Chapter 4
Examples of Indexed PIP-Spaces

This chapter is devoted to a detailed analysis of various concrete examples of PIP-spaces. We will explore sequence spaces, spaces of measurable functions, and spaces of analytic functions. Some cases have already been presented in Chapters 1 and 2. We will of course not repeat these discussions, except very briefly. In addition, various functional spaces are of great interest in signal processing (amalgam spaces, modulation spaces, Besov spaces, coorbit spaces). These will be studied systematically in a separate chapter (Chapter 8).

4.1 Lebesgue Spaces of Measurable Functions

4.1.1 L^p Spaces on a Finite Interval

Our first example is the family of Lebesgue spaces over the interval [0, 1] with their usual norm topology:

$$\mathcal{I} = \{L^p := L^p([0,1]; dx),\ 1 < p < \infty\}.$$

These spaces form a chain: $p > q$ implies $L^p \subset L^q$, the embedding is continuous and has dense image. With the involution

$$L^p \leftrightarrow (L^p)^\times = L^{\overline{p}},\ p^{-1} + \overline{p}^{-1} = 1,$$

$\mathcal{I}$ is an involutive covering of the space $V = \bigcup_{1<p<\infty} L^p = \sum_{1<p<\infty} L^p$.[1] Given the corresponding compatibility $\#$, i.e., $(L^p)^\# = L^{\overline{p}}$, we can compute explicitly the complete involutive lattice $\mathcal{F}(V, \#)$.

[1] Technically, the inductive limit $\overrightarrow{\lim}_{1<p<\infty} L_p$. See Appendix B.

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Remark: Notice that we do not include the space L^1. By symmetry this would demand inclusion of L^∞ as well, which would invalidate some of the statements made below about duality properties.

First we evaluate "elements of the first generation" of $\mathcal{F}$. Given an arbitrary subset S of $(1,\infty)$, define the spaces

$$L^P(S) := \bigcap_{p\in S} L^p, \qquad L^I(S) := \bigcup_{p\in S} L^p = \sum_{p\in S} L^p,$$

Introducing $r = \inf S, t = \sup S$, and defining $\widehat{S} = (1,t)\bigcup S$, $\widetilde{S} = (r,\infty) \bigcup S$, one shows that:

$$L^P(S) = L^P(\widehat{S}), \qquad L^I(S) = L^I(\widehat{S}). \tag{4.1}$$

This leaves us with four possible cases:

(i) $t \in S \Rightarrow \widehat{S} = (1,t]$ and $L^P(S) = \bigcap_{1<q\leqslant t} L^q = L^t$;
(ii) $t \notin S \Rightarrow \widehat{S} = (1,t)$ and $L^P(S) = \bigcap_{1<q<t} L^q := L^{t-}$;
(iii) $r \in S \Rightarrow \widetilde{S} = [r,\infty)$ and $L^I(S) = \bigcup_{r\leqslant q<\infty} L^q = L^r$;
(iv) $r \notin S \Rightarrow \widetilde{S} = (r,\infty)$ and $L^I(S) = \bigcup_{r<q<\infty} L^q := L^{r+}$.

Thus we get two new types of spaces,[2] namely the spaces $L^{p\pm}$. Their topological properties are based on the observation that, in the definition of L^{p-}, it is enough [by Eq.(4.1)] to consider a cofinal countable subset of the V_q's. Therefore, we get

(i) For $1 < p \leqslant \infty$, $L^{p-} := \bigcap_{1<q<p} L^q$, with the projective topology, is a non-normable, reflexive Fréchet space, hence barreled and complete, with conjugate dual $(L^{p-})^\times = (L^{p-})^\# = L^{\overline{p}+}$. In particular, $L^{\infty-}$ coincides with the space L^ω of Arens (also called the Arens algebra).
(ii) For $1 \leqslant p < \infty, L^{p+} := \bigcup_{p<q<\infty} L^q$, with the inductive topology, is a nonmetrizable (Mackey) complete, barreled topological vector space, with conjugate dual $(L^{p+})^\times = (L^{p+})^\# = L^{\overline{p}-}$.
(iii) Furthermore, the following inclusions are proper: $L^{p+} \subset L^p \subset L^{p-}$ $(1 < p < \infty)$, the embeddings are continuous and have dense range.

Proposition 4.1.1. *Let $\mathcal{I}$ be the chain $\{L^p, 1 < p < \infty\}$. Then the complete lattice $\mathcal{F}(V,\#)$ generated by $\mathcal{I}$ is also a chain, obtained by replacing each L^p in $\mathcal{I}$ by the triplet $L^{p+} \subset L^p \subset L^{p-}$, and adding the smallest element $L^{\infty-}$ and the largest element L^{1+}.*

Proof. First we evaluate elements of the form $\{f\}^{\#\#}$, then $\{f\}^\#$, and finally arbitrary elements of $\mathcal{F}(V,\#)$ through the usual relation:

$$\mathcal{F}(V,\#) \ni V_r = \bigcap_{f\in V_{\overline{r}}} \{f\}^\#.$$

[2] Nonstandard analysis could also be used here: V_{p+} is really $V_{p+\epsilon}$, ϵ infinitesimal.

Let $f \in V$. Then, by Eq.(1.9),

$$\{f\}^{\#\#} = \bigcap_{L^p \ni f} L^p = L^{\overline{q}} \text{ or } L^{\overline{q}-}, \text{ for some } \overline{q} \in (1, \infty).$$

Therefore $\{f\}^{\#} = L^q$ or L^{q+} for some $q \in (1, \infty)$. Finally

$$V_r = \bigcap_{f \in V_r} \{f\}^{\#} = \Big(\bigcap_{q \in S} L^q\Big) \cap \Big(\bigcap_{r \in T} L^{r+}\Big),$$

where S, T are some subsets of $(1, \infty)$, to be replaced by $\widehat{S}, \widehat{T}$ respectively.

For the first term on the right-hand side, we get

$$\bigcap_{q \in S} L^q = \bigcap_{q \in \widehat{S}} L^q = L^s \text{ or } L^{s-}, \quad \text{with } s = \sup S.$$

As for the second term, we observe that $r_1 < r_2$ implies $L^{r_2+} \subset L^{r_1+}$ with continuous embedding; the set inclusion is obvious, and the embedding $L^{r_2+} \hookrightarrow L^{r_1+}$ can be factorized continuously through L^r, where r is any real number such that $r_1 < r < r_2$. Therefore, all the spaces L^{r+}, $1 \leqslant r < \infty$, form a chain with continuous embeddings. Thus we get

$$\bigcap_{r \in T} L^{r+} = L^{t+}, \text{ where } t = \sup T.$$

Finally, V_r is either of the form $L^s \cap L^{t+}$, or of the form $L^{s-} \cap L^{t+}$. For $s \leqslant t$, we have $L^{s-} \supset L^s \supset L^{t+}$ and $V_r = L^{t+}$. For $s > t$, we have $L^{t+} \supset L^{s-} \supset L^s$, so that $V_r = L^s$ or $V_r = L^{s-}$. This concludes the proof. ■

Notice that, in addition to its PIP-space structure, the family $\mathcal{I}$ generates a partial *-algebra under pointwise multiplication.

4.1.2 The Spaces $L^p(\mathbb{R}, dx)$

We turn now to the L^p spaces on $\mathbb{R}$. If we consider the family $\{L^p(\mathbb{R}) \cap L^1(\mathbb{R}), 1 \leqslant p \leqslant \infty\}$, we obtain a scale similar to the previous one (except that the individual spaces are not complete), which may be used to endow $L^1(\mathbb{R})$ with a PIP-space structure.

However, the spaces $L^p(\mathbb{R})$ themselves no longer form a chain, no two of them being comparable. We have only

$$L^p \cap L^q \subset L^s, \text{ for all } s \text{ such that } p < s < q.$$

Hence we have to take the lattice generated by $\mathcal{I} = \{L^p(\mathbb{R}, dx), 1 \leqslant p \leqslant \infty\}$, that we call $\mathcal{J}$. The extreme spaces of the lattice are, respectively:

$$V_{\mathrm{J}}^{\#} = \bigcap_{1 \leqslant q \leqslant \infty} L^q, \quad \text{and} \quad V_{\mathrm{J}} = \bigcup_{1 \leqslant q \leqslant \infty} L^q = \sum_{1 \leqslant q \leqslant \infty} L^q. \tag{4.2}$$

Here too, the lattice structure allows to give to V_{J} a structure of a PIP-space and of a locally convex partial *-algebra.

The lattice operations on $\mathcal{J}$ are easily described:

- $L^p \wedge L^q = L^p \cap L^q$ is a Banach space for the projective norm $\|f\|_{p \wedge q} = \|f\|_p + \|f\|_q$.
- $L^p \vee L^q = L^p + L^q$ is a Banach space for the inductive norm

$$\|f\|_{p \vee q} = \inf_{f=g+h} \{\|g\|_p + \|h\|_q ; g \in L^p,\, h \in L^q\}.$$

- For $1 < p, q < \infty$, both spaces $L^p \wedge L^q$ and $L^p \vee L^q$ are reflexive and $(L^p \wedge L^q)^{\times} = L^{\overline{p}} \vee L^{\overline{q}}$.

At this stage, it is convenient to introduce a unified notation:

$$L^{(p,q)} = \begin{cases} L^p \wedge L^q, & \text{if } p \geqslant q, \\ L^p \vee L^q, & \text{if } p \leqslant q. \end{cases}$$

Thus, for $1 < p, q < \infty$, each space $L^{(p,q)}$ is a reflexive Banach space, with conjugate dual $L^{(\overline{p},\overline{q})}$. The modifications when p, q equal 1 or ∞ are obvious.

Next, if we represent (p, q) by the point of coordinates $(1/p, 1/q)$, we may associate all the spaces $L^{(p,q)}$ $(1 \leqslant p, q \leqslant \infty)$ in a one-to-one fashion with the points of a unit square $\mathrm{J} = [0, 1] \times [0, 1]$ (see Fig. 4.1). Thus, in this picture, the spaces L^p are on the main diagonal, intersections $L^p \cap L^q$ above it and sums $L^p + L^q$ below. The space $L^{(p,q)}$ is contained in $L^{(p',q')}$ if (p, q) is on the left and/or above (p', q'). Thus the smallest space is

$$V_{\mathrm{J}}^{\#} = L^{(\infty,1)} = L^{\infty} \cap L^1$$

and it corresponds to the upper left corner, the largest one is

$$V_{\mathrm{J}} = L^{(1,\infty)} = L^1 + L^{\infty},$$

corresponding to the lower right corner. Inside the square, duality corresponds to (geometrical) symmetry with respect to the center (1/2,1/2) of the square, which represents the space L^2. The ordering of the spaces corresponds to the following rule:

$$L^{(p,q)} \subset L^{(p',q')} \iff (p, q) \leqslant (p', q') \iff p \geqslant p' \text{ and } q \leqslant q'. \tag{4.3}$$

By the way, this rule shows that the spaces L^p on the main diagonal are not comparable, as we know.

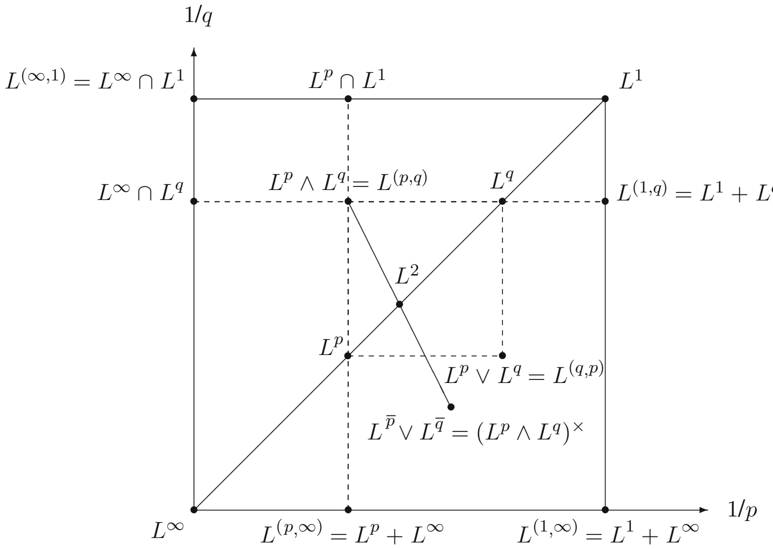


Fig. 4.1 The unit square describing the lattice J

For $\infty \geqslant q_o \geqslant 1$, consider now the horizontal row $q = q_o$, $\{L^{(p,q_o)} : \infty \geqslant p \geqslant 1\}$. It corresponds to the chain:

$$L^\infty \cap L^{q_o} \subset \ldots \subset L^r \cap L^{q_o} \subset \ldots \subset L^{q_o} \subset \ldots \subset L^s + L^{q_o} \subset \ldots \subset L^1 + L^{q_o}. \quad (4.4)$$
$$(\infty > r > q_o > s > 1)$$

The point is that all the embeddings in the chain (4.4) are continuous and have dense range. The same holds true for a vertical row $p = p_o$, $\{L^{(p_o,q)} : 1 \leqslant q \leqslant \infty\}$:

$$L^{p_o} \cap L^s \subset \ldots \subset L^{p_o} \cap L^s \subset \ldots \subset L^{p_o} \subset \ldots \subset L^{p_o} + L^r \subset \ldots \subset L^{p_o} + L^\infty. \quad (4.5)$$
$$(1 < s < p_o < r < \infty)$$

Combining these two facts, we see that the partial order extends to the spaces $L^{(p,q)}$ $(1 \leqslant p, q \leqslant \infty)$, inclusion meaning now continuous embedding with dense range.

Now the set of points contained in the square J may be considered as an involutive lattice with respect to the partial order (4.3), with operations:

$$\begin{aligned}(p,q) \wedge (p',q') &= (p \vee p', q \wedge q')\\ (p,q) \vee (p',q') &= (p \wedge p', q \vee q')\\ \overline{(p,q)} &= (\overline{p}, \overline{q}),\end{aligned}$$

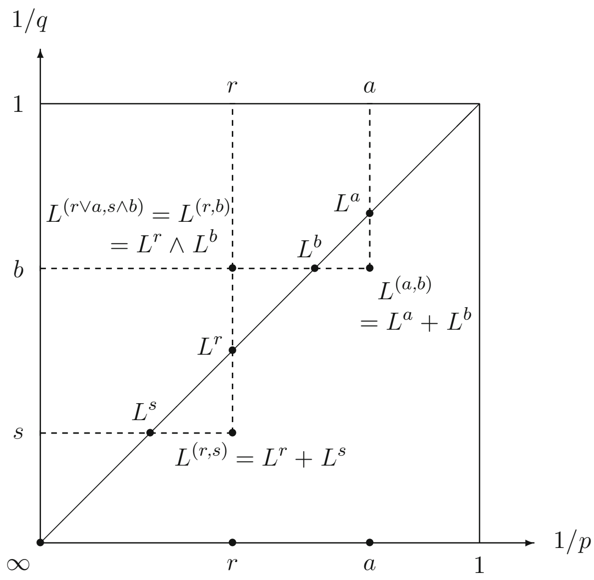


Fig. 4.2 The intersection of two spaces from J

where, as usual, $p \wedge p' = \min\{p, p'\},\ p \vee p' = \max\{p, p'\}$.

The considerations made above imply that the lattice $\mathcal{J}$ generated by $\mathcal{I} = \{L^p\}$ is already obtained at the first generation. For example, $L^{(r,s)} \wedge L^{(a,b)} = L^{(r\vee a, s\wedge b)}$ (see Fig. 4.2), and the latter may be either above, on or below the diagonal, depending on the values of the indices. For instance, if $p < q < s$, then $L^{(p,q)} \wedge L^{(q,s)} = L^q$, both as sets and as topological vector spaces.

The conclusion is that, using this language, the only difference between the two cases $\{L^p([0,1])\}$ and $\{L^p(\mathbb{R})\}$ lies in the type of order obtained: a chain I (total order) or a partially ordered lattice J. From this remark, the lattice completion of $\mathcal{J}$ can be obtained exactly as before. This introduces again Fréchet and DF-spaces, all reflexive if we start from $1 < p < \infty$, and in natural duality as in the previous case. In particular, for the spaces of the first generation, it suffices to consider intervals $S \subset [1, \infty]$ and define the spaces

$$L^{\mathrm{proj}}(S) = \bigcap_{q\in S} L^q, \quad L^{\mathrm{ind}}(S) = \bigcup_{q\in S} L^q.$$

Then:

- If S is a closed interval $S = [p, q]$, with $p < q$, then $L^{\mathrm{proj}}(S) = L^p \wedge L^q = L^{(q,p)}$ and $L^{\mathrm{ind}}(S) = L^p \vee L^q = L^{(p,q)}$ are Banach spaces. If S is a semi-open or open interval, $L^{\mathrm{proj}}(S)$ is a non-normable Fréchet space and $L^{\mathrm{ind}}(S)$ a DF-space.
- Let $S \subset (1, \infty)$ and define $\overline{S} = \{\overline{q} : q \in S\}$. Then $(L^{\mathrm{proj}}(S))^\times = L^{\mathrm{ind}}(\overline{S})$ and $(L^{\mathrm{ind}}(S))^\times = L^{\mathrm{proj}}(\overline{S})$.

A special rôle will be played in the sequel by the spaces L^{ind} corresponding to semi-infinite intervals, namely:

$$L^{(p,\infty)} = L^{\text{ind}}([p,\infty]) = \bigcup_{p \leqslant s \leqslant \infty} L^s = L^p + L^\infty, \text{ a nonreflexive Banach space,}$$
$$L^{(p,\omega)} = L^{\text{ind}}([p,\infty)) = \bigcup_{p \leqslant s < \infty} L^s, \quad \text{a reflexive DF-space.}$$

As for the lattice completion $\mathcal{F}_{\mathrm{J}}$, one can build an 'enriched' or 'nonstandard' square F, exactly as in the previous section. Take first $1 < q < \infty$, that is, the interior $\mathrm{J_o}$ of the square J. The extreme spaces of the corresponding complete lattice $\mathcal{F}_{\mathrm{o}}$ are:

$$V^{\#}_{\mathrm{J_o}} = \bigcap_{1<q<\infty} L^q, \quad \text{and} \quad V_{\mathrm{J_o}} = \bigcup_{1<q<\infty} L^q = \sum_{1<q<\infty} L^q,$$

with their projective and inductive topologies, respectively. All embeddings are continuous and have dense range.

Similar results are valid when one includes L^1 and L^∞, except for the obvious modifications concerning duality. The extreme spaces of the full lattice $\mathcal{F}_{\mathrm{J}}$ then are $V_{\mathrm{J}} = L_G := L^1 + L^\infty$ and $V_{\mathrm{J}}{}^{\#} = L_G^{\#} = L^1 \cap L^\infty$, with their inductive and projective norms, respectively, which make them into nonreflexive Banach spaces (none of them is the dual of the other). Notice that the space L_G, known as the space of Gould, contains strictly all the L^p, $1 \leqslant p \leqslant \infty$.

Proposition 4.1.2. *The space $L_G := L^1(\mathbb{R}, dx) + L^\infty(\mathbb{R}, dx)$ is a nonreflexive LBS, generated by the family $\mathcal{I} = \{L^p(\mathbb{R}, dx), 1 \leqslant p \leqslant \infty\}$ and the corresponding compatibility $(L^p)^{\#} = L^{\overline{p}}$.*

Exactly as in the case of a finite interval, we may restrict the generating spaces to $\{L^s,\ p \leqslant s \leqslant q)$, which amounts to take a subsquare $\mathrm{J}^{(p,q)}$ of J. The rest is obvious.

4.1.3 Reflexive Chains of Banach Spaces

The chain $\mathcal{I} = \{L^p := L^p([0,1]; dx),\ 1 < p < \infty\}$ of Lebesgue spaces over the interval [0, 1] is the prototype of a whole class of chains of reflexive Banach spaces, with exactly the same properties.

Let $I_o = (a, b)$ be an open interval of $\mathbb{R}$, and for each $p \in I_o$ let there be a reflexive Banach space V_p. We say that the family $\mathcal{I} = \{V_p\}_{p \in I_o}$ is *a (continuous) reflexive chain of Banach spaces* if the two following conditions hold:

(i) $p < q \ \Rightarrow\ V_p \subsetneqq V_q$, the inclusion map is injective and continuous;[3]

[3] *Warning:* The order here is direct: $p \mapsto V_p$ is monotone *increasing*, whereas it was *decreasing* in the Lebesgue case (inverse order).

(ii) I_o carries an involution $p \leftrightarrow \overline{p}$ such that $V_{\overline{p}}$ is the conjugate dual of V_p, that is, the norm $\| \cdot \|_{\overline{p}}$ on $V_{\overline{p}}$ is the conjugate of the norm $\| \cdot \|_p$ on V_p.

In other words, a reflexive chain of Banach spaces is a totally ordered LBS. The following properties follow easily from the definition:

(a) $\mathcal{I}$ is an involutive covering of $V := \bigcup_{p \in I_o} V_p = \sum_{p \in I_o} V_p$, corresponding to the compatibility $(V_p)^\# = V_{\overline{p}}$.
(b) Whenever $p < q$, the inclusion map $V_p \to V_q$ has dense range and each V_p is dense in V (considered as the inductive limit $\varinjlim_{p \in I_o} V_p$).
(c) $V^\# := \bigcap_{p \in I_o} V_p$ is a dense subspace of every V_p, and of V as well.

