

Preface

*Were I to take an iron gun,
And fire it off towards the sun;
I grant 'twould reach its mark at last,
But not till many years had passed.*

*But should that bullet change its force,
And to the planets take its course,
'Twould never reach the nearest star,
Because it is so very far.*

from FACTS by Lewis Carroll [55]

Let me begin by describing the two purposes which prompted me to write this monograph. This is a book about algebraic topology and more especially about homotopy theory. Since the inception of algebraic topology [217] the study of homotopy classes of continuous maps between spheres has enjoyed a very exceptional, central role. As is well known, for homotopy classes of maps $f : S^n \longrightarrow S^n$ with $n \geq 1$ the sole homotopy invariant is the degree, which characterises the homotopy class completely. The search for a continuous map between spheres of different dimensions and not homotopic to the constant map had to wait for its resolution until the remarkable paper of Heinz Hopf [111]. In retrospect, finding an example was rather easy because there is a canonical quotient map from S^3 to the orbit space of the free circle action $S^3/S^1 = \mathbb{CP}^1 = S^2$. On the other hand, the problem of showing that this map is not homotopic to the constant map requires either ingenuity (in this case Hopf's observation that the inverse images of any two distinct points on S^2 are linked circles) or, more influentially, an invariant which does the job (in this case the Hopf invariant). The Hopf invariant is an integer which is associated to any continuous map of the form $f : S^{2n-1} \longrightarrow S^n$ for $n \geq 1$. Hopf showed that when n is even, there exists a continuous map whose Hopf invariant is equal to any even integer. On the other hand the homotopy classes of continuous maps $g : S^m \longrightarrow S^n$ in almost all cases with $m > n \geq 1$ form a finite abelian group. For the study of the 2-Sylow subgroup of these groups the appropriate invariant is the Hopf invariant modulo 2. With the construction of mod p cohomology operations by Norman Steenrod it became possible to define

the mod 2 Hopf invariant for any g but the only possibilities for non-zero mod 2 Hopf invariants occur when $m - n + 1$ is a power of two ([259] p. 12).

As described in Chapter 1, § 1, when $n \gg 0$ the homotopy classes of g 's form the stable homotopy group $\pi_{m-n}(\Sigma^\infty S^0)$, which is a finite group when $m > n$. The p -Sylow subgroups of stable homotopy groups were first organised systematically by the mod p Adams spectral sequence, constructed by Frank Adams in [1]. Historically, the case when $p = 2$ predominates. As one sees from Chapter 1, Theorem 1.1.2, on the line $s = 1$, four elements exist denoted by h_0, h_1, h_2, h_3 in the 2-Sylow subgroups of $\pi_j(\Sigma^\infty S^0)$ when $j = 0, 1, 3$ and 7 , respectively. In positive dimensions the homotopy classes with non-zero mod 2 Hopf invariant would all be represented on the $s = 1$ line in dimensions of the form $j = 2^k - 1$, by Steenrod's result ([259] p. 12). However, a famous result due originally to Frank Adams ([2]; see also [11], [248] and Chapter 6, Theorem 6.3.2) shows that only h_1, h_2, h_3 actually correspond to homotopy classes with non-zero mod 2 Hopf invariant.

From this historical account one sees that the resolution of the behaviour on the $s = 2$ line of the mod 2 Adams spectral sequence qualifies as a contender to be considered the most important unsolved problem in 2-adic stable homotopy theory. This basic unsolved problem has a history extending back over fifty years. Inspection of the segment of the spectral sequence which is given in Chapter 1, Theorem 1.1.2 correctly gives the impression that this problem concerns whether or not the classes labelled h_i^2 represent elements of $\pi_{2^{i+1}-2}(\Sigma^\infty S^0)$. The invariant which is capable of detecting homotopy classes represented on the $s = 2$ line is due to Michel Kervaire [138] as generalised by Ed Brown Jr. [50]. Bill Browder discovered the fundamental result [47], the analogue of Steenrod's result about the Hopf invariant, that the Arf-Kervaire invariant could only detect stable homotopy classes in dimensions of the form $2^{i+1} - 2$.

