Introduction: Lattices of Hilbert or Banach Spaces and Operators on Them

I.1 Motivation

It is a fact that many function spaces that play a central role in analysis come in the form of families, indexed by one or several parameters that characterize the behavior of functions (smoothness, behavior at infinity, ...). The simplest structure is a *chain of Hilbert or (reflexive) Banach spaces.* Let us give two familiar examples.

(i) *The Lebesgue L^p spaces on a finite interval*, e.g. $\mathcal{I} = \{L^p([0,1], dx), 1 \leqslant p \leqslant \infty\}$:

$$L^{\infty} \subset \ldots \subset L^{\overline{q}} \subset L^{\overline{r}} \subset \ldots \subset L^2 \subset \ldots \subset L^r \subset L^q \subset \ldots \subset L^1, \tag{I.1}$$

where $1 < q < r < 2$. Here L^q and $L^{\overline{q}}$ are dual to each other $(1/q + 1/\overline{q} = 1)$, and similarly $L^r, L^{\overline{r}}$ $(1/r + 1/\overline{r} = 1)$. By the Hölder inequality, the (L^2) inner product

$$\langle f|g \rangle = \int_0^1 \overline{f(x)}\, g(x)\, dx \tag{I.2}$$

is well-defined if $f \in L^q$, $g \in L^{\overline{q}}$. However, it is *not* well-defined for two arbitrary functions $f, g \in L^1$. Take for instance, $f(x) = g(x) = x^{-1/2}$: $f \in L^1$, but $fg = f^2 \notin L^1$. Thus, on L^1, (I.2) defines only a *partial* inner product. The same result holds for any finite interval of $\mathbb{R}$ instead of [0,1].

(ii) *The chain of Hilbert spaces built on the powers of a positive self-adjoint operator* $A \geqslant 1$ in a Hilbert space $\mathcal{H}_0$. Let $\mathcal{H}_n$ be $D(A^n)$, the domain of A^n, equipped with the graph norm $\|f\|_n = \|A^n f\|$, $f \in D(A^n)$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathcal{H}_{\overline{n}} := \mathcal{H}_{-n} = \mathcal{H}_n^{\times}$ (conjugate dual):

$$\mathcal{D}^{\infty}(A) := \bigcap_n \mathcal{H}_n \subset \ldots \subset \mathcal{H}_2 \subset \mathcal{H}_1 \subset \mathcal{H}_0 \subset \mathcal{H}_{\overline{1}} \subset \mathcal{H}_{\overline{2}} \ldots \subset \mathcal{D}_{\overline{\infty}}(A) := \bigcup_n \mathcal{H}_n. \tag{I.3}$$

Note that here the index n may be integer or real, the link between the two cases being established by the spectral theorem for self-adjoint operators.

Here again the inner product of $\mathcal{H}_0$ extends to each pair $\mathcal{H}_n, \mathcal{H}_{\overline{n}}$, but on $\mathcal{D}_{\overline{\infty}}(A)$ it yields only a *partial* inner product. The following examples, all three in $\mathcal{H}_0 = L^2(\mathbb{R}, dx)$ are standard:

- $(A_{\rm p} f)(x) = (1 + x^2)^{1/2} f(x)$.
- $(A_{\rm m} f)(x) = (1 - \frac{d^2}{dx^2})^{1/2} f(x)$.
- $(A_{\rm osc} f)(x) = (1 + x^2 - \frac{d^2}{dx^2}) f(x)$.

