Chapter 1
The Case of Manifolds

Abstract In this chapter we review briefly some of the fundamental results of the classical theory of indices of vector fields and characteristic classes of smooth manifolds. These were first defined in terms of obstructions to the construction of vector fields and frames. In the case of a vector field the Poincaré–Hopf Theorem says that Euler–Poincaré characteristic is the obstruction to constructing a nonzero vector field tangent to a compact manifold. Extension of this result to frames yields to the definition of Chern classes from the viewpoint of obstruction theory.

There is another important point of view for defining characteristic classes on the differential geometry side, this is the Chern–Weil theory. Sections 3 and 4 provide an introduction to that theory and the corresponding definition of Chern classes.

Finally, Sect. 5 sets up one of the key features of this monograph: the interplay between localization via obstruction theory, which yields to the classical relative characteristic classes, and localization via Chern–Weil theory, which yields to the theory of residues. This is one way of thinking of the Poincaré–Hopf Theorem and its generalizations.

Throughout the book, M will denote either a complex manifold of (complex) dimension m, or a C^∞ manifold of (real) dimension m'.

1.1 Poincaré–Hopf Index Theorem

1.1.1 Poincaré–Hopf Index at Isolated Points

Let $v = \sum_{i=1}^{m'} f_i \partial/\partial x_i$ be a vector field on an open set $U \subset \mathbb{R}^{m'}$ with coordinates $\{(x_1, \ldots, x_{m'})\}$. The vector field is said to be continuous, smooth, analytic, etc., according as its components $\{f_1, \ldots, f_{m'}\}$ are continuous, smooth, analytic, etc., respectively (here "smooth" means C^∞, however in most cases C^1 is sufficient). A *singularity* a of v is a point where all of its components vanish, *i.e.*, $f_i(a) = 0$ for all $i = 1, \ldots, m'$. The singularity is *isolated* if at every point x near a there is at least one component of v which is not zero.

J.-P. Brasselet et al., *Vector Fields on Singular Varieties*,
Lecture Notes in Mathematics 1987, DOI 10.1007/978-3-642-05205-7_1,

The Poincaré–Hopf index of a vector field at an isolated singularity is its most basic invariant, and it has many interesting properties. To define it, let v be a continuous vector field on U with an isolated singularity at a, and let $\mathbb{S}_\varepsilon$ be a small sphere in U around a. Then the (local) Poincaré–Hopf index of v at a, denoted by $\mathrm{Ind}_{\mathrm{PH}}(v, a)$ (if there is no fear of confusion, we will denote it simply by $\mathrm{Ind}(v, a)$), is the degree of the Gauss map $\frac{v}{||v||}$ from $\mathbb{S}_\varepsilon$ into the unit sphere in $\mathbb{R}^{m'}$.

If v and v' are two such vector fields, then their local indices at a coincide if and only if their Gauss maps are homotopic (special case of Hopf Theorem [120]). That is equivalent to saying that their restrictions to the sphere $\mathbb{S}_\varepsilon$ are homotopic.

Let us consider now an m'-dimensional smooth manifold M, then a *vector field* on M is a section of its tangent bundle TM. Giving a local chart $(x_1, \ldots, x_{m'})$ on M, a vector field on M is locally expressed as above and the definition of the local index at an isolated singularity extends in the obvious way. The index does not depend on the local chart.

Definition 1.1.1. The *total index* of v, denoted

$$\mathrm{Ind}_{\mathrm{PH}}(v, M),$$

is the sum of all its local indices at the singular points.

A fundamental property of the total index is the following classical theorem:

Theorem 1.1.1. (Poincaré–Hopf) *Let M be a closed, oriented manifold and v a continuous vector field on M with finitely many isolated singularities. Then one has*

$$\mathrm{Ind}_{\mathrm{PH}}(v, M) = \chi(M),$$

independently of v, where $\chi(M)$ denotes the Euler–Poincaré characteristic of M.

If M is now an oriented manifold with boundary, one has a similar theorem:

Theorem 1.1.2. *Let M be a compact, oriented m'-manifold with boundary ∂M, and let v be a nonsingular vector field on a neighborhood U of ∂M. Then:*

(1) *v can be extended to the interior of M with finitely many isolated singularities.*
(2) *The total index of v in M is independent of the way we extend it to the interior of M. In other words, the total index of v is fully determined by its behavior near the boundary.*
(3) *If v is everywhere transverse to the boundary and pointing outwards from M, then one has $\mathrm{Ind}_{\mathrm{PH}}(v, M) = \chi(M)$. If v is everywhere transverse to ∂M and pointing inwards M, then $\mathrm{Ind}_{\mathrm{PH}}(v, M) = \chi(M) - \chi(\partial M)$.*

Remark 1.1.1. It is worth saying that although $\mathrm{Ind}_{\mathrm{PH}}(v, M)$ is determined by its behavior near the boundary, it does depend on the topology of the interior of M. In fact a formula of Morse and Pugh (c.f. [124, 133, 145]) provides an explicit way to compute the index out of boundary data, generalizing a classical formula of Poincaré for vector fields on the plane.

We remark also that one of the basic properties of the index is its stability under perturbations. In other words, if v has an isolated singularity at a point a in a manifold M of index $\mathrm{Ind}(v, a)$ and we make a small perturbation of v to get a new vector field $\hat{v}$ with isolated singularities, then $\mathrm{Ind}(v, a)$ will be the sum of the local indices of $\hat{v}$ at its singular points near a. In fact it is well-known that every vector field can be *morsified*, *i.e.*, approximated by vector fields whose singularities are nondegenerate. Each such singularity has local index ± 1 and the number of such points, counted with signs, equals the index of v at a. In short, the local index of v at a is the number of singularities, counted with sign, into which a splits under a morsification of v. We will see later that this basic property has its analogues in the case of vector fields on singular varieties.

This stability of the index is also preserved for vector fields with nonisolated singularities. To make this precise we need to introduce a few concepts, which will also be used later.

The following property of the local index is well-known and we leave the proof as an exercise:

Proposition 1.1.1. *Let v be a vector field around $0 \in \mathbb{R}^{m'}$ with an isolated singularity at 0 of index $\mathrm{Ind}(v, 0)$, and let w be a vector field around $0 \in \mathbb{R}^{n'}$ with an isolated singularity at 0 of index $\mathrm{Ind}(w, 0)$. Then the direct product $v \oplus w$ is a vector field in $\mathbb{R}^{m'+n'}$ with an isolated singularity of index $\mathrm{Ind}(v, 0) \cdot \mathrm{Ind}(w, 0)$.*

A consequence of this result is the well-known fact that if M, N are closed, oriented manifolds, then $\chi(M \times N) = \chi(M) \cdot \chi(N)$. Another consequence of 1.1.1 that will be used later is that if we have a vector field v in $\mathbb{R}^{m'}$ with an isolated singularity at 0 of index $\mathrm{Ind}(v, 0)$, and if we extend it to $\mathbb{R}^{m'} \times \mathbb{R}^{n'}$ by taking the vector field $w = \sum_{i=1}^{n'} y_i \partial/\partial y_i$ in $\mathbb{R}^{n'}$, then the index does not change, where $(y_1, \ldots, y_{n'})$ are the coordinates on $\mathbb{R}^{n'}$. If we took the vector field $-\sum_{i=1}^{r} y_i \partial/\partial y_i + \sum_{i=r+1}^{n'} y_i \partial/\partial y_i$ in $\mathbb{R}^{n'}$, then the index in $\mathbb{R}^{m'} \times \mathbb{R}^{n'}$ would be $\pm\mathrm{Ind}(v, 0)$, depending on the parity of the number r of negative signs.

1.1.2 Poincaré–Hopf Index at Nonisolated Points

In the following, singularities of the vector field v are not necessarily isolated points. We still define a Poincaré–Hopf index in that case.

Let M be a manifold with boundary ∂M. Let us consider a triangulation (K) of M compatible with the boundary and (K') a barycentric subdivision of (K). Using (K') one constructs the associated cellular dual decomposition (D) of M: given a simplex σ in (K) of dimension s, its *dual* $d(\sigma)$ is the union of all simplices τ in (K') whose closure meets σ exactly at its barycenter $\hat{\sigma}$, that is $\overline{\tau} \cap \sigma = \hat{\sigma}$. If σ is in the interior of M, that is a cell, if σ is in the boundary of M, that is a "half-cell." It is an exercise to see that the dimension of $d(\sigma)$ is $m' - s$. Taking the union of all these dual cells (or half-cells) we get the dual decomposition (D) of (K); by construction its cells and half-cells are all transverse to (K) (we refer to [25] for details including orientation notions).

Let S be a compact connected (K)-subcomplex of the interior of M.

Definition 1.1.2. A *cellular tube* $\mathcal{T}$ around S in M is the union of cells (D) which are dual of simplices in S.

This notion generalizes the concept of tubular neighborhood of a submanifold S. If S is a submanifold without boundary, then $\mathcal{T}$ is a bundle on S, whose fibers are discs. In general, that is not the case.

Remark 1.1.2. A cellular tube $\mathcal{T}$ around S has the following properties :
(1) $\mathcal{T}$ is a compact neighborhood of S, containing S in its interior and $\partial\mathcal{T}$ is a retract of $\mathcal{T} \setminus S$.
(2) $\mathcal{T}$ is a *regular* neighborhood of S, thus $\mathcal{T}$ retracts to S.
(3) We can assume the cellular tubes in M have smooth boundary [83].

Let us denote by U a neighborhood of S in M. If the triangulation is sufficiently "fine," then we can assume $\mathcal{T} \subset U$.

According to Theorem 1.1.2, a nonsingular continuous vector field v on a neighborhood of $\partial\mathcal{T}$ can be extended to the interior of $\mathcal{T}$ with finitely many isolated singularities. The total index of v on $\mathcal{T}$ is defined as the sum of the indices of the extension of v at these points.

Definition 1.1.3. Let v be a continuous vector field on a neighborhood U of S in M, nonsingular on $U \setminus S$, then the *Poincaré–Hopf index* of v at S, denoted $\mathrm{Ind}_{\mathrm{PH}}(v, S)$ (or simply by $\mathrm{Ind}(v, S)$, if there is no ambiguity), is defined as $\mathrm{Ind}_{\mathrm{PH}}(v, \mathcal{T})$.

