathsf{R}_3(\varphi_3)\ , \tag{2.26}$$

whereas $\mathsf{R}_1(\varphi_1)$, $\mathsf{R}_2(\varphi_2)$, $\mathsf{R}_3(\varphi_3)$ are the matrices of successive rotations around ξ, η, and ζ axes.

The definition of Euler angles is not unique. In works of various authors different sets of angles describe the body orientations and other naming conventions for the same angles are used. These conventions depend on the axes about which the rotations are carried out and on the rotation sequences. Basic definitions, names, and expressions used in this work are briefly presented below. For the rotation of the body of value φ_i around the ξ axis (Fig. 2.8) the rotation matrix has the following form:

$$\mathsf{R}_\xi(\varphi_i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi_i & -\sin\varphi_i \\ 0 & \sin\varphi_i & \cos\varphi_i \end{bmatrix}. \tag{2.27}$$

For the rotation of value φ_i around the η axis

$$\mathsf{R}_\eta(\varphi_i) = \begin{bmatrix} \cos\varphi_i & 0 & \sin\varphi_i \\ 0 & 1 & 0 \\ -\sin\varphi_i & 0 & \cos\varphi_i \end{bmatrix}. \tag{2.28}$$

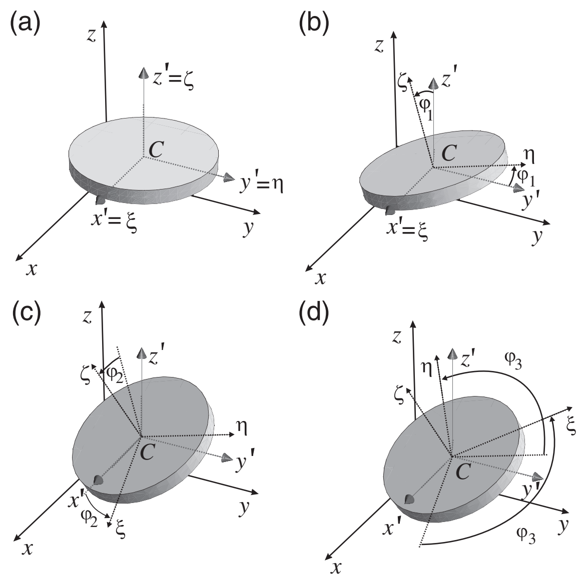


Fig. 2.8 Initial position of a body (**a**) and the rotation sequence: φ_1 – around the ξ (**b**), φ_2 – around the η (**c**), φ_3 – around the ζ (**d**)

The rotation of value φ_i around the ζ axis leads to

$$\mathsf{R}_\zeta(\varphi_i) = \begin{bmatrix} \cos\varphi_i & -\sin\varphi_i & 0 \\ \sin\varphi_i & \cos\varphi_i & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{2.29}$$

Depending on the order of rotations around the sequence of chosen axes there are 12 possible variants of this method determining the position of a body in space. Denoting the axes $\xi\eta\zeta$ of the body embedded frame by symbols *1, 2, 3* ($\xi \to$ *1*, $\eta \to$ *2*, $\zeta \to$ *3*), possible rotation sequences can be represented as follows: *121* (ξ-η-ξ), *123* (ξ-η-ζ), *131* (ξ-ζ-ξ), *132* (ξ-ζ-η), *212* (η-ξ-η), *213* (η-ξ-ζ), *231* (η-ζ-ξ), *232* (η-ζ-η), *312* (ζ-ξ-η), *313* (ζ-ξ-ζ), *321* (ζ-η-ξ), *323* (ζ-η-ζ).