(The notation is suggested by the operators of position, momentum and harmonic oscillator energy in quantum mechanics, respectively). In the case of $A_{\rm m}$, the intermediate spaces are the Bessel potential (or Sobolev) spaces $H^s(\mathbb{R})$, $s \in \mathbb{Z}$ or $\mathbb{R}$. Note that both $\mathcal{D}^\infty(A_{\rm p}) \cap \mathcal{D}^\infty(A_{\rm m})$ and $\mathcal{D}^\infty(A_{\rm osc})$ coincide with the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth functions of fast decay, and $\mathcal{D}_{\overline{\infty}}(A_{\rm osc})$ with the space $\mathcal{S}^\times(\mathbb{R})$ of tempered distributions.[1]

However, a moment's reflection shows that the total order relation inherent in a chain is in fact an unnecessary restriction, partially ordered structures are sufficient, and indeed necessary in practice. For instance, in order to get a better control on the behavior of individual functions, one may consider the lattice built on the powers of $A_{\rm p}$ and $A_{\rm m}$ simultaneously. Then the extreme spaces are still $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}^\times(\mathbb{R})$. Similarly, in the case of several variables, controlling the behavior of a function in each variable separately requires a nonordered set of spaces. This is in fact a statement about tensor products (remember that $L^2(X \times Y) \simeq L^2(X) \otimes L^2(Y)$). Indeed a glance at the work of Palais on chains of Hilbert spaces shows that the tensor product of two chains of Hilbert spaces, $\{\mathcal{H}_n\} \otimes \{\mathcal{K}_m\}$ is naturally a lattice $\{\mathcal{H}_n \otimes \mathcal{K}_m\}$ of Hilbert spaces. For instance, in the example above, for two variables x, y, that would mean considering intermediate Hilbert spaces corresponding to the product of two operators, $\big(A_{\rm m}(x)\big)^n \big(A_{\rm m}(y)\big)^m$.

Thus the structure we want to analyze is that of *lattices of Hilbert or Banach spaces*. Many examples are around us, for instance the lattice generated by the spaces $L^p(\mathbb{R}, dx)$, the amalgam spaces $W(L^p, \ell^q)$, the mixed norm spaces $L^{p,q}_m(\mathbb{R}, dx)$, and many more (these spaces will be discussed in detail in Chapters 4 and 8, where the references to original papers will be given). In all these cases, which contain most families of function spaces of interest in analysis and in signal processing, a common structure emerges for the "large" space V, defined as the union of all individual spaces. There is a lattice of Hilbert or reflexive Banach spaces V_r, with an (order-reversing) involution $V_r \leftrightarrow V_{\overline{r}}$, where $V_{\overline{r}} = V_r^\times$ (the space of continuous antilinear functionals on V_r), a central Hilbert space $V_o \simeq V_{\overline{o}}$, and a partial inner product on V that extends the inner product of V_o to pairs of dual spaces $V_r, V_{\overline{r}}$.

Actually, in many cases, it is the family $\{V_r\}$ as a whole that is meaningful, not the individual spaces. The spaces $L^p(\mathbb{R})$ are a good example. Therefore,

[1] considered here as continuous *conjugate linear* functionals on $\mathcal{S}$. See the Notes to Chapter 1, Section 1.1.

many operators should be considered globally, for the whole chain or lattice, instead of on individual spaces. For instance, in many spaces of interest in signal processing, this would apply to operators implementing translations ($x \mapsto x-y$) or dilations ($x \mapsto x/a$), convolution operators, Fourier transform, etc. In the same spirit, it is often useful to have a *common* basis for the whole family of spaces, such as the Haar basis for the spaces $L^p(\mathbb{R})$, $1 < p < \infty$. Thus we need a notion of operator and basis defined globally for the chain or lattice itself.

The subject matter of the present volume is to present a formalism that answers these questions, namely, the theory of *partial inner product spaces* or PIP-*spaces*. However, before analyzing in detail the general theory, we will concentrate in this introductory chapter on the simple case of lattices of Hilbert or Banach spaces and operators on them. These are indeed the most useful families of spaces for the applications.