This number $\mathrm{Ind}_{\mathrm{PH}}(v, S)$ depends only on the behavior of v near S and not on the choice of the neighborhood U, or of the tube $\mathcal{T}$. Moreover, for this index it does not matter what actually happens on S, we only care what happens around S, but away from S. In particular, if v is "radial" from S, *i.e.*, if it is transverse to the boundary of a cellular tube around S pointing outward, then $\mathrm{Ind}_{\mathrm{PH}}(v, S) = \chi(S)$.

Now let M be a compact oriented C^∞ manifold possibly with boundary ∂M and v a continuous vector field on M, nonsingular on the boundary. From the above considerations, we may assume that the set $S(v)$ of singular points of v has only a finite number of components $\{S_\lambda\}$.

If M has no boundary, the Poincaré–Hopf Theorem implies that

$$\sum_\lambda \mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda) = \chi(M). \tag{1.1.3}$$

If M has a boundary, the sum $\sum_\lambda \mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda)$ depends only on the behavior of v near ∂M. For example, if v is pointing outwards everywhere on ∂M, then we have the same formula (1.1.3). If v is pointing inwards everywhere on ∂M, the right hand side becomes $\chi(M) - \chi(\partial M)$. In particular, if the (real) dimension of M is even (as it will usually be the case in this book) and if v is everywhere transverse to ∂M, then we have again the same formula (1.1.3).

Here we introduce the concept of *the difference* which will be used in the rest of the book. For this we let v and v' be continuous vector fields on a neighborhood U of S in M, nonsingular on $U \setminus S$. Let $\mathcal{T}$ and $\mathcal{T}'$ be cellular tubes around S in U such that interior of $\mathcal{T}$ contains the closure of $\mathcal{T}'$ and denote $X = \mathcal{T} \setminus \mathcal{T}'$. Let us consider w a vector field on X with isolated singularities which restricts to v on $\partial\mathcal{T}$ and to v' on $\partial\mathcal{T}'$; such a vector field w always exists by Theorem 1.1.2. We may denote by $d(v, v') = \mathrm{Ind}_{\mathrm{PH}}(w, X)$ *the difference* between v and v'. Then one has:

$$\mathrm{Ind}_{\mathrm{PH}}(v, S) = \mathrm{Ind}_{\mathrm{PH}}(v', S) + d(v, v'). \tag{1.1.4}$$

One can easily prove the following result that will be used later.

Proposition 1.1.2. *Let M_1 and M_2 be compact oriented m'-manifolds, $m' > 1$, with the same boundary $N = \partial M_1 = \partial M_2$, and let v be a nonsingular vector field defined on a neighborhood of N. Then one has:*

$$\mathrm{Ind}_{\mathrm{PH}}(v, M_1) - \mathrm{Ind}_{\mathrm{PH}}(v, M_2) = \chi(M_1) - \chi(M_2).$$

1.2 Poincaré and Alexander Dualities

We briefly review the classical case, which will be generalized to the case of singular varieties in Sect. 10.4 below. In either case, we follow the descriptions given in [25].

Let M be an oriented manifold of real dimension m'. We take a triangulation (K) of M and the cellular decomposition (D) dual to (K), as before. The groups of chains relative to (K) and (D) are denoted by $C_*^{(K)}(M)$ and $C_*^{(D)}(M)$, respectively. Also, the groups of cochains relative to (K) and (D) are denoted by $C^*_{(K)}(M)$ and $C^*_{(D)}(M)$, respectively. The intersection of an

i-simplex σ and its dual $(m'-i)$-cell $d(\sigma)$ is transverse and consists of one point, the barycenter $\hat{\sigma}$ of σ.

First, if M is compact, we define a homomorphism

$$P : C_{(D)}^{m'-i}(M) \longrightarrow C_i^{(K)}(M) \qquad \text{by} \qquad P(c) = \sum_{\sigma} \langle c, d(\sigma) \rangle \, \sigma \tag{1.2.1}$$

for an $(m'-i)$-cochain c, where the sum is taken over all i-simplices σ of M (we follow the orientation conventions in [25]). This induces the Poincaré isomorphism

$$P_M : H^{m'-i}(M) \xrightarrow{\sim} H_i(M).$$

Next, let S be a (K)-subcomplex of M whose geometric realization is also denoted by S. Let $C_{(D)}^*(M, M \setminus S)$ denote the subgroup of $C_{(D)}^*(M)$ consisting of cochains which are zero on the cells not intersecting with S.

Suppose S is compact (M may not be compact). Then we may define a homomorphism

$$A : C_{(D)}^{m'-i}(M, M \setminus S) \longrightarrow C_i^{(K)}(S)$$

taking, in the sum in (1.2.1), only i-simplices of S. This induces the Alexander isomorphism

$$A_{M,S} : H^{m'-i}(M, M \setminus S) \xrightarrow{\sim} H_i(S).$$

From the construction, we have the following

Proposition 1.2.1. *If M is compact, we have the commutative diagram*

$$\begin{array}{ccc} H^{m'-i}(M, M \setminus S) & \xrightarrow{j^*} & H^{m'-i}(M) \\ \wr \big\downarrow A_{M,S} & & \wr \big\downarrow P_M \\ H_i(S) & \xrightarrow{i_*} & H_i(M). \end{array}$$

1.3 Chern Classes via Obstruction Theory

1.3.1 Chern Classes of Almost Complex Manifolds

Let us recall the definition of the Chern classes via obstruction theory [28, 89, 123, 153]. This can be done in full generality, however for simplicity we consider first the case of Chern classes of almost-complex manifolds, and later in this section we indicate how this generalizes to complex vector bundles in general.

Now we assume we are given an almost complex $m' = 2m$-manifold M, so its tangent bundle TM is endowed with the structure of a complex vector bundle of rank m.

Definition 1.3.1. An *r-field* on a subset A of M is a set $v^{(r)} = \{v_1, \dots, v_r\}$ of r continuous vector fields defined on A. A singular point of $v^{(r)}$ is a point where the vectors (v_i) fail to be linearly independent. A nonsingular r-field is also called an *r-frame.*

Let $W_{r,m}$ be the Stiefel manifold of complex r-frames in $\mathbb{C}^m$. Notice that we will use r-frames which are not necessarily orthonormal, but this does not change the results, because every frame is homotopic to an orthonormal one. We know (see [153]) that $W_{r,m}$ is $(2m-2r)$-connected and its first nonzero homotopy group is $\pi_{2m-2r+1}(W_{r,m}) \simeq \mathbb{Z}$. The bundle of r-frames on M, denoted by $W_r(TM)$, is the bundle associated with the tangent bundle and whose fiber over $x \in M$ is the set of r-frames in T_xM (diffeomorphic to $W_{r,m}$). In the following, we fix the notation $q = m - r + 1$.

The Chern class $c^q(M) \in H^{2q}(M)$ is the first possibly nonzero obstruction to constructing a section of $W_r(TM)$. Let us recall the standard obstruction theory process to construct this class. Let σ be a k-cell of the given cellular decomposition (D), contained in an open subset $U \subset M$ on which the bundle $W_r(TM)$ is trivialized. If the section $v^{(r)}$ of $W_r(TM)$ is already defined over the boundary of σ, it defines a map:

$$\partial\sigma \simeq \mathbb{S}^{k-1} \xrightarrow{v^{(r)}} W_r(TM)|_U \simeq U \times W_{r,m} \xrightarrow{pr_2} W_{r,m},$$

thus an element of $\pi_{k-1}(W_{r,m})$.

If $k \leq 2m-2r+1$, this homotopy group is zero, so the section $v^{(r)}$ can be extended to σ without singularity. It means that we can always construct a section $v^{(r)}$ of $W_r(TM)$ over the $(2q-1)$-skeleton of (D).

If $k = 2(m-r+1) = 2q$, we meet an obstruction. The r-frame on the boundary of each cell σ defines an element, denoted by $\mathrm{Ind}(v^{(r)}, \sigma)$, in the homotopy group $\pi_{2q-1}(W_{r,m}) \simeq \mathbb{Z}$.

Definition 1.3.2. The integer $\mathrm{Ind}(v^{(r)}, \sigma)$ is *the (Poincaré–Hopf) index* of the r-frame $v^{(r)}$ on the cell σ.

Notice that for this index, to be well defined, we need that the cell σ has the correct dimension. This will be essential for our considerations in Chap. 10.

The generators of $\pi_{2q-1}(W_{r,m})$ being consistent (see [153]), this defines a cochain

$$\gamma \in C^{2q}(M; \pi_{2q-1}(W_{r,m})),$$

by setting $\gamma(\sigma) = \mathrm{Ind}(v^{(r)}, \sigma)$, for each $2q$-cell σ, and then by extending it linearly. This cochain is actually a cocycle and the cohomology class that it represents is the q-th *Chern class* $c^q(M)$ of M in $H^{2q}(M)$.

The class one gets in this way is independent of the various choices involved in its definition. Note that $c^m(M)$ coincides with the Euler class of the underlying real tangent bundle $T_{\mathbb{R}}M$, so these classes are natural generalization of the Euler class.

There is another useful definition of the index $\mathrm{Ind}(v^{(r)}, \sigma)$: let us write the frame $v^{(r)}$ as $(v^{(r-1)}, v_r)$, where the last vector is individualized, and suppose that $v^{(r)}$ is already defined on $\partial\sigma$. There is no obstruction to extending the $(r-1)$-frame $v^{(r-1)}$ from $\partial\sigma$ to σ because the dimension of the obstruction for such an extension is $2(m-(r-1)+1) = \dim\sigma + 2$. The $(r-1)$-frame $v^{(r-1)}$, defined on σ, generates a complex subbundle G^{r-1} of rank $(r-1)$ of $TM|_\sigma$ and one can write

$$TM|_\sigma \simeq G^{r-1} \oplus Q^q ,$$

where Q^q is an orthogonal complement of (complex) rank $q = m-(r-1)$.

The obstruction to extending the last vector v_r inside a $2q$-simplex σ as a nonvanishing section of Q^q is given by an element of $\pi_{2q-1}(\mathbb{C}^q \setminus \{0\}) \simeq \mathbb{Z}$ corresponding to the composition of the map $v_r : \partial\sigma \simeq \mathbb{S}^{2q-1} \longrightarrow Q^q|_U$ with the projection on the fiber $\mathbb{C}^q \setminus \{0\}$. Let us denote by $\mathrm{Ind}_{Q^q}(v_r, \sigma)$ the integer so obtained. The obstruction to extending the r-frame $v^{(r)}|_{\partial\sigma}$ inside σ as an r-frame tangent to M is the same as the obstruction to extending the last vector v_r inside σ as a non zero section of Q^q. In fact there is a natural isomorphism $\pi_{2q-1}(W_{r,m}) \simeq \pi_{2q-1}(\mathbb{C}^q \setminus \{0\})$ (for compatible orientations) and by this isomorphism we have the equality of integers

$$\mathrm{Ind}(v^{(r)}, \sigma) = \mathrm{Ind}_{Q^q}(v_r, \sigma).$$

A different choice of $v^{(r-1)}$ gives another choices of v_r and of Q^q, however all such bundles Q^q are homotopic and the index we obtain is the same.

