

# Introduction

The present book is based on the course given by the author at the Silesian University in the academic year 1974/75, entitled Additive Functions and Convex Functions. Writing it, we have used excellent notes taken by Professor K. Baron.

It may be objected whether an exposition devoted entirely to a single equation (Cauchy's Functional Equation) and a single inequality (Jensen's Inequality) deserves the name An introduction to the Theory of Functional Equations and Inequalities. However, the Cauchy equation plays such a prominent role in the theory of functional equations that the title seemed appropriate. Every adept of the theory of functional equations should be acquainted with the theory of the Cauchy equation. And a systematic exposition of the latter is still lacking in the mathematical literature, the results being scattered over particular papers and books. We hope that the present book will fill this gap.

The properties of convex functions (i.e., functions fulfilling the Jensen inequality) resemble so closely those of additive functions (i.e., functions satisfying the Cauchy equation) that it seemed quite appropriate to speak about the two classes of functions together.

Even in such a large book it was impossible to cover the whole material pertinent to the theory of the Cauchy equation and Jensen's inequality. The exercises at the end of each chapter and various bibliographical hints will help the reader to pursue further his studies of the subject if he feels interested in further developments of the theory. In the theory of convex functions we have concentrated ourselves rather on this part of the theory which does not require regularity assumptions about the functions considered. Continuous convex functions are only discussed very briefly in Chapter 7.

The emphasis in the book lies on the theory. There are essentially no examples or applications. We hope that the importance and usefulness of convex functions and additive functions is clear to everybody and requires no advertising. However, many examples of applications of the Cauchy equation may be found, in particular, in books Aczél [5] and Dhombres [68]. Concerning convex functions, numerous examples are scattered throughout almost the whole literature on mathematical analysis, but especially the reader is referred to special books on convex functions quoted in 5.3.

We have restricted ourselves to consider additive functions and convex functions defined in (the whole or subregions of)  $N$ -dimensional euclidean space  $\mathbb{R}^N$ . This gives the exposition greater uniformity. However, considerable parts of the theory presented

can be extended to more general spaces (Banach spaces, topological linear spaces). Such an approach may be found in some other books (Dhombres [68], Roberts-Varberg [267]). Only occasionally we consider some functional equations on groups or related algebraic structures.

We assume that the reader has a basic knowledge of the calculus, theory of Lebesgue's measure and integral, algebra, topology and set theory. However, for the convenience of the reader, in the first part of the book we present such fragments of those theories which are often left out from the university courses devoted to them. Also, some parts which are usually included in the university courses of these subjects are also very shortly treated here in order to fix the notation and terminology.

In the notation we have tried to follow what is generally used in the mathematical literature<sup>1</sup>. The cardinality of a set  $A$  is denoted by  $\text{card } A$ . The word *countable* or *denumerable* refers to sets whose cardinality is exactly  $\aleph_0$ . The topological closure and interior of  $A$  are denoted by  $\text{cl } A$  and  $\text{int } A$ . Some special letters are used to denote particular sets of numbers. And so  $\mathbb{N}$  denotes the set of positive integers, whereas  $\mathbb{Z}$  denotes the set of all integers.  $\mathbb{Q}$  stands for the set of all rational numbers,  $\mathbb{R}$  for the set of all real numbers, and  $\mathbb{C}$  for the set of all complex numbers. The letter  $N$  is reserved to denote the dimension of the underlying space. The end of every proof is marked by the sign  $\square$ . Other symbols are introduced in the text, and for the convenience of the reader they are gathered in an index at the end of the volume.

The book is divided in chapters, every chapter is divided into sections. When referring to an earlier formula, we use a three digit notation: (X.Y.Z) means formula Z in section Y in Chapter X. The same rule applies also to the numbering of theorems and lemmas. When quoting a section, we use a two digit notation: X.Y means section Y in Chapter X. The same rule applies also to exercises at the end of each chapter. The book is also divided in three parts, but this fact has no reflection in the numeration.

Many colleagues from Poland and abroad have helped us with bibliographical hints and otherwise. We do not endeavour to mention all their names, but nonetheless we would like to thank them sincerely at this place. But at least two names must be mentioned: Professor R. Ger, and above all, Professor K. Baron, whose help was especially substantial, and to whom our debt of gratitude is particularly great. We thank also the authorities of the Silesian University in Katowice, which agreed to publish this book. We hope that the mathematical community of the world will find it useful.

Katowice, July 1979

*Marek Kuczma*

---

<sup>1</sup>The notation in the second edition has been slightly changed. The following sentences are modified accordingly.

An Introduction to the Theory of Functional Equations  
and Inequalities

Cauchy's Equation and Jensen's Inequality

Kuczma, M. - Gilányi, A. (Ed.)

2009, XIV, 595 p., Softcover

ISBN: 978-3-7643-8748-8

A product of Birkhäuser Basel