Let now $I = [a, b]$ be a closed interval. The analogous definition is obvious, the only difference being that now $V^\# = V_a, V = V_b$. Within this context, a proposition similar to Proposition 4.1.1. can be formulated. As before we define

$$V_{p-} := \bigcup_{q<p} V_q, \quad V_{p0} := V_p, \quad V_{p+} := \bigcap_{p<q} V_q.$$

Let F_0 be the totally ordered set $F_0 = (a, b) \times \{-, 0, +\}$ with its lexicographic order $(- < 0 < +)$. Then we have:

Proposition 4.1.3. *Let $\mathcal{I} = \{V_p\}_{p \in I}$ be a reflexive chain of Banach spaces. Then the complete involutive lattice $\mathcal{F} = \mathcal{F}(V, \#)$ is a chain given explicitly as follows:*

(i) If $I = (a, b)$ is an open interval, $\mathcal{F}$ consists of $V^\# := V_{a+}, \ \mathcal{F}_0, \ V := V_{b-}$, where $\mathcal{F}_0$ is a chain indexed by F_0.
(ii) If $I = [a, b]$ is a closed interval, $\mathcal{F}$ is the chain $V^\# := V_a, V_{a+}, \ \mathcal{F}_0, \ V_{b-}$ $V := V_b$.

Proof. The argument is exactly the same as the one given in Proposition 4.1.1 for the case of Lebesgue spaces. First one shows (using the involution $p \leftrightarrow \overline{p}$) that the inclusions $V_{p-} \subset V_p \subset V_{p+}$ are proper. Then, one proves that V_{p+} , with the projective topology, is a reflexive Fréchet space, with dual $V_{\overline{p}-}$, and V_{p-} , with the inductive topology, is a reflexive nonmetrizable, barreled topological vector space, with dual V_{p+}. The rest of the argument is then identical and gives (i); as for (ii), it is obvious. ■

The situation described above is in fact extremely frequent in applications. The following examples of reflexive chains of Banach spaces are all well-known (the first two have the direct order, the other two the inverse order):

(1) The chain of sequence spaces $\{\ell^p, \ 1<p<\infty\}$ (Example (ii) in Section 1.1);
(2) The chain of ideals $\{\mathcal{C}^p(\mathcal{H}), \ 1 < p < \infty\}$ of compact operators in a Hilbert space $\mathcal{H}$, which is isomorphic to (1) (Example (v) in Section 1.1);

(3) The chain of (generalized) Sobolev spaces (also called Bessel potential spaces) H^s $(-\infty < s < \infty)$ defined as follows : A tempered distribution $f \in \mathcal{S}^\times(\mathbb{R}^n)$ belongs to H^s if its Fourier transform $\widehat{f}$ verifies

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 (1 + |\xi|^2)^s \, d\xi < \infty;$$

the involution here is $(H^s)^\# = (H^s)^\times = H^{-s}$.

(4) The chain of Banach spaces that generalizes the Lebesgue spaces L^p to operator algebras, namely, the noncommutative L^p spaces associated to a Hilbert algebra (with unit), introduced by Segal.

Among reflexive chains of Banach spaces, we find, of course, chains of hilbertian or Hilbert spaces. We have already encountered several examples, such as the canonical scale built on the powers of a self-adjoint operator in a Hilbert space (Section I.1) or the chain of Hilbert spaces $\mathcal{F}^\rho$ $(-\infty < \rho < \infty)$ in Bargmann's space $\mathcal{E}^\times$ (Example (v) in Section 1.1). The latter will be studied in detail in Section 4.6.2 below. As for the former, we will revisit and refine it in Section 5.2.1.

4.2 Locally Integrable Functions

4.2.1 *A Generating Subset of Locally Integrable Functions*

In this section, we will study in detail the PIP-space $L^1_{\mathrm{loc}}(X, d\mu)$ of locally integrable functions on a measure space (X, μ). First we take $V = L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$, the space of all functions on $\mathbb{R}^n$ that are locally integrable with respect to the Lebesgue measure. With the compatibility.

$$f \# g \iff \int_{\mathbb{R}^n} |f(x)\, g(x)| \, dx < \infty,$$

one has, as usual, $V^\# = L^\infty_{\mathrm{c}}(\mathbb{R}^n, dx)$, the essentially bounded functions of compact support (Section I.2.3 (ii)).

Let $r : \mathbb{R}^n \to \mathbb{R}^+$ be a measurable, a.e. positive function, such that both r and $\overline{r} := r^{-1}$ are locally square integrable. Denote by $L^2(r)$ the Hilbert space of measurable functions $f : \mathbb{R}^n \to \mathbb{C}$ such that fr^{-1} is square integrable. Then we claim:

Proposition 4.2.1. *(i)* $L^\infty_{\mathrm{c}} \subset L^2(r) \subset L^1_{\mathrm{loc}}$.
(ii) The family $\mathcal{I} := \{L^2(r)\}$ *of all such subspaces is a generating sublattice of* $\mathcal{F}(V, \#)$.

Proof. Part (i) is a straightforward verification. As for (ii), let $\{K_j, j = 1, 2 \ldots\}$ be a covering of $\mathbb{R}^n$ by an increasing sequence of compact subsets (e.g. closed balls of radius j), and write $\Omega_j = K_j \setminus K_{j-1}$, so that each Ω_j is relatively compact, $\Omega_j \cap \Omega_k = 0$ if $j \neq k, \mathbb{R}^n = \bigcup_j \Omega_j$. Let $f, g \in V, f\#g$, and $\{\alpha_j\}, \{\beta_j\}$ two arbitrary sequences of positive numbers such that:

$$\sum_{j=1}^{\infty} \alpha_j \int_{\Omega_j} |f(x)|\, dx < \infty, \quad \sum_{j=1}^{\infty} \beta_j \int_{\Omega_j} |g(x)|\, dx < \infty.$$

Define

$$r(x) = \sum_{j=1}^{\infty} \chi_j(x) r_j(x),$$

where

$$r_j(x)^2 = \max(\beta_j, |f(x)|)/\max(\alpha_j, |g(x)|)$$

and χ_j is the characteristic function of Ω_j. Thus $r^{-2}(x) = \sum_{j=1}^{\infty} \chi_j(x) r_j^{-2}(x)$, and both r and r^{-1} are locally square integrable.[4] Furthermore, it is easily shown that $f \in L^2(r)$, $g \in L^2(\overline{r})$. For instance:

$$\begin{aligned}
\int_{\mathbb{R}^n} |f|^2 r^{-2}\, dx &= \sum_{j=1}^{\infty} \int_{\Omega_j} |f|^2 \, \frac{\max(\alpha_j, |g|)}{\max(\beta_j, |f|)}\, dx \\
&\leqslant \sum_{j=1}^{\infty} \int_{\Omega_j} |f| \max(\alpha_j, |g|)\, dx \\
&= \sum_{j=1}^{\infty} \left(\int_{\Omega_j \cap \{x \,|\, |g(x)| \geqslant \alpha_j\}} |fg|\, dx + \int_{\Omega_j \cap \{x \,|\, |g(x)| < \alpha_j\}} \alpha_j |f|\, dx \right) \\
&\leqslant \sum_{j=1}^{\infty} \left(\int_{\Omega_j} |fg| dx + \alpha_j \int_{\Omega_j} |f|\, dx \right) \\
&= \int_{\mathbb{R}^n} |fg| dx + \sum_{j=1}^{\infty} \alpha_j \int_{\Omega_j} |f|\, dx < \infty.
\end{aligned}$$

[4] *Remark*: Elements of V are in fact classes of equivalent functions; if in the definition of r_j, f or g is replaced by a equivalent function, so is r_j, and nothing changes in the argument.

Thus the family $\mathcal{I} = \{L^2(r)\}$ is generating in $L^1_{\rm loc}(\mathbb{R}^n, dx)$. Moreover it is an involutive sublattice of $\mathcal{F}(V, \#)$ with the following lattice operations:

- $L^2(r) \subseteq L^2(s) \Longleftrightarrow r(x) \leqslant s(x)$ a.e.,
- $L^2(r) \wedge L^2(s) = L^2(p), \quad$ with $p(x) = \min\{r(x), s(x)\}$,
- $L^2(r) \vee L^2(s) = L^2(q), \quad$ with $q(x) = \max\{r(x), s(x)\}$,
- $[L^2(r)]^\# = L^2(\overline{r}), \quad$ with $\overline{r}(x) = r^{-1}(x)$.

Remark 4.2.2. These results generalize to the space $L^1_{\rm loc}(X, d\mu)$, where X is a locally compact and σ-compact space (i.e., $X = \bigcup_n K_n, K_n \subset K_{n+1}, K_j$ relatively compact) and μ is a non-negative Radon measure on X. Let

$$L^2(r) := \{f : f \text{ is } \mu\text{-measurable}, \int_X |f|^2 r^{-2} d\mu < \infty, \text{ with } r \text{ measurable}, r > 0 \text{ a.e.}.\}$$

Then one has, with continuous injections :

$$\begin{aligned} &\text{(i)}\;\; L^\infty_{\rm c}(X, d\mu) \subset L^2(r) \;\text{ if and only if } r^{-1} \in L^2_{\rm loc}(X, d\mu), \\ &\text{(ii)}\; L^2(r) \subset L^1_{\rm loc}(X, d\mu) \text{ if and only if } \;\; r \in L^2_{\rm loc}(X, d\mu). \end{aligned} \tag{4.6}$$

We denote by I_1 the set of weight functions r such that $r^{\pm 1} \in L^2_{\rm loc}(X, d\mu)$. Then we have a LHS realization of $L^1_{\rm loc}(X, d\mu)$ (the proof given in Proposition 4.2.1 extends easily to the general case):

$$L^1_{\rm loc}(X, d\mu) = \bigcup_{r \in I_1} L^2(r), \quad L^\infty_{\rm c}(X, d\mu) = \bigcap_{r \in I_1} L^2(r). \tag{4.7}$$

On $L^1_{\rm loc}(X, d\mu)$, the inductive limit of the norm topologies of $L^2(r)$ coincides with the Mackey topology $\tau(L^1_{\rm loc}, L^\infty_{\rm c})$. However, on $L^\infty_{\rm c}(X, d\mu)$, the projective limit $\mathsf{t}_{\rm proj}$ of the norm topologies is coarser than the Mackey topology $\tau(L^\infty_{\rm c}, L^1_{\rm loc})$. Of course, the continuity of the embeddings in (4.6) refers to the Mackey topologies.

Similar results hold with $V = L^2_{\rm loc}(X, d\mu), V^\# = L^2_{\rm c}(X, d\mu)$ and $r^{\pm 1} \in L^\infty_{\rm loc}(X, d\mu)$. Also if X is compact, i.e., $V = L^1(X, d\mu), V^\# = L^\infty(X, d\mu)$, the statement is simply that the family $\{L^2(r), r^{\pm 2} \in L^1(X, d\mu)\}$ is generating.

Let us come back to the simpler case $(X, d\mu) = (\mathbb{R}^n, dx)$. Now, combining (4.7) with the decomposition $\mathbb{R}^n = \bigcup_j \Omega_j$, $\Omega_j = K_j \setminus K_{j-1}$, we obtain, with obvious identifications, three different realizations:

$$L^1_{\rm loc}(\mathbb{R}^n, dx) = \bigcup_{r \in I_1} L^2(r) = \bigcap_{j=1}^{\infty} L^1(K_j) = \bigcap_{j=1}^{\infty} L^1(\Omega_j), \tag{4.8}$$

$$L^\infty_{\rm c}(\mathbb{R}^n, dx) = \bigcap_{r \in I_1} L^2(r) = \bigcup_{j=1}^{\infty} L^\infty(K_j) = \bigcup_{j=1}^{\infty} L^\infty(\Omega_j). \tag{4.9}$$

In these relations, each intersection is meant to carry the projective limit topology and each union the inductive limit topology. Thus, $L^1_{\rm loc}$, as countable projective limit of the spaces $L^1(K_j)$, becomes a nonreflexive Fréchet space, with dual $L^\infty_{\rm c}$. The latter, with its strong topology $\beta(L^\infty_{\rm c}, L^1_{\rm loc})$, is the inductive limit of the spaces $L^\infty(K_j)$, with their norm topologies. Hence it is an (LF)- and a (DF)-space (see Section B.6).

The Mackey topology $\tau(L^\infty_{\rm c}, L^1_{\rm loc})$ coincides with the inductive limit of the Mackey topologies $\tau(L^\infty(K_j), L^1(K_j))$, but it is strictly coarser than the strong topology $\beta(L^\infty_{\rm c}, L^1_{\rm loc})$, since the space is nonreflexive.

In addition, one may equip $L^\infty_{\rm c}$ with the topology γ of compact (or precompact since $L^1_{\rm loc}$ is Fréchet) convergence, i.e., the uniform convergence on the compact subsets of $L^1_{\rm loc}$. Then the various topologies on $L^\infty_{\rm c}$ are related to one another as follows (σ denotes the weak topology).

Proposition 4.2.3. *Let $L^\infty_{\rm c}(\mathbb{R}^n)$ denote the space of measurable, essentially bounded functions with compact support on $\mathbb{R}^n$. Then, with the notations above, one has the following relationships among the various topologies on $L^\infty_{\rm c}$:*

$$\beta \succ \tau \succcurlyeq \mathsf{t}_{\rm proj} \succ \gamma \succ \sigma. \tag{4.10}$$

The space $L^\infty_{\rm c}(\mathbb{R}^n)$ is complete for all topologies, except the weak one σ.

Proof. We know already that $\beta \succ \tau \succcurlyeq \mathsf{t}_{\rm proj} \succ \sigma$ and it is a standard result that $\tau \succcurlyeq \gamma \succ \sigma$. Also $\tau \succ \gamma$, since $L^1_{\rm loc}$ contains convex balanced subsets that are weakly compact but not compact. So it remains to prove that $\mathsf{t}_{\rm proj} \succ \gamma$. By Eq.(4.8) and Tychonoff's theorem, a basis of compact subsets of $L^1_{\rm loc}$ is given by sets of the form $M = \prod_{j=1}^\infty M_j$, where M_j is a compact subset of $L^1(\Omega_j)$. Each M_j is bounded and therefore contained in a ball $B_j := \{f \in L^1(\Omega_j) : \int_{\Omega_j} |f| dx < c_j, c_j > 0\}$. Define

$$r(x) = \sum_{j=1}^\infty \chi_j(x) r_j(x),$$

where

$$r_j(x)^2 = 2^j c_j \max(1, |f(x)|)$$

and χ_j is the characteristic function of Ω_j. Then $r^{\pm 1} \in L^1_{\rm loc}$, and we have, for every $f \in B := \prod_{j=1}^\infty B_j$:

$$\begin{aligned}\int |f|^2 r^{-2} dx &= \sum_{j=1}^\infty \int_{\Omega_j} |f|^2 r_j^{-2} dx \\ &\leqslant \sum_{j=1}^\infty 2^{-j} (c_j)^{-1} \int_{\Omega_j} |f| dx \ \leqslant \ \sum_{j=1}^\infty 2^{-j} = 1.\end{aligned}$$

Thus every compact set $M = \prod_j M_j$ is contained in the unit ball of $L^2(r)$ for some r. Taking polars, we see that every neighborhood of zero in $L^\infty_c[\gamma]$ contains the unit ball of some $L^2(\overline{r})$, which is a neighborhood of zero in $L^\infty_c[\mathsf{t}_{\rm proj}]$. Thus $\mathsf{t}_{\rm proj} \succcurlyeq \gamma$. Conversely, since the unit ball of $L^2(r)$ contains $B = \prod_{j=1}^\infty, B_j$ which is not weakly compact, *a fortiori* not compact, in $L^1_{\rm loc}$, it cannot be contained in a compact set, and therefore there are neighborhoods of 0 in $L^\infty_c[\mathsf{t}_{\rm proj}]$ which contain no neighborhoods of 0 of $L^\infty_c[\gamma]$, i.e., $\mathsf{t}_{\rm proj} \succ \gamma$.

Concerning completeness, since $L^1_{\rm loc}$ is Fréchet, its dual L^∞_c is complete in the topology γ, and also for the topologies β and τ (see [Köt69] §21.6 (4)). It is also complete for the projective topology $\mathsf{t}_{\rm proj}$, as projective limit of the Hilbert spaces $L^2(r)$. ■□

As for the last inequality, $\tau \succcurlyeq \mathsf{t}_{\rm proj}$, it is not known whether it is proper or not. Anyway, it follows from Proposition 4.2.3 that L^∞_c, with any of the topologies $\tau, \mathsf{t}_{\rm proj}, \gamma, \sigma$ is semi-reflexive, but nonreflexive, hence not (quasi)-barreled.

We turn now to the pair $\langle L^2_c, L^2_{\rm loc}\rangle$. Denote by I_2 the set of weight functions r such that $r^{\pm 1} \in L^\infty_{\rm loc}$. Then, exactly as before, we have both a LHS realization and a natural one:

$$L^2_{\rm loc}(\mathbb{R}^n, dx) = \bigcup_{r\in I_2} L^2(r) = \bigcap_{j=1}^\infty L^2(K_j) = \bigcap_{j=1}^\infty L^2(\Omega_j),$$
$$L^2_c(\mathbb{R}^n, dx) = \bigcap_{r\in I_2} L^2(r) = \bigcup_{j=1}^\infty L^2(K_j) = \bigcup_{j=1}^\infty L^2(\Omega_j).$$

As before $L^2_{\rm loc}$ is a Fréchet space, but this one is reflexive, as the inductive limit of a sequence of reflexive (DF)-spaces. Also its strong dual is $L^2_c[\mathsf{t}_{\rm proj}]$, i.e., the three topologies β, τ and $\mathsf{t}_{\rm proj}$ coincide on L^2_c This can also be seen directly in two ways. First, the identity map $i : L^2_c[\mathsf{t}_{\rm proj}] \to L^2_c[\tau] = \operatorname{ind.lim} L^2(K_j)$ factorizes continuously through any $L^2(r)$ with $r \in L^\infty_{\rm loc}$ and some $L^2(K_j)$ (there are plenty of such functions r in I_2). Second, one may repeat the argument of Proposition 4.2.3, replacing γ by τ or β, i.e., starting from sets $B = \prod_{j=1}^\infty B_j$, with B_j a closed ball in $L^2(\Omega_j)$, thus bounded and weakly compact. It follows that $\beta \preccurlyeq \mathsf{t}_{\rm proj}$, and so they must coincide. So we have:

Proposition 4.2.4. *On $L^2_c(\mathbb{R}^n, dx)$, the space of square integrable functions with compact support, the following relationship holds for the various topologies:*

$$\beta = \tau = \mathsf{t}_{\rm proj} \succ \gamma \succ \sigma.$$

Again the space $L^2_c(\mathbb{R}^n, dx)$ is complete for the first four topologies.

4.2.2 Functions or Sequences of Prescribed Growth

The results of Section 4.2.1 can be improved if V is restricted to those locally integrable functions which satisfy a growth condition at infinity (such as functions, or sequences, of polynomial or exponential growth), in the sense that the weight functions $r(x)$ can now be assumed to have the same type of growth.

More precisely, for X and μ as above, let A be a partially ordered set and let $\{F^{(\alpha)}, \alpha \in A\}$ be a family of positive μ-locally integrable functions, indexed by A and monotonically increasing in α:

$$\alpha \leqslant \beta \Longleftrightarrow F^{(\alpha)}(x) \leqslant F^{(\beta)}(x) \quad (\mu\text{–a.e.}).$$

Assume furthermore that, given $\alpha, \beta \in A$, there exists a positive square integrable function $s_{\alpha\beta}$ which verifies the following inequality for some positive constant $C_{\alpha\beta}$:

$$0 < s^2_{\alpha\beta}(x) \leqslant C_{\alpha\beta} F^{(\alpha)}(x) F^{(\beta)}(x) \quad (\mu\text{–a.e.}). \tag{4.11}$$

Define V as the vector space of those functions $f \in L^1_{\rm loc}(X, d\mu)$ which grow no faster than the functions $F^{(\alpha)}$, i.e., $f \in V$ if there exists $\alpha \in A$ and a constant $c > 0$ such that:

$$|f(x)| \leqslant c F^{(\alpha)}(x) \quad (\mu\text{–a.e.}).$$

Equip V with the compatibility $\#$ inherited from $L^1_{\rm loc}$. Then:

Proposition 4.2.5. *Let V as above. Consider the family $\mathcal{I}_A := \{L^2(r)\}$, where each r is a weight function that verifies the following inequalities for some $\alpha, \beta \in A$ and positive constants c', c'':*

$$c'[s_{\alpha\beta}(x)/F^{\beta}(x)] \leqslant r(x) \leqslant c''[F^{(\alpha)}(x)/s_{\alpha\beta}(x)] \tag{4.12}$$

Then the family $\mathcal{I}_A$ is a generating subset of $\mathcal{F}(V, \#)$.

Proof. Let $|f(x)| \leqslant cF^{(\alpha)}(x), |g(x)| \leqslant c' F^{(\beta)}(x)$, and $\int_x |f(x)g(x)| d\mu < \infty$. We choose a function $s_{\alpha\beta} \in L^2(X, d\mu)$ that verifies Eq. (4.11). Then we proceed as in Section 1.4.1, dividing X into four disjoint subsets $X_j, j = 1, ..., 4$, depending on whether

$$|f(x)| - [s^2_{\alpha\beta}(x)/F^{(\beta)}(x)] \geqslant 0 \; (X_1 \,\text{and}\, X_3) \,\text{or}\, \leqslant 0 \; (X_2 \,\text{and}\, X_4),$$
$$|g(x)| - [s^2_{\alpha\beta}(x)/F^{(\alpha)}(x)] \geqslant 0 \; (X_1 \,\text{and}\, X_2) \,\text{or}\, \leqslant 0 \; (X_3 \,\text{and}\, X_4).$$

We define a weight function $r_{\alpha\beta}$ as follows:

$$r_{\alpha\beta}(x) = \begin{cases} |f(x)|^{1/2}\,|g(x)|^{-1/2}, \text{ for } x \in X_1, \\ s_{\alpha\beta}(x)|\,|g(x)|^{-1}, \text{ for } x \in X_2, \\ |f(x)|\,s_{\alpha\beta}^{-1}(x), \text{ for } x \in X_3, \\ \text{arbitrary for } x \in X_4, \text{ provided Eq.(4.12) is verified.} \end{cases}$$

It is then straightforward to verify that $f \in L^2(r_{\alpha\beta})$, $g \in L^2(\overline{r_{\alpha\beta}})$, and also that $r_{\alpha\beta}$ verifies Eq. (4.12) on all of X. ◼

This proposition covers many cases of interest. Let us give a few examples.

