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Numerical Scheme for Laplacian Growth Models Based on the Helmholtz–Kirchhoff Method

A.S. Demidov and J.-P. Lohéac

Abstract. The Helmholtz–Kirchhoff method is an efficient tool for analysing bi-dimensional problems in fluid mechanics. It especially allows to transform a free boundary problem in a fixed boundary problem by introducing a convenient parametrization of the free boundary.

In this paper, it will be shown how it also leads to build numerical schemes. The case of Hele-Shaw flows will be especially studied.

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Keywords. Hele-Shaw flows, free boundary problems, integro-differential equations.

1. Hele-Shaw problems

The core of the Hele-Shaw device is a blob of fluid moving between two glass plates. We are interested in the motion of such a blob when fluid is injected or sucked between the plates. The air which surrounds the blob is another fluid with negligible viscosity. In 1934, L.S. Leibenson gave a simplified mathematical model for this problem when the source of the flow is punctual.

We here consider a class of free boundary problems derived from this so-called Stokes-Leibenson problem.

Let $\Omega_0 \subset \mathbb{R}^2$ be a bounded simply connected domain such that its boundary Γ_0 is smooth enough. This domain will be deformed according to the following law: at time t, we obtain a domain $\Omega_t = \Omega$ of boundary $\Gamma_t = \Gamma$ such that the normal velocity of each point $\mathbf{s} \in \Gamma$ is given by the following kinetic condition,

$$\dot{\mathbf{s}}.\boldsymbol{\nu} = \partial_\nu u \,, \tag{1.1}$$

where u is the solution of some Laplace problem.

Above we denote by $\boldsymbol{\nu}$ the normal unit outwards vector at $\mathbf{s} \in \Gamma$ and by $\partial_\nu u$ the normal derivative of u at this point. We will consider three cases (see Fig. 1).

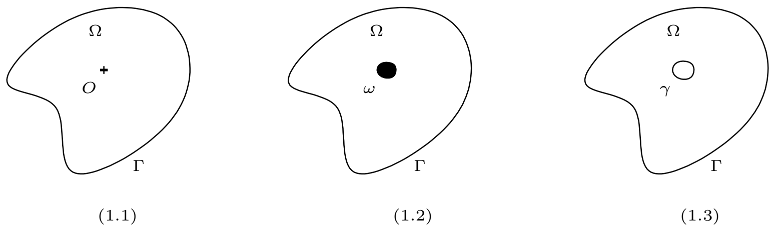

FIGURE 1. The three considered cases: (1.1) punctual source, (1.2) distributed internal source, (1.3) distributed boundary source

1.1. Punctual source

Here u is the solution of the following problem,

$$\begin{cases} \Delta u = q\,\delta\,, & \text{in } \Omega\,, \\ u = 0\,, & \text{on } \Gamma\,, \end{cases} \tag{1.2}$$

where δ is the Dirac distribution at some point $O \in \Omega_0$ and q represents the power of the source.

Observe that by an elementary computation, one can obtain the increasing rate of the fluid domain Ω:

$$\int_\Gamma \dot{\mathbf{s}}.\boldsymbol{\nu}\,d\sigma = \int_\Gamma \partial_\nu u\,d\sigma = q\,.$$

1.2. Distributed internal source

In this case, u satisfies

$$\begin{cases} \Delta u = f\,, & \text{in } \Omega\,, \\ u = 0\,, & \text{on } \Gamma\,. \end{cases} \tag{1.3}$$

Here, the support of the right-hand side f is contained in an open set ω such that $\overline{\omega} \subset \Omega_0$.

As well as above, the increasing rate of the fluid domain Ω can be computed

$$\int_\Gamma \dot{\mathbf{s}}.\boldsymbol{\nu}\,d\sigma = \int_\Gamma \partial_\nu u\,d\sigma = \int_\omega f\,d\mathbf{x}\,.$$

1.3. Distributed boundary source

Here, u satisfies

$$\begin{cases} \Delta u = 0\,, & \text{in } \Omega\,, \\ -\partial_\nu u = g\,, & \text{on } \gamma\,, \\ u = 0\,, & \text{on } \Gamma\,, \end{cases} \tag{1.4}$$

and we can write

$$\int_\Gamma \dot{\mathbf{s}}.\boldsymbol{\nu}\,d\sigma = \int_\Gamma \partial_\nu u\,d\sigma = \int_\gamma g\,d\sigma\,.$$

2. Helmholtz–Kirchhoff method

In each case, we consider u as an harmonic function on some subdomain Ω' of Ω and we introduce its harmonically conjugate function v. At every point of Ω', level curves of u and v are orthogonal.

Let us set: $z = x + \imath y$ and: $w = u + \imath v$. the function $z \longmapsto w$ is analytic and univalent from Ω' onto a fixed domain $\Pi \subset \mathbb{C}$.