Classical Euler angles are: $\varphi_1 = \psi$, $\varphi_2 = \vartheta$, $\varphi_3 = \varphi$, that indicate consecutively the rotations: by angle $\varphi_1 = \psi$ around the ζ axis, by $\varphi_2 = \vartheta$ around the new position of the ξ axis, and by $\varphi_3 = \varphi$ around the ζ (rotation sequence is abbreviated as ζ-ξ-ζ or *313*). The rotation matrix R for such angles is obtained by substituting into (2.26) the following rotational matrices:

$$\mathsf{R}_1(\varphi_1) = \mathsf{R}_\zeta(\psi) , \quad \mathsf{R}_2(\varphi_2) = \mathsf{R}_\xi(\vartheta) , \quad \mathsf{R}_3(\varphi_3) = \mathsf{R}_\zeta(\varphi) , \tag{2.30}$$

that means

$$\mathsf{R} = \mathsf{R}_\zeta(\psi)\mathsf{R}_\xi(\vartheta)\mathsf{R}_\zeta(\varphi) . \tag{2.31}$$

In Table 2.1 the set of 12 possible rotation sequences and the singularity condition for each case and values of angles causing singularities in numerical solutions are presented. These singularities arise in the inversion process of the matrix B, which is used to calculate the generalized velocities $\dot{\mathsf{q}}$ on the basis of body angular velocity $\tilde{\boldsymbol{\omega}}$ ($\dot{\mathsf{q}} = \mathsf{B}^{-1}\tilde{\boldsymbol{\omega}}$).

Table 2.1 Singularity condition in the numerical analysis of the rigid body dynamics ($k = 0, 1, \ldots$)

Rotation sequence	Singularity condition	Angle indicating singularity
121	$-\sin\varphi_2 = 0$	$\varphi_2 = \pm k\pi$
123	$\cos\varphi_2 = 0$	$\varphi_2 = \frac{\pi}{2} \pm k\pi$
131	$-\sin\varphi_2 = 0$	$\varphi_2 = \pm k\pi$
132	$-\cos\varphi_2 = 0$	$\varphi_2 = \frac{\pi}{2} \pm k\pi$
212	$-\sin\varphi_2 = 0$	$\varphi_2 = \pm k\pi$
213	$-\cos\varphi_2 = 0$	$\varphi_2 = \frac{\pi}{2} \pm k\pi$
231	$\cos\varphi_2 = 0$	$\varphi_2 = \frac{\pi}{2} \pm k\pi$
232	$-\sin\varphi_2 = 0$	$\varphi_2 = \pm k\pi$
312	$\cos\varphi_2 = 0$	$\varphi_2 = \frac{\pi}{2} \pm k\pi$
313	$-\sin\varphi_2 = 0$	$\varphi_2 = \pm k\pi$
321	$-\cos\varphi_2 = 0$	$\varphi_2 = \frac{\pi}{2} \pm k\pi$
323	$-\sin\varphi_2 = 0$	$\varphi_2 = \pm k\pi$

2.3.2 Euler's Parameters

An alternative to Euler's angles and similar conventions of body orientation description are Euler's parameters (also called Euler symmetric parameters and known in mathematics as normalized quaternions) [9, 10, 12]. They are very useful in representing the rotations due to some advantages in comparison to other representations. The main advantage of Euler parameters is that they do not produce any singularities in numerical solutions of body motion equations.

In the matrix notation Euler's parameters are represented by the column matrix

$$\mathsf{p} = \begin{bmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{bmatrix} = \begin{bmatrix} \cos\frac{\phi}{2} \\ v_1 \sin\frac{\phi}{2} \\ v_2 \sin\frac{\phi}{2} \\ v_3 \sin\frac{\phi}{2} \end{bmatrix} , \tag{2.32}$$

or, shortly

$$\mathsf{p} = \begin{bmatrix} e_0 \\ \mathsf{e} \end{bmatrix} = \begin{bmatrix} \cos\frac{\phi}{2} \\ \mathsf{v}^o \sin\frac{\phi}{2} \end{bmatrix} . \tag{2.33}$$

The basis of this method of body orientation description is well known as the Euler theorem stating that any rotation of the rigid body can be expressed as a single rotation about some axis. The axis can be represented by 3D vector $\mathbf{v^o}$ (Fig. 2.9). The vector $\mathbf{v^o}$ is a unit vector and it remains unchanged during the body rotation. The rotation angle ϕ is a scalar value.