(i) *Functions of polynomial growth in* $L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$

$$F^{(\alpha)}(x) = (1+|x|^2)^{\alpha/2}, \quad \alpha \in \mathbb{Z} \text{ or } \mathbb{R},$$
$$s_{\alpha\beta}(x) = (1+|x|^2)^{-\gamma/2} \text{ with } \gamma > n/2 \text{ and } \gamma \geqslant -\frac{1}{2}(\alpha+\beta).$$

The assaying subsets $L^2(r_{\alpha\beta})$ obtained in this example actually form an involutive sublattice of $\mathcal{F}(V, \#)$.
(ii) *Slowly increasing sequences (in* ω*)*
By the same reasoning, we find that the family $\{\ell^2(r)\}$, where $r = (r_n)$ is a sequence of tempered weights, i.e.,

$$c'(1+n)^{-\beta} \leqslant r_n \leqslant c''(1+n)^{\alpha}, \; c', c'' > 0,$$

is generating in the space $s^\times$ of slowly increasing sequences, equipped with the standard compatibility from ω.
(iii) *Functions of exponential growth in* $L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$

$$F^{(\alpha)}(x) = e^{\alpha|x|}, \; \alpha \in \mathbb{R},$$
$$s_{\alpha\beta}(x) = e^{\gamma|x|} \text{ with } \gamma \in \mathbb{R}, \; \gamma > \alpha+\beta.$$

(iv) *Entire functions of order 2 in Bargmann's space* $\mathcal{E}^\times$
As in Example (v) in Section 1.1, take $X = \mathbb{C}^n$, with Gaussian measure $d\mu(z) = e^{-|z|^2}\,dz$. Then $\mathcal{E}^\times$ consists also of those Borel functions with growth indexed by the following family:

$$F^{(\alpha)}(z) = e^{1/2|z|^2}(1+|z|^2)^{-\alpha/2}, \; \alpha \in \mathbb{R} \text{ or } \mathbb{Z},$$
$$s_{\alpha\beta}(z) = 1, \; \forall\, \alpha, \beta.$$

4.2.3 Operators on Spaces of Locally Integrable Functions

Finally we discuss operators on $L^1_{\rm loc}(\mathbb{R}^n, dx)$ (the same analysis applies to $L^2_{\rm loc}(\mathbb{R}^n, dx)$). This is a case where the various classes of operators can be characterized reasonably well, hence it is worth treating it in detail. More precisely, we would like to give conditions under which an operator on the PIP-space $L^1_{\rm loc}$ is regular or belongs to one of the three algebras $\mathfrak{A}, \mathfrak{B}$ or $\mathfrak{C}$.

The crucial remark is that each of $L^1_{\rm loc}, L^\infty_{\rm c}[\tau], L^\infty_{\rm c}[\gamma]$ is a normal space of distributions,[5] i.e., a subspace E of $\mathcal{D}^\times(\mathbb{R}^n)$ such that $\mathcal{D}(\mathbb{R}^n) \subset E \subset \mathcal{D}^\times(\mathbb{R}^n)$, where both inclusions are continuous and the first one has dense range. This fact allows us to use Schwartz's theory of kernels. Indeed every continuous linear map $A : L^\infty_{\rm c}[\tau] \to L^1_{\rm loc}[\tau]$ is then the extension of a continuous map $A : \mathcal{D}(\mathbb{R}^n) \to \mathcal{D}^\times(\mathbb{R}^n)$ which is uniquely determined by a kernel $A(x, y) \in \mathcal{D}^\times(\mathbb{R}^{2n})$ i.e., a distribution in two variables. Consequently, every operator on the PIP-space $L^1_{\rm loc}(\mathbb{R}^n)$ is given by a unique kernel $A(x, y)$. The problem is to identify the corresponding class of kernels, and here we may exploit the results of Schwartz. But we will get in general sufficient conditions only since Schwartz uses the topology γ on $L_{\rm c}$. A continuous map $A : L^\infty_{\rm c}[\gamma] \to L^1_{\rm loc}$ is *a fortiori* continuous on $L^\infty_{\rm c}[\tau]$. Thus it defines an PIP-space operator, but the converse need not be true. For regular operators, however, the two classes of maps coincide: a map $B : L^\infty_{\rm c} \to L^\infty_{\rm c}$ is continuous for τ if and only if it is for γ (since $B = B^{\times\times}$).

We may distinguish several classes of operators:

(1) *General operators,* given by kernels $A(x, y) \in \mathcal{L}(L^\infty_{\rm c}, L^1_{\rm loc})$, the space of all linear maps from $L^\infty_{\rm c}$ into $L^1_{\rm loc}$.

In fact, we will consider only *integral* operators given by a kernel $A(x, y) \in L^1_{\rm loc}(\mathbb{R}^{2n})$. This is *a priori* a subset of $\mathcal{L}(L^\infty_{\rm c}, L^1_{\rm loc}) \simeq {\rm Op}(L^1_{\rm loc})$, since neither $L^\infty_{\rm c}[\tau]$, nor $L^1_{\rm loc}$ is a *nuclear* space. Note that the kernel of $A^\times$ is $A^\times(x, y) := \overline{A(y, x)}$.

(2) *Convolution operators,* given by kernels $A(x, y) = C(x - y)$.

(3) *Multiplication operators* given by diagonal kernels

$$A(x, y) = \delta(x - y) f(y),\ f \in L^1_{\rm loc}.$$

(4) *Finite rank operators*, given by separable kernels

$$A(x, y) = \sum_{j=1}^{n} \overline{f_j(y)} g_j(x),\ f_j, g_j \in L^1_{\rm loc}.$$

[5] This notion will be generalized in Section 5.4.1 under the name of *interspaces.*

Table 4.1 Various classes of operators on $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. The symbol $\star$ in a box means that the condition stated is necessary and sufficient for the operator to belong in the corresponding class

kernel	Op	Reg	$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{C}$
integral operator $A(x,y)$	$L^1_{\mathrm{loc}} \otimes L^1_{\mathrm{loc}}$	$L^\infty_{\mathrm{c}}(L^\infty_{\mathrm{c}})$	$\Rightarrow L^\infty_{\mathrm{c}}(L^\infty_{\mathrm{c}})$	?	$\times$
convolution operator $C(x-y)$	L^1_{loc}	L^1_{c}	$\Rightarrow L^1_{\mathrm{c}}$	?	$\times$
multiplication operator $\delta(x-y)\, f(y)$	$^\star L^1_{\mathrm{loc}}$	$^\star L^\infty_{\mathrm{loc}}$	$^\star L^\infty$	$\Rightarrow^\star L^\infty$	$\Rightarrow^\star L^\infty$
rank 1 operator $\overline{f} \otimes g$	$^\star L^1_{\mathrm{loc}} \otimes L^1_{\mathrm{loc}}$	$^\star L^\infty_{\mathrm{c}} \otimes L^\infty_{\mathrm{c}}$	$\Rightarrow^\star L^\infty_{\mathrm{c}} \otimes L^\infty_{\mathrm{c}}$	(?)	$\times$

For each class we will state sufficient conditions on the kernel for the operator to be in Op, Reg, $\mathfrak{A}, \mathfrak{B}$ or $\mathfrak{C}$. The results are summarized in Table 4.1 below that we now comment (the algebra $\mathfrak{C}$ will be discussed separately).

(1) *Integral operators:* every function $A(x,y)$, locally integrable in both variables, defines an operator on L^1_{loc}, by [Sch57, Prop.33]. By the remark following the same proposition, such an operator is regular when the following two conditions are satisfied:

   (i) the kernel $A(x,y)$ is compact, i.e., for every compact subset K of $\mathbb{R}^n$, $\operatorname{supp} A \cap (\mathbb{R}^n \times K)$ and $\operatorname{supp} A \cap (K \times \mathbb{R}^n)$ are compact (note that $\operatorname{supp} A$ itself need not be compact, as the example of convolution operators shows), which implies that both A and $A^\times$ map L^∞_{c} into L^1_{c}.
   (ii) For every compact $K \subset \mathbb{R}^n$, $\int_K |A(x,y)| dx$ is bounded in y and $\int_K |A(x,y)| dy$ is bounded in x (both functions are in L^1_{c} by (i)).

   We denote by $L^\infty_{\mathrm{c}}(L^\infty_{\mathrm{c}})$ the class of kernels that verify those two conditions. In fact, the resulting regular operators are automatically in $\mathfrak{A}$ (symbol $\Rightarrow$) and they are even Hilbert-Schmidt operators in every $L^2(r)$. For $\mathfrak{B}$, however, we have no result.

(2) *Convolution operators*: the results follow immediately from the two well-known facts:

   (a) $\operatorname{supp}(f * g) \subset \operatorname{supp} f + \operatorname{supp} g$;
   (b) $L^1 * L^p \subset L^p, 1 \leqslant p \leqslant \infty$.

   For $\mathfrak{B}$, we have again no result.

(3) *Multiplication operators*: all results are obvious, and all conditions are actually necessary *and* sufficient (symbol $\star$)

(4) *Finite rank operators*: obvious again. As for $\mathfrak{B}$, there are finite rank operators that do not belong to it. Let $\overline{f} \otimes g \equiv |g\rangle\langle f|$ be of rank one, with $f, g \in L^\infty_{\rm c}$ and normalized in L^2. Its norm in $L^2(r)$ is:

$$\|\overline{f} \otimes g\|_{rr} = \sup_{h \in L^2(r)} \frac{|\langle f|h\rangle| \|g\|_r}{\|h\|_r} = \|g\|_r \|f\|_{\overline{r}}.$$

Then $\overline{f} \otimes g \in \mathfrak{B}$ if and only if $\sup_r \|g\|_r \; \|f\|_{\overline{r}} < \infty$. Consider, for instance, the family of weights $b(x) = \exp bx, \;\; b \in \mathbb{R}$ (we work in one dimension for simplicity). So:

$$\|g\|_b = \int_{\mathbb{R}} |g(x)|^2 e^{bx} dx \geqslant \exp\left(b \int_{\mathbb{R}} |g(x)|^2 x \, dx\right)$$

by Jensen's inequality. Thus we have:

$$\begin{aligned}\sup_{r \in I_1} \|g\|_r \, \|f\|_{\overline{r}} &\geqslant \sup_{b \in \mathbb{R}} \|g\|_b \, \|f\|_{\overline{b}} \\ &\geqslant \sup_{b \in \mathbb{R}} \, \exp\left(b \int_{\mathbb{R}} \left(|g(x)|^2 - |f(x)|^2\right) \, x \, dx\right)\end{aligned}$$

and the last supremum is infinite, unless the integral vanishes. Therefore $\overline{f} \otimes g$ cannot belong to $\mathfrak{B}$, except perhaps for special choices of f and g. We may observe also that $\|g\|_b$ is a convex function of b, and often $\|g\|_b = \|g\|_{\overline{b}}$ (for instance when g is even, or odd). In such a case, again $\overline{g} \otimes g \notin \mathfrak{B}$. We do not know, however, if $\mathfrak{B}$ contains any finite rank operator at all.

We finally come to the algebra $\mathfrak{C}$, and here of course we need the Riesz operators $U_{(r)} : L^2(r) \to L^2(\overline{r})$. Clearly $U(r)$ is the multiplication by $r^{-1}(x)$, and it is regular if and only if $r^{\pm 1} \in L^\infty_{\rm loc}$. Thus some of the Riesz operators are regular, the others are not. Note also that these spaces $L^2(r)$ with $r^{\pm 1} \in L^\infty_{\rm loc}$ are precisely those for which $r^{\pm n} \in L^\infty_{\rm loc}$ for all n, so that the whole scale $V(r)$ generated by each such multiplication operator is contained in $\mathcal{I}$. In other words, since all those operators $U(r), r^{\pm 1} \in L_{\rm loc}$ commute, they generate an abelian *-subalgebra of $\mathrm{Reg}(L^1_{\rm loc})$. On the other hand, the family $\mathcal{R}$ of all Riesz operators is not an algebra.

Now a regular operator belongs to $\mathfrak{C}$ if and only if it commutes with every element of $\mathcal{R}$. But it is easy to check that no integral operator can commute with all multiplication operators by $r \in L^\infty_{\rm loc}$ unless it is itself a multiplication operator. This explains the last column of Table 1. Of course we have not proved that $\mathfrak{B} = \mathfrak{C}$, but all indications point into that direction.

The analysis of the various classes of operators on $L^2_{\rm loc}$ is entirely parallel to the one above, so we will not do it explicitly.

4.3 Köthe Sequence Spaces

As in Section 1.1, consider $V = \omega$, the space of all complex sequences, with the compatibility

$$(x_n)\#(y_n) \iff \sum_n |x_n y_n| < \infty.$$

This is called *α-duality* and the corresponding assaying subsets are called Köthe's *perfect* sequence spaces. This example shows how big and unpractical the complete lattice $\mathcal{F}(V,\#)$ can be. Indeed, this set $\mathcal{F}(\omega,\#)$ contains almost all possible types of topological vector spaces, many of them with rather awkward properties (it is Köthe's main source of counterexamples!). Thus it is imperative to restrict ourselves to suitable subsets of $\mathcal{F}(\omega,\#)$.

4.3.1 Weighted ℓ^2 Spaces

Our familiar example is the family $\mathcal{I}$ of all assaying subsets of the form

$$\ell^2(r) = \{(x_n) \in \omega : (x_n r_n^{-1}) \in \ell^2\},$$

where $r = (r_n)$ is an arbitrary sequence of positive numbers. The set $\mathcal{I}$ is an involutive covering of ω. Indeed, given $(x_n) \in \omega$ there is a weight sequence $r = (r_n)$ such that $(x_n) \in \ell^2(r)$. Take, for instance, the following weights:

- $r_n = n\,|x_n|$, whenever $x_n \neq 0$,
- r_n arbitrary, whenever $x_n = 0$.

Then we have shown in Chapter 1 that $\mathcal{F}(\omega,\#)$ is an involutive lattice and that $\mathcal{I}$ is a generating subset of it (see Section 1.4.1), and in fact an involutive sublattice.

Indeed, $\mathcal{I}$ is a lattice for the following operations:

$$\begin{aligned}
\ell^2(r) \wedge \ell^2(s) &= \ell^2(u), \quad \text{where } \ u_n = \min\{r_n, s_n\},\\
\ell^2(r) \vee \ell^2(s) &= \ell^2(v), \quad \text{where } \ v_n = \max\{r_n, s_n\}.
\end{aligned}$$

Let us show that the norms of $\ell^2(u)$ and $\ell^2(v)$ are equivalent, respectively, to the projective and inductive norms defined in (I.4), (I.5).

First, on $\ell^2(u) = \ell^2(r) \cap \ell^2(s)$, the norm $\|\cdot\|_u$ defines the same topology as the projective topology given by $\|x\|_r^2 + \|x\|_s^2$. One has indeed:

$$\begin{aligned}
\|x\|_u^2 = \sum_n \frac{|x_n|^2}{\min\{r_n^2, s_n^2\}} &= \sum_n |x_n|^2 \max\left\{\frac{1}{r_n^2}, \frac{1}{s_n^2}\right\}\\
&\leqslant \sum_n |x_n|^2 \left(\frac{1}{r_n^2} + \frac{1}{s_n^2}\right) = \|x\|_{r\wedge s}^2\\
&\leqslant \sum_n |x_n|^2 \; 2 \; \max\left\{\frac{1}{r_n^2}, \frac{1}{s_n^2}\right\} = 2\|x\|_u^2.
\end{aligned}$$

Then, on $\ell^2(v) = \ell^2(r)+\ell^2(s)$, the norm $\|\cdot\|_v$ defines a topology equivalent to the inductive topology given by $\|x\|^2_{r\vee s} = \inf_{x=y+z}(\|y\|^2_r + \|z\|^2_s)$. One has indeed:

$$\|x\|^2_{r\vee s} = \inf_{x=y+z}\Big(\sum_n \frac{|y_n|^2}{r_n^2} + \sum_n \frac{|z_n|^2}{s_n^2}\Big)$$
$$= \sum_n \inf_{x_n=y_n+z_n}\Big(\frac{|y_n|^2}{r_n^2} + \frac{|z_n|^2}{s_n^2}\Big) = \sum_n \inf_{y_n}\Big(\frac{|y_n|^2}{r_n^2} + \frac{|x_n-y_n|^2}{s_n^2}\Big).$$

Since all terms are non-negative, we may compute the minimum of each term separately, by equating to zero the partial derivatives with respect to y_n, z_n, where $x_n = y_n + z_n$. An easy computation shows that this infimum is reached for

$$y_n = x_n\, \frac{r_n^2}{r_n^2+s_n^2}, \quad z_n = x_n\, \frac{s_n^2}{r_n^2+s_n^2}.$$

From this, we get:

$$\|x\|^2_{r\vee s} = \inf_{x=y+z}(\|y\|^2_r + \|z\|^2_s) = 2\, \sum_n \frac{|x_n|^2}{r_n^2+s_n^2}$$

and

$$\frac{1}{2}\|x\|^2_v = \sum_n \frac{|x_n|^2}{2\, \max\{r_n^2, s_n^2\}} \leqslant \sum_n \frac{|x_n|^2}{r_n^2+s_n^2} = \frac{1}{2}\|x\|^2_{r\vee s} \leqslant \sum_n \frac{|x_n|^2}{\max\{r_n^2, s_n^2\}} = \|x\|^2_v\ .$$

Thus finally

$$\|x\|_u \asymp \|x\|_{r\wedge s}, \qquad \|x\|_v \asymp \|x\|_{r\vee s},$$

where the symbol $\asymp$ denotes equivalence of norms.

4.3.2 Norming Functions and the ℓ_ϕ Spaces

Besides the ℓ^p and the $\ell^2(r)$ spaces, there are many other types of perfect spaces. In the sequel of this section, we will describe an interesting class of them, which constitutes a PIP-space of type (B) (and in fact a LBS). Throughout the following sections, sequences are added and multiplied componentwise: for $x = (x_n), y = (y_n) \in \omega$, we write $x + y = (x_n + y_n)$, $\alpha x = (\alpha x_n)\, (\alpha \in \mathbb{C})$ and $x \cdot y = (x_n y_n)$. Also, we quote the results without proofs, which may be found in the references given in the Notes.

Definition 4.3.1. A real-valued function ϕ defined on the space φ of finite sequences is said to be a *norming function* if

(n_1) $\phi(x) > 0$ for $x \neq 0$;
(n_2) $\phi(\alpha x) = |\alpha|\phi(x),\ \forall\, \alpha \in \mathbb{C}$;

(n_3) $\phi(x+y) \leqslant \phi(x) + \phi(y)$;
(n_4) $\phi(1,0,0,0,\ldots) = 1$.

A norming function ϕ is *symmetric* if

(n_5) $\phi(x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \phi(|x_{j_1}|, |x_{j_2}|, \ldots, |x_{j_n}|, 0, 0, \ldots)$,

where $j_1, j_2, \ldots, j_n$ is an arbitrary permutation of $1, 2, \ldots, n$.

From property (n_5), it is clear that a symmetric norming function ϕ is entirely determined by its values on the set $[\varphi]$ of finite, positive, nonincreasing sequences. Hence, from conditions (n_2) and (n_4), we deduce that

$$\phi_\infty(x) \;\leqslant\; \phi(x) \;\leqslant\; \phi_1(x),\ \forall\, x \in \varphi,$$

where $\phi_\infty(x) = \max_{i=1,\ldots,n} |x_i|$ and $\phi_1(x) = \sum_{i=1}^n |x_i|$.

Lemma 4.3.2. *The set of all symmetric norming functions possesses a maximal element, ϕ_1, and a minimal one, ϕ_∞.*

To every symmetric norming function ϕ, one can associate a Banach space ℓ_ϕ as follows. Given a sequence $x \in \omega$, define its n^{th} section as $x^{(n)} = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$. Then the sequence $\big(\phi(x^{(n)})\big)$ is nondecreasing, so that one can define

$$\ell_\phi = \{x \in \omega : \sup_n \phi(x^{(n)}) < \infty\}$$

and then extend the norming function ϕ to the whole of ℓ_ϕ by putting $\phi(x) = \lim_n \phi(x^{(n)})$. This relation defines a norm ϕ on ℓ_ϕ, for which it is complete, hence a Banach space. In other words, we can also say that $\ell_\phi = \{x \in \omega : \phi(x) < \infty\}$ is the natural domain of definition of the extended norming function ϕ.

Clearly, one has $\ell_{\phi_\infty} = \ell^\infty$ and $\ell_{\phi_1} = \ell^1$. Similarly, $\ell^p = \ell_{\phi_p}$, where $\phi_p(x) = \big(\sum_n |x_n|^p\big)^{1/p}$. Thus every space ℓ_ϕ contains ℓ^1 and is contained in ℓ^∞. Moreover, every space ℓ_ϕ is a two-sided ideal in ℓ^∞. Indeed, one has, for every $x \in \ell_\phi$ and $y \in \ell^\infty$,

$$\begin{aligned} x \cdot y \in \ell_\phi \;\text{ and }\; \phi(x \cdot y) \leqslant \phi(x)\, \|y\|_\infty\,, \\ y \cdot x \in \ell_\phi \;\text{ and }\; \phi(y \cdot x) \leqslant \phi(x)\, \|y\|_\infty\,. \end{aligned} \tag{4.13}$$

In addition, the set of Banach spaces ℓ_ϕ constitutes a lattice. Given two symmetric norming functions ϕ and ψ, one defines their infimum and supremum, exactly as for the general case:

- $\phi \wedge \psi := \max\{\phi, \psi\}$, which defines on the space $\ell_{\phi\wedge\psi} := \ell_\phi \cap \ell_\psi$ a norm equivalent to $\phi(x) + \psi(x)$;
- $\phi \vee \psi := \min\{\phi, \psi\}$, which defines on the space $\ell_{\phi\vee\psi} := \ell_\phi + \ell_\psi$ a norm equivalent to $\inf_{x=y+z}\{\phi(y) + \psi(z)\}, x \in \ell_\phi + \ell_\psi, y \in \ell_\phi, z \in \ell_\psi$.