Stable homotopy classes with Hopf invariant one (mod 2) only exist for dimensions 1, 3, 7 and currently (see Chapter 1, Sections One and Eight) stable homotopy classes with Arf-Kervaire invariant one (mod 2) have only been constructed in dimensions 2, 6, 14, 30 and 62 (see [247] and [145] – I believe that [183] has a gap in its construction). Accordingly the following conjecture seems reasonable:

Conjecture. Stable homotopy classes with Arf-Kervaire invariant one (mod 2) exist only in dimensions 2, 6, 14, 30 and 62.

This brings me to my first purpose. As ideas for progress on a particular mathematics problem atrophy and mathematicians at the trend-setting institutions cease to direct their students to study the problem, then it can disappear. Accordingly I wrote this book in order to stem the tide of oblivion in the case of the problem of the existence of framed manifolds of Arf-Kervaire invariant one. During the 1970's I had heard of the problem – during conversations with Ib Madsen, John Jones and Elmer Rees in Oxford pubs and lectures at an American Mathematical Society Symposium on Algebraic Topology at Stanford University

in 1976. However, the problem really came alive for me during February of 1980. On sabbatical from the University of Western Ontario, I was visiting Princeton University and sharing a house with Ib Madsen and Marcel Böksted. I went to the airport with Wu Chung Hsiang one Sunday to collect Ib from his plane. During dinner somewhere in New York's Chinatown Ib explained to Wu Chung his current work with Marcel, which consisted of an attempt to construct new framed manifolds of Arf-Kervaire invariant one. I did not understand the sketch given that evening but it was very inspirational. Fortunately the house we were sharing had a small blackboard in its kitchen and, by virtue of being an inquisitive pest day after day, I received a fascinating crash course on the Arf-Kervaire invariant one problem. A couple of months later, in April 1980, in that same sabbatical my family and I were enjoying a month's visit to Aarhus Universitet, during the first two days of which I learnt a lot more about the problem from Jorgen Tornehave and [247] was written. For a brief period overnight during the writing of [247] we were convinced that we had the method to make all the sought-after framed manifolds – a feeling which must have been shared by lots of topologists working on this problem. All in all, the temporary high of believing that one had the construction was sufficient to maintain in me at least an enthusiastic spectator's interest in the problem, despite having moved away from algebraic topology as a research area.

In the light of the above conjecture and the failure over fifty years to construct framed manifolds of Arf-Kervaire invariant one this might turn out to be a book about things which do not exist. This goes some way to explain why the quotations which preface each chapter contain a preponderance of utterances from the pen of Lewis Carroll (aka Charles Lutwidge Dodgson [55]).

My second purpose is to introduce a new technique, which I have christened upper triangular technology, into 2-adic classical homotopy theory. The method derives its name from the material of Chapter 3 and Chapter 5, which gives a precise meaning to the Adams operation ψ^3 as an upper triangular matrix. Briefly this is a new and easy to use method to calculate the effect of the unit maps

$$\pi_*(bo \wedge X) \longrightarrow \pi_*(bu \wedge bo \wedge X)$$

and

$$\pi_*(bu \wedge X) \longrightarrow \pi_*(bu \wedge bu \wedge X)$$

induced by the unit $\eta : S^0 \longrightarrow bu$ from the map on $bo_*(X)$ (respectively, $bu_*(X)$) given by the Adams operation ψ^3 . Here bu and bo denote the 2-adic, connective complex and real K-theory spectra (see Chapter 1, § 1.3.2(v)).

There is a second point of view concerning upper triangular technology. Upper triangular technology is the successor to the famous paper of Michael Atiyah concerning operations in periodic unitary K-theory [25]. The main results of [25] are (i) results about the behaviour of operations in KU -theory with respect to the filtration which comes from the Atiyah-Hirzebruch spectral sequence and (ii) results which relate Adams operations in KU -theory to the Steenrod operations of

[259] in mod p singular cohomology in the case of spaces whose integral cohomology is torsion free. As explained in the introduction to Chapter 8, Atiyah's result had a number of important applications. Generalising Atiyah's results to spaces with torsion in their integral cohomology has remained unsolved since the appearance of [25]. At least when X is the mapping cone of $\Theta_{2n} : \Sigma^\infty S^{2n} \longrightarrow \Sigma^\infty \mathbb{R}P^{2n}$, which is a spectrum with lots of 2-ary torsion in its K-theory, I shall apply the upper triangular technology of Chapters 3 and 5 to offer a solution to this problem, although the solution will look at first sight very different from [25]. As explained in Chapter 1, § 8 the existence of framed manifolds of Arf-Kervaire invariant one may be equivalently rephrased in terms of the behaviour of the mod 2 Steenrod operations on the mapping cone of Θ_{2n} . By way of application, in Chapter 8, the upper triangular technology is used to give a new, very simple (here I use the word "simple" in the academic sense of meaning "rather complicated" and I use the phrase "rather complicated" in the non-academic sense of meaning "this being mathematics, it could have been a lot worse"!) proof of a conjecture of Barratt-Jones-Mahowald, which rephrases K-theoretically the existence of framed manifolds of Arf-Kervaire invariant one.