Remark 1.3.1. The Chern classes of complex vector bundles in general are defined in essentially the same way as above. If E is a complex vector bundle of rank $k > 0$ over a locally finite simplicial complex B of dimension $n \geq k$, then one has *Chern classes* $c^i(E) \in H^{2i}(B;\mathbb{Z}), i = 1, \ldots, k$. The class $c^i(E)$ is by definition the primary obstruction to constructing $(k-i+1)$ linearly independent sections of E.

The class $c^0(E)$ is defined to be 1 and one has *the total Chern class* of E defined by:

$$c^*(E) = 1 + c^1(E) + \cdots + c^k(E)$$

This can be regarded as an element in the cohomology ring $H^*(B)$ and it is invertible in this ring.

1.3.2 Relative Chern Classes

Suppose now that (L) is a sub-complex of (D) whose geometric realization $|L|$ is also denoted by L. Assume that we are already given an r-frame $v^{(r)}$ on the $2q$-skeleton of L, denoted by $L^{(2q)}$. The same arguments as before say that we can always extend $v^{(r)}$ without singularity to $L^{(2q)} \cup D^{(2q-1)}$. If we wish to extend this frame to the $2q$-skeleton of (D) we meet an obstruction for each corresponding cell which is not in (L). This gives rise to a cochain which vanishes on L and is a cocycle in $H^{2q}(M, L)$.

Definition 1.3.3. The *relative Chern class*

$$c^q(M, L; v^{(r)}) \in H^{2q}(M, L),$$

is the class represented by the previous cocycle.

The image of $c^q(M, L; v^{(r)})$ by the natural map in $H^{2q}(M)$ is the usual Chern class but as a relative class it does depend on the choice of the frame $v^{(r)}$ on L. Let us discuss how the relative Chern class varies as we change the r-frame.

If we have two frames $v_1^{(r)}$ and $v_2^{(r)}$ on $L^{(2q)}$ the difference between the corresponding classes is given by the difference cocycle of the frames on L; in the product $L \times I$, suppose $v_1^{(r)}$ is defined at the level $L \times \{0\}$ and $v_2^{(r)}$ is defined at the level $L \times \{1\}$, then the difference cocycle $d(v_1^{(r)}, v_2^{(r)})$ is well defined in

$$H^{2q}(L \times I, L \times \{0\} \cup L \times \{1\}) \simeq H^{2q-1}(L),$$

as the obstruction to the extension of the given sections on the boundary of $L \times I$ ([153] Sect. 33.3). As shown in [153], we have the following formula:

$$c^q(M, L; v_2^{(r)}) = c^q(M, L; v_1^{(r)}) + \delta d(v_1^{(r)}, v_2^{(r)}),$$

where $\delta : H^{2q-1}(L) \to H^{2q}(M, L)$ is the connecting homomorphism. Also, for three frames $v_1^{(r)}$, $v_2^{(r)}$, and $v_3^{(r)}$ as above, we have

$$d(v_1^{(r)}, v_3^{(r)}) = d(v_1^{(r)}, v_2^{(r)}) + d(v_2^{(r)}, v_3^{(r)}) \tag{1.3.1}$$

For $r = 1$ the frames consist of a single vector field and the difference above corresponds, via Poincaré duality, to the one previously defined for vector fields (cf. 1.1.4).

In the sequel, we will show that the relative Chern class allows us to define Chern class in homology.

Let S be a compact (K)-subcomplex of M, and U a neighborhood of S. Let $\mathcal{T}$ be a cellular tube in U around S. Take an r-field $v^{(r)}$ defined on $D^{(2q)}$, possibly with singularities. We suppose that the only singularities inside U are located in S. This implies that $v^{(r)}$ has no singularities on $(\partial\mathcal{T})^{(2q)}$ so there is a well defined relative Chern class (see 1.3.3)

$$c^q(\mathcal{T}, \partial\mathcal{T}; v^{(r)}) \in H^{2q}(\mathcal{T}, \partial\mathcal{T}).$$

Definition 1.3.4. The *Poincaré–Hopf class* of $v^{(r)}$ at S, which is denoted by $\mathrm{PH}(v^{(r)}, S)$, is the image of $c^q(\mathcal{T}, \partial\mathcal{T}; v^{(r)})$ by the isomorphism $H^{2q}(\mathcal{T}, \partial\mathcal{T}) \simeq H^{2q}(\mathcal{T}, \mathcal{T} \setminus S)$ followed by the Alexander duality (see [25])

$$A_M : H^{2q}(\mathcal{T}, \mathcal{T} \setminus S) \xrightarrow{\sim} H_{2r-2}(S). \tag{1.3.2}$$

For $r = 1$ the frame consists of a single vector field v and the class $\mathrm{PH}(v, S) \in H_0(S)$ is identified with the Poincaré–Hopf index of v at S, $\mathrm{Ind}_{\mathrm{PH}}(v, S)$, previously defined (Definition 1.1.3).

Note that if $\dim S < 2r - 2$, then $\mathrm{PH}(v^{(r)}, S) = 0$.

The relation between the Poincaré–Hopf class of $v^{(r)}$ and the index we defined above is the following:

$$\mathrm{PH}(v^{(r)}, S) = \sum \mathrm{Ind}(v^{(r)}, d(\sigma))\, \sigma\,,$$

where the sum runs over the $2(r-1)$-simplices σ of the triangulation of S and $d(\sigma)$ is the dual cell of σ (of dimension $2q$).

Let us consider now the case of manifolds with boundary. Let M be a compact almost complex $2m$-manifold, with nonempty boundary ∂M. Let (K) be a triangulation of M compatible with ∂M. The union of all "half-cells" dual to simplices in ∂M, denoted by $\mathcal{U}$ is a regular neighborhood of ∂M. Its boundary is denoted by $\partial\mathcal{U}$, which is a union of (D)-cells and is homeomorphic to ∂M. The pair $(M \setminus (\mathrm{Int}\,\mathcal{U}), \partial\mathcal{U})$ is homeomorphic to $(M, \partial M)$ and one can apply the previous construction.

Let $v^{(r)}$ be an r-field on the $(2q)$-skeleton of (D), with singularities located on a compact subcomplex S in $M \setminus (\mathrm{Int}\,\mathcal{U})$. On the $(2q)$-skeleton of $\mathcal{U}$, we have a well defined r-frame $v^{(r)}$. Let $\{S_\lambda\}$ be the connected components of S. Then, by setting $c_{r-1}(M; v^{(r)}) = c^q(M, \partial M; v^{(r)}) \frown [M, \partial M]$, we have

$$\sum_\lambda (i_\lambda)_* \mathrm{PH}(v^{(r)}, S_\lambda) = c_{r-1}(M; v^{(r)}) \qquad \text{in} \quad H_{2r-2}(M), \tag{1.3.3}$$

where $i_\lambda : S_\lambda \hookrightarrow M$ is the inclusion.

In particular, the sum of the Poincaré–Hopf classes is determined by the behavior of $v^{(r)}$ near ∂M and does not depend on the extension to the interior of M. Note that we may assume that $v^{(r)}$ is nonsingular on $D^{(2q-1)}$.

If $r = 1$ and $v^{(1)} = \{v\}$, the relative Chern class is also called the *Euler class of* M *relative to* v and its evaluation on the orientation cycle of $(M, \partial M)$ gives the index of v on M. Thus, if v is everywhere transverse to the boundary, the formula (1.3.3) reduces to (1.1.3).

Remark 1.3.2. In the sequel we often speak of *localizing* Chern classes, which can be done by two different methods: either obstruction theory or Chern–Weil theory. The obstruction theoretical viewpoint comes from the above concept of relative Chern classes: if S is a compact sub-complex of M, U a tubular neighborhood of S, and we are given an r-frame on the intersection with $U \setminus S$ with the appropriate skeleton (for some triangulation or cellular decomposition of M), then the cycle that represents the corresponding Chern class c^q vanishes on ∂U. Hence we have a contribution for c^q that is localized in S, and another contribution in the complement of U. In the following sections the geometric counterpart for making these localizations will be to consider connections that are flat in the linear subspaces determined by the frame. If $r = 1$ and S is a point, the "localization" one gets is simply the contribution to $\chi(M)$ given by the local Poincaré–Hopf index of a vector field at the isolated singularity.

1.4 Chern–Weil Theory of Characteristic Classes

In this section, we briefly review how to define characteristic classes of complex vector bundles using connections. This approach allows us to obtain precise results. If we combine this with the Čech-de Rham cohomology, this method is particularly effective when we deal with the "localization problem."

Let M be a C^∞ manifold of dimension m'. For an open set U in M, we denote by $A^p(U)$ the complex vector space of complex valued C^∞ p-forms on U. Also, for a C^∞ complex vector bundle E of rank k on M, we let $A^p(U, E)$ be the vector space of "E-valued p-forms" on U, *i.e.*, C^∞ sections of the bundle $\bigwedge^p (T^c_{\mathbb{R}} M)^* \otimes E$ on U, where $(T^c_{\mathbb{R}} M)^*$ denotes the dual of the complexification of the real tangent bundle $T_{\mathbb{R}} M$ of M. Thus $A^0(U)$ is the ring of C^∞ functions and $A^0(U, E)$ is the $A^0(U)$-module of C^∞ sections of E on U.

Definition 1.4.1. A *connection* for E is a $\mathbb{C}$-linear map

$$\nabla : A^0(M, E) \longrightarrow A^1(M, E)$$

satisfying the "Leibniz rule":

$$\nabla(fs) = df \otimes s + f\nabla(s) \quad \text{for} \quad f \in A^0(M) \text{ and } s \in A^0(M, E).$$

Example 1.4.1. The exterior derivative

$$d : A^0(M) \longrightarrow A^1(M)$$

is a connection for the trivial line bundle $M \times \mathbb{C}$.