It remains to analyze the relationship of the spaces ℓ_ϕ and the PIP-space structure of ω. Given a symmetric norming function ϕ and a finite sequence y, consider the function

$$F_y(x) := \frac{\langle x|y\rangle}{\phi(x)} = \frac{\sum_n \overline{x_n} y_n}{\phi(x)} \qquad \text{(for } x = 0\text{, one puts } F_y(0) = 0).$$

This function is bounded on $[\varphi]$ and reaches is maximum, since

$$F_y(x) \leqslant F_y((x_1, x_2, \ldots, x_n, 0, 0, \ldots)),$$

where n is the largest index such that $y_n \neq 0$. Thus we may define, for $y \in [\varphi]$:

$$\overline{\phi}(y) := \max_{x \in [\varphi]} \frac{\sum_n \overline{x_n} y_n}{\phi(x)} = \max_{x \in [\varphi]} \frac{\langle x|y\rangle}{\phi(x)}.$$

The function $\overline{\phi}$ thus defined is a symmetric norming function, hence it can be extended to the corresponding Banach space $\ell_{\overline{\phi}}$. The function $\overline{\phi}$ is said to be *conjugate* to ϕ and the space $\ell_{\overline{\phi}}$ is the dual of ℓ_ϕ with respect to the partial inner product, i.e., $\ell_{\overline{\phi}} = (\ell_\phi)^\#$. Clearly one has $\overline{\overline{\phi}} = \phi$, hence $\ell_{\overline{\overline{\phi}}} = (\ell_\phi)^{\#\#} = \ell_\phi$.

In addition, it is easy to show that $\ell_{\overline{\phi \wedge \psi}} = \ell_{\overline{\phi} \vee \overline{\psi}}$ and $\ell_{\overline{\phi \vee \psi}} = \ell_{\overline{\phi} \wedge \overline{\psi}}$. In other words,

Proposition 4.3.3. *The family of Banach spaces ℓ_ϕ, where ϕ is a symmetric norming function, is an involutive sublattice of the lattice $\mathcal{F}(\omega, \#)$ and a LBS.*

Actually, since every ϕ satisfies the inclusions $\ell^1 \subset \ell_\phi \subset \ell^\infty$, the family $\{\ell_\phi\}$ is also an involutive sublattice of the lattice $\mathcal{F}(\ell^\infty, \#)$ obtained by restricting to ℓ^∞ the PIP-space structure of ω.

The interest of the spaces ℓ_ϕ is their close connection with ideals of compact operators. Let indeed A be a compact operator in a Hilbert space $\mathcal{H}$. The *singular values* of A are the (positive) eigenvalues $s_j, j = 1, 2, \ldots$ of the positive self-adjoint operator $|A| := (A^*A)^{1/2}$. Then the set of all compact operators A for which the sequence (s_j) belongs to ℓ_ϕ constitutes a two-sided ideal $\mathcal{C}_\phi$ of the space $\mathcal{C}^\infty$ of all compact operators. In particular, ℓ^1 corresponds to the ideal $\mathcal{C}^1$ of nuclear or trace-class operators, ℓ^2 to the ideal $\mathcal{C}^2$ of Hilbert-Schmidt operators, and ℓ^∞ to $\mathcal{C}^\infty$ itself.

As a final remark, we may also notice that the family of Banach spaces $\ell_\phi \subset \omega$ has a counterpart in the space $L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$ of locally integrable functions, namely the so-called Köthe function spaces, that we shall discuss in detail in Section 4.4 below.

More general sequence spaces may be defined, in terms of normed ideals. These are defined as follows.

Definition 4.3.4. A *normed ideal* is a couple (l, ν) where

(ni_1) l is a subspace of ω such that $\varphi \subseteq l \subseteq \ell^\infty$;

(ni_2) ν is a norm on l and one has $\nu(1, 0, \ldots) = 1$;

(ni_3) for every $x \in l$ and $y \in \ell^\infty$, one has

$$\begin{aligned} x \cdot y \in l \text{ and } \quad \nu(x \cdot y) \leqslant \nu(x)\, \|y\|_\infty \,, \\ y \cdot x \in l \text{ and } \quad \nu(y \cdot x) \leqslant \nu(x)\, \|y\|_\infty \,; \end{aligned}$$

(ni_4) if $x \in l$, then $(x_{j_1}, x_{j_2}, \ldots, x_{j_n}, \ldots) \in l$ for any permutation of the indices and $\nu(x) = \nu(x_{j_1}, x_{j_2}, \ldots, x_{j_n}, \ldots)$.

If, in addition, l is complete for the norm ν, it is called a *Banach ideal*.

An order relation may be defined on the set of all Banach ideals as follows. We say that $(l, \nu) \prec (m, \mu)$ if $l \subset m$ and the induced norm $\mu\restriction l$ is equivalent to ν.

Definition 4.3.5. A Banach ideal (l, ν) is said to be *maximal* if there exists no Banach ideal (m, μ) such that $(l, \nu) \prec (m, \mu)$.

To make contact with the previous case, take an arbitrary normed ideal (l, ν) and consider the restriction of the norm ν to the space φ, which is obviously a symmetric norming function. Define, as above, a function $\underline{\nu}$ on ℓ^∞ by $\underline{\nu}(x) = \lim_n \nu(x^{(n)})$. This limit exists, but is not necessarily finite. In addition, one has $\underline{\nu}(x) \leqslant \nu(x)$, $\forall\, x \in l$.

Definition 4.3.6. ν is said to be a *strong norm* if $\underline{\nu}(x) = \nu(x)$, $\forall\, x \in l$. Then the ideal (l, ν) is called a strong normed ideal, and a strong Banach ideal if it is complete.

As before, we may extend $\underline{\nu}$ to its natural domain $\ell_\nu := \ell_{\underline{\nu}} = \{x \in \ell^\infty : \underline{\nu}(x) < \infty\}$. Then ℓ_ν is a maximal strong Banach ideal. The interest of this notion in the present context is the next result.

Proposition 4.3.7. *For every normed ideal (l, ν), the following three properties are equivalent and each of them implies that (l, ν) is a strong ideal:*

(i) (l, ν) *is maximal;*

(ii) $(l, \nu) = \ell_\nu$*;*

(iii) (l, ν) *is perfect, i.e.,* $(l, \nu)^{\#\#} = (l, \nu)$*, i.e.,* (l, ν) *is an assaying subspace in* $\mathcal{F}(\ell^\infty, \#)$.

For a counterexample, take the space c_o of sequences converging to 0. The space c_o is a Banach ideal for the norm $\phi_\infty(x) = \sup_n |x_n|$, but it is not maximal, since ℓ^∞ is a (maximal) ideal containing c_o. It is not perfect either, thus not an assaying subspace of ℓ^∞, since one has $(c_o)^\# = \ell^1$, $(\ell^1)^\# = \ell^\infty$, $(\ell^\infty)^\# = \ell^1$, hence $c_o \subsetneqq (c_o)^{\#\#} = \ell^\infty$.

4.3.3 Other Types of Sequence Ideals

Besides the ideals ℓ_ϕ, there exist many other types of sequence ideals, perfect or not. We mention three different types, namely, quotient ideals, root ideals and weighted function spaces.

(i) Quotient ideals

Definition 4.3.8. Given two normed ideals (l, ν) and (m, μ), consider the pair $(l : m, \nu : \mu)$, where $l : m$ is the set:

$$l : m \ := \{x \in \ell^\infty : x \cdot m \subseteq l \text{ and } x : (m, \mu) \to (l, \nu) \text{ is a bounded map}\}$$

and $\nu : \mu$ is the norm

$$(\nu : \mu)(x) := \sup_{y \in m, \mu(y) \leqslant 1} \nu(x \cdot y).$$

Then $(l : m, \nu : \mu)$ is a normed ideal called the *quotient ideal*.

This quotient ideal has some interesting properties.

(i) If l is complete for ν, then $l : m$ is complete for $\nu : \mu$.
(ii) If ν is a strong norm on l, $\nu : \mu$ is a strong norm on $l : m$.
(iii) If (l, ν) is a maximal strong Banach ideal, so is $(l : m, \nu : \mu)$.
(iv) If (l, ν) is perfect (= assaying), so is $(l : m, \nu : \mu)$, for any normed ideal (m, μ), and one has $l : m = (l^\# \cdot m)^\#$.
(v) If $l = \ell^1$, then $\ell^1 : m = m^\#$ and $\nu_1 : \mu = \overline{\mu}$, the norming function conjugate to ν.

Thus this notion of quotient allows one to construct many assaying subspaces in $(\ell^\infty, \#)$. Indeed, if l is assaying, $l^{\#\#} = l$, then the quotient $l : m$ is assaying for any normed ideal m, i.e., $(l : m)^{\#\#} = l : m$.

(ii) Root ideals

Definition 4.3.9. Given a normed ideal (l, ν), its p^{th} *root* $(l^p, \nu_{(p)})$ $(1 < p < \infty)$ is the normed ideal defined as follows:

$$l^p = \{x \in \ell^\infty : |x|^p \in l\}\,,$$
$$\nu_{(p)}(x) = \big(\nu(|x|^p)\big)^{1/p}\,.$$

Then one shows that $(l^p, \nu_{(p)})$ is perfect if and only if (l, ν) is perfect.

As examples, we may mention $\ell^p = (\ell^1)^p$ and $(\ell_\phi)^p = \{x \in \ell^\infty : |x|^p \in \ell_\phi\}$. We note the relations

$$\begin{aligned}
\ell_\phi : (\ell_\phi)^p &= (\ell_\phi)^q, \text{ where } \tfrac{1}{p} + \tfrac{1}{q} = 1\,, \\
(\ell_\phi)^p : (\ell_\phi)^q &= (\ell_\phi)^r, \text{ where } \tfrac{1}{p} + \tfrac{1}{q} = \tfrac{1}{r}\,, \\
(\ell_\phi)^{p\#} &= (\ell_{\overline{\phi}})^q\,.
\end{aligned}$$

(iii) Weighted sequence spaces

Following the familiar example of the weighted Hilbert spaces $\ell^2(r)$, we may introduce weighted ℓ_ϕ spaces as follows. Given an arbitrary sequence $r = (r_n)$ of positive numbers, we define

$$\ell_\phi(r) := \{x \in \omega : \phi(xr^{-1}) < \infty\}, \quad \text{where} \;\; xr^{-1} = \bigl(x_n r_n^{-1}\bigr).$$

These spaces generalize the ℓ_ϕ spaces, but they need not be contained in ℓ^∞. Yet they are assaying subsets of $(\omega, \#)$, since one has

$$\begin{aligned}
\ell_\phi(r)^{\#} &= \ell_{\overline{\phi}}(r^{-1}), \\
\ell_\phi(r)^{\#\#} &= \ell_\phi(r).
\end{aligned}$$

The most common example, which will exploited systematically in Chapter 8, is that of the weighted ℓ^p spaces, called $\ell^p_m \equiv \ell^p(m^{-1})$, for a given sequence $m := (m_k)$, with norm

$$\|x\|_{\ell^p_m} = \Bigl(\sum_k |x_k|^p\, m_k^p\Bigr)^{1/p}.$$

This notion allows one to compute explicitly the assaying subspaces generated by a single element $x \in \omega$. Let first $x_n \neq 0$ for every n. Then one has

$$\begin{aligned}
\{x\}^{\#} &= \{y \in \omega : \sum_n |x_n|\,|y_n| < \infty\} = \ell^1(|x|^{-1}), \\
\{x\}^{\#\#} &= \{z \in \omega : |z_n| \leqslant c|x_n|\} = \ell^\infty(|x|),
\end{aligned}$$

which is the smallest assaying subspace containing x. On the other hand, if x is a finite sequence, then $\{x\}^{\#} = \omega$ and $\{x\}^{\#\#} = \varphi$. Thus, in general, $\{x\}^{\#}$ is the direct sum of a weighted ℓ^1 space and ω, whereas $\{x\}^{\#\#}$ is the direct sum of a weighted ℓ^∞ space and φ.

An interesting example of that construction is given by the so-called echelon spaces. Let $a_{(1)}, a_{(2)}, \ldots \in \omega$ be an increasing sequence of positive sequences, called *steps*. Then one calls *echelon space* the set

$$l := \bigcap_k \bigl\{a_{(k)}\bigr\}^{\#} = \bigcap_k \ell^1(a_{(k)}^{-1})$$

and *co-echelon space* the #-dual

$$l^{\#} := \bigcup_k \{a_{(k)}\}^{\#\#} = \bigcup_k \ell^\infty(a_{(k)}),$$

both of which are assaying subspaces of ω. An example is given by the spaces on entire analytic functions Exp and $\mathfrak{Z}$ described in Section 4.6.1.

4.4 Köthe Function Spaces

A nontrivial (i.e., not a chain) example of indexed PIP-space of type (B), actually a LBS, is the family of the so-called Köthe function spaces. Since it is highly instructive, we feel it worthwhile to discuss it extensively. We will begin by repeating the basic definitions.

Let (X, μ) be a σ-finite measure space, M^+ the set of all measurable, non-negative functions on X, where two functions are identified if they differ at most on a μ-null set. A *function norm* is a mapping $\rho : M^+ \to \overline{\mathbb{R}}$ such that:

(i) $0 \leqslant \rho(f) \leqslant \infty$, $\forall\, f \in M^+$ and $\rho(f) = 0$ if and only if $f = 0$;
(ii) $\rho(f_1 + f_2) \leqslant \rho(f_1) + \rho(f_2)$, $\forall f_1, f_2 \in M^+$;
(iii) $\rho(af) = a\rho(f)$, $\forall\, f \in M^+$, $\forall\, a \geqslant 0$;
(iv) $f_1 \leqslant f_2 \Rightarrow \rho(f_1) \leqslant \rho(f_2)$, $\forall\, f_1, f_2 \in M^+$.

A function norm ρ is said to have the *Fatou property* if and only if $0 \leqslant f_1 \leqslant f_2 \leqslant \ldots, f_n \in M^+$ and $f_n \to f$ pointwise, implies $\rho(f_n) \to \rho(f)$.

Given a function norm ρ, it can be extended to all complex measurable functions on X by defining $\rho(f) = \rho(|f|)$ (for simplicity, we keep the same notation). Denote by L_ρ the set of all measurable f such that $\rho(f) < \infty$. With the norm $\|f\| = \rho(f)$, L_ρ is a normed space and a subspace of the vector space V of all measurable, μ-a.e. finite, functions on X. In addition, the space L_ρ is *solid*, that is, if $f \in V$, $g \in L_\rho$ and $|f(x)| \leqslant |g(x)|$ a.e., then $f \in L_\rho$ and $\|f\| \leqslant \|g\|$. Furthermore, if ρ has the Fatou property, L_ρ is complete, i.e., a Banach space. This is a generalization of the spaces $L^p(X, \mu)$, which correspond to $\rho(f) = (\int_X |f|^p d\mu)^{1/p}$ for $1 \leqslant p < \infty$ and $\rho(f) = \sup |f|$ for $p = \infty$.

A function norm ρ is said to be *saturated* if, for any measurable set $E \subset X$ of positive measure, there exists a measurable subset $F \subset E$ such that $\mu(F) > 0$ and $\rho(\chi_F) < \infty$ (χ_F is the characteristic function of F).

Let ρ be a saturated function norm with the Fatou property. Define:

$$\rho'(f) = \sup\left\{ \int_X |f\, g|\, d\mu : \rho(g) \leqslant 1 \right\} \tag{4.14}$$

Then ρ' is a saturated function norm with the Fatou property and $\rho'' \equiv (\rho')' = \rho$. Hence, $L_{\rho'}$ is a Banach space. Moreover, one has also:

$$\rho'(f) = \sup\left\{\left|\int_X f\, g\, d\mu\right| : \rho(g) \leqslant 1\right\} \tag{4.15}$$

In our language these results can be restated as follows. The vector space V of all measurable, a.e.-finite functions on X carries a natural PIP-space structure, with compatibility

$$f \# g \iff \int_X |f g|\, d\mu < \infty \tag{4.16}$$

and partial inner product

$$\langle f|g\rangle = \int_X \overline{f}\, g\, d\mu\,. \tag{4.17}$$

V is clearly the largest space on which the partial inner product (4.17) may be defined, but it is too large. Indeed, $V^{\#} = \{0\}$ and the partial inner product is degenerate. However, there are plenty of subspaces of V which are nondegenerate, such as $L^1_{\text{loc}}, L^2_{\text{loc}}$ or the space L_{ρ_0} to be defined below. Furthermore, for each ρ as above, L_ρ is a Banach space and $L_{\rho'} = (L_\rho)^{\#}$, i.e., each L_ρ is assaying. The pair $\langle L_\rho, L_{\rho'}\rangle$ is actually a dual pair, although $\langle V^{\#}, V\rangle$ is not. The space $L_{\rho'}$ is called the *Köthe dual* or *α-dual* of L_ρ and denoted by $(L_\rho)^\alpha$. However, $L_{\rho'}$ is in general only a closed subspace of the Banach conjugate dual $(L_\rho)^\times$, thus the Mackey topology $\tau(L_\rho, L_{\rho'})$ is coarser than the ρ-norm topology, which is $\tau(L_\rho, (L_\rho)^\times)$. This defect can be remedied by further restricting ρ. A function norm ρ is called *absolutely continuous* if $\rho(f_n) \searrow 0$ for every sequence $f_n \in L_\rho$ such that $f_1 \geqslant f_2 \geqslant \ldots \searrow 0$ pointwise a.e. on X. For instance, the Lebesgue L^p-norm is absolutely continuous for $1 \leqslant p < \infty$ but the L^∞-norm is *not*! Also, even if ρ is absolutely continuous, ρ' need not be. Yet, this is the appropriate concept, in view of the following results:

(i) $L_{\rho'} = (L_\rho)^\alpha = (L_\rho)^\times$ if and only if ρ is absolutely continuous;
(ii) L_ρ is reflexive if and only if ρ *and* ρ' are absolutely continuous and ρ has the Fatou property.

Let us denote by J the set of saturated, absolutely continuous function norms ρ on X, with the Fatou property and such that ρ' is also absolutely continuous. Then, for every $\rho \in J, \langle L_\rho, L_{\rho'}\rangle$ is a reflexive dual pair of Banach assaying subspaces of $(V, \#, \langle\cdot|\cdot\rangle)$. All that remains to do in order to get an indexed PIP-space of type (B) is to restrict the partial inner product to a nondegenerate subspace and perform the lattice construction of Section 2.2. Now the last point is in fact already done:

Lemma 4.4.1. *The set J is an involutive lattice with respect to the following partial order: $\rho_1 \leqslant \rho_2$ if and only if $\rho_1(f) \leqslant \rho_2(f)$, $\forall\, f \in V$. The lattice operations are the following:*

- $(\rho_1 \vee \rho_2)(f) = \max\{\rho_1(f), \rho_2(f)\}$,
- $(\rho_1 \wedge \rho_2)(f) = \inf\{\rho_1(f_1) + \rho_2(f_2); f_1, f_2 \in M^+, f_1 + f_2 = |f|\}$,
- *involution :* $\rho \leftrightarrow \rho'$.

Proof. Let $\rho_1, \rho_2 \in J$; so are ρ_1', ρ_2'. First we show that $\rho_1 \vee \rho_2$, which is obviously a norm, is saturated. Suppose it is not saturated, i.e., there exists a measurable set E of positive measure, such that $(\rho_1 \vee \rho_2)(\chi_F) = \infty$ for every measurable subset $F \subset E$ of positive measure. Thus, for every such $F \subset E,\ \rho_1(\chi_F) = \infty$ or $\rho_2(\chi_F) = \infty$. Since ρ_1 is saturated, there is a set $G \subset E$ such that $\rho_1(\chi_G) < \infty$ and for every $G_1 \subset G,\ \rho_1(\chi_{G_1}) < \infty$. This implies that $\rho_2(\chi_{G_1}) = \infty$ for every such G_1 and this is impossible for a saturated ρ_2.

Next it is always true [Zaa61, Problem 71.2] that $(\rho_1 \wedge \rho_2)' = \rho_1' \vee \rho_2'$, although $\rho_1 \wedge \rho_2$ as defined could be only a function *seminorm* [i.e., $p(f) = 0 \nRightarrow f = 0$]. However, since $\rho_1' \vee \rho_2'$ is a saturated norm, it follows from this equality that $\rho_1 \wedge \rho_2$ is also a saturated norm [Zaa61, Theorem 71.4]. Since ρ_1 and ρ_2 are absolutely continuous, so are all the others. Thus $L_{(\rho_1 \wedge \rho_2)'} = (L_{\rho_1 \wedge \rho_2})^{\times}$ is reflexive, and, therefore, $L_{\rho_1 \wedge \rho_2}$, is also reflexive, which implies that $\rho_1 \wedge \rho_2$ has the Fatou property. Since $\rho_1 \vee \rho_2$ also has the Fatou property, like any supremum [Zaa61, Theorem 65.4)], the proof is complete. ■

It is clear from the construction that we have recovered the general situation, for we have the relations

$$L_{\rho_1 \vee \rho_2} = (L_{\rho_1} \cap L_{\rho_2})_{\mathrm{proj}}, \quad L_{\rho_1 \wedge \rho_2} = (L_{\rho_1} + L_{\rho_2})_{\mathrm{ind}}.$$

It is interesting to notice that for any $\rho_1, \rho_2 \in J$, the ρ_1 norm and the ρ_2 norm are always consistent (see Proposition 2.2.1) on $L_{\rho_1} \cap L_{\rho_2}$ since $\rho_1 \wedge \rho_2$ is a norm.