I imagine that the upper triangular technology will be developed for p -adic connective K-theory when p is an odd prime. This will open the way for the applications to algebraic K-theory, which I describe in Chapter 3, and to motivic cohomology, which I mention in Chapter 9. The connection here is the result of Andrei Suslin ([264], [265]) which identifies the p -local algebraic K-theory spectrum of an algebraically closed field containing $1/p$ as that of p -local bu -theory.

The contents of the book are arranged in the following manner.

Chapter 1 contains a sketch of the algebraic topology background necessary for reading the rest of the book. Most likely many readers will choose to skip this, with the exception perhaps of § 8. § 1 covers some history of computations of the stable homotopy groups of spheres and § 2 describes how the Pontrjagin-Thom construction rephrases these stable homotopy groups in terms of framed manifolds. § 3 describes the classical stable homotopy category and gives several examples of spectra which will be needed later. § 4 introduces the classical mod 2 Adams spectral sequence which ever since its construction in [1] has remained (along with its derivatives) the central computational tool of stable homotopy theory. § 5 introduces the Snaith splitting and the Kahn-Priddy Theorem. The latter is needed to rephrase the Arf-Kervaire invariant one problem in terms of $\Theta_{2n} : \Sigma^\infty S^{2n} \longrightarrow \Sigma^\infty \mathbb{R}P^{2n}$ and the former is needed both to prove the latter and to establish in Chapter 3 the first episode of the upper triangular saga. This section introduces the space $QX = \Omega^\infty \Sigma^\infty X$ as well as finitely iterated loopspaces $\Omega^r \Sigma^r X$. § 6 recapitulates the properties of the mod 2 Steenrod algebra, which is needed in connection with the Adams spectral sequences we shall use and to prove the results of Chapter 2. § 7 introduces the Dyer-Lashof algebra, which is also needed to prove the results of Chapter 2. § 7 describes the Arf-Kervaire invariant one problem and its equivalent reformulations which are either used or established

in the course of this book – in particular, the upper triangular technology is used in Chapter 8 to prove the reformulation which is given in Theorem 1.8.10.

Chapter 2 considers the adjoint $\text{adj}(\Theta_{2n}) : S^{2n} \longrightarrow Q\mathbb{RP}^{2n}$ and derives a number of formulae for the Arf-Kervaire invariant in this formulation. In particular, the equivalent formulation in terms of a framed manifold M^{2n} together with an orthogonal vector bundle gives rise to a formula for the Arf-Kervaire invariant (Theorems 2.2.2 and 2.2.3) which leads to a very easy construction in § 3 of a framed manifold of Arf-Kervaire invariant one in dimension 30 (and, of course, in dimensions 2, 6 and 14). The material of Chapter 2 is taken from my joint paper with Jorgen Tornehave [247] which I mentioned above.

Chapter 3 introduces the upper triangular technology. § 1 describes left bu -module splittings of the 2-local spectra $bu \wedge bu$ and $bu \wedge bo$ which were originally discovered by Mark Mahowald – although I learnt about their proof from Don Anderson, via his approach, just prior to a lecture at the Vancouver ICM in 1974 in which Jim Milgram talked about their proof and properties via his approach! My account tends to follow the account given by Frank Adams in [9] except that I make use of the multiplicative properties of the Snaith splitting of $\Omega^2 S^3$. In § 2, Mahowald’s splittings are used to show that the group of homotopy classes of left- bu -module automorphisms of the 2-local spectrum $bu \wedge bo$ which induce the identity on mod 2 homology is isomorphic to the group of upper triangular matrices with entries in the 2-adic integers. In § 3 I explain how this result may be applied to the construction of operations in algebraic K-theory and Chow groups (or even Spencer Bloch’s higher Chow groups – see Chapter 9). The material of Chapter 3 is taken from [252].