I.2 Lattices of Hilbert or Banach Spaces

I.2.1 Definitions

Let thus $\mathcal{J} = \{\mathcal{H}_p,\, p \in J\}$ be a family of Hilbert spaces, partially ordered by inclusion (the index set J has the same order structure). Then $\mathcal{J}$ generates a lattice $\mathcal{I}$, indexed by I, by the operations:

- $\mathcal{H}_{p\wedge q} = \mathcal{H}_p \cap \mathcal{H}_q$, with the projective norm

$$\|f\|^2_{p\wedge q} = \|f\|^2_p + \|f\|^2_q, \; ; \tag{I.4}$$

- $\mathcal{H}_{p\vee q} = \mathcal{H}_p + \mathcal{H}_q$, the vector sum, with the inductive norm

$$\|f\|^2_{p\vee q} = \inf_{f=g+h} \left(\|g\|^2_p + \|h\|^2_q\right), \; g \in \mathcal{H}_p, f \in \mathcal{H}_q \, . \tag{I.5}$$

It turns out that both $\mathcal{H}_{p\wedge q}$ and $\mathcal{H}_{p\vee q}$ are Hilbert spaces, that is, they are complete with the norms indicated. These statements will be proved, and the corresponding mathematical structure analyzed, in Chapter 2, Section 2.2.

Assume that the original index set J has an involution $q \leftrightarrow \overline{q}$, with $\mathcal{H}_{\overline{q}} = \mathcal{H}_q^\times$ (by an involution, we mean a one-to-one correspondence such that $p \leqslant q$ implies $\overline{q} \leqslant \overline{p}$ and $\overline{\overline{p}} = p$). Then the lattice $\mathcal{I}$ inherits the same duality structure, with $\mathcal{H}_{p\wedge q} \leftrightarrow \mathcal{H}_{\overline{p}\vee\overline{q}}$ (it is then called an *involutive lattice*; a precise definition will be given in Section 1.1). Finally, we assume the family $\mathcal{J}$ contains a unique self-dual space $V_o = V_{\overline{o}}$. The resulting structure is called a *lattice of Hilbert spaces* or LHS.

In addition to the family $\mathcal{I} = \{V_r,\, r \in I\}$, it is convenient to consider the two spaces $V^\#$ and V defined as

$$V = \sum_{q \in I} \mathcal{H}_q, \quad V^{\#} = \bigcap_{q \in I} \mathcal{H}_q. \tag{I.6}$$

These two spaces themselves usually do *not* belong to $\mathcal{I}$.

The concept of LHS is closely related to that of nested Hilbert space, which will be discussed in Section 2.4). More important, this construction is the basic structure of interpolation theory.

A similar construction can be performed with a family $\mathcal{J} = \{V_p,\, p \in J\}$ of reflexive Banach spaces, the resulting structure being then a *lattice of Banach spaces* or LBS. In this case, one considers the following norms, which are usual in interpolation theory.

- $V_{p\wedge q} = V_p \cap V_q$, with the *projective* norm

$$\|f\|_{p\wedge q} = \|f\|_p + \|f\|_q\,; \tag{I.7}$$

- $V_{p\vee q} = V_p + V_q$, with the *inductive* norm

$$\|f\|_{p\vee q} = \inf_{f=g+h} \left(\|g\|_p + \|h\|_q\right), g \in V_p, f \in V_q. \tag{I.8}$$

Here too, we assume the family $\mathcal{J}$ contains a unique self-dual space $V_o = V_{\overline{o}}$, which is a Hilbert space.

I.2.2 Partial Inner Product on a LHS/LBS

The basic question is how to generate such structures in a systematic fashion. In order to answer it, we may reformulate it as follows: given a vector space V and two vectors $f, g \in V$, when does their inner product make sense? A way of formalizing the answer is given by the idea of *compatibility*.

Let $\mathcal{I} := \{V_r,\, r \in I\}$ be a LHS or a LBS and $f, g \in V$ two vectors. Then we say that f and g are *compatible*, which we note $f\#g$, if the following relation holds:

$$f\#g \;\Leftrightarrow\; \exists\, r \in I \text{ such that } f \in V_r, g \in V_{\overline{r}}\,. \tag{I.9}$$

Clearly the relation $\#$ is a symmetric binary relation which preserves linearity:

$$\begin{aligned} f\#g &\Longleftrightarrow g\#f, \;\forall\, f, g \in V, \\ f\#g, f\#h &\Longrightarrow f\#(\alpha g + \beta h), \;\forall\, f, g, h \in V, \forall\, \alpha, \beta \in \mathbb{C}. \end{aligned}$$

From now on, we write $\mathcal{I} = (V, \#)$. A formal definition will be given in Section 1.1.