Finally, for any sublattice I of J, define the space $V_I \equiv \sum_{\rho \in I} L_\rho$; thus $V_I^{\#} = \bigcap_{\rho \in I} L_\rho$. Then we have:

Proposition 4.4.2. *Let V be the vector space of all measurable, a.e. finite functions on the σ-finite measure space (X, μ). With the compatibility (4.16) and partial inner product (4.17), V becomes a degenerate* PIP*-space. Denote by J the involutive lattice of all saturated, absolutely continuous function norms ρ on X, which have the Fatou property and are such that ρ' is also absolutely continuous. Let I be any involutive sublattice of J such that:*

(i) $V_I \equiv \sum_{\rho \in I} L_\rho$ *is an assaying subset of* V*;*
(ii) $(V_I^{\#})^{\perp} = \{0\}$.

Then V_I with the PIP*-space structure induced by V is a nondegenerate indexed* PIP*-space of type (B) and, actually, a LBS.*

Example 4.4.3. The space of locally integrable functions
An interesting example is our familiar space $V_I = L^1_{\mathrm{loc}}(\mathbb{R}^n, dx)$. Then $V_I^{\#} = L^\infty_{\mathrm{c}}(\mathbb{R}^n, dx)$ and the two conditions above are verified. The corresponding set I is easily characterized: $\rho \in J$ belongs to I if and only if

$L_\rho \subset L^1_{\rm loc}(\mathbb{R}^n, dx)$ with continuous injection. If we write $X = \bigcup_j \Omega_j$, Ω_j compact, as in the proof of Proposition 4.2.1, we have the representations (4.8) and (4.9). Then $L_\rho \subset L^1_{\rm loc}(\mathbb{R}^n, dx)$ with continuous injection, if and only if ρ satisfies the following set of conditions:

$$\int_{\Omega_j} |f| dx \leqslant c_j \rho(f) \quad \text{for each } j = 1, 2, \ldots .$$

For instance, a weighted L^2-space $L^2(r)$ satisfies this condition if $r \in L^2_{\rm loc}(\mathbb{R}^n, dx)$ with $c_j = \left[\int_{\Omega_j} r^2 \, dx \right]^{1/2}$. It is thus clear that the family $\{L_\rho, \rho \in I\}$ is generating in $L^1_{\rm loc}(\mathbb{R}^n, dx)$, since it contains the generating subfamily $\{L^2(r), r \text{ and } r^{-1} \in L^2_{\rm loc}(\mathbb{R}^n, dx)\}$.

Example 4.4.4. L^p-spaces

Another example of the construction given in Proposition 4.4.2 is the lattice generated by the spaces $L^p(X, \mu)$, $1 < p < \infty$, where (X, μ) be a σ-finite measure space. Indeed, each L^p norm is saturated and absolutely continuous and has the Fatou property. Thus the family $\{L^p(X, \mu), 1 < p < \infty\}$ generates a LBS with the compatibility (4.16) and the L^2 inner product.

Remember that if $\mu(X) = \infty$ and μ has no atoms, no two L^p spaces are contained in each other, but $L^p \cap L^r \subset L^q$ for all q such that $p \leqslant q \leqslant r$. Hence, we get a genuine lattice in that case, as discussed in Section 4.1.2.

What about the spaces L^1 and L^∞? They are not reflexive, since the L^∞-norm is not absolutely continuous; hence they do not belong to any V_I. But if they are added by hand, one gets an interesting result. Consider the following norm:

$$\rho_0(f) = \sup_E \left\{ \int_E |f| d\mu : \mu(E) = 1 \right\}.$$

Then:

(i) ρ_0 has the Fatou property (since it is a supremum of function norms that have it), hence L_{ρ_0} is a Banach space and, in fact, $L_{\rho_0} = L_G$, the Gould space described in Section 4.1.2;
(ii) $L_{\rho_0} = (L^1 + L^\infty)_{\rm ind}, \quad (L_{\rho_0'} = L^1 \cap L^\infty)_{\rm proj}$;
(iii) $\bigcup_{1 \leqslant p \leqslant \infty} L^p$ is properly included in L_{ρ_0}.

So, exactly as for the chain $\{L^p[0,1], \ 1 \leqslant p \leqslant \infty\}$ discussed in Section 4.1, the family $\{L^p(X, \mu), \ 1 \leqslant p \leqslant \infty\}$ generates a lattice with extreme elements L_{ρ_0} and $L_{\rho_0'}$. The completion of that lattice can easily be described along the same lines.

A particular case is that of *weighted L^p spaces*, denoted $L^p_m(\mathbb{R}^d)$, corresponding to $X = \mathbb{R}^d$ and $d\mu(x) = m(x)\,dx$, for some positive, measurable, locally integrable weight function m. The corresponding norm reads

$$\|f\|_{L^p_m} := \left(\int_{\mathbb{R}^d} |f(x)|^p \, m(x)^p \, dx \right)^{1/p}, \;\; f \in L^p_m(\mathbb{R}^d). \tag{4.18}$$

These spaces will play a central role in Chapter 8, in the applications to signal processing.

Example 4.4.5. Mixed-norm spaces

Among the Köthe function spaces, an interesting class consists in the so-called L^P spaces with mixed norm. Let (X, μ) and (Y, ν) be two σ-finite measure spaces and $1 \leqslant p, q \leqslant \infty$ (in the general case, one considers n such spaces and n-tuples $P := (p_1, p_2, \ldots, p_n)$). Then, a function $f(x, y)$ measurable on the product space $X \times Y$ is said to belong to $L^{(p,q)}(X \times Y)$ if the number obtained by taking successively the p-norm in x and the q-norm in y, in that order, is finite (exchanging the order of the two norms leads in general to a different space). If $p, q < \infty$, the norm reads

$$\|f\|_{(p,q)} = \left(\int_Y \left(\int_X |f(x,y)|^p \, d\mu(x) \right)^{q/p} d\nu(y) \right)^{1/q}. \tag{4.19}$$

The analogous norm for p or $q = \infty$ is obvious. For $p = q$, one gets the usual space $L^p(X \times Y)$.

These spaces enjoy a number of properties similar to those of the L^p spaces.

(i) Completeness: each space $L^{(p,q)}$ is a Banach space and it is reflexive if and only if $1 < p, q < \infty$.

(ii) Duality: the conjugate dual of $L^{(p,q)}$ is $L^{(\overline{p},\overline{q})}$, where, as usual, $p^{-1} + \overline{p}^{-1} = 1$, $q^{-1} + \overline{q}^{-1} = 1$. Thus the topological conjugate dual coincides with the Köthe dual, as for all Köthe function spaces.

(iii) Generalized Hölder inequality:

$$\left| \iint_{X \times Y} f_1 f_2 \ldots f_m \, d\mu \, d\nu \right| \leqslant \|f_1\|_{(p_1,q_1)} \, \|f_2\|_{(p_2,q_2)} \cdots \|f_m\|_{(p_m,q_m)},$$

whenever $\sum_{i=1}^m \frac{1}{p_i} = 1, \sum_{i=1}^m \frac{1}{q_i} = 1$.

(iv) Interpolation: if $f \in L^{(p_1,p_2)}$ and $f \in L^{(q_1,q_2)}$, then $f \in L^{(r_1,r_2)}$, where $\frac{1}{r_i} = \frac{t}{p_i} + \frac{1-t}{q_i}$, $i = 1, 2,\; 0 \leqslant t \leqslant 1$, and

$$\|f\|_{(r_1,r_2)} \leqslant \left(\|f\|_{(p_1,p_2)} \right)^t \left(\|f\|_{(q_1,q_2)} \right)^{1-t}.$$

Notice that there is in general no inclusion relation between two different spaces $L^{(p,q)}$.

When $X = Y = \mathbb{R}$ with Lebesgue measure, additional theorems hold true.

(v) Young's theorem: Let $\frac{1}{p_i} + \frac{1}{q_i} = 1 + \frac{1}{r_i}$, $i = 1, 2$. If $f \in L^{(p_1,p_2)}$, $g \in L^{(q_1,q_2)}$, then $f * g \in L^{(r_1,r_2)}$ and

$$\|f * g\|_{(r_1,r_2)} \leqslant \|f\|_{(p_1,p_2)} \, \|g\|_{(q_1,q_2)}.$$

(vi) Hausdorff-Young theorem: Let $f \in L^{(p,q)}$, with $1 \leqslant q \leqslant p \leqslant 2$, and let $\mathcal{F}f(\xi, \eta)$ be the Fourier transform of $f(x, y)$. Then $\mathcal{F}f \in L^{(\overline{p},\overline{q})}$, where $p^{-1} + \overline{p}^{-1} = 1$, $q^{-1} + \overline{q}^{-1} = 1$, and

$$\|\mathcal{F}f\|_{(\overline{p},\overline{q})} \leqslant \|f\|_{(p,q)}.$$

On the other hand, the assertion is false if $p < q$.

We have given here the general definition, but in fact only some particular subclasses of these mixed-norm spaces have found applications in signal processing. These spaces will thus be described in Chapter 8 which is entirely devoted to that topic.

4.5 Analyticity/Trajectory Spaces

A different kind of PIP-space is given by the so-called analyticity/trajectory spaces, which were conceived as a substitute to distribution spaces. These spaces are of two types, which are in duality. We treat them briefly in succession.

Let $\mathcal{H}$ be a separable Hilbert space and let A be a non-negative self-adjoint operator in $\mathcal{H}$. The *analyticity space* associated to A, denoted $\mathcal{S}_{\mathcal{H},A}$, consists of all vectors $v \in \mathcal{D}^\infty(A) = \bigcap_{n=1}^\infty D(A^n)$ for which there exists constants a, b such that $\|A^n v\| \leqslant n! a^n b$, $\forall\, n \in \mathbb{N}$. Such vectors are called *analytic* vectors for A, hence the name of the space (a different, equivalent definition of analytic vectors is given in Section 7.1.2). Let $\{e^{-tA},\ t > 0\}$ be the semigroup generated by $-A$. Then the vector v is analytic for A if and only if there exists a $t > 0$ such that $v \in e^{-tA}(\mathcal{H})$. Thus we may write

$$\mathcal{S}_{\mathcal{H},A} = \bigcup_{t>0} e^{-tA}(\mathcal{H}). \tag{4.20}$$

For each $t > 0$, the space $e^{-tA}(\mathcal{H})$ is a Hilbert space for the inner product

$$\langle v|w\rangle_t := \langle e^{tA}v|e^{tA}w\rangle_{\mathcal{H}},\ v, w \in e^{-tA}(\mathcal{H}).$$

Since $e^{-tA}(\mathcal{H}) \subset e^{-sA}(\mathcal{H})$ for $0 < s < t$, with continuous embedding, we obtain a continuous scale of Hilbert spaces.

In view of the definition (4.20), one naturally puts on $\mathcal{S}_{\mathcal{H},A}$ the inductive limit topology (but the inductive limit is not strict, see Appendix B). However, one can put on $\mathcal{S}_{\mathcal{H},A}$ an (uncountable) family of seminorms that define the same inductive limit topology, and this fact ensures that the space has nice properties. In particular, $\mathcal{S}_{\mathcal{H},A}$ is complete, bornological and barreled. In addition, $\mathcal{S}_{\mathcal{H},A}$ is nuclear if and only if every operator $e^{-tA}, t > 0$, is Hilbert-Schmidt on $\mathcal{H}$.

Next we introduce the second type of spaces. A *trajectory* is a map $F : (0, \infty) \to \mathcal{H}$ such that

$$F(t+s) = e^{-sA}F(t), \quad \text{for all } t > 0, s \geqslant 0. \tag{4.21}$$

We denote by $\mathcal{T}_{\mathcal{H},A}$ the space of all trajectories. This is in fact the space of all solutions of the evolution equation

$$\frac{dF}{dt} = -AF, \ t > 0,$$

such that $F(t) \in \mathcal{H}$ for all $t > 0$ (but not necessarily $F(0)$). The trajectory space can be given a natural topology with the seminorms $q_t(F) := \|F(t)\|_{\mathcal{H}}$, and for this topology $\mathcal{T}_{\mathcal{H},A}$ is a Fréchet space. As a consequence, $\mathcal{T}_{\mathcal{H},A}$ is complete, bornological and barreled. In addition, $\mathcal{T}_{\mathcal{H},A}$ is nuclear if and only if every operator $e^{-tA}, t > 0$, is Hilbert-Schmidt on $\mathcal{H}$.

The Hilbert space $\mathcal{H}$ can be embedded into $\mathcal{T}_{\mathcal{H},A}$ by the map $\iota : \mathcal{H} \to \mathcal{T}_{\mathcal{H},A}$ defined as $\iota(x) : t \mapsto e^{-tA}x, \ t > 0, x \in \mathcal{H}$. Thus, putting all together, we obtain the following triplet, with continuous embeddings and dense images,

$$\mathcal{S}_{\mathcal{H},A} \subset \mathcal{H} \subset \mathcal{T}_{\mathcal{H},A} \ . \tag{4.22}$$

Finally one turns $\langle \mathcal{S}_{\mathcal{H},A}, \mathcal{T}_{\mathcal{H},A} \rangle$ into a dual pair, with the sesquilinear form

$$\langle w, F \rangle := \langle e^{tA} w | F(t) \rangle_{\mathcal{H}}, \ w \in \mathcal{S}_{\mathcal{H},A}, \ F \in \mathcal{T}_{\mathcal{H},A}. \tag{4.23}$$

Here t has to be chosen so small that $w \in e^{-tA}(\mathcal{H})$, and then the definition of $\langle w, F \rangle$ is independent of the choice of $t > 0$.

It is easy to see that the sequilinear form $\langle \cdot, \cdot \rangle$ defined in (4.23) is nondegenerate, so that $\langle \mathcal{S}_{\mathcal{H},A}, \mathcal{T}_{\mathcal{H},A} \rangle$ is indeed a dual pair. In addition, both spaces being barreled, the topologies described above coincide with the respective Mackey topologies $\tau(\mathcal{S}_{\mathcal{H},A}, \mathcal{T}_{\mathcal{H},A})$ and $\tau(\mathcal{T}_{\mathcal{H},A}, \mathcal{S}_{\mathcal{H},A})$, respectively, and the same holds for the corresponding strong topologies.

In view of the last result, it is clear that the triplet (4.22) is a RHS of the standard form, including nuclearity in some cases. Thus it may serve as a substitute to the standard RHSs of distributions of Schwartz and Gel'fand (another triplet, based on the so-called Feichtinger algebra, will be described in Section 8.3.2). As such, it is also a PIP-space. But one can also define, in several ways, a compatibility relation directly on $\mathcal{T}_{\mathcal{H},A}$. For instance, one can declare $F, G \in \mathcal{T}_{\mathcal{H},A}$ compatible whenever $\lim_{t\to 0}\langle F(t)|G(t)\rangle_{\mathcal{H}}$ exists. It would be interesting to explore the resulting PIP-space structure.

The formalism of analyticity/trajectory spaces covers a large number of interesting situations. We refer to the literature quoted in the Notes for more information. Let us just give a few examples.

(i) Spherical harmonics on the unit sphere

Let $\mathcal{H} = L^2(S^{n-1})$, where S^{n-1} is the unit sphere in $\mathbb{R}^n$, and write the Laplacian operator in spherical coordinates (r, ϖ):

$$\Delta = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\,\Delta_{\mathrm{LB}},$$

where Δ_{LB} is the Laplace-Beltrami operator on S^{n-1}. Then the spherical harmonics are eigenfunctions of Δ_{LB} and the operators $e^{-t\Delta_{\mathrm{LB}}^{1/2}}$ are Hilbert-Schmidt for all $t > 0$. Hence the space $\mathcal{S}_{L^2(S^{n-1}),\,\Delta_{\mathrm{LB}}^{1/2}}$ is nuclear.

(ii) Gel'fand-Shilov spaces

For $\mathcal{H} = L^2(\mathbb{R}, dx)$, certain Gel'fand-Shilov spaces S_α^β (see Section 5.4.3) are analyticity spaces, in particular the spaces S_α^α for all $\alpha \geqslant \frac{1}{2}$ (see next item).

(iii) Analyticity spaces based on classical polynomials

A general technique for generating analyticity spaces is to start from an orthonormal basis $\{v_n,\ n \in \mathbb{N}\}$ in $\mathcal{H}$ and the corresponding unitary (Fourier) map $f \mapsto (\langle v_n|f\rangle)$ from $\mathcal{H}$ onto ℓ^2. Let $0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \ldots$ be a sequence of real numbers such that $\sum_{n=1}^{\infty} e^{-\lambda_n t}$ converges for all $t > 0$. Consider the space $\mathcal{T}$ of formal series $\sum_{n=1}^{\infty} a_n v_n$ where the coefficients a_n satisfy the following condition:

$$\sup_{n\in\mathbb{N}} |a_n|\, e^{-\lambda_n t} < \infty,\ \forall\, t > 0.$$

Then the (Köthe) α-dual of $\mathcal{T}$, denoted $\mathcal{S}$, can be represented by elements in $\mathcal{H}$ of the form $\sum_{n=1}^{\infty} b_n v_n$, where the coefficients b_n satisfy the condition

(s) There exists $s > 0$ such that $\sup_{n\in\mathbb{N}} |b_n|\, e^{\lambda_n s} < \infty,\ \forall\, t > 0.$

Consider now the operator $A = \sum_{n=1}^{\infty} \lambda_n \langle v_n|f\rangle) v_n$ on the domain $D(A) = \{f \in \mathcal{H} : \sum_{n=1}^{\infty} \lambda_n^2 |\langle v_n|f\rangle) v_n|^2\}$. Clearly A is a non-negative self-adjoint operator in $\mathcal{H}$, with eigenvalues λ_n and eigenvectors v_n. Then one has $\mathcal{S} = \mathcal{S}_{\mathcal{H},A}$ and $\mathcal{T} = \mathcal{T}_{\mathcal{H},A}$.

If we take $\mathcal{H} = L^2(\mathbb{R}, dx)$ and particularize the operator A to $H_{\rm osc} = \frac{1}{2}\big(-\frac{d^2}{dx^2} + x^2 \ + 1\big)$ (the Hamiltonian of the quantum harmonic oscillator), so that v_n is the $n^{\rm th}$ Hermite function, one obtains an analyticity space based on Hermite polynomials. In particular, one can show that $\mathcal{S}_{L^2(\mathbb{R}),H_{\rm osc}^{1/2\alpha}}$ coincides with the Gel'fand-Shilov space S_α^α for all $\alpha \geqslant \frac{1}{2}$. Similar constructions lead to analyticity spaces based on other orthogonal families of classical polynomials, such as the Laguerre or the Jacobi polynomials.

The analyticity/trajectory spaces have been designed as a mathematical tool for obtaining a rigorous formulation of Dirac's formalism of quantum mechanics. We will describe this application in Section 7.1.1 (ii).

4.6 PIP-Spaces of Analytic Functions

Our last class of examples is of a totally different nature. Namely, the spaces to be considered consist of analytic functions, of three different types. In the first case, the space V_I consists of entire functions, the central Hilbert space is the Fock-Bargmann space $\mathfrak{F} \equiv \mathfrak{F}^0$ introduced in Section 1.1.3, Example (v), and the indexing parameter is the rate of growth at infinity. The second class of PIP-spaces is made of functions analytic in a sector and there the indexing parameter is the opening angle of that sector. The interesting feature of these last examples is that the "large" space V is itself a Hilbert space. Finally we turn to spaces of functions analytic in the unit disk. In the sequel we will state the results in substantial details, but omitting all proofs, for which we refer to the original references quoted in the Notes.

4.6.1 A RHS of Entire Functions

There are several situations where one considers Hilbert spaces consisting of analytic functions of a complex variable z. The prime example is the Fock-Bargmann or phase space representation of quantum mechanics, in which $z = q + ip$ (q denotes position and p momentum).

For convenience, we repeat the definition, specializing to one dimension. The Fock-Bargmann Hilbert space is

$$\mathfrak{F} = \{f(z) \text{ entire} : \int_{\mathbb{C}} |f(z)|^2 \, d\mu(z) < \infty\}, \tag{4.24}$$

with inner product

$$\langle f|g\rangle = \int_{\mathbb{C}} \overline{f(z)} g(z) \, d\mu(z). \tag{4.25}$$

Here $d\mu(z) = \pi^{-1} e^{-|z|^2} d\nu(z)$, where $d\nu(z) = \frac{i}{2} dz \wedge d\overline{z} = dxdy$ is the Lebesgue measure on $\mathbb{C}$. The Hilbert space $\mathfrak{F}$ possesses several interesting properties.