From the definition we have the following:

Lemma 1.4.1. *A connection ∇ is a local operator, i.e., if a section s is identically* 0 *on an open set U, so is $\nabla(s)$.*

Thus the restriction of ∇ to an open set U makes sense and it is a connection for $E|_U$.

Definition 1.4.2. Let ∇ be a connection for E on U. For a nonvanishing section s of E on U, we say that ∇ is *s-trivial*, if $\nabla(s) = 0$. More generally, for an r-frame $\mathbf{s} = (s_1, \dots, s_r)$, ∇ is *$\mathbf{s}$-trivial*, if $\nabla(s_i) = 0,\ i = 1, \dots, r$.

Thus in Example 1.4.1, ∇ is trivial with respect to an arbitrary (nonzero) constant section. From the definition we also have the following lemma.

Lemma 1.4.2. *Let $\nabla_1, \dots, \nabla_\ell$ be connections for E and $f_1, \dots, f_\ell$ C^∞ functions on M with $\sum_{i=1}^{\ell} f_i \equiv 1$. Then $\sum_{i=1}^{\ell} f_i \nabla_i$ is a connection for E.*

One of the consequences of the above lemmas is that every vector bundle admits a connection. This can be shown by taking an open covering $\mathcal{U}$ of M so that E is trivial on each open set in $\mathcal{U}$, choosing a connection on each open set trivial with respect to some frame of E, and then patching them together by a partition of unity subordinate to $\mathcal{U}$.

If ∇ is a connection for E, it induces a $\mathbb{C}$-linear map

$$\nabla : A^1(M, E) \longrightarrow A^2(M, E)$$

satisfying

$$\nabla(\omega \otimes s) = d\omega \otimes s - \omega \wedge \nabla(s) \quad \text{for} \quad \omega \in A^1(M) \text{ and } s \in A^0(M, E).$$

The composition

$$K = \nabla \circ \nabla : A^0(M, E) \longrightarrow A^2(M, E)$$

is called the *curvature* of ∇. It is not difficult to see that

$$K(fs) = fK(s) \quad \text{for} \quad f \in A^0(M) \text{ and } s \in A^0(M, E).$$

The fact that a connection is a local operator allows us to obtain local representations of it and its curvature by matrices whose entries are differential forms. Thus suppose that ∇ is a connection for a vector bundle E of rank k and that E is trivial on U. If $\mathbf{e} = (e_1, \dots, e_k)$ is a frame of E on U, we may write, for $i = 1, \dots, k$,

$$\nabla(e_i) = \sum_{j=1}^{k} \theta_{ji} \otimes e_j, \quad \theta_{ji} \in A^1(U).$$

We call $\theta = (\theta_{ij})$, the matrix whose (i, j) entry is θ_{ij}, the connection matrix of ∇ with respect to $\mathbf{e}$. For an arbitrary section s on U, we may write $s = \sum_{i=1}^{k} f_i e_i$ where the f_i are C^∞ functions on U and we compute

$$\nabla(s) = \sum_{i=1}^{k}(df_i + \sum_{j=1}^{k}\theta_{ij}f_j) \otimes e_i.$$

Note that the connection ∇ is $\mathbf{e}$-trivial if and only if $\theta = 0$. Thus in this case we have $\nabla(s) = \sum_{i=1}^{k} df_i \otimes e_i$. Also, from the definition we get

$$K(e_i) = \sum_{j=1}^{k}\kappa_{ji} \otimes e_j, \qquad \kappa_{ij} = d\theta_{ij} + \sum_{\ell=1}^{k}\theta_{i\ell} \wedge \theta_{\ell j}.$$

We call $\kappa = (\kappa_{ij})$ the curvature matrix of ∇ with respect to $\mathbf{e}$. If $\mathbf{e}' = (e'_1 \ldots, e'_k)$ is another frame of E on U', we have $e'_i = \sum_{j=1}^{k} a_{ji}e_j$ for some C^∞ functions a_{ji} on $U \cap U'$. The matrix $A = (a_{ij})$ is nonsingular at each point of $U \cap U'$. If we denote by θ' and κ' the connection and curvature matrices of ∇ with respect to $\mathbf{e}'$,

$$\theta' = A^{-1} \cdot dA + A^{-1}\theta A \quad \text{and} \quad \kappa' = A^{-1}\kappa A \quad \text{in} \quad U \cap U'. \tag{1.4.1}$$

Let $m = [m'/2]$ and, for each $i = 1, \ldots, m$, let σ_i denote the ith elementary symmetric function in m variables $X_1, \ldots, X_m$, *i.e.*, $\sigma_i(X_1, \ldots, X_m)$ is a polynomial of degree i defined by

$$\prod_{i=1}^{m}(1 + X_i) = 1 + \sigma_1(X_1, \ldots, X_m) + \cdots + \sigma_m(X_1, \ldots, X_m).$$

Since differential forms of even degrees commute with one another with respect to the exterior product, we may treat κ as an ordinary matrix whose entries are numbers. We define a $2i$-form $\sigma_i(\kappa)$ on U by

$$\det(I + \kappa) = 1 + \sigma_1(\kappa) + \cdots + \sigma_m(\kappa),$$

where I denotes the identity matrix of rank k. Note that $\sigma_i(\kappa) = 0$ for $i = k+1, \ldots, m$, and in particular, $\sigma_1(\kappa)$ is the trace $\mathrm{tr}(\kappa)$ and $\sigma_k(\kappa)$ is the determinant $\det(\kappa)$. Although $\sigma_i(\kappa)$ depends on the connection ∇, by (1.4.1), it does not depend on the choice of the frame of E and it defines a global $2i$-form on M, which we denote by $\sigma_i(\nabla)$. It is shown that the form is closed ([75, Ch.3, 3 Lemma], [123, Appendix C, Fundamental Lemma]). We set

$$c^i(\nabla) = \left(\frac{\sqrt{-1}}{2\pi}\right)^i \sigma_i(\nabla)$$

and call it the i-th Chern form.

If we have two connections ∇ and ∇' for E, there is a $(2i-1)$-form $c^i(\nabla, \nabla')$ satisfying

$$c^i(\nabla,\nabla') = -c^i(\nabla',\nabla) \qquad \text{and} \qquad d\,c^i(\nabla,\nabla') = c^i(\nabla') - c^i(\nabla). \tag{1.4.2}$$

In fact the form $c^i(\nabla,\nabla')$ is constructed as follows. We consider the vector bundle $E\times\mathbb{R}\to M\times\mathbb{R}$ and define the connection $\widetilde{\nabla}$ for it by

$$\widetilde{\nabla} = (1-t)\nabla + t\nabla',$$

where t denotes a coordinate on $\mathbb{R}$. Denoting by $[0,1]$ the unit interval and by $\pi : M\times[0,1]\to M$ the projection, we have the integration along the fiber

$$\pi_* : A^{2i}(M\times[0,1]) \longrightarrow A^{2i-1}(M).$$

Then we set

$$c^i(\nabla,\nabla') = \pi_*(c^i(\widetilde{\nabla})). \tag{1.4.3}$$

A similar construction works for an arbitrary collection of finite number of connections and the resulting differential form is called the *Bott difference form* ([19, p. 65]).

From the above, we see that the class $[c^i(\nabla)]$ of the closed $2i$-form $c^i(\nabla)$ in the de Rham cohomology $H^{2i}(M,\mathbb{C})$ depends only on E and not on the choice of the connection ∇. We denote this class by $c^i(E)$ and call it the i-th Chern class $c^i(E)$ of E via the Chern–Weil theory. We call

$$c(E) = 1 + c^1(E) + \cdots + c^k(E)$$

the total Chern class of E, which is considered as an element in the cohomology ring $H^*(M,\mathbb{C})$. Note that the class $c(E)$ is invertible in $H^*(M,\mathbb{C})$.

Remark 1.4.1. 1. It is known (see, e.g., [123]) that the class $c^i(E)$ defined as above is the image of the class $c^i(E)$ in $H^{2i}(M,\mathbb{Z})$ defined via the obstruction theory by the canonical homomorphism

$$H^{2i}(M,\mathbb{Z}) \longrightarrow H^{2i}(M,\mathbb{C}).$$

This fact can also be proved directly using an expression of the mapping degree in terms of connections (see, e.g., [161]).

2. Let H be a hyperplane in the projective space $\mathbb{CP}^m$. For the hyperplane bundle L_H, the line bundle determined by H, we have

$$c(L_H) = 1 + h_m,$$

where h_m denotes the canonical generator of $H^2(\mathbb{CP}^m,\mathbb{C})$ (the Poincaré dual of the homology class $[\mathbb{CP}^{m-1}]$). See Sect. 1.6.4 for the proof of a more precise statement.

More generally, if we have a symmetric polynomial φ, we may write $\varphi = P(\sigma_1, \sigma_2, \dots)$ for some polynomial P. We define, for a connection ∇ for E, the characteristic form $\varphi(\nabla)$ for φ by $\varphi(\nabla) = P(c^1(\nabla), c^2(\nabla), \dots)$, which is a closed form and defines the characteristic class $\varphi(E)$ of E for φ in the de Rham cohomology. We may also define the difference form $\varphi(\nabla, \nabla')$ by a construction similar to the one for the Chern polynomials.

1.5 Čech-de Rham Cohomology

In the subsequent sections, we discuss "localizations of characteristic classes" and for this purpose, the Chern–Weil theory adapted to the Čech-de Rham cohomology is particularly relevant. The Čech-de Rham cohomology is defined for an arbitrary covering of a manifold M, however for simplicity here we only consider coverings of M consisting of two open sets. For details, we refer to [20] and [156].

Let M be a C^∞ manifold of dimension m' and $\mathcal{U} = \{U_0, U_1\}$ an open covering of M. We set $U_{01} = U_0 \cap U_1$. Define the vector space $A^p(\mathcal{U})$ as

$$A^p(\mathcal{U}) = A^p(U_0) \oplus A^p(U_1) \oplus A^{p-1}(U_{01}).$$

Thus an element ξ in $A^p(\mathcal{U})$ is given by a triple $\xi = (\xi_0, \xi_1, \xi_{01})$ with ξ_0 a p-form on U_0, ξ_1 a p-form on U_1 and ξ_{01} a $(p-1)$-form on U_{01}.