Now we introduce the basic notions of our structure, namely, a partial inner product and a partial inner product space.

A *partial inner product* on $(V, \#)$ is a hermitian form $\langle\cdot|\cdot\rangle$ defined exactly on compatible pairs of vectors, that is, on $\Delta = \left(\bigcup_{V_r\in\mathcal{I}} V_r \times V_{\overline{r}}\right) \bigcup \left(V^{\#} \times V\right)$. A *partial inner product space (*PIP*-space)* is a vector space V equipped with a linear compatibility and a partial inner product.

In general, the partial inner product is not required to be positive definite, but it will be in all the examples given in this chapter. In the present case of a LHS or a LBS $\mathcal{I} = \{V_r,\, r \in I\}$, this simply means that the partial inner product is defined between elements of two spaces in duality $(V_r, V_{\overline{r}})$; in particular, for $r = o$, it is a Hermitian form on the Hilbert space V_o, which we take as equal to the inner product. Thus, the partial inner product may be seen as the extension of the inner product of V_o to the whole of V, whenever possible. Clearly, with the compatibility (I.9) and this definition of partial inner product, the LHS/LBS $\mathcal{I}$ is a PIP-space, that we henceforth denote as $(V, \#, \langle\cdot|\cdot\rangle)$.

The partial inner product defines a notion of *orthogonality* : $f \perp g$ if and only if $f \# g$ and $\langle f|g\rangle = 0$. Then we say that the PIP-space $(V, \#, \langle\cdot|\cdot\rangle)$ is *nondegenerate* if $(V^{\#})^{\perp} = \{0\}$, that is, if $\langle f|g\rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$.

In this introductory chapter, we will assume that our PIP-space $(V, \#, \langle\cdot|\cdot\rangle)$ is nondegenerate. This assumption has important topological consequences, that will be explored at length in Chapter 2. In a nutshell, $(V^{\#}, V)$, like every couple $(V_r, V_{\overline{r}})$, $r \in I$, is a dual pair in the sense of topological vector spaces. Furthermore, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r.

I.2.3 Two Examples of LHS

Let us give two simple examples of LHS, thus of PIP-spaces as well.

(i) Sequence spaces

Let V be the space ω of *all* complex sequences $x = (x_n)$. Consider the following family of weighted Hilbert spaces, which are obviously subspaces of ω:

$$\ell^2(r) = \{(x_n) \in \omega : (x_n/r_n) \in \ell^2, i.e., \sum_{n=1}^{\infty} |x_n|^2\, r_n^{-2} < \infty\}, \tag{I.10}$$

where $r = (r_n)$, $r_n > 0$, is a sequence of positive numbers. The family possesses an involution:

$$\ell^2(r) \leftrightarrow \ell^2(\overline{r}) = \ell^2(r)^{\times}, \text{ where } \overline{r}_n = 1/r_n.$$

In addition, there is a central, self-dual Hilbert space, namely, $\ell^2(1) = \ell^2(\overline{1}) = \ell^2$, where 1 denotes the unit sequence, $r_n = 1$, for all n.

As a matter of fact, the collection $\mathcal{I} := \{\ell^2(r)\}$ of those vector subspaces of ω is an involutive lattice.

(i) $\mathcal{I}$ is a lattice for the following operations:

$$\begin{aligned}\ell^2(r) \wedge \ell^2(s) &= \ell^2(u), \quad \text{where} \quad u_n = \min\{r_n, s_n\},\\ \ell^2(r) \vee \ell^2(s) &= \ell^2(v), \quad \text{where} \quad v_n = \max\{r_n, s_n\}.\end{aligned}$$

Indeed one shows easily that the norms of $\ell^2(u)$ and $\ell^2(v)$ are equivalent, respectively, to the projective and inductive norms defined in (I.4), (I.5) above (proofs will be given in Section 4.3).