- *Orthonormal basis:* $u_n(z) = (n!)^{-1/2}\, z^n$, $n = 0, 1, 2, \ldots$. In this basis, the inner product (4.25) reads

$$\langle f|g\rangle = \sum_n n!\, \overline{f_n}\, g_n, \text{ for } f(z) = \sum_n f_n\, z^n,\ g(z) = \sum_n g_n\, z^n. \tag{4.26}$$

- *Principal vectors* (coherent states): $e_w(z) = e^{\overline{w}z}$, $w, z \in \mathbb{C}$, so that $f(z) = \langle e_z|f\rangle$, that is, e_z is an *evaluation functional* (the equivalent of a delta function, but here e_z is a bona fide vector of the Hilbert space).
- *Reproducing kernel:* $K(w, z) = e^{w\overline{z}} = \langle e_w|e_z\rangle$, which means that

$$f(w) = \int_{\mathbb{C}} K(w, z) f(z)\, d\mu(z),\ \forall\, f \in \mathfrak{F}. \tag{4.27}$$

The question that arises now is, how to build a RHS or a LHS around the Fock-Bargmann space $\mathfrak{F}$. Since $\mathfrak{F}$ consists of entire functions, one may identify immediately possible extreme spaces:

- The *maximal* space $\mathfrak{Z}$, consisting of *all* entire functions. With uniform convergence on compact sets, the space $\mathfrak{Z}$ is a nuclear Fréchet space. Its conjugate dual $\mathfrak{Z}^\times$, the space of *antianalytic functionals*, is thus a nuclear, complete DF-space, exactly what is needed for applying the nuclear spectral theorem.
- The *minimal* space Exp consisting of entire functions of exponential type:

$$\mathrm{Exp} = \{f \in \mathfrak{Z} : \exists\, a, c > 0 \text{ such that } |f(z)| \leqslant c\, e^{a|z|},\ \forall\, z \in \mathbb{C}\}. \tag{4.28}$$

It is a standard result that $\mathfrak{Z}^\times$ is isomorphic to Exp. The correspondence is

$$\mu \in \mathfrak{Z}^\times \mapsto \widehat{\mu} \in \mathrm{Exp},$$

where $\widehat{\mu}(w) = \langle \mu, e_w\rangle$, the Fourier-Borel transform. Thus we get a natural RHS around $\mathfrak{F}$:

$$\mathfrak{Z}^\times \simeq \mathrm{Exp}\ \subset\ \mathfrak{F}\ \subset\ \mathfrak{Z}. \tag{4.29}$$

The duality between Exp and $\mathfrak{Z}$ is given indeed by a natural extension of the inner product of $\mathfrak{F}$:

$$\langle \overline{f}, g\rangle = \langle f|g\rangle = \sum_n n!\, \overline{f_n}\, g_n, f \in \mathrm{Exp},\ g \in \mathfrak{Z}. \tag{4.30}$$

This answers the first question. But how to enrich the RHS (4.29) into a LHS?

4.6.2 A LHS of Entire Functions Around $\mathfrak{F}$

On the Hilbert space $\mathfrak{F}$, the two equivalent forms (4.25) and (4.26) of the inner product define two natural linear compatibilities (see Example 1.5.5):

$$f \#_1 g \iff \int_{\mathbb{C}} |f(z)\, g(z)|\, d\mu(z) < \infty, \tag{4.31}$$

$$f \#_2 g \iff \sum_n n! |f_n\, g_n| < \infty. \tag{4.32}$$

Of course, $\#_1$ and $\#_2$ coincide on $\mathfrak{F}$, but they are not comparable on $\mathfrak{Z}$! In fact, $\#_1$ is too general, and somewhat pathological, since one has $\mathfrak{Z}^{\#_1} = \{0\}$, whereas $\#_2$ is more regular, and indeed $\mathrm{Exp} \overset{\#_2}{\longleftrightarrow} \mathfrak{Z}$, but it applies only to sequences, not analytic functions. Actually, $\mathfrak{Z}$ may be considered as an echelon space (see Section 4.3.3) and Exp as the corresponding dual co-echelon space. Indeed, define the steps $a_{(k)}$ by $(a_{(k)})_n = k^n,\ k = 1, 2, \ldots$. Then one has

$$\mathfrak{Z} = \bigcap_{k=1}^{\infty} \left\{a_{(k)}\right\}^{\#_2}, \quad \mathrm{Exp} = \bigcup_{k=1}^{\infty} \left\{a_{(k)}\right\}^{\#_2\#_2}.$$

The solution is to restrict the large space $\mathfrak{Z}$ to some smaller, more manageable space. This may be done in two ways.

(i) A chain of Hilbert spaces

For every $\rho \in \mathbb{R}$, consider again the Hilbert space $\mathfrak{F}^\rho$ defined in Section 1.1.3, Example (iv):

$$\mathfrak{F}^\rho = \{f \in \mathfrak{Z} : \|f\|_\rho^2 := \int_{\mathbb{C}} |f(z)|^2\, (1 + |z|^2)^\rho\, d\mu(z) < \infty\}. \tag{4.33}$$

The vectors $u_m^\rho(z) := (\eta_m^\rho)^{-1/2} u_m(z)$, $m = 0, 1, 2, \ldots$ form an orthonormal basis of $\mathfrak{F}^\rho$, where $\eta_m^\rho = \|u_m\|_\rho^2$. Thus each space $\mathfrak{F}^\rho$ may be realized as a sequence space, namely, a weighted ℓ^2 space.

The family $\{\mathfrak{F}^\rho,\ \rho \in \mathbb{R}\}$ is a chain of Hilbert spaces, corresponding to:

$$\mathfrak{F}^\rho \subset \mathfrak{F}^\sigma \quad \iff \quad \rho > \sigma, \tag{4.34}$$

and one has $\mathfrak{F}^0 = \mathfrak{F}$, $(\mathfrak{F}^\rho)^\times = \mathfrak{F}^{-\rho} = (\mathfrak{F}^\rho)^{\#_1} = (\mathfrak{F}^\rho)^{\#_2}$ (as vector spaces) and $e_w \in \mathfrak{F}^\rho,\ \forall\, \rho \in \mathbb{R}, \forall\, w \in \mathbb{C}$. Defining the extreme spaces

$$\mathfrak{E} = \bigcap_{\rho \in \mathbb{R}} \mathfrak{F}^\rho, \quad \mathfrak{E}^\times = \bigcup_{\rho \in \mathbb{R}} \mathfrak{F}^\rho. \tag{4.35}$$

one recovers Bargmann's space $\mathfrak{E}^\times$. Then one gets a Hilbert chain, with the structure ($\rho > 0$):

$$\mathrm{Exp} \subset \mathfrak{E} \subset \ldots \mathfrak{F}^\rho \ldots \subset \mathfrak{F} \subset \ldots \mathfrak{F}^{-\rho} \ldots \subset \mathfrak{E}^\times \subset \mathfrak{Z}, \tag{4.36}$$

and, by restriction, a RHS isomorphic to the Schwartz triplets $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})^\times$ and $s \subset \ell^2 \subset s^\times$, namely,

$$\mathfrak{E} \subset \mathfrak{F} \subset \mathfrak{E}^\times. \tag{4.37}$$

A word of caution is in order here concerning duality. Indeed, in the chain (4.36) above, the space $\mathfrak{F}^\rho$ carries the norm $\| \cdot \|_\rho$ and $\mathfrak{F}^{-\rho}$ the norm $\| \cdot \|_{-\rho}$. The two spaces are in duality, in virtue of the Hölder inequality

$$|\langle g|f\rangle| \leqslant \|g\|_{-\rho} \|f\|_\rho, \ g \in \mathfrak{F}^{-\rho}, f \in \mathfrak{F}^\rho,$$

where

$$\|g\|^2_{-\rho} = \sum_m \eta_m^{-\rho} \, |\langle u_m|g\rangle|^2, \qquad \|f\|^2_\rho = \sum_m \eta_m^\rho \, |\langle u_m|f\rangle|^2.$$

Thus, if we define the weights $r_m(\rho) := \left(\eta_m^\rho\right)^{1/2}$, we may identify

$$\mathfrak{F}^\rho \sim \ell^2[r(\rho)], \quad \mathfrak{F}^{-\rho} \sim \ell^2[r(-\rho)].$$

Notice that the "natural" bijection $f(z) \mapsto (1+|z|^2)^\rho f(z)$ does *not* map $\mathfrak{F}^\rho$ onto $\mathfrak{F}^{-\rho}$, because of the requirement of analyticity. Hence we have to make the detour via sequence spaces for identifying the conjugate dual $(\mathfrak{F}^\rho)^\times$ of $\mathfrak{F}^\rho$. The latter is the set of all continuous, conjugate linear, functionals on $\mathfrak{F}^\rho$. For such a functional $L \in (\mathfrak{F}^\rho)^\times$, define the (dual) norm

$$\|L\|^\times_\rho := \sup_{\|f\|_\rho \leqslant 1} |L(f)|, \ f \in \mathfrak{F}^\rho. \tag{4.38}$$

By the Riesz Lemma, $L(f)$ is of the form $L(f) = \langle f|h\rangle_\rho$, for a unique element $h \in \mathfrak{F}^\rho$ and

$$\|L\|^\times_\rho = \|h\|_\rho = \Big(\sum_m (\eta_m^\rho)^{-1} \, |L(u_m)|^2 \Big)^{1/2}. \tag{4.39}$$

Thus $(\mathfrak{F}^\rho)^\times$ is a Hilbert space for the norm $\| \cdot \|^\times_\rho$ and one may identify it with $\ell^2[r(\rho)^{-1}]$. $(\mathfrak{F}^\rho)^\times$ coincides with $\mathfrak{F}^{-\rho}$ as a vector space, but *not* as Hilbert space. Indeed, every functional $L \in (\mathfrak{F}^\rho)^\times$ is of the form $L(f) = \langle f|g\rangle$, for a unique element $g \in \mathfrak{F}^{-\rho}$. Then the identity

$$(\eta_m^\rho)^{-1} \leqslant \eta_m^{-\rho} \leqslant c_\rho \eta_m^\rho \quad (c_\rho > 0)$$

implies

$$\|L\|_\rho^\times \leqslant \|g\|_{-\rho} \leqslant c_\rho \|L\|_\rho^\times,$$

so that the dual norm (4.38) is equivalent to, but different from the norm $\|\cdot\|_{-\rho}$. Since (4.34) implies also $\rho > \sigma \Leftrightarrow (\mathfrak{F}^\rho)^\times \subset (\mathfrak{F}^\sigma)^\times$, we obtain the following results.

Proposition 4.6.1. *(i) The family $\{\mathfrak{F}^\rho,\ \rho \in \mathbb{R}\}$, is a chain of hilbertian spaces, but not a LHS.*
(ii) The family $\{\mathfrak{F}^\rho, (\mathfrak{F}^\rho)^\times,\ \rho \geqslant 0\}$, is a LHS.
(iii) The same holds true for the family $\{(\mathfrak{F}^\rho)^\times, \mathfrak{F}^\rho,\ \rho \leqslant 0\}$.

The obvious conclusion of this proposition is that the notion of LHS is too restrictive in the case of spaces of analytic functions, the structure of a chain of hilbertian spaces is more natural.

(ii) A genuine lattice of hilbertian spaces

One may go one step further and construct a lattice of hilbertian spaces by considering more general weights in (4.33):

$$\mathfrak{F}(\rho) = \{f \in \mathfrak{Z} : \|f\|^2_{(\rho)} := \int_{\mathbb{C}} |f(z)|^2\, e^{-\rho(z)}\, d\mu(z) < \infty\}, \tag{4.40}$$

where $\rho : \mathbb{C} \to \mathbb{R}$ is a measurable function. Clearly the space $\mathfrak{F}(\rho)$ reduces to $\mathfrak{F}^\rho$ if $\rho(z) = \ln(1 + |z|^2)^{-\rho}$, $\rho \in \mathbb{R}$. In order to obtain a lattice of hilbertian spaces around $\mathfrak{F}$, we must require that the space $\mathfrak{F}(\rho)$ satisfies several conditions, each of which imposes some restrictions to the weight function ρ (all of them are satisfied by every $\mathfrak{F}^\rho$, $\rho \in \mathbb{R}$):

(i) $\mathfrak{F}(\rho)$ *is a Hilbert space*, i.e., it is complete, which holds true if ρ is locally bounded, that is, bounded on compact sets.

(ii) *The set of polynomials is dense in* $\mathfrak{F}(\rho)$. This is a restriction on the growth of ρ: if ρ grows too fast at infinity, $\mathfrak{F}(\rho)$ may become trivial and $\mathfrak{F}(-\rho)$ too large. When (ii) holds, the monomials $\{u_m(z) = (m!)^{-1/2} z^m,\ m = 1, 2, \ldots\}$ form a basis of $\mathfrak{F}(\rho)$. If ρ is *radial*, i.e. $\rho(z) = \rho(|z|)$, then the functions

$$u_m^{(\rho)}(z) = (\eta_m^{(\rho)})^{-1/2}\, u_m(z),\ m = 1, 2, \ldots$$

form an orthonormal basis of $\mathfrak{F}(\rho)$. The normalization coefficients, defined as

$$\eta_m^{(\rho)} = \int_{\mathbb{C}} |u_m(z)|^2\, e^{-\rho(z)}\, d\mu(z) = \frac{1}{m!} \int_0^\infty t^m\, e^{-\rho(t) - t}\, dt, \tag{4.41}$$

are clearly related to a moment problem. In that case, all spaces $\mathfrak{F}(\rho)$ may be realized as weighted ℓ^2 sequence spaces and the compatibility $\#_2$ is easy to handle.

(iii) *Duality:* $\mathfrak{F}(\rho)^{\times}$ *is isomorphic to* $\mathfrak{F}(-\rho)$ *(as locally convex spaces).* This is the crucial condition, and it is difficult to verify, unless ρ is radial. There are several partial results:

(a) If ρ is locally bounded, then $\mathfrak{F}(-\rho) = \mathfrak{F}(\rho)^{\#_1} \subseteq \mathfrak{F}(\rho)^{\times}$;
(b) If ρ is locally bounded and radial, then one has, in addition, $\mathfrak{F}(\rho)^{\times} = \mathfrak{F}(\rho)^{\#_2}$;
(c) If ρ is locally bounded and radial, then $\mathfrak{F}(-\rho) = \mathfrak{F}(\rho)^{\times}$ if and only if there exists a positive constant $c(\rho)$ such that

$$1 \leqslant \eta_m^{(\rho)} \, \eta_m^{(-\rho)} \leqslant c(\rho), \text{ for all } m = 0, 1, 2, \ldots .$$

Thus we conclude that the duality condition is equivalent to the condition $\mathfrak{F}(\rho)^{\#_1} = \mathfrak{F}(\rho)^{\#_2}$.

(iv) *Principal vectors:* $e_w \in \mathfrak{F}(\rho), \forall\, w \in \mathbb{C}$, which implies that the principal vectors e_w generate a dense subspace of $\mathfrak{F}(\rho)$, and therefore every operator A in the resulting PIP-space is an integral operator, with kernel $A(w,z) = \langle e_w | A e_z \rangle$.

Now we have to examine the relationship between weights ρ and subspaces $\mathfrak{F}(\rho)$. We say that two weights ρ_1, ρ_2 are *equivalent* ($\rho_1 \approx \rho_2$) if $\mathfrak{F}(\rho_1) = \mathfrak{F}(\rho_2)$ as vector spaces. This happens if and only if there exists a constant C such that $|\rho_1(z) - \rho_2(z)| \leqslant C$ almost everywhere or, equivalently, there exist two constants A, B such that

$$A \leqslant e^{\rho_1(z) - \rho_2(z)} \leqslant B, \quad \text{a.e.} \tag{4.42}$$

In that case, the identity map $\mathfrak{F}(\rho_1) \to \mathfrak{F}(\rho_2)$ is bicontinuous, i.e., the two norms are equivalent.

Thus the family of Hilbert spaces $\{\mathfrak{F}(\rho)\}$ is indexed by $L^{\infty}_{\text{loc}}(\mathbb{C})/{\approx}$, which is an involutive lattice for the pointwise partial order ($\rho_1 \leqslant \rho_2$ if and only if $\rho_1(z) \leqslant \rho_2(z)$ a.e.) modulo $\approx$. We notice, in particular, the obvious equivalences

$$\begin{aligned} \rho_1 \wedge \rho_2 &:= \min(\rho_1, \rho_2) \approx -\ln(e^{-\rho_1} + e^{-\rho_2}), \\ \rho_1 \vee \rho_2 &:= \max(\rho_1, \rho_2) \approx \ln(e^{\rho_1} + e^{\rho_2}). \end{aligned}$$

Although some results may be obtained for general weights ρ, a complete answer may be formulated for radial weights only.

Theorem 4.6.2. *Let I be the set of weight functions $\rho(z)$ that satisfy the following three conditions:*

(i) ρ is locally bounded and radial, $\rho(z) = \rho(|z|)$.
(ii) e_w belongs to $\mathfrak{F}(\rho) \cap \mathfrak{F}(-\rho)$, for any $w \in \mathbb{C}$.
(iii) There are positive constants A, B such that

$$A \leqslant \eta_m^{(\rho)} \eta_m^{(-\rho)} \leqslant B, \text{ for all } m = 0, 1, 2, \ldots, \tag{4.43}$$

where $\eta_m^{(\rho)}$ is given by (4.41).

Then the family $\{\mathfrak{F}(\rho), \rho \in I\}$ is a lattice of hilbertian spaces with central Hilbert space $\mathfrak{F}(0) = \mathfrak{F}$ and lattice operations

$$\begin{aligned}\mathfrak{F}(\rho_1 \wedge \rho_2) &= \mathfrak{F}(\rho_1) \cap \mathfrak{F}(\rho_2), \\ \mathfrak{F}(\rho_1 \vee \rho_2) &= \mathfrak{F}(\rho_1) + \mathfrak{F}(\rho_2), \\ \mathfrak{F}(\rho)^\times &= \mathfrak{F}(-\rho)\,.\end{aligned}$$

The lattice is indexed by the set $I/\approx$, where the equivalence relation $\approx$ is defined by (4.42). The compatibility $\#$ coincides with both $\#_1$ and $\#_2$, and the partial inner product with that induced by $\mathfrak{F}$ and ℓ^2, respectively.

For the proof, it suffices to notice that condition (4.43) is necessary and sufficient for the isomorphism $\mathfrak{F}(\rho)^\times \simeq \mathfrak{F}(-\rho)$ as locally convex spaces.

In this discussion, one takes each Hilbert space $\mathfrak{F}(\rho)$ with its norm $\|\cdot\|_{(\rho)}$, but the dual norm is only equivalent to the norm $\|\cdot\|_{(-\rho)}$, as a consequence of the inequalities (4.43). This may be seen exactly as in the case of the spaces $\mathfrak{F}^\rho$ discussed above. If we denote by $\mathfrak{F}(\widehat{\rho})$ the dual Hilbert space $\mathfrak{F}(\rho)^\times$, equipped with the dual norm, then we can identify the three Hilbert spaces $\mathfrak{F}(\pm\rho), \mathfrak{F}(\widehat{\rho})$ with weighted ℓ^2 spaces and the conclusion follows.

In particular, the problem with this construction is that the space $\mathfrak{F}(\widehat{\rho})$ is a space of functions analytic in $\mathbb{C}$, but need not be associated to a weight. The existence of the latter is a moment problem, for which some results are known, but we shall not pursue the point.

In other words, the structure described in Theorem 4.6.2 is that of a lattice of hilbertian spaces and cannot lead to a LHS.

4.6.3 Functions Analytic in a Sector

In this section, we shall describe another LHS of analytic functions, in which the order parameter is the opening angle of a sector, instead of the rate of growth at infinity. This LHS simplifies considerably the formulation of scattering theory, as we shall see in Section 7.2.

Define $G(a,b)$ $(-\pi < a < b < \pi)$ as the space of all functions $f(z)$, $z = re^{i\varphi}$, which are analytic in the open sector $S_{a,b} := \{z = re^{i\varphi}, a < \varphi < b\}$, and such that the integral $\int_0^\infty |f(re^{i\varphi})|^2 \, dr < \infty$ is uniformly bounded in $\varphi \in (a,b)$. Then it is known that every function $f(re^{i\varphi}) \in G(a,b)$ possesses well-defined limiting values (in the sense of L^2 limits) $f(re^{ia}), f(re^{ib})$ on the boundaries of $S_{a,b}$, and furthermore that $G(a,b)$ is complete, thus a Hilbert space, for the inner product

$$\langle f|g\rangle_{ab} = \int_0^\infty \overline{f(re^{ia})}\, g(re^{ia})\; dr + \int_0^\infty \overline{f(re^{ib})}\, g(re^{ib})\; dr. \tag{4.44}$$

Among the bounded linear operators on $G(a,b)$, a distinguished role is played by the van Winter class $\mathfrak{W}$ of those operators A for which the quantity

$$\mathfrak{a}(A,\varphi) = \sup_{f\in G(a,b)} \left[\int_0^\infty |(Af)(re^{i\varphi})|^2\; dr\right] \left[\int_0^\infty |f(re^{i\varphi})|^2\; dr\right]^{-1/2} \tag{4.45}$$

is uniformly bounded in $\varphi \in (a,b)$. It turns out that the class $\mathfrak{W}$ is a Banach *-algebra for the norm $\alpha(A) = \sup_{a<\varphi<b} \mathfrak{a}(A,\varphi)$. Furthermore, $\mathfrak{W}$ contains a two-sided ideal $\mathfrak{K}$ of integral operators which are all Hilbert-Schmidt. Next, we define the Hilbert space

$$\widetilde{G}(a,b) := \{\widetilde{f}(t,\varphi) = e^{\varphi t - i\varphi/2}\widetilde{f}(t),\; t\in\mathbb{R},\; a \leqslant \varphi \leqslant b, \tag{4.46}$$

$$\text{with} \int_{-\infty}^{+\infty} (e^{2at} + e^{2bt})\, |\widetilde{f}(t)|^2\, dt < \infty\}, \tag{4.47}$$

with inner product

$$\langle f|g\rangle_{\widetilde{ab}} = \int_{-\infty}^{\infty} (e^{2at} + e^{2bt})\, \overline{\widetilde{f}(t)}\widetilde{g}(t)\, dt.$$

Introduce the Mellin transform of f:

$$\widetilde{f}(t) = (2\pi)^{-1/2} \int_0^{+\infty} f(x)\, x^{-it-\frac{1}{2}}\, dx$$

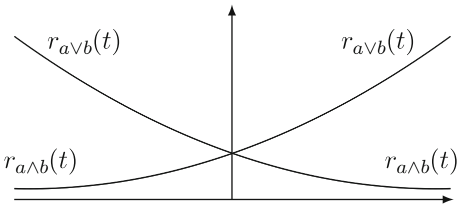
and its inverse

$$f(re^{i\varphi}) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \widetilde{f}(t)\, (re^{i\varphi})^{it-\frac{1}{2}}\, dt.$$

Then it turns out that $f(re^{i\varphi}) \in G(a,b)$ if and only if $\widetilde{f}(t,\varphi) \in \widetilde{G}(a,b)$. Furthermore the Mellin transform $f \mapsto \widetilde{f}$ is a unitary map from $G(a,b)$ onto $\widetilde{G}(a,b)$. One has indeed, by a straightforward calculation,

$$\begin{aligned}\langle f|g\rangle_{ab} &= \int_0^\infty \overline{f(re^{ia})}\, g(re^{ia})\, dr + \int_0^\infty \overline{f(re^{ib})}\, g(re^{ib})\, dr \\ &= \int_{-\infty}^\infty (e^{2at} + e^{2bt})\, \overline{\widetilde{f}(t)}\widetilde{g}(t)\, dt = \langle \widetilde{f}|\widetilde{g}\rangle_{\widetilde{ab}}.\end{aligned}$$

Fig. 4.3 The functions $r_{a\wedge b}$ and $r_{a\vee b}$ for $a<0<b$



In addition, the Fourier transform is unitary from $\widetilde{G}(a,b)$ onto $\widetilde{G}(-b,-a)$.