We define the operator $D : A^p(\mathcal{U}) \to A^{p+1}(\mathcal{U})$ by

$$D\xi = (d\xi_0, d\xi_1, \xi_1 - \xi_0 - d\xi_{01}).$$

Then it is not difficult to see that $D \circ D = 0$. This allows us to define a cohomological complex, *the Čech-de Rham complex* :

$$\cdots \longrightarrow A^{p-1}(\mathcal{U}) \xrightarrow{D^{(p-1)}} A^p(\mathcal{U}) \xrightarrow{D^{(p)}} A^{p+1}(\mathcal{U}) \longrightarrow \cdots$$

Set $Z^p(\mathcal{U}) = \mathrm{Ker} D^p$, $B^p(\mathcal{U}) = \mathrm{Im} D^{p-1}$ and

$$H^p_D(\mathcal{U}) = Z^p(\mathcal{U})/B^p(\mathcal{U}),$$

which is called the p-th Čech-de Rham cohomology of $\mathcal{U}$. We denote the image of ξ by the canonical surjection $Z^p(\mathcal{U}) \to H^p_D(\mathcal{U})$ by $[\xi]$.

Theorem 1.5.1. *The map $A^p(M) \to A^p(\mathcal{U})$ given by $\omega \mapsto (\omega, \omega, 0)$ induces an isomorphism*

$$\alpha : H^p_{dR}(M) \xrightarrow{\sim} H^p_D(\mathcal{U}).$$

Proof. It is not difficult to show that α is well-defined. To prove that α is surjective, let $\xi = (\xi_0, \xi_1, \xi_{01})$ be such that $D\xi = 0$. Let $\{\rho_0, \rho_1\}$ be a partition of unity subordinated to the covering $\mathcal{U}$. Define $\omega = \rho_0\xi_0 + \rho_1\xi_1 - d\rho_0 \wedge \xi_{01}$. Then it is easy to see that $d\omega = 0$ and $[(\omega, \omega, 0)] = [\xi]$. The injectivity of α is not difficult to show.

We define the "cup product"

$$A^p(\mathcal{U}) \times A^q(\mathcal{U}) \longrightarrow A^{p+q}(\mathcal{U})$$

by assigning to ξ in $A^p(\mathcal{U})$ and η in $A^q(\mathcal{U})$ the element $\xi \smile \eta$ in $A^{p+q}(\mathcal{U})$ given by

$$(\xi \smile \eta)_i = \xi_i \wedge \eta_i,\ i = 0,\ 1, \quad (\xi \smile \eta)_{01} = (-1)^p \xi_0 \wedge \eta_{01} + \xi_{01} \wedge \eta_1. \quad (1.5.2)$$

Then we have $D(\xi \smile \eta) = D\xi \smile \eta + (-1)^p \xi \smile D\eta$. Thus it induces the cup product

$$H^p_D(\mathcal{U}) \times H^q_D(\mathcal{U}) \longrightarrow H^{p+q}_D(\mathcal{U})$$

compatible, via the isomorphism of 1.5.1, with the cup product in the de Rham cohomology.

1.5.1 Integration on the Čech-de Rham Cohomology

Now we recall the integration on the Čech-de Rham cohomology (cf. [109]). Suppose that the m'-dimensional manifold M is oriented and compact and let $\mathcal{U} = \{U_0, U_1\}$ be a covering of M. Let $R_0, R_1 \subset M$ be two compact manifolds of dimension m' with C^∞ boundary with the following properties:
(1) $R_j \subset U_j$ for $j = 0, 1$,
(2) $\mathrm{Int}R_0 \cap \mathrm{Int}R_1 = \emptyset$ and
(3) $R_0 \cup R_1 = M$.

Let $R_{01} = R_0 \cap R_1$ and give R_{01} the orientation as the boundary of R_0; $R_{01} = \partial R_0$, equivalently give R_{01} the orientation opposite to that of the boundary of R_1; $R_{01} = -\partial R_1$. We define the integration

$$\int_M : A^{m'}(\mathcal{U}) \longrightarrow \mathbb{C} \quad \text{by} \quad \int_M \xi = \int_{R_0} \xi_0 + \int_{R_1} \xi_1 + \int_{R_{01}} \xi_{01}.$$

Then by the Stokes theorem, we see that if $D\xi = 0$ then $\int_M \xi$ is independent of $\{R_0, R_1\}$ and that if $\xi = D\eta$ for some $\eta \in A^{p-1}(\mathcal{U})$ then $\int_M \xi = 0$. Thus we may define the integration

$$\int_M : H^{m'}_D(\mathcal{U}) \longrightarrow \mathbb{C},$$

which is compatible with the integration on the de Rham cohomology via the isomorphism of 1.5.1.

1.5.2 Relative Čech-de Rham Cohomology – Alexander Duality

Next we define the relative Čech-de Rham cohomology and describe the Alexander duality. Let M be an m'-dimensional oriented manifold (not necessarily compact) and S a compact subset of M. Let $U_0 = M \setminus S$ and let U_1 be an open neighborhood of S. We consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. We set

$$A^p(\mathcal{U}, U_0) = \{\, \xi = (\xi_0, \xi_1, \xi_{01}) \in A^p(\mathcal{U}) \mid \xi_0 = 0 \,\}.$$

Then we see that if ξ is in $A^p(\mathcal{U}, U_0)$, $D\xi$ is in $A^{p+1}(\mathcal{U}, U_0)$. This gives rise to another complex, called the relative Čech-de Rham complex, and we may define the p-th relative Čech-de Rham cohomology of the pair $(\mathcal{U}, U_0)$ as

$$H^p_D(\mathcal{U}, U_0) = \mathrm{Ker} D^p / \mathrm{Im} D^{p-1}.$$

By the five lemma, we see that there is a natural isomorphism

$$H^p_D(\mathcal{U}, U_0) \simeq H^p(M, M \setminus S; \mathbb{C}).$$

Let R_1 be a compact manifold of dimension m' with C^∞ boundary such that $S \subset \mathrm{Int} R_1 \subset R_1 \subset U_1$. Let $R_0 = M \setminus \mathrm{Int} R_1$. Note that $R_0 \subset U_0$. The integral operator $\int_M$ (which is not defined in general for $A^{m'}(\mathcal{U})$ unless M is compact) is well defined on $A^{m'}(\mathcal{U}, U_0)$:

$$\int_M : A^{m'}(\mathcal{U}, U_0) \longrightarrow \mathbb{C}, \qquad \int_M \xi = \int_{R_1} \xi_1 + \int_{R_{01}} \xi_{01},$$

and induces an operator $\int_M : H^{m'}_D(\mathcal{U}, U_0) \to \mathbb{C}$.

In the cup product $A^p(\mathcal{U}) \times A^{m'-p}(\mathcal{U}) \to A^{m'}(\mathcal{U})$ given as (1.5.2), we see that if $\xi_0 = 0$, the right hand side depends only on ξ_1, ξ_{01}, and η_1. Thus we have a pairing $A^p(\mathcal{U}, U_0) \times A^{m'-p}(U_1) \to A^{m'}(\mathcal{U}, U_0)$, which, followed by the integration, gives a bilinear pairing

$$A^p(\mathcal{U}, U_0) \times A^{m'-p}(U_1) \longrightarrow \mathbb{C}.$$

If we further assume that U_1 is a regular neighborhood of S, this induces the Alexander duality (cf 1.3.2 and [25])

$$A : H^p(M, M \setminus S; \mathbb{C}) \simeq H^p_D(\mathcal{U}, U_0) \xrightarrow{\sim} H^{m'-p}(U_1, \mathbb{C})^* \simeq H_{m'-p}(S, \mathbb{C}). \tag{1.5.3}$$

Proposition 1.5.1. *[25] If M is compact, we have the commutative diagram*

$$\begin{array}{ccc} H^p(M, M \setminus S; \mathbb{C}) & \xrightarrow{j^*} & H^p(M, \mathbb{C}) \\ \wr\downarrow A & & \wr\downarrow P \\ H_{m'-p}(S, \mathbb{C}) & \xrightarrow{i_*} & H_{m'-p}(M, \mathbb{C}), \end{array}$$

where i and j denote, respectively, the inclusions $S \hookrightarrow M$ and $(M, \emptyset) \hookrightarrow (M, M \setminus S)$.

We finish this section by giving a fundamental example of computation of relative Čech-de Rham cohomology.

Example 1.5.1. Let $M = \mathbb{R}^{m'}$ and $S = \{0\}$ with $m' \geq 2$. In this case, $U_0 = \mathbb{R}^{m'} \setminus \{0\}$, which retracts to $\mathbb{S}^{m'-1}$. Let $U_1 = \mathbb{R}^{m'}$. In this situation, we compute $H^p_D(\mathcal{U}, U_0)$. For $p = 0$, each element ξ in $A^0(\mathcal{U}, U_0)$ can be written as $\xi = (0, f, 0)$ for some C^∞ function f on U_1. If $D\xi = 0$, we have $f \equiv 0$ and therefore $H^0_D(\mathcal{U}, U_0) = \{0\}$. Next, an element ξ in $A^1(\mathcal{U}, U_0)$ can be written as $\xi = (0, \xi_1, f)$ with ξ_1 a 1-form on U_1 and f a C^∞ function on $U_0 \cap U_1$. If ξ is a cocycle then $d\xi_1 = 0$ on U_1 and $df = \xi_1$ on $U_0 \cap U_1$. By the Poincaré lemma the first condition implies that $\xi_1 = dg$ for some C^∞ function g on U_1 and the second condition implies that $f \equiv g + c$ for some $c \in \mathbb{C}$. Therefore f has a C^∞ extension, still denoted by f, over $\{0\}$ and $\xi = (0, df, f) = D(0, f, 0)$. Hence $H^1_D(\mathcal{U}, U_0) = \{0\}$. For $p \geq 2$ the map

$$H^{p-1}_{dR}(U_0) \longrightarrow H^p_D(\mathcal{U}, U_0) \qquad \text{given by} \quad [\omega] \mapsto [(0, 0, -\omega)]$$

can be shown to be an isomorphism (we leave the details to the reader) and we have

$$H^p_D(\mathcal{U}, U_0) \simeq H^{p-1}_{dR}(U_0) \simeq H^{p-1}(\mathbb{S}^{m'-1}) = \begin{cases} \mathbb{C}, & \text{for} \quad p = m', \\ 0, & \text{for} \quad p = 2, \dots, m' - 1. \end{cases}$$