(ii) $\mathcal{I}$ is an involutive lattice, with the involution $r \leftrightarrow \overline{r} \equiv (r_n^{-1})$. Indeed:

$$\begin{aligned}[\ell^2(u)]^{\#} &= \ell^2(\overline{u}) = \ell^2(\overline{r}) \vee \ell^2(\overline{s}),\\ [\ell^2(v)]^{\#} &= \ell^2(\overline{v}) = \ell^2(\overline{r}) \wedge \ell^2(\overline{s}).\end{aligned}$$

Actually, $\mathcal{I}$ is a sublattice of $\mathcal{L}(\omega)$, the lattice of all vector subspaces of ω, i.e.,

$$\begin{aligned}\ell^2(r) \wedge \ell^2(s) &= \ell^2(r) \cap \ell^2(s),\\ \ell^2(r) \vee \ell^2(s) &= \ell^2(r) + \ell^2(s).\end{aligned}$$

As for the extreme spaces, it is easy to see that the family $\{\ell^2(r)\}$ generates the space ω of *all* complex sequences, while the intersection is the space φ of all *finite* sequences:

$$\bigcup_{r \in I} \ell^2(r) = \omega, \quad \bigcap_{r \in I} \ell^2(r) = \varphi.$$

Thus, with the partial inner product $\langle x|y\rangle = \sum_{n=1}^{\infty} \overline{x_n}\, y_n$, inherited from $V_o = \ell^2$, the family $\mathcal{I} = \{\ell^2(r)\}$ is a nondegenerate LHS.

(ii) Spaces of locally integrable functions

Instead of sequences, we consider locally integrable functions, i.e., Lebesgue measurable functions, integrable over compact subsets, $f \in L^1_{\mathrm{loc}}(\mathbb{R}, dx)$, and define again weighted Hilbert spaces:

$$L^2(r) = \{f \in L^1_{\mathrm{loc}}(\mathbb{R}, dx) : fr^{-1} \in L^2, i.e. \int_{\mathbb{R}} |f(x)|^2\, r(x)^{-2}\, dx < \infty\}, \quad \text{(I.11)}$$

with $r, r^{-1} \in L^2_{\mathrm{loc}}(\mathbb{R}, dx), r(x) > 0$ a.e. The family $\mathcal{I} = \{L^2(r)\}$ has an involution, $L^2(r) \leftrightarrow L^2(\overline{r})$, with $\overline{r} = r^{-1}$, and a central, self-dual Hilbert space, $L^2(\mathbb{R}, dx)$. This is, of course, the continuous analogue of the preceding example. Thus we get exactly the same structure as in (i), namely the family $\mathcal{I} = \{L^2(r)\}$ is an involutive lattice, for the operations:

- infimum: $L^2(p \wedge q) = L^2(p) \wedge L^2(q) = L^2(r)$, $r(x) = \min(p(x), q(x))$;
- supremum: $L^2(p \vee q) = L^2(p) \vee L^2(q) = L^2(s)$, $s(x) = \max(p(x), q(x))$;
- duality: $L^2(p \wedge q) \leftrightarrow L^2(\overline{p} \vee \overline{q})$, $L^2(p \vee q) \leftrightarrow L^2(\overline{p} \wedge \overline{q})$.