The rotation matrix (2.25) can be expressed by Euler parameters as [10]

$$\mathsf{R} = (2e_0^2 - 1)\mathsf{I} + 2\mathsf{e}\mathsf{e}^\mathsf{T} + 2e_0\mathsf{E} , \tag{2.34}$$

in which I is the identity matrix of dimensions (3×3) and the matrix E has the form

$$\mathsf{E} = \begin{bmatrix} 0 & -e_3 & e_2 \\ e_3 & 0 & -e_1 \\ -e_2 & e_1 & 0 \end{bmatrix} . \tag{2.35}$$

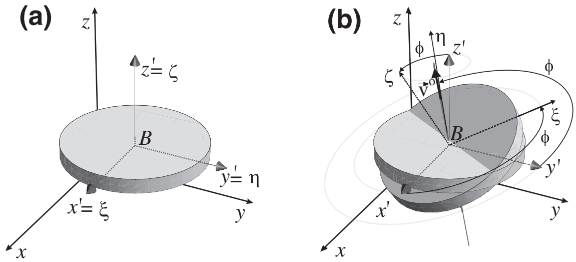


Fig. 2.9 The body rotation by ϕ – around the vector $\mathbf{v^o}$

The expanded form of the matrix R (2.34) – expressed by unit quaternions $(e_0, \ldots, e_3)$ – has the following form:

$$\mathsf{R} = \begin{bmatrix} -1 + 2e_0^2 + 2e_1^2 & 2e_1e_2 - 2e_0e_3 & 2e_0e_2 + 2e_1e_3 \\ 2e_1e_2 + 2e_0e_3 & -1 + 2e_0^2 + 2e_2^2 & -2e_0e_1 + 2e_2e_3 \\ -2e_0e_2 + 2e_1e_3 & 2e_0e_1 + 2e_2e_3 & -1 + 2e_0^2 + 2e_3^2 \end{bmatrix}. \tag{2.36}$$

The antisymmetric matrix $\tilde{\boldsymbol{\Omega}}_\xi$ ($\tilde{\boldsymbol{\Omega}}_\xi = \mathsf{R}^\mathsf{T}\dot{\mathsf{R}}$) containing scalar components of the coin angular velocity vector – in the body embedded frame $\xi\eta\zeta$ – has the form

$$\tilde{\boldsymbol{\Omega}}_\xi = 2\begin{bmatrix} 0 & -\dot{e}_3e_0 + \dot{e}_2e_1 - \dot{e}_1e_2 + \dot{e}_0e_3 & \dot{e}_2e_0 + \dot{e}_3e_1 - \dot{e}_0e_2 - \dot{e}_1e_3 \\ & 0 & -\dot{e}_1e_0 + \dot{e}_0e_1 + \dot{e}_3e_2 - \dot{e}_2e_3 \\ & \boldsymbol{asym.} & 0 \end{bmatrix}. \tag{2.37}$$

The angular velocity vector of the coin $\tilde{\boldsymbol{\omega}}_\xi$ in the body embedded frame $\xi\eta\zeta$ is expressed by the column matrix

$$\tilde{\boldsymbol{\omega}}_\xi = \begin{bmatrix} \omega_\xi \\ \omega_\eta \\ \omega_\zeta \end{bmatrix} = 2\begin{bmatrix} \dot{e}_1e_0 - \dot{e}_0e_1 - \dot{e}_3e_2 + \dot{e}_2e_3 \\ \dot{e}_2e_0 + \dot{e}_3e_1 - \dot{e}_0e_2 - \dot{e}_1e_3 \\ \dot{e}_3e_0 - \dot{e}_2e_1 + \dot{e}_1e_2 - \dot{e}_0e_3 \end{bmatrix}, \tag{2.38}$$