We claim that the family $\{G(a,b),\, -\frac{\pi}{2} \leqslant a < b \leqslant \frac{\pi}{2}\}$ may be identified with a part of a LHS of weighted L^2 spaces. Indeed, for $-\frac{\pi}{2} \leqslant a \leqslant \frac{\pi}{2}$, define the Hilbert space

$$L^2(a) := \left\{ \widetilde{f} : \int_{-\infty}^{+\infty} e^{2at}\, |\widetilde{f}(t)|^2 \, dt < \infty \right\} = L^2(r_a), \quad \text{with } r_a(t) = e^{-2at}. \tag{4.48}$$

Then consider the lattice generated by the family $\{L^2(a),\, -\frac{\pi}{2} \leqslant a \leqslant \frac{\pi}{2}\}$. The infimum is $L^2(a) \wedge L^2(b) = L^2(a) \cap L^2(b) = L^2(a \wedge b)$, with $r_{a\wedge b}(t) = \min(r_a(t), r_b(t))$, and the supremum $L^2(a) \vee L^2(b) = L^2(a) + L^2(b) = L^2(a \vee b)$, $r_{a\vee b}(t) = \max(r_a(t), r_b(t))$ (see Fig. 4.3 for the case $a < 0 < b$). As usual, these norms are equivalent to the projective, resp. inductive, norms. Duality is given by $L^2(a \wedge b) \Longleftrightarrow L^2(-a \vee -b)$.

Thus we obtain a LHS, with extreme spaces $V^{\#} = L^2(-\frac{\pi}{2}) \cap L^2(\frac{\pi}{2})$, $V = L^2(-\frac{\pi}{2}) + L^2(\frac{\pi}{2})$, which are themselves Hilbert spaces. In addition, all spaces are obtained at the first generation, i.e., they are all of the form $L^2(c \wedge d)$ or $L^2(c \vee d)$. One has, for instance, with $|c| = \min(|a|, |b|)$:

$$L^2(a \wedge b) \wedge L^2(-b \vee -a) = \begin{cases} L^2(-|c| \vee |c|), & \text{if } a, b \text{ have the same sign,} \\ L^2(-|c| \wedge |c|), & \text{if } a, b \text{ have opposite signs.} \end{cases}$$

Therefore, all spaces may be obtained from $L^2(-\frac{\pi}{2})$ and $L^2(\frac{\pi}{2})$ by interpolation (see Section 5.1.2).

In the case $0 < a < b$, one gets the picture shown in Fig. 4.4. Duality corresponds to symmetry with respect to the center (i.e., L^2): $a \wedge b \iff -b \vee -a$. Notice that one can work with fixed b also, using only the sublattice generated (in the case depicted in Fig. 4.4) by $L^2(b), L^2(-b)$.

As for operators on this LHS, one may show, using again interpolation theory, that the two operator algebras $\mathfrak{A}$ and $\mathfrak{B}$ defined in Section 3.3.3 coincide, i.e., there is a unique Banach *-algebra of bounded operators:

$$\mathfrak{A} = \mathfrak{B} = \{A \text{ linear } : A \text{ maps } L^2(-\frac{\pi}{2}) \text{ and } L^2(\frac{\pi}{2}) \text{ into themselves continuously}\}. \tag{4.49}$$

Furthermore, this algebra coincides with the Mellin transform of the van Winter algebra $\mathfrak{W}$, which contains $\mathfrak{K}$ as an ideal.

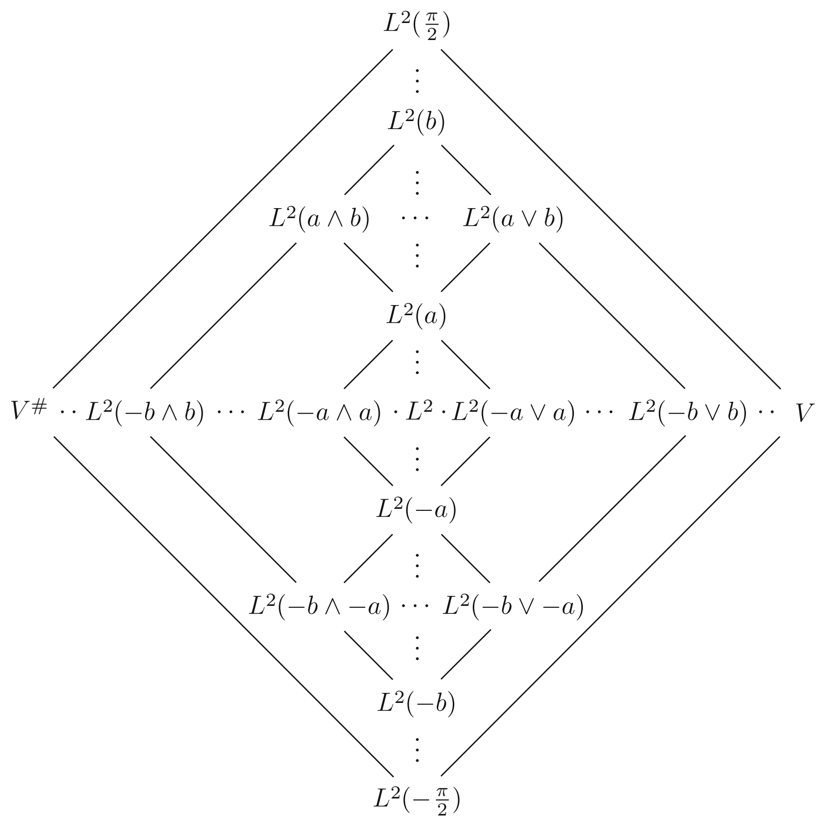


Fig. 4.4 The van Winter LHS

In terms of these spaces, the statement above reads $f(re^{i\varphi}) \in G(a,b)$ if and only if $\widetilde{f}(t,\varphi) = e^{\varphi t - i\varphi/2}\widetilde{f}(t)$, with $\widetilde{f}(t) \in L^2(a) \cap L^2(b) = L^2(a \wedge b)$. As a consequence, although the inverse Mellin transform is unitary from $G(a,b)$ onto $\widetilde{G}(a,b)$, it does not preserve the lattice structure of the left-hand side of the LHS, because of the prefactor in the definition of $\widetilde{f}(t,\varphi)$.

It is true that, given $\varphi \in [a,b]$, every function $f(re^{i\varphi}) \in L^2([0,\infty), dr)$ may be extended to a function $f(re^{i\varphi}) \in G(a,b)$, that is, analytic in the sector $a < \varphi < b$, provided its Mellin transform is $e^{at-ia/2}\widetilde{f}(t)$, where $\widetilde{f}(t) \in L^2(a \wedge b)$. Thus, if the sectors $[a,b]$ and $[c,d]$ have a nontrivial overlap, then $L^2(a \wedge b) \cap L^2(c \wedge d) = L^2(a \wedge d)$ and $G(a,b) \cap G(c,d) = G(a,d)$, by the uniqueness of analytic continuation. But if we take two disjoint sectors $[a,b]$ and $[c,d]$, corresponding to $-\frac{\pi}{2} < a < b < c < d < \frac{\pi}{2}$, there is *a priori* no way of extending analytically a function from one sector to the other, so that we get only $G(a,d) \subseteq G(a,b) \cap G(c,d)$. Nevertheless one still has $L^2(a \wedge d) = L^2(a \wedge b) \cap L^2(c \wedge d)$.

There is a partial solution to this difficulty, however. Given $a < b$, consider the sector $T_{a,b} := \{z : -b < \arg z < \pi - a\}$. Notice that

$$T_{a,b} = \bigcup_{a<\varphi<b} T_{\varphi,\varphi} = \bigcup_{a<\varphi<b} e^{-i\varphi}\mathbb{C}^+, \ \text{ where } \mathbb{C}^+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}.$$

Thus, since the opening angle is larger than π, all the sectors $T_{a,b}$ overlap. Then, for a fixed number $\alpha > -1$, define $D(a,b)$ as the space of all functions $f(z)$ analytic in $T_{a,b}$ and such that

$$\int_{\mathbb{C}^+} |f(ze^{-i\varphi})|^2 \, (\operatorname{Im} z)^\alpha \, d\nu(z) \tag{4.50}$$

is uniformly bounded in $\varphi \in (a,b)$. The integral in (4.50) makes sense, since $z \in \mathbb{C}^+$ and $a < \varphi < b$ imply $ze^{-i\varphi} \in T_{a,b}$. It turns out that $D(a,b)$ is a Hilbert space, with inner product

$$\langle f|g\rangle_{D(a,b)} = \int_{\mathbb{C}^+} \overline{f(ze^{-ia})} g(ze^{-ia}) (\operatorname{Im} z)^\alpha d\nu(z) + \int_{\mathbb{C}^+} \overline{f(ze^{-ib})} g(ze^{-ib}) (\operatorname{Im} z)^\alpha d\nu(z). \tag{4.51}$$

For $\psi \in G(a/2, b/2)$ and $a/2 < \varphi < b/2$, consider the function $V_\alpha\psi$ given by

$$(V_\alpha\psi)(z,\varphi) = \big(2\pi\,\Gamma(\alpha+1)\big)^{-1/2} \int_0^\infty (re^{i\varphi})^{\alpha+3/2} \, e^{iz(re^{i\varphi})^2/2} \, \psi(re^{i\varphi}) \, e^{i\varphi} \, dr. \tag{4.52}$$

This function is analytic in z in the half-plane $R_\varphi := T_{2\varphi,2\varphi} = e^{-2i\varphi}\mathbb{C}^+$. Next one defines a function $f(z) := (\widehat{V}_\alpha\psi)(z)$ as the function analytic in $T_{a,b}$, whose restriction to R_φ is exactly $(V_\alpha\psi)(z,\varphi)$. Then one shows that the function $\widehat{V}_\alpha\psi$ belongs to $D(a,b)$, and moreover, that the map $\widehat{V}_\alpha$ is unitary from $G(a/2, b/2)$ onto $D(a,b)$. The inverse map $\widehat{W}_\alpha : D(a,b) \to G(a/2, b/2)$ reads

$$(\widehat{W}_\alpha f)(re^{i\varphi}) = \big(2\pi\Gamma(\alpha+1)\big)^{-1/2} \int_{\mathbb{C}^+} (e^{-2i\varphi})^{\alpha/2+5/4} r^{\alpha+3/2} e^{-i\overline{z}r^2/2} f(e^{-2i\varphi}z)(\operatorname{Im} z)^\alpha d\nu(z). \tag{4.53}$$

Thus, for fixed a, b, we get three unitary equivalent Hilbert spaces, where the unitary map T may be computed explicitly from the other two, as shown in Fig. 4.5.

What about the LHS structure of these three families? We have seen above that $\{\widetilde{G}(a,b)\}$ may be identified with the van Winter lattice generated

$$\begin{tikzcd}
\widetilde{G}(a,b) \arrow[rr, leftrightarrow, "\text{Mellin}"] \arrow[dr, leftrightarrow, "T"'] & & G(a,b) \\
& D(2a,2b) \arrow[ur, leftrightarrow, "\widehat{V}_\alpha"'] &
\end{tikzcd}$$

Fig. 4.5 Three unitary equivalent Hilbert spaces by the spaces $L^2(c)$, shown in Fig. 4.4, whereas this is impossible for the family $\{G(a,b)\}$. As for the third family, one obtains only the left-hand part of a LHS. Indeed, for any $a \in (-\frac{\pi}{4}, \frac{\pi}{4})$, define the space $D(2a) = D(2a, 2a)$, namely the space of complex-valued functions, analytic in the half-plane $e^{-2ia}\mathbb{C}^+ = T_{2a,2a}$ and such that the following integral converges:

$$\int_{\mathbb{C}^+} |f(ze^{-2ia})|^2 \, (\operatorname{Im} z)^\alpha \, d\nu(z) < \infty.$$

This space is a Hilbert space for the inner product

$$\int_{\mathbb{C}^+} \overline{f(ze^{-2ia})} \, g(ze^{-2ia}) (\operatorname{Im} z)^\alpha \, d\nu(z).$$

Thus we may conclude that $D(2a, 2b) = D(2a) \cap D(2b)$, with the projective norm. Thus the left-hand side of the van Winter lattice may be transported by the map $T^{-1} : \widetilde{G}(a,b) \to D(2a, 2b)$ and one gets the half-lattice generated, by intersection, by the family $\{D(c), -\frac{\pi}{2} \leqslant c \leqslant \frac{\pi}{2}\}$. However, the identification of the right-hand side of this lattice is problematic, since it involves the duals of the spaces $D(a,b)$, which are not easy to characterize explicitly (of course, since $D(a,b)$ is a Hilbert space, it is anti-isomorphic to its conjugate dual, but we don't know how to characterize the elements of that dual as functions or functionals).

4.6.4 A Link with Bergman Spaces of Analytic Functions in the Unit Disk

However, there is another direction to be explored. Indeed, in the whole analysis so far, the parameter α was fixed. What happens if we let it vary? The starting point is the Bergman space $\mathcal{H}_\alpha := D(0) \equiv A^{2,\alpha+2}$, where the space $A^{p,\beta}$ is defined in (8.23), that is, the space of complex-valued functions f analytic in $\mathbb{C}^+$ and such that

$$\|f\|^2_{\mathcal{H}_\alpha} := \int_{\mathbb{C}^+} |f(z)|^2 \, (\operatorname{Im} z)^\alpha \, d\nu(z) < \infty. \tag{4.54}$$

With the inner product

$$\langle f|g\rangle_{\mathcal{H}_\alpha} := \int_{\mathbb{C}^+} \overline{f(z)} \, g(z) (\operatorname{Im} z)^\alpha \, d\nu(z),$$

$\mathcal{H}_\alpha$ is a Hilbert space, with reproducing kernel

$$\rho_z^\alpha(w) = \frac{\alpha+1}{4\pi}\Big(\frac{w-\overline{z}}{2i}\Big)^{-\alpha-2}, \ z, w \in \mathbb{C}^+.$$

The following functions constitute an orthonormal basis of $\mathcal{H}_\alpha$:

$$v_n^\alpha(z) := \frac{2^{\alpha+1}}{\pi^{1/2}}\left(\frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)\Gamma(n+1)}\right)^{1/2} \frac{(z-i)^n}{(z+i)^{n+\alpha+2}}, \ n \in \mathbb{N}.$$

The interesting point is that each space $\mathcal{H}_\alpha$ is unitarily equivalent to a space of analytic functions in the unit disk $\mathbb{D} := \{w \in \mathbb{C} : |w| < 1\}$, namely, the Bergman space $\mathcal{K}_\alpha$ of all functions g analytic in $\mathbb{D}$ such that the following integral converges:

$$\|g\|^2_{\mathcal{K}_\alpha} := \int_{\mathbb{D}} |g(w)|^2 \left(\frac{1-|w|^2}{2}\right)^\alpha d\nu(w) < \infty. \tag{4.55}$$

With the norm $\|\cdot\|_{\mathcal{K}_\alpha}$ and the corresponding inner product, $\mathcal{K}_\alpha$ is a Hilbert space with reproducing kernel

$$\widehat{\rho}_z^\alpha(w) = \frac{\alpha+1}{4\pi}\Big(\frac{1-\overline{z}w}{2}\Big)^{-\alpha-2}, \ z, w \in \mathbb{D}.$$

Here again there is an orthonormal basis, consisting of the functions:

$$u_n^\alpha(w) := \frac{2^{\alpha/2}}{\pi^{1/2}}\left(\frac{\Gamma(n+\alpha+2)}{\Gamma(\alpha+1)\Gamma(n+1)}\right)^{1/2} w^n, \ n \in \mathbb{N}.$$

The unitary maps connecting the two spaces, namely, $B_\alpha : \mathcal{H}_\alpha \to \mathcal{K}_\alpha$ and $B_\alpha^{-1} : \mathcal{K}_\alpha \to \mathcal{H}_\alpha$, read, respectively:

$$B_\alpha f(w) = 2^{\alpha+1/2}\Big(\frac{1-w}{i}\Big)^{-\alpha-2} f\Big(i\frac{1+w}{1-w}\Big), \ f \in \mathcal{H}_\alpha\,, \tag{4.56}$$

$$B_\alpha^{-1} g(z) = 2^{\alpha+1/2}(z+i)^{-\alpha-2}\, g\Big(i\frac{z-i}{z+i}\Big), \ g \in \mathcal{K}_\alpha\,. \tag{4.57}$$

The sequel of the analysis follows closely that of Section 4.6.2, the spaces $\mathcal{K}_\alpha$ playing the role of the $\mathfrak{F}^\rho$. In particular, on the space $\mathcal{K}(\mathbb{D})$ of all functions which are analytic in $\mathbb{D}$, as for entire functions of Bargmann type, one has two natural linear compatibilities:

$$f \,\#_1\, g \iff \int_{\mathbb{D}} |f(z)\, g(z)|\, d\nu(z) < \infty, \tag{4.58}$$

$$f \,\#_2\, g \iff \sum_n \frac{\pi}{n+1} |f_n\, g_n| < \infty. \tag{4.59}$$

Then, given a Lebesgue-measurable function p on $\mathbb{D}$, one defines the space $\mathcal{K}(p)$ as the space of functions f analytic in $\mathbb{D}$ and such that the following integral converges:

$$\|f\|^2_{\mathcal{K}(p)} := \int_{\mathbb{D}} |f(z)|^2 e^{-p(z)}\, d\nu(z) < \infty \tag{4.60}$$

With the associated inner product, the space $\mathcal{K}(p)$ is a pre-Hilbert space, which obviously reduces to $\mathcal{K}_\alpha$ for the weight $p_\alpha(z) = -\alpha \ln\left(\frac{1-|z|^2}{2}\right)$. In order to proceed, we have to impose several conditions on the spaces $\mathcal{K}(p)$, which are translated into restrictions on the weights p:

(i) *Completeness:* $\mathcal{K}(p)$ is complete, thus a Hilbert space, if p is locally bounded (that is, bounded on compact sets), i.e., $p \in L^\infty_{\mathrm{loc}}(\mathbb{D})$.

(ii) *Density of the set of polynomials in* $\mathcal{K}(p)$*:* Consider the monomials $\{u_n(z) = (\frac{n+1}{\pi})^{1/2} z^n,\ n = 1, 2, \ldots\}$. Assume that p is radial, $p(z) = p(|z|)$, and that

$$\eta_n^{(p)} := \int_{\mathbb{D}} |u_n(z)|^2\, e^{-p(z)}\, d\nu(z) < \infty,\ n = 1, 2, \ldots. \tag{4.61}$$

Then the functions

$$u_n^{(p)}(z) = (\eta_n^{(p)})^{-1/2}\, u_n(z),\ n = 1, 2, \ldots$$

form an orthonormal basis of $\mathcal{K}(p)$. In that case, all spaces $\mathcal{K}(p)$ may be realized as weighted ℓ^2 sequence spaces. Thus, for the spaces $\mathcal{K}(p)$ the compatibility $\#_2$ is easy to handle.

(iii) *Duality:* $\mathcal{K}(p)^\times$ is isomorphic to $\mathcal{K}(-p)$. Here too there are several results:

(a) if p is locally bounded, then $\mathcal{K}(-p) = \mathcal{K}(p)^{\#_1} \subseteq \mathcal{K}(p)^\times$;

(b) if p is locally bounded and radial, and $\eta_n^{(p)} < \infty, \forall n \in \mathbb{N}$, then $\mathcal{K}(p)^\times = \mathcal{K}(p)^{\#_2}$;

(c) if p is locally bounded and radial, and $\eta_n^{(\pm p)} < \infty, \forall n \in \mathbb{N}$, then $\mathcal{K}(p)^\times = \mathcal{K}(-p)$ if and only if there exists a positive constant $c(p)$ such that

$$1 \leqslant \eta_n^{(p)}\, \eta_n^{(-p)} \leqslant c(p),\ \text{ for all } n = 0, 1, 2, \ldots. \tag{4.62}$$

Considering now the family of Hilbert spaces $\mathcal{K}(p)$, we may state a theorem exactly parallel to Theorem 4.6.2

Theorem 4.6.3. *Let I be the set of weight functions $p(z)$ that satisfy the following three conditions:*

(i) p is locally bounded and radial, $p(z) = p(|z|)$.
(ii) $u_n^{(p)}(z)$ belongs to $\mathcal{K}(p)$ and $\mathcal{K}(-p)$, for every $n \in \mathbb{N}$.
(iii) There are positive constants A, B such that

$$A \leqslant \eta_n^{(p)} \eta_n^{(-p)} \leqslant B, \text{ for all } n = 0, 1, 2, \dots, \tag{4.63}$$

where $\eta_n^{(p)}$ is given by (4.61).