Here too, it is easily shown that the lattice $\mathcal{I}$ generates the extreme spaces:

$$\bigcup_{r\in I} L^2(r) = L^1_{\rm loc}(\mathbb{R}, dx), \qquad \bigcap_{r\in I} L^2(r) = L^\infty_{\rm c}(\mathbb{R}),$$

where $L^\infty_{\rm c}(\mathbb{R})$ denotes the space of essentially bounded measurable functions of compact support. With the partial inner product inherited from the central space L^2,

$$\langle f|g\rangle = \int_{\mathbb{R}} \overline{f(x)} g(x)\, dx,$$

the family $\mathcal{I} = \{L^2(r)\}$ becomes a nondegenerate LHS. The construction extends trivially to $\mathbb{R}^n$, or to any manifold (X, μ). It may also be done around Fock space, instead of L^2 (see Section 1.1.3, Example (iv)).

I.3 Operators on a LHS/LBS

As follows from the compatibility relation (I.9), the basic idea of LHS/LBS (and, more generally, PIP-spaces, as we shall see in the next chapter) is that vectors should not be considered individually, but only in terms of the subspaces V_r $(r \in I)$, the building blocks of the structure. Correspondingly, an operator on such a space should be defined in terms of the defining subspaces only, with the proviso that only *bounded* operators between Hilbert or Banach spaces are allowed. Thus an operator is a *coherent collection* of bounded operators. More precisely,

Definition I.3.1. Given a LHS or LBS $V_I = \{V_r,\, r \in I\}$, an *operator* on V_I is a map A from a subset $\mathcal{D} \subseteq V$ into V, where

(i) $\mathcal{D}$ is a nonempty union of defining subspaces of V_I;
(ii) for every defining subspace V_q contained in $\mathcal{D}$, there exists a $p \in I$ such that the restriction of A to V_q is linear and continuous into V_p (we denote this restriction by A_{pq});
(iii) A has no proper extension satisfying (i) and (ii), i.e., it is maximal.

According to Condition (iii), the domain $\mathcal{D}$ is called the *natural domain* of A and denoted $\mathcal{D}(A)$.

The linear bounded operator $A_{pq} : V_q \to V_p$ is called a *representative* of A. In terms of the latter, the operator A may be characterized by the set $\mathsf{j}(A) = \{(q,p) \in I \times I : A_{pq} \text{ exists}\}$. Thus the operator A may be identified with the collection of its representatives,

$$A \simeq \{A_{pq} : V_q \to V_p : (q,p) \in \mathsf{j}(A)\}.$$

We also need the two sets obtained by projecting $\mathsf{j}(A)$ on the "coordinate" axes, namely,

$$\mathsf{d}(A) = \{q \in I : \text{there is a } p \text{ such that } A_{pq} \text{ exists}\},$$
$$\mathsf{i}(A) = \{p \in I : \text{there is a } q \text{ such that } A_{pq} \text{ exists}\}.$$

The following properties are immediate:

- $\mathsf{d}(A)$ is an initial subset of I: if $q \in \mathsf{d}(A)$ and $q' < q$, then $q' \in \mathsf{d}(A)$, and $A_{pq'} = A_{pq}E_{qq'}$, where $E_{qq'}$ is a representative of the unit operator (this is what we mean by a 'coherent' collection).
- $\mathsf{i}(A)$ is a final subset of I: if $p \in \mathsf{i}(A)$ and $p' > p$, then $p' \in \mathsf{i}(A)$ and $A_{p'q} = E_{p'p}A_{pq}$.
- $\mathsf{j}(A) \subset \mathsf{d}(A) \times \mathsf{i}(A)$, with strict inclusion in general.

We denote by $\mathrm{Op}(V_I)$ the set of all operators on V_I. Of course, a similar definition may be given for operators $A : V_I \to Y_K$ between two LHSs or LBSs.