whereas

$$\dot{\tilde{\boldsymbol{\omega}}}_\xi = \begin{bmatrix} \dot{\omega}_\xi \\ \dot{\omega}_\eta \\ \dot{\omega}_\zeta \end{bmatrix} = 2\begin{bmatrix} -e_1\ddot{e}_0 + e_0\ddot{e}_1 + e_3\ddot{e}_2 - e_2\ddot{e}_3 \\ -e_2\ddot{e}_0 - e_3\ddot{e}_1 + e_0\ddot{e}_2 + e_1\ddot{e}_3 \\ -e_3\ddot{e}_0 + e_2\ddot{e}_1 - e_1\ddot{e}_2 + e_0\ddot{e}_3\,. \end{bmatrix}. \tag{2.39}$$

The column matrix containing xyz scalar components of the coin angular velocity vector (i.e., the components in fixed spatial frame) has the form

$$\tilde{\boldsymbol{\omega}} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix} = 2\begin{bmatrix} \dot{e}_1e_0 - \dot{e}_0e_1 + \dot{e}_3e_2 - \dot{e}_2e_3 \\ \dot{e}_2e_0 - \dot{e}_3e_1 - \dot{e}_0e_2 + \dot{e}_1e_3 \\ \dot{e}_3e_0 + \dot{e}_2e_1 - \dot{e}_1e_2 - \dot{e}_0e_3 \end{bmatrix}. \tag{2.40}$$

Using Euler's parameters allows to avoid singularities in numerical solutions of body rotation problems. The coin dynamics is one of such problems. The equations of coin dynamics are presented in the following sections. Special attention is paid to the coin dynamics description in quaternions.

2.4 Air Resistance Forces and Moments

To define the right-hand sides of (2.3)–(2.4) or (2.8)–(2.9) the components of the forces acting on the body and moments of these forces with respect to the $C\xi\eta\zeta$ frame axis should be determined.

The determination of the air resistance forces acting on the body is cumbersome due to its variable velocity and a change in the Reynolds number ($Re = \frac{2rv}{\nu}$) that follows during the toss [13]. (A range of changes in the Reynolds number is wide because of variations in the body velocity: $Re << 1$ in the initial stage of its motion (during its fall with a zero initial velocity), $Re \cong 10^4$ – at the velocity of the mass center $v = 10$ m/s and the coin radius $r = 0.01$ m.)

For detailed information on the aerodynamics and fluid mechanics pertinent to this section, see [13–17].

On the basis of kinematic relations for a rigid body, a distribution of velocities on outer surfaces of the body is determined. It has been assumed that the air resistance occurs only on this part of the body surface on which the velocity vectors of its points have a sense compliant with the normal vector to this surface and directed outward from the body. The air resistance force vector $\mathbf{f}_r$ will be determined as a sum of resistance forces ($\mathbf{f}_i$) on all planes of a body

$$\mathbf{f}_r = \sum_{i=1}^{n_s} \mathbf{f}_i \ . \tag{2.41}$$

Each of the vectors $\mathbf{f}_i$ is treated as a sum of two components originating from tangential (parallel) and normal (perpendicular) forces to the body surface (S_i). This results in the necessity of using various air resistance coefficients λ_τ for the forces $\mathbf{f}_{i\tau}$ (of the tangential direction (air friction forces)) and λ_n – for the forces $\mathbf{f}_{in}$ in the normal direction (air pressure forces). Hence

$$\mathbf{f}_i = \mathbf{f}_{i\tau} + \mathbf{f}_{in} \ , \tag{2.42}$$

where $\mathbf{f}_{1\tau}$ and $\mathbf{f}_{2\tau}$ are determined from the relations

$$\mathbf{f}_{i\tau} = -\lambda_\tau \int_{S_i} |\mathbf{v}_{Ai\tau}|^b \, \mathbf{v}_{Ai\tau} \, dS_i \ , \tag{2.43}$$

$$\mathbf{f}_{in} = -\lambda_n \int_{S_i} |\mathbf{v}_{Ain}|^b \, \mathbf{v}_{Ain} \, s_i \, dS_i \ . \tag{2.44}$$