An explicit generator of $H^{m'-1}(\mathbb{S}^{m'-1})$ is given as follows ([75, p. 370]). For $x = (x_1, \dots, x_{m'})$ in $\mathbb{R}^{m'}$, we set $\Phi(x) = dx_1 \wedge \cdots \wedge dx_{m'}$ and

$$\Phi_i(x) = (-1)^{i-1} x_i \, dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{m'}.$$

Also, let $C_{m'}$ be the constant given by

$$C_{m'} = \begin{cases} \frac{(\ell-1)!}{2\pi^\ell}, & \text{for } m' = 2\ell \\ \frac{(2\ell)!}{2^{2\ell+1}\pi^\ell \ell!}, & \text{for } m' = 2\ell+1. \end{cases}$$

Then the form

$$\psi_{m'} = C_{m'} \frac{\sum_{i=1}^{m'} \Phi_i(x)}{\|x\|^{m'}}$$

is a closed $(m'-1)$-form on $\mathbb{R}^{m'} \setminus 0$ whose integral on the unit sphere $\mathbb{S}^{m'-1}$ (in fact a sphere of arbitrary radius) is 1. Now we identify $\mathbb{C}^m$ with $\mathbb{R}^{2m}$, then $\psi_{2m} = (\beta_m + \overline{\beta_m})/2$, where

$$\beta_m = C'_m \frac{\sum_{i=1}^{m} \overline{\Phi_i(z)} \wedge \Phi(z)}{\|z\|^{2m}}, \qquad C'_m = (-1)^{\frac{m(m-1)}{2}} \frac{(m-1)!}{(2\pi\sqrt{-1})^m}. \tag{1.5.4}$$

Then β_m is a closed $(m, m-1)$-form on $\mathbb{C}^m \setminus 0$, real on $\mathbb{S}^{2m-1}$ and $\int_{\mathbb{S}^{2m-1}} \beta_m = 1$. We call β_m the Bochner–Martinelli kernel on $\mathbb{C}^m$. Note that

$$\beta_1 = \frac{1}{2\pi\sqrt{-1}} \frac{dz}{z},$$

is the Cauchy kernel on $\mathbb{C}$.

1.6 Localization of Chern Classes

In a previous section we described the topological viewpoint for localizing Chern classes on a given compact subset S of a manifold M, taking an appropriate frame in the appropriate skeleton of a neighborhood of S. This gives an explicit representative of the Chern class which represents it as a relative cohomology class, with a specific contribution localized at S. We also know (see for instance [14,19,123]) that Chern classes of manifolds and vector bundles in general can be defined via Chern–Weil theory, using the curvature tensor of a connection. To describe the localization of Chern classes, we modify the Chern–Weil theory so that it is adapted to the Čech-de Rham cohomology.

1.6.1 Characteristic Classes in the Čech-de Rham Cohomology

Let M be a C^∞ manifold and $\mathcal{U} = \{U_0, U_1\}$ an open covering of M. For a vector bundle E over M, we take a connection ∇_j on U_j, $j = 0, 1$, and let $c^i(\nabla_*)$ be the element of $A^{2i}(\mathcal{U})$ given by

$$c^i(\nabla_*) = (c^i(\nabla_0), c^i(\nabla_1), c^i(\nabla_0, \nabla_1)). \tag{1.6.1}$$

Then we see that $Dc^i(\nabla_*) = 0$ and this defines a class $[c^i(\nabla_*)]$ in $H^{2i}_D(\mathcal{U})$. It is not difficult to show the following

Theorem 1.6.2. *The class $[c^i(\nabla_*)] \in H^{2i}_D(\mathcal{U})$ corresponds to the Chern class $c^i(E) \in H^{2i}_{dR}(M)$ under the isomorphism of Theorem 1.5.1.*

By a similar construction, we may define the characteristic class $\varphi(E)$ for a polynomial φ in the Chern polynomials in the Čech-de Rham cohomology. It can be done also for virtual bundles (see Chap. 5).

Using Bott difference forms, we may define characteristic classes in the Čech-de Rham cohomology for an arbitrary open covering of M.

This way of representing characteristic classes is particularly useful in dealing with the “localization problem,” which we explain in the next subsection. This theory involves vanishing theorems, one of which is given as follows.

Let E be a complex vector bundle of rank k on a C^∞ manifold M. Let $\mathbf{s} = (s_1, \dots, s_r)$ be an r-frame of E on an open set U. Recall that (Definition 1.4.2) a connection ∇ for E on U is $\mathbf{s}$-trivial, if $\nabla(s_i) = 0$ for $i = 1, \dots, r$.

Proposition 1.6.1. *If ∇ is $\mathbf{s}$-trivial, then*

$$c^j(\nabla) \equiv 0 \qquad \textit{for} \quad j \geq k - r + 1.$$

Proof. For simplicity, we prove the proposition when $r = 1$. Let $U \subset M$ be an open set such that $E|_U \simeq U \times \mathbb{C}^k$. Since $s_1 \neq 0$ everywhere on M, we may take a frame $\mathbf{e} = (e_1, \dots, e_k)$ on U so that $e_1 = s_1$. Then all the entries of the first row of the curvature matrix κ of ∇ with respect to $\mathbf{e}$ are zero. Since $c^k(\nabla) = \det \kappa$, up to a constant, we have $c^k(\nabla) = 0$.

1.6.2 Localization of Characteristic Classes of Complex Vector Bundles

In this subsection, we explain how we obtain indices and residues in the subsequent sections.

Let M be an oriented C^∞ manifold of dimension m' and E a C^∞ complex vector bundle of rank k over M. Also, let S be a closed set in M and U_1 a neighborhood of S in M. Setting $U_0 = M \setminus S$, we consider the covering $\mathcal{U} = \{U_0, U_1\}$ of M. For a homogeneous symmetric polynomial φ of degree d, the characteristic class $\varphi(E)$ is represented by the cocycle $\varphi(\nabla_*)$ in $A^{2d}(\mathcal{U})$ given by

$$\varphi(\nabla_*) = (\varphi(\nabla_0), \varphi(\nabla_1), \varphi(\nabla_0, \nabla_1)),$$

where ∇_0 and ∇_1 denote connections for E on U_0 and U_1 respectively. Sometimes, it happens that we have a “vanishing theorem” on U_0 for some polynomials φ. Namely, there is some "geometric object" γ on U_0, to which is associated a class $\mathcal{C}$ of connections for E on U_0 such that, for a connection ∇_0 belonging to $\mathcal{C}$ and for a certain polynomial φ, we have

$$\varphi(\nabla_0) \equiv 0.$$

We call a connection belonging to $\mathcal{C}$ *special* and a polynomial φ as above *adapted* to γ. As we see below, this kind of vanishing defines a localization of the relevant characteristic class. Moreover, if we have also the vanishing of the Bott difference forms for families of special connections, we may show that the localization does not depend on the connections involved. This is the case in all the cases we consider below and we assume this hereafter.

Thus, if ∇_0 is special and if φ is adapted to γ, then the above cocycle $\varphi(\nabla_*)$ is in $A^{2d}(\mathcal{U}, U_0)$ and it defines a class in $H^{2d}(M, M \setminus S; \mathbb{C})$, which is denoted by $\varphi_S(E, \gamma)$. It is sent to the class $\varphi(E)$ by the canonical homomorphism $j^* : H^{2d}(M, M \setminus S; \mathbb{C}) \to H^{2d}(M, \mathbb{C})$. It is not difficult to see that the class $\varphi_S(E, \gamma)$ does not depend on the choice of the special connection ∇_0 or the connection ∇_1.

We call $\varphi_S(E, \gamma)$ the *localization of* $\varphi(E)$ *at* S *by* γ. Suppose S is a compact set admitting a regular neighborhood. Then we have the Alexander duality (1.5.3)

$$A : H^{2d}(M, M \setminus S; \mathbb{C}) \xrightarrow{\sim} H_{m'-2d}(S, \mathbb{C}).$$

Thus the class $\varphi_S(E, \gamma)$ defines a class in $H_{m'-2d}(S, \mathbb{C})$, which we call the residue of γ for the class $\varphi(E)$ at S and denote by $\mathrm{Res}_\varphi(\gamma, E; S)$.

We suppose that U_1 is a regular neighborhood and let R_1 be an m'-dimensional manifold with C^∞ boundary in U_1 containing S in its interior and we set $R_{01} = -\partial R_1$. Then the residue $\mathrm{Res}_\varphi(\gamma, E; S)$ is represented by an $(m' - 2d)$-cycle C in S such that

$$\int_C \eta = \int_{R_1} \varphi(\nabla_1) \wedge \eta + \int_{R_{01}} \varphi(\nabla_0, \nabla_1) \wedge \eta \tag{1.6.3}$$

for every closed $(m' - 2d)$-form η on U_1. In particular, if $2d = m'$, the residue is a complex number given by

$$\mathrm{Res}_\varphi(\gamma, E; S) = \int_{R_1} \varphi(\nabla_1) + \int_{R_{01}} \varphi(\nabla_0, \nabla_1). \tag{1.6.4}$$

Suppose moreover that S has a finite number of connected components $(S_\lambda)_\lambda$. Then we have a decomposition

$$H_{m'-2d}(S, \mathbb{C}) = \bigoplus_\lambda H_{m'-2d}(S_\lambda, \mathbb{C})$$

and accordingly, we have the residue $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ in $H_{m'-2d}(S_\lambda, \mathbb{C})$ for each λ. Replacing U_1 by a regular neighborhood U_λ of S_λ, disjoint from the other components, and R_1 by an m'-dimensional manifold R_λ with boundary in U_λ containing S_λ in its interior, we have an expression $(1.6.3)_\lambda$ or $(1.6.4)_\lambda$ for the residue $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ similar to (1.6.3) or (1.6.4).

From the above considerations and Proposition 1.5.1, we have the following "residue theorem."

Theorem 1.6.5. *In the above situation,*

(1) *For each connected component S_λ of S, we have the residue* $\mathrm{Res}_\varphi(\gamma, E; S_\lambda)$ *in the homology $H_{m'-2d}(S_\lambda, \mathbb{C})$, which is determined by the local behavior of γ near S_λ and is expressed as $(1.6.3)_\lambda$ or $(1.6.4)_\lambda$.*

(2) *If M is compact,*

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_\varphi(\gamma, E; S_\lambda) = \varphi(E) \frown [M] \qquad \text{in} \quad H_{m'-2d}(M, \mathbb{C}),$$

where $i_\lambda : S_\lambda \hookrightarrow M$ denotes the inclusion.