Since $V^\#$ is dense in V_r, for every $r \in I$, an operator may be identified with a sesquilinear form on $V^\# \times V^\#$. Indeed, the restriction of any representative A_{pq} to $V^\# \times V^\#$ is such a form, and all these restrictions coincide (these sesquilinear forms are even separately continuous for appropriate topologies on $V^\#$, see Chapter 3). Equivalently, an operator may be identified with a linear map from $V^\#$ into V (here also continuity may be obtained). But the idea behind the notion of operator is to keep also the *algebraic operations* on operators, namely:

(i) *Adjoint $A^\times$:* every $A \in \mathrm{Op}(V_I)$ has a unique adjoint $A^\times \in \mathrm{Op}(V_I)$, defined by the relation

$$\langle A^\times x | y \rangle = \langle x | Ay \rangle, \text{ for } y \in V_r,\ r \in \mathsf{d}(A), \text{ and } x \in V_{\overline{s}},\ s \in \mathsf{i}(A),$$

that is, $(A^\times)_{\overline{r}\overline{s}} = (A_{sr})^*$ (usual Hilbert/Banach space adjoint).

It follows that $A^{\times\times} = A$, for every $A \in \mathrm{Op}(V_I)$: no extension is allowed, by the maximality condition (iii) of Definition I.3.1.

(ii) *Partial multiplication:* AB is defined if and only if there is a $q \in \mathsf{i}(B) \cap \mathsf{d}(A)$, that is, if and only if there is a continuous factorization through some V_q :

$$V_r \xrightarrow{B} V_q \xrightarrow{A} V_s, \quad \text{i.e.,} \quad (AB)_{sr} = A_{sq}B_{qr}\,.$$

It is worth noting that, for a LHS/LBS, the natural domain $\mathcal{D}(A)$ is always a vector subspace of V (this is not true for a general PIP-space). Therefore, $\mathrm{Op}(V_I)$ is a vector space and a *partial *-algebra*.

The concept of PIP-space operator is very simple, yet it is a far reaching generalization of bounded operators. It allows indeed to treat on the same footing all kinds of operators, from bounded ones to very singular ones. By this, we mean the following, loosely speaking. Given $A \in Op(V_I)$, when looked at from the central Hilbert space $V_o = \mathcal{H}$, there are three possibilities:

- if $(o, o) \in \mathsf{j}(A)$, i.e., A_{oo} exists, then A corresponds to a bounded operator $V_o \to V_o$;
- if $(o, o) \notin \mathsf{j}(A)$, but there is an $r < o$ such that $(r, o) \in \mathsf{j}(A)$, i.e., A_{oo} does not exist, but only $A_{or} : V_r \to V_o$, with $r < o$, then A corresponds to an unbounded operator A_{or}, with hilbertian domain containing V_r;
- if $(r, o) \notin \mathsf{j}(A)$, for any $r \leqslant o$, i.e., no A_{or} exists, then A is a sesquilinear form on some V_s, $s \leqslant o$, and, as an operator on $\mathcal{H}$, its domain does not contain any V_r (it may be reduced to $\{0\}$): then A corresponds to a singular operator; this happens, for instance, if $(r, s) \in \mathsf{j}(A)$ with $r < o < s$, i.e., there exists $A_{sr} : V_r \to V_s$.

Exactly as for Hilbert or Banach spaces, one may define various types of operators between PIP-spaces, in particular LBS/LHS, such that regular operators, orthogonal projections, homomorphisms and isomorphisms, symmetric operators, unitary operators, etc. We will describe those classes in detail in Chapter 3.

In the following chapters, we will extend this discussion to a general PIP-space and operators between two PIP-spaces. A slightly more restrictive structure, called an *indexed* PIP-*space*, will also be introduced. Many concrete examples will be discussed in detail.

Notes

Section I.1. For unbounded operators, see Reed-Simon I [RS72, Section VIII. 2]. For the spectral theorem, see Kato [Kat76, RS72] or Reed-Simon I [RS72, Section VIII. 1]. The work of Palais may be found in [162, Chap.XIV].

Section I.2. Nested Hilbert spaces were introduced by Grossmann [114].

- For interpolation theory, one may consult the monographs of Bergh-Löfström [BL76] or Triebel [Tri78a].
- Our standard references for topological vector spaces are the monographs of Köthe [Köt69] and Schaefer [Sch71].

Section I.3. Partial *-algebras are studied in detail in the monograph of Antoine-Inoue-Trapani [AIT02].

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