The symbols used in formulas (2.43)–(2.44) denote, respectively, $\mathbf{v}_{Ai\tau}$ – vector comprising the velocity components tangential to the surface on which the point A_i is situated, $\mathbf{v}_{Ain}$ – vector of velocity components perpendicular to this surface, $|\mathbf{v}_{Ai\tau}|$ and $|\mathbf{v}_{Ain}|$ refer to the values of velocity vectors, and b is a real number that belongs to the range $< 0, 1 >$ (for $b = 0$, the air resistance is linearly dependent on velocity, whereas for $b = 1$, resistance depends on the square of velocity). The functions s_i that occur in (2.44) are described by the following relation:

$$s_i = \frac{1}{2} \operatorname{sign}(\mathrm{v}_{Ain}^T \tilde{\boldsymbol{\eta}}_i^o) \left(1 + (\mathrm{v}_{Ain}^T \tilde{\boldsymbol{\eta}}_i^o)\right) \quad (i = 1, 2, 3) \ , \tag{2.45}$$

where $\boldsymbol{\eta}_i^o$ are unit vectors, normal to the body surfaces under consideration, directed outward from the body.

The functions s_i allow for the determination of regions in which the air resistance components normal to the surface act. It has been assumed that normal components of resistance forces do not occur in these regions where the velocity vector of points situated on the surface is directed inward the body. The air resistance forces perpendicular to the body surface occur in these body points in which the following condition is satisfied:

$$\mathsf{v}_{Ain}^T \tilde{\boldsymbol{\eta}}_i^o > 0 \quad (i = 1, 2, 3)\ . \tag{2.46}$$

The effect of application of the functions s_i is illustrated in Fig. 3.5.

After the determination of forces, the moments of these forces with respect to the center of mass C (or the chosen pole B) should be determined. The total moment of air resistance forces with respect to the point B has been presented as a sum of the moments m_i originating from resistance forces on all surfaces of the body

$$\mathsf{m}_{rB} = \sum_{i=1}^{n_s} \mathsf{m}_i\ , \tag{2.47}$$

where m_i is determined as follows:

$$\begin{aligned} \mathsf{m}_i = &-\lambda_n \int_{S_i} |\mathsf{v}_{Ain}|^b\, \mathsf{R}^\mathsf{T} \mathsf{R}_{BAi} \mathsf{v}_{Ain}\, s_i\, dS_i - \\ &- \lambda_\tau \int_{S_i} |\mathsf{v}_{Ai\tau}|^b\, \mathsf{R}^\mathsf{T} \mathsf{R}_{BAi} \mathsf{v}_{Ai\tau}\, dS_i\ . \end{aligned} \tag{2.48}$$

R_{BAi} in (2.48) denotes antisymmetrical matrix including the components of the vector[5] r_{BAi}.

2.5 Modeling of Bodies Impact

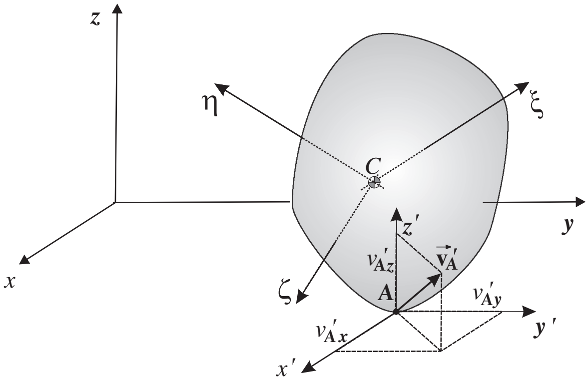
Let us consider that the body collides with a floor when the point A on its surface touches the floor as shown in Fig. 2.10. In the rigid body dynamics the most common model of collision of bodies is based on Newton's hypothesis [5, 7]

$$-\chi = \frac{v'_{Az}}{v_{Az}}\ , \tag{2.49}$$

where χ is the coefficient of restitution ($0 < \chi < 1$), A stands for the body point which makes contact with the floor at the instant of impact (Fig. 2.10), v'_{Az} and