Remark 1.6.1. If $2d = m'$, we do not have to assume that S admits a regular neighborhood. Simply take an arbitrary open neighborhood as U_1 and define $\mathrm{Res}_\varphi(\gamma, E; S)$ by (1.6.4) with R_1 as above, then Theorem 1.6.5 is still valid.

1.6.3 Localization of the Top Chern Class

Let E be a C^∞ complex vector bundle of rank k over an oriented C^∞ manifold M of dimension m'. Let s be a nonvanishing section of E on some open set U. Recall that a connection ∇ for E on U is s-trivial, if $\nabla(s) = 0$. If ∇ is an s-trivial connection, we have the vanishing (Proposition 1.6.1)

$$c^k(\nabla) = 0. \tag{1.6.6}$$

Let S be a closed set in M and suppose we have a C^∞ nonvanishing section s of E on $M \setminus S$. Then, from the above fact, applying the arguments in Sect. 1.6.2 taking c^k as φ and s-trivial connections as special connections, we see that there is a natural lifting $c^k(E, s)$ in $H^{2k}(M, M \setminus S; \mathbb{C})$ of the top Chern class $c^k(E)$ in $H^{2k}(M, \mathbb{C})$. We call $c^k(E, s)$ the localization of $c^k(E)$ with respect to the section s at S.

Also, if S is a compact set admitting a regular neighborhood, the class $c^k(E, s)$ defines a class in $H_{m'-2k}(S, \mathbb{C})$, which we call the residue of s for E at S with respect to c^k and denote by $\mathrm{Res}_{c^k}(s, E; S)$. This residue corresponds to what is called the "localized top Chern class" of E with respect to s in [59, Sect. 14.1].

The residue $\mathrm{Res}_{c^k}(s, E; S)$ is represented by an $(m' - 2k)$-cycle C in S_λ satisfying (1.6.3). In particular, if $2k = m'$, the residue is a complex number given by (1.6.4) with $\varphi = c^k$. If S has a finite number of connected components $(S_\lambda)_\lambda$, we have the residue $\mathrm{Res}_{c^k}(s, E; S_\lambda)$ in $H_{m'-2k}(S_\lambda, \mathbb{C})$ for each λ. Moreover, Theorem 1.6.5 becomes

Theorem 1.6.7. *In the above situation,*

(1) *For each connected component S_λ of S, we have the residue* $\mathrm{Res}_{c^k}(s, E; S_\lambda)$ *in the homology $H_{m'-2k}(S_\lambda, \mathbb{C})$.*

(2) *If M is compact,*

$$\sum_\lambda (i_\lambda)_* \mathrm{Res}_{c^k}(s, E; S_\lambda) = c^k(E) \frown [M] \qquad \text{in} \;\; H_{m'-2k}(M, \mathbb{C}).$$

Remark 1.6.2. 1. In fact it can be shown that the above residues are in the integral homology and the equality in Theorem 1.6.7 holds in the integral homology (see [161]).
2. A localization theory of Chern classes, other than the top one, by a finite number of sections can be developed similarly (see [159–161]).

1.6.4 Hyperplane Bundle

As a basic example of the theory developed in the previous subsections, we prove that the Poincaré dual of the first Chern of the hyperplane bundle L_H on a projective space is (the homology class of) the hyperplane H. In fact, we prove a more precise statement that the Alexander dual of the localization of the first Chern of L_H by the canonical section is the fundamental class of H in the homology of H. Note that the essential point in the proof is the Cauchy integral formula; $\frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{dz}{z} = 1$.

Let $\mathbb{CP}^m$ be the m-dimensional complex projective space with homogeneous coordinates $[\zeta_0, \dots, \zeta_m]$. We denote by W_i the open set in $\mathbb{CP}^m$ defined by $\zeta_i \neq 0$, $i = 0, \dots, m$. Let H denote the hyperplane defined by $\zeta_0 = 0$ and L_H the line bundle determined by H. Recall that L_H is defined by the system of transition functions h_{ij}, $h_{ij} = \zeta_j/\zeta_i$. The canonical section s is represented by the collection (s_i), where s_i is a holomorphic function on W_i given by $s_i = \zeta_0/\zeta_i$. Since the zero set of s is H, we have the localization $c^1(L_H, s)$ of $c^1(L_H)$ in $H^2(\mathbb{CP}^m, \mathbb{CP}^m \setminus H)$.

Theorem 1.6.8. *The image of $c^1(L_H, s)$ by the Alexander isomorphism*

$$H^2(\mathbb{CP}^m, \mathbb{CP}^m \setminus H) \xrightarrow{\sim} H_{2m-2}(H)$$

is the fundamental class $[H]$, i.e., $\mathrm{Res}_{c^1}(s, L_H; H) = [H]$.

Proof. Let $\mathcal{U} = \{U_0, U_1\}$ be the covering of $\mathbb{CP}^m$ consisting of $U_0 = \mathbb{CP}^m \setminus H$ and a tubular neighborhood U_1 of H with a C^∞ retraction $\rho : U_1 \to H$. Let ∇_0 be an s-trivial connection for L_H on U_0 so that $c^1(\nabla_0) = 0$ and ∇_1 an arbitrary connection for L_H on U_1. Then the class $c^1(L_H, s)$ is represented by the cocycle $(0, c^1(\nabla_1), c^1(\nabla_0, \nabla_1))$ in $A^2(\mathcal{U}, U_0)$. Let R_1 be a closed tubular neighborhood of H in U_1 and $R_{01} = -\partial R_1$. Our aim is to show that (cf. (1.6.3))

$$\int_H \eta = \int_{R_1} c^1(\nabla_1) \wedge \eta + \int_{R_{01}} c^1(\nabla_0, \nabla_1) \wedge \eta \tag{1.6.9}$$

for every closed $(2m-2)$-form η on U_1.

Since the retraction map ρ induces an isomorphism $\rho^* : H^{2m-2}_{dR}(H) \xrightarrow{\sim} H^{2m-2}_{dR}(U_1)$, we see that there exist a closed $(2m-2)$-form θ on H and a $(2m-3)$-form τ on U_1 with $\eta = \rho^*\theta + d\tau$. By the Stokes theorem and the property of the difference form $c^1(\nabla_0, \nabla_1)$, we see that it suffices to prove (1.6.9) for $\eta = \rho^*\theta$. For the left hand side, we have $\int_H \rho^*\theta = \int_H \theta$. To compute the right hand side, we note that $L_H|_{U_1} \simeq \rho^*(L_H|_H)$. Let ∇ be a connection for $L_H|_H$ and take as ∇_1 the connection corresponding to $\rho^*\nabla$. Then we have $c^1(\nabla_1) \wedge \rho^*\theta = \rho^*(c^1(\nabla) \wedge \theta) = 0$, since $c^1(\nabla) \wedge \theta$ is a $2m$-form on H. In the second term of the right hand side, R_{01} is an S^1 bundle over H with the orientation opposite to the natural one. Let ρ_{01} denote the restriction of ρ to R_{01}. Then by the projection formula, we have

$$\int_{R_{01}} c^1(\nabla_0, \nabla_1) \wedge \rho^*\theta = -\int_H (\rho_{01})_* c^1(\nabla_0, \nabla_1) \cdot \theta,$$

where $(\rho_{01})_*$ denotes the integration along the fiber of ρ_{01} so that the form $(\rho_{01})_* c^1(\nabla_0, \nabla_1)$ is in fact a function on H. It suffices to prove that this function is identically equal to -1. Let p be an arbitrary point in H and suppose it is in W_i, $i \neq 0$. In the sequel, we identify $L_H|_{W_i}$ with $W_i \times \mathbb{C}$. On W_i, the section s is represented by the function $s_i = \zeta_0/\zeta_i$, which can also be thought of as a fiber coordinate of the retraction ρ. Let ∇' denote the connection for $L_H|_H$ on $W_i \cap H$ trivial with respect to the frame ℓ given by $\ell(q) = (q, 1)$ for q in $W_i \cap H$. We may modify ∇' away from a neighborhood of p to obtain a connection ∇ for $L_H|_H$ on H. The pullback $\nabla_1 = \rho^*\nabla$ is a connection for L_H which is trivial with respect to the frame ℓ_1 given by $\ell_1(q) = (q, 1)$ for q in a neighborhood W of p in W_i. Now we try to find $c^1(\nabla_0, \nabla_1)$ on $W \cap U_{01} = W \setminus H$ (cf. (1.4.3)). For this, let $\tilde{\nabla}$ be the connection for $L_H \times \mathbb{R}$ given by $\tilde{\nabla} = (1-t)\nabla_0 + t\nabla_1$. Let θ_i be the connection form of ∇_i with respect to the frame ℓ_1, $i = 0, 1$. Then $\theta_1 = 0$ and, since θ_0 is s-trivial and $\ell_1 = \frac{1}{z}s$, $z = \zeta_0/\zeta_i$, by (1.4.1), we have $\theta_0 = d\left(\frac{1}{z}\right)/\frac{1}{z} = -\frac{dz}{z}$. Thus the connection form $\tilde{\theta}$ of $\tilde{\nabla}$ is given by

$$\tilde{\theta} = -(1-t)\frac{dz}{z}.$$

Hence the curvature form $\tilde{\kappa}$ of $\tilde{\nabla}$ is given by $\tilde{\kappa} = d\tilde{\theta} = dt \wedge \frac{dz}{z}$ and we get

$$c^1(\nabla_0, \nabla_1) = \frac{\sqrt{-1}}{2\pi}\pi_*\tilde{\kappa} = -\frac{1}{2\pi\sqrt{-1}}\frac{dz}{z},$$

where π_* denotes the integration along the fiber of the projection map $\pi : W \setminus H \times [0,1] \to W \setminus H$. Therefore, by the Cauchy integral formula, we have

$$(\rho_{01})_* c^1(\nabla_0, \nabla_1) = -1$$

in a neighborhood of p.

See [157] and [161] for more general results and thorough discussions in this direction.

1.6.5 Grothendieck Residues

As we have seen in the previous subsection and will see also in the sequel, the residues of characteristic classes are deeply related to Grothendieck residues. In this subsection, we briefly review this subject. For details, we refer to, e.g., [75].