Then the family $\{\mathcal{K}(p),\, p \in I\}$ is a chain of hilbertian spaces with central Hilbert space $\mathcal{K}(0)$. The compatibility $\#$ coincides with both $\#_1$ and $\#_2$, and the partial inner product with that induced by $\mathcal{K}(\mathbb{D})$ and ℓ^2, respectively.

The order structure and the lattice operations on this chain of hilbertian spaces are exactly the same as in the previous case, replacing weights ρ by weights p. In particular, the same difficulty arises with duality. Let us indeed identify $\mathcal{K}(p)$ with a sequence space:

$$f \in \mathcal{K}(p) \Longleftrightarrow f = \sum_n a_n u_n^{(p)}, \text{ with } \|f\|^2_{\mathcal{K}(p)} := \sum_n {\eta_n^{(p)}}^{-1} |a_n|^2 < \infty,$$

where the series representing f converges uniformly on compact subsets of $\mathbb{D}$. Then the dual space, denoted $\mathcal{K}(\widehat{p})$, corresponds to the dual sequence space

$$f \in \mathcal{K}(\widehat{p}) \Longleftrightarrow f = \sum_n b_n u_n^{(p)}, \text{ with } \|f\|^2_{\mathcal{K}(\widehat{p})} := \sum_n \eta_n^{(p)} |b_n|^2 < \infty.$$

The point is that, if the condition (4.63) is satisfied, then the norm $\|\cdot\|^2_{\mathcal{K}(\widehat{p})}$ is equivalent to, but different from the norm $\|\cdot\|^2_{\mathcal{K}(-p)}$. Hence we have again to make a distinction between Hilbert spaces and hilbertian spaces, thus between a chain of hilbertian spaces and a LHS.

Once again, the problem is that the space $\mathcal{K}(\widehat{p})$ is a space of functions analytic in $\mathbb{D}$, but need not be associated to a weight. The existence of the latter is a moment problem, as in the previous case. This is why we prefer to stay with the spaces $\mathcal{K}(p)$ and the chain of hilbertian spaces they constitute.

If we particularize these results to the spaces $\mathcal{K}_\alpha$, $-1 < \alpha < 1$, we obtain the following results. First we notice the equivalences

$$\alpha_2 > \alpha_1 \Longleftrightarrow \mathcal{K}_{\alpha_1} \subset \mathcal{K}_{\alpha_2} \Longleftrightarrow \mathcal{K}_{\widehat{\alpha}_2} \subset \mathcal{K}_{\widehat{\alpha}_1},$$

where $\mathcal{K}_{\widehat{\alpha}}$ denotes the dual of $\mathcal{K}_\alpha$ as above. Then we have

Theorem 4.6.4. *(i) The family $\{\mathcal{K}_\alpha,\, -1 < \alpha < 1\}$, is a chain of hilbertian spaces.*

(ii) The family $\{\mathcal{K}_\alpha, \mathcal{K}_{\widehat{\alpha}},\, 0 \leqslant \alpha < 1\}$, *is a LHS:*

$$\ldots \mathcal{K}_{\widehat{\alpha}_2} \subset \mathcal{K}_{\widehat{\alpha}_1} \subset \mathcal{K}_{\widehat{0}} \simeq \mathcal{K}_0 \subset \mathcal{K}_{\alpha_1} \subset \mathcal{K}_{\alpha_2} \ldots \quad (\alpha_2 > \alpha_1).$$

Finally we come back to the original spaces $\mathcal{H}_\alpha$. Since the unitary map $B_\alpha : \mathcal{H}_\alpha \to \mathcal{K}_\alpha$ linking the two spaces depends on α, the following result does not contradict Theorem 4.6.4.

Proposition 4.6.5. *Let* $\alpha_2 > \alpha_1 > -1$. *Then the two spaces* $\mathcal{H}_{\alpha_1}$ *and* $\mathcal{H}_{\alpha_2}$ *are not comparable.*

However, there *is* a continuous embedding $U_{\alpha_2\alpha_1} : \mathcal{H}_{\alpha_1} \to \mathcal{H}_{\alpha_2}$. Consider the maps $V_\alpha : L^2(\mathbb{R}^+) \to \mathcal{H}_\alpha$ and $W_\alpha : \mathcal{H}_\alpha \to L^2(\mathbb{R}^+)$ defined from (4.52) and (4.53), respectively, for $\varphi = 0$. Then one defines

$$U_{\alpha_2\alpha_1} = V_{\alpha_2} W_{\alpha_1} \;\text{ and } U^{-1}_{\alpha_2\alpha_1} = V_{\alpha_1} W_{\alpha_2} = U_{\alpha_1\alpha_2}.$$

The map $U_{\alpha_2\alpha_1}$ may be computed explicitly. Given $f_1 \in \mathcal{H}_{\alpha_1}$, one has

$$\begin{aligned} f_2(z) :&= [U_{\alpha_2\alpha_1} f_1](z) \\ &= \frac{1}{4\pi} \left(\frac{\Gamma(\frac{\alpha_1+\alpha_2}{2}+2)}{\Gamma(\alpha_1+1)\Gamma(\alpha_2+1)} \right)^{1/2} \int_{\mathbb{C}^+} \left(\frac{z-\overline{w}}{2i} \right)^{-(\frac{\alpha_1+\alpha_2}{2}+2)} f_1(w)\, (\operatorname{Im} w)^{\alpha_1}\, d\nu(w). \end{aligned}$$

Equivalently, if $f_1 = V_{\alpha_1}\psi$, with $\psi \in L^2(\mathbb{R}^+)$, that is,

$$f_1(z) = \big(2\pi\, \Gamma(\alpha_1+1)\big)^{-1/2} \int_0^\infty r^{\alpha_1+3/2}\, e^{izr^2/2}\, \psi(r)\, dr,$$

then

$$f_2(z) := [U_{\alpha_2\alpha_1} f_1](z) = \big(2\pi\, \Gamma(\alpha_1+1)\big)^{-1/2} \int_0^\infty r^{\alpha_1+3/2}\, e^{izr^2/2}\, r^{(\alpha_2-\alpha_1)} \psi(r)\, dr.$$

Combining now the spaces $\mathcal{H}_{\alpha_j}$ and $\mathcal{K}_{\alpha_j}$, we get the following commutative diagram:

$$\alpha_2 \geqslant \alpha_1 > -1: \qquad \begin{array}{ccc} \mathcal{K}_{\alpha_1} & \xrightarrow{\;i_{\alpha_2\alpha_1}\;} & \mathcal{K}_{\alpha_2} \\ {\scriptstyle B_{\alpha_1}}\uparrow & & \downarrow{\scriptstyle B^{-1}_{\alpha_2}} \\ \mathcal{H}_{\alpha_1} & \xrightarrow{\;\widehat{\imath}_{\alpha_2\alpha_1}\;} & \mathcal{H}_{\alpha_2} \end{array}$$

where $\widehat{\imath}_{\alpha_2\alpha_1} = B^{-1}_{\alpha_2} \cdot i_{\alpha_2\alpha_1} \cdot B_{\alpha_1}$ and the maps B_α, B^{-1}_α are defined in (4.56),(4.57). Since $i_{\alpha_2\alpha_1}$, i.e., the natural embedding, is continuous with dense range, so is $\widehat{\imath}_{\alpha_2\alpha_1}$. In addition, these maps satisfy the composition law:

$$\text{For } \alpha_3 \geqslant \alpha_2 \geqslant \alpha_1 > -1, \quad \text{one has } \widehat{\imath}_{\alpha_3\alpha_1} = \widehat{\imath}_{\alpha_3\alpha_2} \widehat{\imath}_{\alpha_2\alpha_1}.$$

Actually, the maps $\widehat{\imath}_{\alpha_2\alpha_1}$ can be calculated explicitly.

Proposition 4.6.6. *Let $\alpha_2 \geqslant \alpha_1 > -1$. Then the injection $\widehat{\imath}_{\alpha_2\alpha_1} : \mathcal{H}_{\alpha_1} \to \mathcal{H}_{\alpha_2}$ is the operator of multiplication by $2^{(\alpha_2-\alpha_1)/2}\,(z+i)^{(\alpha_1-\alpha_2)}$. This operator is injective and continuous, and has dense range.*

In order to understand the structure of the family $\{\mathcal{H}_\alpha\}$, we must introduce a new notion. Given a directed set A, an *inductive spectrum* is a family $\{E_\alpha, \pi\}_{\alpha\in A}$, where each E_α is a locally convex space and π is a collection of continuous linear maps $\pi_{\beta\alpha} : E_\alpha \to E_\beta$, $\alpha \leqslant \beta$, satisfying the composition rule $\pi_{\gamma\beta} \circ \pi_{\beta\alpha} = \pi_{\gamma\alpha}$, $\alpha \leqslant \beta \leqslant \gamma$ (thus $\pi_{\alpha\alpha}$ is the identity on E_α). Given the inductive spectrum $\{E_\alpha, \pi\}_{\alpha\in A}$, one may construct the algebraic inductive limit of the family $\{E_\alpha\}_{\alpha\in A}$ as explained for nestings in Section 2.4.1 (iv). From this, we see that a nesting is an inductive spectrum with an involution $\alpha \leftrightarrow \overline{\alpha}$.

Coming back to the spaces $\{\mathcal{H}_\alpha\}$, we may state

Proposition 4.6.7. *The family $\{\mathcal{H}_\alpha,\, -1 < \alpha < 1,\}$ is an inductive spectrum of Hilbert spaces with respect to the maps $\widehat{\imath}_{\alpha_2\alpha_1}$.*

In order to get a genuine NHS, we need that $\mathcal{H}_{-\alpha}$ be the conjugate dual of $\mathcal{H}_\alpha$, but that is not obvious. In any case, we have here another example of nestings which are not natural embeddings.

4.6.5 Hardy Spaces of Analytic Functions in the Unit Disk

There is another classical family of spaces consisting of functions analytic in the unit disk, the Hardy spaces H^p, $0 < p \leqslant \infty$. For reasons of simplicity, we shall restrict ourselves to the range $1 \leqslant p < \infty$.

Given a function f analytic in the unit disk $\mathbb{D}$, define the quantity

$$M_p(r,f) := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta\right)^{1/p},\ 0 < r < 1,\ 1 \leqslant p < \infty, \qquad (4.64)$$

which is a nondecreasing function of r. Then the function f is said to be of class H^p if $M_p(r,f)$ remains bounded as $r \to 1$.

For any function $f \in H^p$, the limit $f(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere and belongs to the space $L^p(S^1, d\theta)$, where S^1 is the unit circle. Then H^p is a Banach space for the norm

$$\|f\|_p := \lim_{r\to 1} M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^p d\theta\right)^{1/p},\ 1 \leqslant p < \infty.$$

This allows to identify H^p with a closed subspace of $L^p(S^1)$. This subspace consists of all functions $f \in L^p$ such that

$$\int_0^{2\pi} e^{in\theta} f(e^{i\theta})\, d\theta = 0,\ n = 1, 2, 3, \ldots$$

In particular, the polynomials are dense in H^p. Clearly, $H^{p_1} \subset H^{p_2}$ if $p_1 > p_2$, the embedding is continuous and has dense range.

Let us look now at the dual spaces. We know that the conjugate dual $(L^p)^\times$ of L^p is $L^{\overline{p}}$, with $1/p + 1/\overline{p} = 1$. Thus every bounded linear functional ϕ on L^p ($1 \leqslant p < \infty$) has a unique representation

$$\phi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, g(e^{i\theta}) d\theta, \text{ with } g \in L^{\overline{p}}.$$

Since H^p is subspace of L^p, the conjugate dual $(H^p)^\times$ is the quotient $L^{\overline{p}}/H_0^{\overline{p}}$, where $H_0^{\overline{p}}$ is the annihilator of H^p. An element $g(e^{i\theta})$ of that subspace is the boundary function of some $g(z) \in H^{\overline{p}}$ such that $g(0) = 0$. Actually one can replace $H_0^{\overline{p}}$ by $H^{\overline{p}}$ in the quotient, so that finally $(H^p)^\times \simeq L^{\overline{p}}/H^{\overline{p}}$ (isometric isomorphism). In addition, for $1 < p < \infty$, each $\phi \in (H^p)^\times$ is representable by a *unique* function $g \in H^{\overline{p}}$ for which

$$\phi(f) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta})\, \overline{g(e^{i\theta})} d\theta,\ \forall\, f \in H^p.$$

Since the conjugate dual of $L^{\overline{p}}/H_0^{\overline{p}}$ is H^p, the latter is reflexive. The conclusion is that the family $\{H^p,\, 1 < p < \infty\}$, is an inductive spectrum and a chain of reflexive Banach spaces, but not a LBS.

There are two variants of the Hardy spaces that ought to be mentioned. First one can consider functions analytic in the upper half-plane $\mathbb{C}^+$, as for the Bergman spaces. Thus one defines the space $\mathfrak{H}_+^p$ ($1 \leqslant p < \infty$) of functions analytic in $\mathbb{C}^+$, such that $|f(x+iy)|^p$ is integrable for each $y > 0$ and

$$\mathfrak{M}_p^{(+)}(y, f) := \Big(\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \Big)^{1/p} \tag{4.65}$$

is bounded for $0 < y < \infty$. The space $\mathfrak{H}_+^p$ is again a reflexive Banach space, which has the same properties as H^p. In particular, the limit property takes the following form. If $f \in \mathfrak{H}^p$, then the boundary function $f(x) = \lim_{y\to 0} f(x+iy)$ exists almost everywhere and belongs to $L^p(\mathbb{R})$. For duality, it turns out that the dual of $\mathfrak{H}_+^p$ is isometrically isomorphic to $L^{\overline{p}}/\mathfrak{H}_+^{\overline{p}}$ and the same conclusion follows.

Similar considerations apply to the Hardy spaces $\mathfrak{H}_-^p$ ($1 \leqslant p < \infty$) of functions analytic in the lower half-plane $\mathbb{C}^-$.

Another case of interest is the space $L^p_a(\mathbb{D})$ of L^p functions analytic in the unit disk $\mathbb{D}$. This is a closed subspace of $L^p(\mathbb{D})$, hence a Banach space. Its dual is simply $L^{\overline{p}}_a(\mathbb{D})$, so that here we get a genuine reflexive lattice of Banach spaces.

As a final remark, we may add that most of the results of this section are valid for Hardy spaces of harmonic functions.

Notes for Chapter 4

Section 4.1. The results of this section are based on the extensive study by Davis *et al.* [68, 69]. Notice that we differ slightly from these authors, in that we do not include the space L^1. However, their results apply to our case also, because the first Eq. (4.1) (shown in [68, 69]) holds true also with the definition $\widehat{S} = (1,t) \bigcup S$.

- In the terminology of Floret–Wloka [FW68, §9], L^{p-} is a strict FG-space.
- The space L^ω has been introduced by Arens [33]. It is usually called the Arens algebra in the literature.
- A thorough analysis of partial *algebras may be found in the monograph of Antoine–Inoue–Trapani [AIT02]. The case of the L^p spaces is discussed in Section 6.3.1, that we follow closely. Note that there are some slight errors in the computation of the so-called multiplier spaces. They have been corrected in Antoine [16].
- Further information about Sobolev spaces may be found in the books of Hörmander [Hör63], Trêves [Tre67], or Adams [Ada75].
- The noncommutative L^p spaces associated to a von Neumann algebra were introduced by Segal [176], and they were later studied by Inoue in the context of algebras of unbounded operators [127, 128].
- Chains of hilbertian spaces were defined and studied by Palais [162] and Krein–Petunin [132]. Concerning the distinction between chains and scales, we follow the terminology of Palais [162]. On the contrary, Krein–Petunin [132] use the term 'scales of Banach spaces' for the general case, Hilbert scales being distinguished by their interpolation properties.
- Bargmann's space $\mathcal{E}^\times$ was introduced in [42, 43] for the purpose of the formulation of quantum mechanics in the phase space representation and the consequences of the latter for the theory of distributions (see the note under Section 4.6).

Section 4.2.

- For the Schwartz theory of kernels, see [175, Sch57]. Further information on nuclear spaces may also be found in Trêves [Tre67].
- For the assaying subsets $L^2(r_{\alpha\beta})$ of $L^1_{\rm loc}(\mathbb{R}^n, dx)$, see Hörmander [Hör63, Theorem 2.1.1].
- For the topology γ of compact convergence and for standard results on the various topologies on $L^1_{\rm loc}$ and $L^\infty_{\rm c}$, see for instance the textbooks of Köthe

[Köt69], Schwartz [Sch57] or Trêves [Tre67]. In particular, LF-spaces are discussed at length in Trêves [Tre67, Chap. 13].
- Related results on the representation of operators on L^2 spaces have been obtained by Ascoli *et al.* [36].

Section 4.3.

- Köthe has considered also another duality on the space ω, corresponding to the so-called β-compatibility, namely

$$(x_n)\#_\beta(y_n) \iff \sum_n \overline{x_n} y_n \text{ converges.}$$

This correspondence associates to each subspace l its *β-dual* [Köt69, §30.10]. The two compatibilities will be studied in more detail in Section 5.5.7.
- In fact, Proposition 4.3.7 allows us to dispense of the notions of strong norm and strong Banach ideals, the notion of perfect spaces or assaying subspaces of ω is equivalent. For further details on normed ideals and their relationship with spaces of compact operators, we refer to the classical texts of Schatten [Sch70], Simon [Sim79], Pietsch [Pie80] or Gohberg–Krein [GK69]. See also the work of Cigler [58] and Oostenbrink [Oos73].
- Echelon and co-echelon spaces have been introduced by Köthe, see [Köt69, §30.8].

Section 4.4. Köthe function spaces have been introduced (and given that name) by Dieudonné [73]. The corresponding lattice has been introduced and studied by Luxemburg–Zaanen (see [Zaa61] and references therein), that we follow here. These spaces are also called normed Köthe spaces or Banach function spaces.

- The space L_G was introduced by Gould [111], who calls it Ω; see also Zaanen [Zaa61, §30, Exercises], who denotes it by L_ρ.

Section 4.5. Analyticity and trajectory spaces were introduced and studied systematically by van Eijndhoven–de Graaf in a series of papers, later synthesized in van Eijndhoven's PhD thesis [Eij83] and two books [EG85,EG86]. The avowed goal of this work was to produce a mathematical framework capable of producing a rigorous version of Dirac's bra-and-ket formalism of quantum mechanics. More about this will be found in Section 7.1.1 (ii). These spaces also play a role in the representation theory of Lie groups, see Section 7.4.

Section 4.6.

- The analysis of this section is largely due to Antoine–Vause [31].
- Besides the familiar position (q) and momentum (p) representations of quantum mechanics, the Fock-Bargmann representation offers an attractive alternative [42, 43]. It is based on the canonical (or oscillator) coherent states and it is characterized by the fact that its wave functions

are entire analytic functions of $z = q + ip$. It is therefore a *phase space* representation. As such it is useful for studying the quantum-to-classical transition, quantum optics, path integrals, geometric quantization, etc. The properties of the Hilbert space $\mathfrak{F}$ are in fact characteristic of all phase space representations. For further details on the latter and, in particular, the Fock-Bargmann representation, see for instance Ali *et al.* [6] or the monograph of Ali–Antoine–Gazeau [AAG00].

- The duality between the spaces $\mathfrak{Z}$ and Exp is discussed in detail in the textbook of Trêves [Tre67, Chap. 22], in particular Exercise 22.5.
- The lattice of hilbertian spaces constituted from the spaces $\mathfrak{F}(\rho)$, which is *not* a scale, has been studied in detail by Antoine–Vause [31]. This lattice of hilbertian spaces has found interesting applications in the study of Weyl quantization by Daubechies [65–67], that is, the setting of a correspondence between functions on phase space $f(q+ip) \equiv \tilde{f}(q,p)$ and operators on $L^2(\mathbb{R}, dx)$.
- The spaces $G(a,b)$ and $\widetilde{G}(a,b)$, for fixed a,b, were introduced and studied systematically by van Winter in a series of papers devoted to a reformulation of quantum scattering theory [188, 189]. The Mellin transform and its inverse are usually defined as follows:

$$M(s) = \int_0^\infty f(x)\, x^{s-1}\, dx, \quad f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} M(s)\, x^{-s}\, ds,$$

  but the change of variables $s = -it + \frac{1}{2}$ leads to the version given in the text.
- The van Winter LHS $\{L^2(a)\}$ and its connection with the spaces $G(a,b)$ and $\widetilde{G}(a,b)$ was first described in the work of Gollier [Gol82]. The space $D(a,b)$ and its connection with the three unitary Hilbert spaces were introduced by Klein [Kle87].
- Part of the van Winter LHS $\{L^2(a)\}$ has been considered, for similar purposes in quantum scattering theory, by Horwitz–Katznelson [125] and by Skibsted [180]. Note, however, that the analysis of Horwitz–Katznelson is not entirely correct.
- The spaces $\mathcal{H}_\alpha$ and $\mathcal{K}_\alpha$, for integer values of α, including their mutual relationship, are standard in the representation theory of the groups SL(2,$\mathbb{R}$) and SU(1,1). The corresponding spaces for arbitrary $\alpha \in (-1,1)$ have been introduced by Paul [163]. As for the spaces $\mathcal{K}(p)$, they have been introduced and studied in detail by Klein [Kle87]. There one may find several examples and counterexamples of weights.
- The notion of inductive spectrum is studied in detail in the monograph of Floret–Wloka [FW68, §23].
- Hardy spaces are essential tools in harmonic analysis. Detailed treatments may be found in the monographs of Duren [Dur70, Chaps. 3,7 and 11] and Koosis [Koo80, Chaps. IV and VII]. The space L^p_a and its generalizations are described by Luecking [144].

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