[5] In many textbooks – for example, in [10] – following notation is used $\mathsf{R}_{BAi} = \tilde{\mathsf{r}}_{BAi}$.

Fig. 2.10 Velocity vector $\mathbf{v}'_A$ of the point A after the collision and its scalar components $v_{A\tau}{}'$, $v_{A\nu}{}'$, $v_{Az}{}'$



v_{Az} are projections of the velocity of the point A on the direction (z) normal to the impact surface, respectively, before and after the impact.

Assumption (2.49) means that we include energy losses during the collision and only the momentum is conserved. (In a purely elastic collision of bodies ($\chi = 1$) kinetic energy, linear and angular momentum are conserved.) On the basis of Newton hypothesis we can calculate z component of the reaction force impulse (S_z).

Besides Newton's hypothesis to analyze the phenomena that accompany the impact the laws of linear momentum and angular momentum theorems of a body, as well as supplementary constraint equations, have been employed.

The position of the colliding point (A) in the body embedded frame is described by ξ_A, η_A, ζ_A. Assuming that the collision time is negligibly short the position and the orientation of body after the impact are the same as before the impact.

To describe the motion after the impacts we will use the additional frame with an origin at point A and axis: z' – parallel to the fixed axis z and axes τ, ν in the ground plane (Fig. 2.10).

Mosekilde [18] and Feldberg et al. [19] proposed to obtain the value of χ considering free vibrations of dumped oscillator described by equation $m\ddot{z} + \kappa\dot{z} + kz = 0$. It means that during the collision the body vibrates on visco-elastic foundation (κ and k are foundation damping and elasticity coefficients). The illustration for such a model is shown in Fig. 2.11 and sample results are presented in Fig. 2.12.

Comparing the oscillations velocity v_z for $t_0 = 0$ and $t_1 = T/2$ (where $T = 4\pi/\sqrt{\frac{4km-\kappa^2}{m^2}}$) one finds (Fig. 2.12)

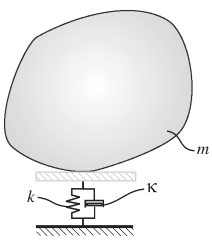


Fig. 2.11 Body on visco-elastic foundation

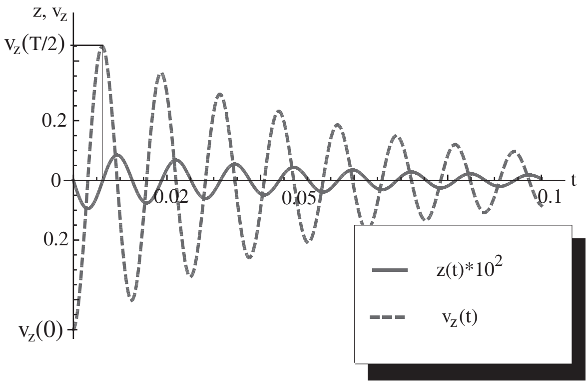


Fig. 2.12 Displacement and velocity of dumped oscillator

$$\chi = \frac{v_z(T/2)}{v_z(0)} = e^{-\frac{2\pi\kappa}{\sqrt{4km-\kappa^2}}} . \tag{2.50}$$

To obtain additional constraint equations we have to consider the model of the contact between the colliding bodies. For specific collision cases (coin, dice, roulette ball) constraint equations are different and are presented in next sections.

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