Preface

The theory of foliations is closely related to that of differential equations: an oriented one-dimensional foliation is equivalent to a nowhere vanishing vector field X on a manifold M, and the integral curves of X are solutions of a system of ordinary differential equations. Higher-dimensional foliations correspond to systems of partial differential equations via the Frobenius theorem. When the solutions (or leaves) are locally equidistant, the foliation is said to be metric. If in addition, the space B of leaves is reasonably well-behaved, the map $M \longrightarrow B$ that sends a point of M to the leaf on which it lies is called a metric fibration or Riemannian submersion.

In the past three or four decades, there has been increasing realization that these foliations play a key role in understanding the structure of Riemannian manifolds, particularly those with positive or nonnegative sectional curvature. In fact, all known such spaces are constructed from only a representative handful by means of metric fibrations or deformations thereof. This is even more pronounced in positive curvature, where every such space is the image of a Riemannian submersion from a nonnegatively curved manifold. Further indication of the key role that submersions play in nonnegative curvature is Perelman's result that all noncompact spaces with curvature ≥ 0 are metric fibrations over compact ones.

This text is an attempt to document some of these constructions, many of which have only appeared in journal form. The emphasis here is less on the fibration itself and more on how to use it to either construct or understand a metric with curvature of fixed sign on a given space. The approach differs in this sense from previous ones in which a typical question would be to ask whether there exists a metric on the ambient space for which a given foliation has this or that property. The reader will in fact find that this work has little intersection with other books on the subject such as Molino's [91] or Tondeur's [124]. In particular, topics such as basic cohomology or Lie foliations are either omitted or only briefly mentioned. It is assumed that the reader has a working knowledge of differentiable manifolds and Riemannian metrics, such as that offered in an intermediate level course in Riemannian geometry.

The first chapter introduces the main concepts and tools that are used throughout, and relates the curvature of the ambient space with that of the base.

It further discusses the relation between the geodesics in both manifolds as well as the Jacobi fields along them, and ends with the description of Wilking's dual foliation.

Chapter 2 begins by studying how warping of the fibers affects the curvature of the ambient space. This is then used to introduce warped products, after which we discuss the main class of Riemannian submersions, namely those generated by isometric group actions. The rest of the chapter is mostly devoted to the construction of spaces of positive or nonnegative curvature by means of submersions. This includes fundamental examples such as Lie groups, as well as more elaborate ones, such as the Allof-Wallach and Eschenburg spaces.

Chapter 3 studies the structure of complete, noncompact manifolds with curvature ≥ 0, beginning with the Cheeger-Gromoll soul construction. We present Perelman's proof of the generalized soul theorem, and Wilking's work on smoothness of the metric projection onto the soul. The converse of the soul theorem, namely the question of which vector bundles admit nonnegatively curved metrics is also discussed.

The last chapter deals with the problem of classifying metric foliations on spaces of constant curvature. Although this is a fundamental question, it is also a surprisingly delicate one which at the time of writing is still not entirely answered.

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On a tragic note, the first named author, Detlef Gromoll, passed away during the final revision phase of this manuscript. He is fondly remembered by his many friends, colleagues, and former students.

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