In Fig. 2, 3, 4, we construct a way composed of level curves of u and v in Ω' and its image in Π.

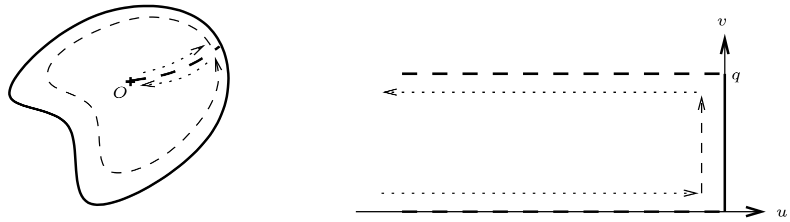

FIGURE 2. The case of a punctual source: $\Omega' = \Omega \setminus \{O\}$ and $\Pi = (-\infty, 0) \times (0, q)$, when $q > 0$.

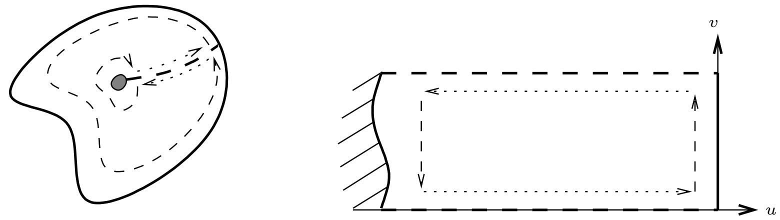

FIGURE 3. The case of an internal distributed source: $\Omega' = \Omega \setminus \overline{\omega}$ and $\Pi \subset (-\infty, 0) \times (0, \int_\omega f\, d\mathbf{x})$, when $\int_\omega f\, d\mathbf{x} > 0$.

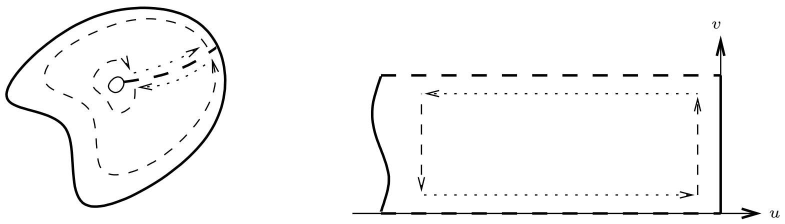

FIGURE 4. The case of a boundary distributed source: $\Omega' = \Omega$ and $\Pi \subset (-\infty, 0) \times (0, \int_\gamma g\, d\sigma)$, when $\int_\gamma g\, d\sigma > 0$.

Now, in order to simplify, we assume that

- there is a constant axis of symmetry,
- the increasing rate of the fluid domain is 2.

Then we can only consider the "upper" part of the fluid domain and we can choose v so that

$$v(t,x,0)=0\,,\ \text{ if }\ x>0\,,\qquad v(t,x,0)=1\,,\ \text{ if }\ x<0\,.$$

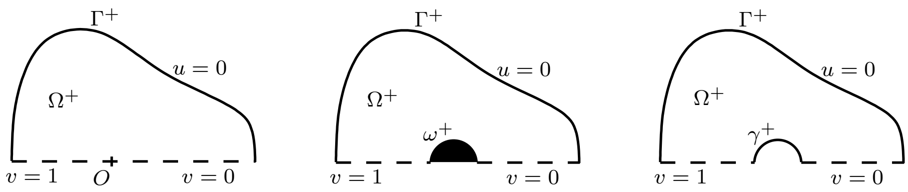

FIGURE 5. The upper part of the domain in the three above cases.

The corresponding domain Π^+ is a simply connected subset of $(-\infty,0)\times(0,1)$ and the boundary of Π^+ contains $\{0\}\times(0,1)$, which corresponds to Γ^+.
Since Γ^+ is a part of a level curve of u, it can be parametrized by the value of v at each point. In Fig. 6, we show an example of computed level curves of u and v (this has been obtained by using a finite elements method).

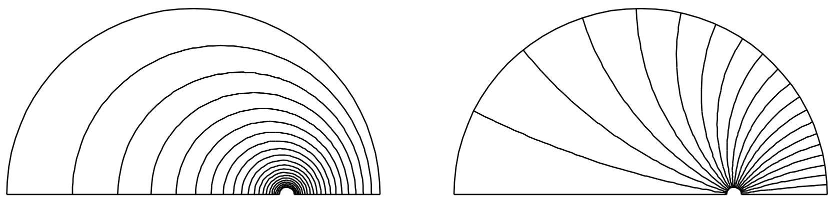

FIGURE 6. Case of an uncentered source in a circular domain: level curves of u (left) and v (right).