Chapter 4 is included in order to amplify the comments in Chapter 3, § 3 concerning the connections between bu and algebraic K-theory. It sketches the ingenious proof of [265]. Suslin’s proof is very sophisticated in its prerequisites and this chapter arose originally as a 10 lecture course which presented the result to an audience with few of the prerequisites. Accordingly, § 1 sketches simplicial sets and their realisations which are then applied in § 2 to construct Quillen’s K-theory space and to introduce K-theory mod m . Sections Three, Four and Five introduce affine schemes, Henselian rings, Henselian pairs and their mod m K-theory. § 6 defines group cohomology and sketches Andrei Suslin’s universal homotopy construction which is the backbone of the proof. § 7 sketches John Milnor’s simplicial construction of the fibre of the map from the classifying space of a discrete Lie group to the classifying space of the Lie group with its classical topology and follows [265] in using it to compute the mod m algebraic K-theory of an Archimedean field. § 8 sketches the analogous result for commutative Banach algebras and § 9 outlines the alternative approach via excision in the case of the C^* -algebra $C(X)$.

Chapter 5 is the second instalment of upper triangular technology. It is taken from the paper by Jon Barker and me [27]. It evaluates the conjugacy class represented by the map $1 \wedge \psi^3 : bu \wedge bo \longrightarrow bu \wedge bo$ in the group of upper triangular matrices with coefficients in the 2-adic integers. The crux of the chapter is that

it suffices to compute the effect of $1 \wedge \psi^3$ on homotopy modulo torsion with respect to a 2-adic basis coming from Mahowald's splitting of Chapter 3. There is a homotopy basis given in [58] with respect to which the matrix of $1 \wedge \psi^3$ is easy to compute – a fact which I learnt from Francis Clarke. Sections Two and Three carefully compare these two bases sufficiently to calculate the diagonal and the super-diagonal of the matrix. § 4 shows (by two methods) that any two matrices in the infinite upper triangular matrix with 2-adic coefficients with this diagonal and super-diagonal are conjugate. § 5 shows by an application, which goes back to [183], the manner in which the upper triangular technology can sometimes reduce homotopy theory to matrix algebra.

Chapter 6 contains various cohomological results related to real projective space and maps involving them. § 1 calculates the MU -theory and KU -theory of \mathbb{RP}^n together with the effect of the Adams operations in these groups. It also calculates the effect of some of the Landweber-Novikov operations on $MU_*(\mathbb{RP}^n)$. § 2 contains a slightly different method for making these calculations, applied instead to $BP_*(\mathbb{RP}^n)$ in preparation for the applications of Chapter 7. § 3 applies these calculations to the study of the Whitehead product $[\iota_n, \iota_n] \in \pi_{2n-1}(S^n)$. The vanishing of the Whitehead product is equivalent to the classical Hopf invariant one problem and Theorem 6.3.2 is equivalent to the (then new) KU_* proof of the non-existence of classes of Hopf invariant one which I published in the midst of the book review [248]. Theorem 6.3.4 gives equivalent conditions, related to the Arf-Kervaire invariant one problem, for $[\iota_n, \iota_n]$ to be divisible by two. This result was proved in one direction in [30]. The converse is proved as a consequence of the main upper triangular technology result of Chapter 8 (Theorem 8.1.2), which is equivalent to the Arf-Kervaire invariant one reformulation of Chapter 1 § 1.8.9 and Theorem 1.8.10 originally conjectured in [30] (see Chapter 7, Theorem 7.2.2). § 4 contains results which relate e-invariants and Hopf invariants – first considered in Corollary D of [30]. In fact, Theorem 6.4.2 and Corollary 6.4.3 imply both Corollary D and its conjectured converse. § 5 contains a miscellany of results which relate the halving of the Whitehead product to MU_* -e-invariants. The material of Chapter 6 is based upon an unpublished 1984 manuscript concerning the MU -theory formulations of the results [30], enhanced by use of Chapter 8, Theorem 8.1.2 in several crucial places.