Let U be a neighborhood of the origin 0 in $\mathbb{C}^m$ and $f_1, \dots, f_m$ holomorphic functions on U such that their common set of zeros consists only of 0. For a holomorphic m-form ω on U, we set

$$\operatorname{Res}_0 \begin{bmatrix} \omega \\ f_1, \dots, f_m \end{bmatrix} = \frac{1}{(2\pi\sqrt{-1})^m} \int_{\Gamma} \frac{\omega}{f_1 \cdots f_m}, \qquad (1.6.10)$$

where Γ is an m-cycle in U defined by

$$\Gamma = \{ z \in U \mid |f_1(z)| = \cdots = |f_m(z)| = \varepsilon \}$$

for a small positive number ε. We orient Γ so that the form $d\theta_1 \wedge \cdots \wedge d\theta_m$ is positive, $\theta_i = \arg f_i$.

Example 1.6.1. If $m = 1$, the above residue 1.6.10 is the usual Cauchy residue at 0 of the meromorphic 1-form ω/f_1.

Example 1.6.2. In the next subsection, we give various expressions for the residue of the top Chern class at an isolated singularity of a section s. If $(f_1, \dots, f_m)$ denote local components of s around the singularity, the Grothendieck residue with $\omega = df_1 \wedge \cdots \wedge df_m$ appears as an "analytic expression" of the residue. Thus we have

$$\mathrm{Res}_0\begin{bmatrix} df_1\wedge\cdots\wedge df_m \\ f_1,\dots,f_m\end{bmatrix} = \dim_{\mathbb{C}}\mathcal{O}_m/(f_1,\dots,f_m) = \mathrm{Ind}_{\mathrm{PH}}(v,0), \qquad (1.6.11)$$

where v denotes the holomorphic vector field $\sum_{i=1}^m f_i\cdot\partial/\partial z_i$. This positive integer is also interpreted as the intersection number $(D_1\cdots D_m)_0$ at 0 of the divisors D_i defined by f_i (cf. [75, Ch.5, 2], [157]).

Example 1.6.3. In particular, if $f_i = \partial f/\partial z_i$ for some f in $\mathcal{O}_m$, then the residue is the Milnor number $\mu(V,0)$ of the hypersurface V defined by f at 0;

$$\mathrm{Res}_0\begin{bmatrix} d\left(\frac{\partial f}{\partial z_1}\right)\wedge\cdots\wedge d\left(\frac{\partial f}{\partial z_m}\right) \\ \frac{\partial f}{\partial z_1},\dots,\frac{\partial f}{\partial z_m}\end{bmatrix} = \mu(V,0).$$

We also call this number the multiplicity of f at 0 and denote it by $m(f,0)$ (cf. Sect. 1.6.7-b below).

1.6.6 Residues at an Isolated Zero

Let E be a holomorphic vector bundle of rank m over a complex manifold M of dimension m. Suppose we have a section s with an isolated zero at p in M. In this situation, we have $\mathrm{Res}_{c^m}(s,E;p)$ in $H_0(\{p\},\mathbb{C})=\mathbb{C}$. In the following, we give explicit expressions of this residue.

Let U be an open neighborhood of p where the bundle E is trivial with holomorphic frame $(e_1,\dots,e_m)$. We write $s=\sum_{i=1}^m f_i\,e_i$ with f_i holomorphic functions on U.

(I) Analytic expression

Theorem 1.6.12. *In the above situation, we have*

$$\mathrm{Res}_{c^m}(s,E;p) = \mathrm{Res}_p\begin{bmatrix} df_1\wedge\cdots\wedge df_m \\ f_1,\dots,f_m\end{bmatrix}.$$

Proof. We indicate the proof for the case $m=1$ (for $m>1$, we use the Čech-de Rham cohomology theory for m open sets, see [157], [160]). Thus $s=fe_1$ for some holomorphic function f on U. Let R be a closed disk about p in U. In the expression (1.6.4) of the residue, we may take as ∇_1 an e_1-trivial connection on U, thus $c^1(\nabla_1)\equiv 0$ and

$$\mathrm{Res}_p(s,E;p) = -\int_{\partial R} c^1(\nabla_0,\nabla_1)$$

with ∇_0 an s-trivial connection on $U'=U\setminus\{p\}$. The Bott difference form $c^1(\nabla_0,\nabla_1)$ can be computed as in the proof of Theorem 1.6.8. If we let θ_i be the connection matrix of ∇_i, $i = 0, 1$, with respect to the frame e_1, we have $\theta_1 = 0$ and $\theta_0 = -\frac{df}{f}$. Thus this time we have

$$c^1(\nabla_0, \nabla_1) = -\frac{1}{2\pi\sqrt{-1}}\frac{df}{f},$$

which proves the theorem (for the case $m = 1$).

Remark 1.6.3. For general m, if we take suitable connections we see that the difference form is given by

$$c^m(\nabla_0, \nabla_1) = -f^*\beta_m,$$

where $f = (f_1, \dots, f_m)$ and β_m denotes the Bochner–Martinelli kernel on $\mathbb{C}^m$ (cf. (1.5.4)). This gives a direct proof of Theorem 1.6.14 below. Thus we reprove the fact that the Grothendieck residue in the above theorem is equal to the mapping degree of f (cf. [75, Ch.5, 1. Lemma]).

(II) Algebraic expression

Theorem 1.6.13. *In the above situation, we have*

$$\mathrm{Res}_{c^m}(s, E; p) = \dim \mathcal{O}_m/(f_1, \dots, f_m).$$

This can be proved, for example, by perturbing the sections and using the theory of Cohen–Macaulay rings (e.g., [160]).

(III) Topological Expression

Let $\mathbb{S}^{2m-1}_\varepsilon$ denote a small $2m-1$ sphere in U with center p. Then we have the mapping

$$\varphi = \frac{f}{\|f\|} : \mathbb{S}^{2m-1}_\varepsilon \longrightarrow \mathbb{S}^{2m-1},$$

where $\mathbb{S}^{2m-1}$ denotes the unit sphere in $\mathbb{C}^m$.

Theorem 1.6.14. *In the above situation, we have*

$$\mathrm{Res}_{c^m}(s, E; p) = \deg \varphi.$$

This can also be proved by perturbing the sections, see [75], [160].

Remark 1.6.4. There are similar expressions as above for the residues of vector bundles on singular varieties with respect to an appropriate number of sections (see [160]).

1.6.7 Examples

(a) Poincaré–Hopf Index Theorem

Let M be a complex manifold of dimension m. We take as E the holomorphic tangent bundle TM. Then a section of TM is a (complex) vector field v. One can check (see, e.g., [161]) that the Poincaré–Hopf index $\mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda)$ of v at a connected component S_λ of its zero set S, that we defined in 1.1.3 can be expressed as

$$\mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda) = \mathrm{Res}_{c^m}(v, TM; S_\lambda).$$

Then, if M is compact, by Theorem 1.6.7, we have

$$\sum_\lambda \mathrm{Ind}_{\mathrm{PH}}(v, S_\lambda) = \int_M c^m(M),$$

where $c^m(M) = c^m(TM)$ and it is known that the right hand side coincides with the Euler–Poincaré characteristic $\chi(M)$ of M ("Gauss–Bonnet formula"). Thus, by Theorem 1.6.7, we recover the Poincaré–Hopf theorem in case v is holomorphic and the zeros are isolated.

(b) Multiplicity Formula

Let M be a complex manifold of dimension m. We take as E the holomorphic cotangent bundle T^*M. For a holomorphic function f on M, its differential df is a section of T^*M. The zero set S of df coincides with the critical set $C(f)$ of f. We define the multiplicity $m(f, S_\lambda)$ of f at a connected component S_λ of $C(f)$ by

$$m(f, S_\lambda) = \mathrm{Res}_{c^m}(df, T^*M; S_\lambda).$$

Note that, if S_λ consists of a point p, it coincides with the multiplicity $m(f, p)$ of f at p described in Example 1.6.3.

Now we consider the global situation. Let $f : M \to C$ be a holomorphic map of M onto a complex curve (Riemann surface) C. The differential

$$df : TM \longrightarrow f^*TC$$

of f determines a section of $T^*M \otimes f^*TC$, which is also denoted by df. The set of zeros of df is the critical set $C(f)$ of f. Suppose $C(f)$ is a compact set with a finite number of connected components $(S_\lambda)_\lambda$. Then we have the residue $\mathrm{Res}_{c^m}(df, T^*M \otimes f^*TC; S_\lambda)$ for each λ. If M is compact, by Theorem 1.6.7, we have

$$\sum_\lambda \mathrm{Res}_{c^m}(df, T^*M \otimes f^*TC; S_\lambda) = \int_M c^m(T^*M \otimes f^*TC).$$

We look at the both sides of the above more closely. In the sequel, we set $D(f) = f(C(f))$, the set of critical values. Then, if M is compact, f defines a C^∞ fiber bundle structure on $M \setminus C(f) \to C \setminus D(f)$.

We refer to [87] for a precise proof of the following

Lemma 1.6.1. *If M is compact, and if $D(f)$ consists of isolated points,*

$$\int_M c^m(T^*M \otimes f^*TC) = (-1)^m(\chi(M) - \chi(\mathbf{F})\,\chi(C)),$$

where $\mathbf{F}$ denotes a general fiber of f.

Suppose that $f(S_\lambda)$ is a point. Taking a coordinate on C around $f(S_\lambda)$, we think of f as a holomorphic function near S_λ. Then we may write

$$\mathrm{Res}_{c^m}(df, T^*M \otimes f^*TC; S_\lambda) = \mathrm{Res}_{c^m}(df, T^*M; S_\lambda) = m(f, S_\lambda),$$

the multiplicity of f at S_λ. Thus we have

Theorem 1.6.15. *Let $f : M \to C$ be a holomorphic map of a compact complex manifold M of dimension m onto a complex curve C. If the critical values $D(f)$ of f consists of only isolated points, then*

$$\sum_\lambda m(f, S_\lambda) = (-1)^m(\chi(M) - \chi(\mathbf{F})\,\chi(C)),$$

where the sum is taken over the connected components S_λ of $C(f)$.

In particular, we have ([86], see also [59, Example 14.1.5]):

Corollary 1.6.1. *In the above situation, if the critical set $C(f)$ of f consists of only isolated points,*

$$\sum_{p \in C(f)} m(f, p) = (-1)^m(\chi(M) - \chi(\mathbf{F})\,\chi(C)).$$

See [87] for the definition of multiplicities of functions on possibly singular varieties and formula similar to the above for these multiplicities.

Vector fields on Singular Varieties
Brasselet, J.-P.; Seade, J.; Suwa, T.
2009, XX, 232 p., Softcover
ISBN: 978-3-642-05204-0