Then we can define the **Helmholtz–Kirchhoff** function [1]:

$$A+\imath B=\ln\frac{\partial z}{\partial w}\,,$$

where A and B are harmonically conjugate real functions. We can write

$$(A+\imath B)(t,w)=a_0(t)+\pi w+\sum_{k=1}^{\infty}\beta_k(t)\,\exp(k\pi w)\,,$$

and define

$$a(t,\eta)=A(t,0,\eta)\,,\qquad b(t,\eta)=B(t,0,\eta)\,.$$

This leads to a parametrization of Γ^+,

$$\mathbf{s}(t,\eta) = \mathbf{s}_0(t) + \int_0^\eta \exp(a(t,v))\, \boldsymbol{\tau}(t,v)\, dv\,, \qquad \text{with } 0 \le \eta \le 1\,,$$

where $\boldsymbol{\tau}$ is the tangential unit vector at some point of the moving upper boundary Γ^+ (see Fig. 7). Since a and b are linked by a Hilbert-type transform, the main unknown of our problem becomes the function b which represents the orientation of normal unit vector $\boldsymbol{\nu}$.

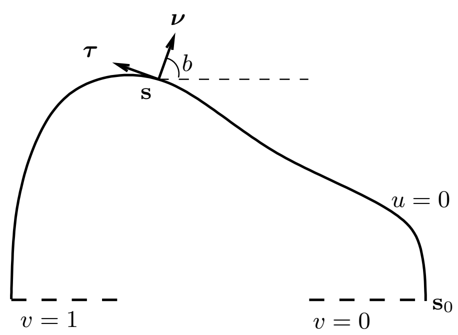

FIGURE 7. Parametrization of Γ^+, unit vectors $\boldsymbol{\nu}$ and $\boldsymbol{\tau}$.

If Γ^+ is regular enough, then we can write, for $t \ge 0$ and $\eta \in [0,1]$,

$$\left(e^a\, \partial_t b\right)(t,\eta) = \left(e^{-a}\, \partial_\eta a\right)(t,\eta) + \partial_\eta b(t,\eta) \int_0^\eta \left(e^a\, \partial_t a - e^{-a}\, \partial_\eta b\right)(t,v)\, dv\,.$$

In other words, we can say that function β such that $\beta(t,\eta) = b(t,\eta) - \pi\eta$, satisfies an integro-differential equation

$$2(t+t_0)\, [\mathbb{K}(\beta)\, \partial_t \beta](t,\eta) = [\mathbb{F}(\beta)](t,\eta)\,, \tag{2.1}$$

where $t_0 > 0$ depends on the measure of the initial domain Ω_0.

This especially leads to prove existence and uniqueness results and qualitative results in some cases of Hele-Shaw flows with a punctual source [2, 3].

3. Numerical model

Our main idea is to restrict above computation to the case of some class of polygonal domains [4].

Hence let us consider the class $\mathcal{P}_m$ ($m \in \mathbb{N}^*$) of simply connected polygonal domains such that

- each of these domains is symmetric with respect to the x-axis and contains the source region,
- its boundary, which we call the "quasi-contour", is a polygonal line with $2m+1$ vertices,

- one vertex belongs to the positive part of the x-axis.

We will compute the behavior of such domains by applying some discrete law inspired by the law of motion of smooth curves in the classical Stokes-Leibenson problem.

For instance, in the case of a punctual source, we obtain the approximated problem:

Let Ω_0^m in $\mathcal{P}_m$ be some "approximation" of Ω_0 and $\Gamma_0^m = \partial\Omega_0^m$.

For $t > 0$, find $\Omega^m = \Omega_t^m$ in $\mathcal{P}_m$ and its boundary $\Gamma^m = \partial\Omega^m$ such that every vertex $\mathbf{p} \in \Gamma^m$ verifies the punctual kinetic law

$$\dot{\mathbf{p}}.\boldsymbol{\nu}(\mathbf{p}) = \int_{\Gamma^m} d_{\mathbf{p}} |\nabla u| \, d\gamma \,, \tag{3.1}$$

with

$$\begin{cases} \Delta u = 2\delta \,, & \text{in } \Omega^m \,, \\ u = 0 \,, & \text{on } \Gamma^m \,. \end{cases} \tag{3.2}$$

As above, we introduce v harmonically conjugate function of u and we use it to get a convenient parametrization of Γ^{m+}.

In Fig. 8, we show an example of computed level curves of u and v for some polygonal domain.

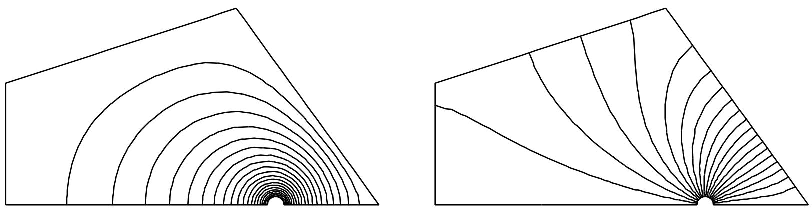

Figure 8. Case of an uncentered source in a pentagonal domain ($m = 2$): level curves of u (left) and v (right).