Chapter 7 applies BP -theory and its related J -theories to analyse the Arf-Kervaire invariant one problem. § 1 contains formulae for the J -theory Hurewicz image of $\Theta_{2^{n+1}-2} : \Sigma^\infty S^{2^{n+1}-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{2^{n+1}-2}$. In § 2 the Hurewicz images in J -theory and ju -theory are related and in Theorem 7.2.2 the conjecture of [30] is proved in its original ju -theory formulation. The material for Chapter 7 is taken from [251]. However, the main result (Theorem 7.2.2) is furnished with three proofs – an outline of the (straightforward but very technical) proof given in [251] and two much simpler proofs which rely on the use of Chapter 8, Theorem 8.1.2.

Chapter 8 is the third and final instalment of upper triangular technology. § 1 describes some historical background and how the method is used to

prove the main technical results (Theorem 8.4.6, Theorem 8.4.7 and Corollary 8.4.8). Then the all important relation is proved, relating the *bu*-e-invariant of $\Theta_{8m-2} : \Sigma^\infty S^{8m-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{8m-2}$ to the mod 2 Steenrod operations on the mapping cone of Θ_{8m-2} . § 2 computes the connective K-theory groups which are needed in the subsequent upper triangular calculations and the effect of the maps which correspond to the super-diagonal entries in the upper triangular matrix corresponding to the Adams operation ψ^3 . § 3 describes the three types of mapping cone to which the method is applied in § 4. In addition to the mapping cone of Θ_{8m-2} , it is also necessary to apply the method to \mathbb{RP}^{8m-1} – the mapping cone of the canonical map from S^{8m-2} to \mathbb{RP}^{8m-2} – in order to estimate the 2-divisibility of some of the terms which appear in the upper triangular equations for the mapping cone of Θ_{8m-2} . In addition we apply the method to the mapping cone of the composition of Θ_{8m-2} with some maps of the form $\Sigma^\infty \mathbb{RP}^{8m-2} \longrightarrow \Sigma^\infty \mathbb{RP}^{8m-8r2}$ constructed by Hirosi Toda [276]. § 4 contains the final details of all these applications and the proofs of the main results – Theorem 8.4.6, Theorem 8.4.7 and Corollary 8.4.8.

The first eight chapters have been dedicated to classical stable homotopy theory in the form of the Arf-Kervaire invariant one problem. Therefore, to leaven the lump, Chapter 9 contains a brief overview of some current themes in stable homotopy theory. Much of this chapter is concerned with the \mathbb{A}^1 -stable homotopy category of Fabien Morel and Vladimir Voevodsky ([281], [203]; see also [180]) because of its spectacular applications, its close relation to algebro-geometric themes in algebraic K-theory (e.g., higher Chow groups) and because several “classical results” with which I am very familiar seem to be given a new lease of life in the \mathbb{A}^1 -stable homotopy category (see Chapter 9 § 9.2.15). For example, I am particularly interested in the potential of the upper triangular technology to establish relations between Adams operations in algebraic K-theory and Steenrod operations in motivic cohomology (see Chapter 9 § 9.2.15(vi)). The chapter closes with a very short scramble through some other interesting and fashionable aspects of stable homotopy theory currently under development and with some “late-breaking news” on recent Arf-Kervaire invariant activity, which was brought to my attention simultaneously on June 9, 2008 by Peter Landweber and Hadi Zare, a student of Peter Eccles at the University of Manchester and appears in the preprints ([16], [17], [18]).

I am very grateful for help and advice to Huajian Yang, my postdoc at McMaster University during 1996–98. It was during that period, when we were discussing the Arf-Kervaire invariant problem extensively, that Huajian convinced me of the utility of the Adams spectral sequence in connective K-theory, which led in particular to the results of Chapter 3, proved in 1998. In addition, I am very grateful to Jonathan Barker, my PhD student at the University of Southampton in 2003–2006, who took an interest in the “ ψ^3 as an upper triangular matrix” project and worked hard in collaboration with me to find the matrix of Chapter 5, Theorem 5.1.2. I am indebted to Francis Clarke who, as a result of a conversation in a queue for coffee at a Glasgow conference in 2001, provided the all-important

guess as to the identity of the matrix we were looking for. In connection with Chapter 9, I am very grateful to Mike Hopkins for a very useful discussion, at the Fields Institute in May 2007, concerning alternative proofs of my classical result (Chapter 1, Theorem 1.3.3) which pointed David Gepner and me to the one which, suitably adapted (see [87]), yields the analogous results in the \mathbb{A}^1 -stable homotopy category described in Chapter 9, §9.2.15.

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