Here, the unknown function b is piecewise constant and can be represented by the finite sequence $\mathbf{N}$ of the orientations of the m normal unit vectors with respect to each edge of Γ^{m+}.

An explicit computation of terms of integro-differential equation (2.1) leads to

$$2(t + t_0) \, [Q(\mathbf{N}) \, \dot{\mathbf{N}}](t) = [P(\mathbf{N})](t) \,, \tag{3.3}$$

where $Q(\mathbf{N})$ is a $m \times m$-matrix and $P(\mathbf{N})$ belongs to $\mathbb{R}^m$.

Then, using initial data, we can perform an explicit Euler scheme for (3.3).

We only present here two examples of computed evolution of quasi-contours: in Fig. 9, the computed evolution of a pentagonal quasi-contour ($m = 2$) and in Fig. 10, the computed evolution of a polygonal quasi-contour ($m = 4$).

In addition, we have got for Hele-Shaw flows with a punctual source the existence of an attractive manifold in the space of quasi-contours. In the case of a

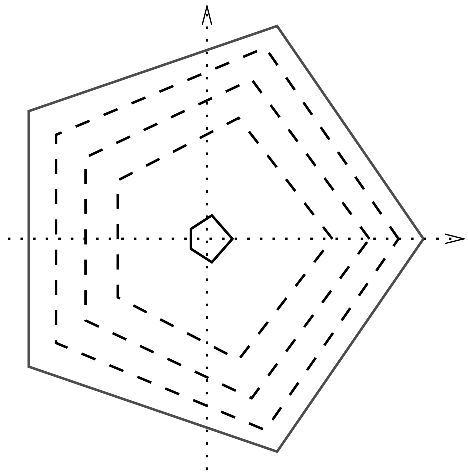

FIGURE 9. An increasing pentagonal quasi-contour: initial shape (center) and shapes at time 1, 2, 3 (dot-lines), 4.

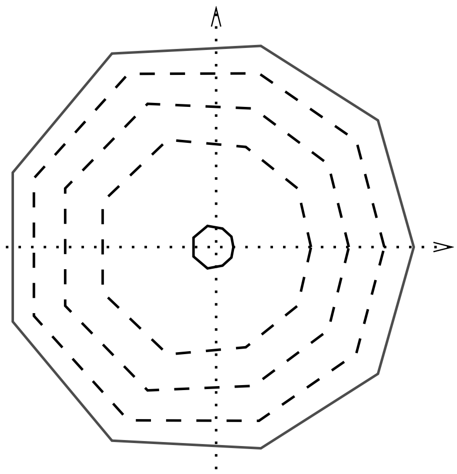

FIGURE 10. An increasing polygonal quasi-contour ($m = 4$): initial shape (center) and shapes at time 1, 2, 3 (dot-lines), 4.

sink, we have to reverse the time-scale. This manifold becomes repulsive and this can explain some fingering phenomenons [4].

For the source case, it has been proved in [2, 3] that the limit of the contour when time tends to ∞ is a circle centered at the source point. Here we obtain in our numerical experiments a similar property: for a fixed m, the quasi-contour tends to a regular polygonal contour centered at the source point when time increases.

In the other hand, the nature of our discrete model does not allow to confirm the property proved in [6] showing that the free boundary becomes instantaneously smooth.

References

[1] A.S. Demidov, *Some Applications of the Helmholtz–Kirchhoff Method.* (*Equilibrium Plasma in Tokamaks, Hele-Shaw Flow, and High-Frequency Asymptotics*). Russian J. Math. Ph. **V. 7, No. 2**, 166–186 (2000).

[2] A.S. Demidov, *Evolution of a perturbation of a circle in a problem for Hele-Shaw flows.* Journ. of Math. Sciences **123, No. 8**, 4381–4403 (2004).

[3] A.S. Demidov, *Evolution of a perturbation of a circle in a problem for Hele-Shaw flows. Part II.* Journ. of Math. Sciences **139, No. 6**, 7064–7078 (2006).

[4] A.S. Demidov, J.-P. Lohéac, *The Stokes–Leibenson Problem for Hele-Shaw Flows.* Patterns and Waves (Eds. A. Abramian, S. Vakulenko, V. Volpert) Saint-Petersburg, 103–124 (2003).

[5] A.S. Demidov, J.-P. Lohéac, *Simulation numérique de phénomènes de digitation dans des écoulements de Hele-Shaw* (in progress).

[6] J.R. King, J.R. Lacey, A.A. Vazquez, *Persistence of corners in free boundaries in Hele-Shaw flow.* Europ. J. Appl. Math. **6** (5), 455–490 (2006).

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