

Preface

Regular rings were originally introduced by John von Neumann to clarify aspects of operator algebras ([33], [34], [9]). A continuous geometry is an indecomposable, continuous, complemented modular lattice that is not finite-dimensional ([8, page 155], [32, page V]). Von Neumann proved ([32, Theorem 14.1, page 208], [8, page 162]): *Every continuous geometry is isomorphic to the lattice of right ideals of some regular ring.* The book of K.R. Goodearl ([14]) gives an extensive account of various types of regular rings and there exist several papers studying modules over regular rings ([27], [31], [15]). In abelian group theory the interest lay in determining those groups whose endomorphism rings were regular or had related properties ([11, Section 112], [29], [30], [12], [13], [24]). An interesting feature was introduced by Brown and McCoy ([4]) who showed that every ring contains a unique largest ideal, all of whose elements are regular elements of the ring. In all these studies it was clear that regularity was intimately related to direct sum decompositions. Ware and Zelmanowitz ([35], [37]) defined regularity in modules and studied the structure of regular modules. Nicholson ([26]) generalized the notion and theory of regular modules.

In this purely algebraic monograph we study a generalization of regularity to the homomorphism group of two modules which was introduced by the first author ([19]). Little background is needed and the text is accessible to students with an exposure to standard modern algebra.

In the following, R is a ring with 1, and A, M are right unital R -modules.

Let $f \in \text{Hom}_R(A, M)$. Then f is **regular** if there exists $g \in \text{Hom}_R(M, A)$ such that $fgf = f$. If so, g is a **quasi-inverse** of f .

The isomorphisms $\text{Hom}_R(R, M) \cong M$ and $\text{Hom}_R(R, R) \cong R$ produce concepts of regularity in M_R and R which are exactly those used by earlier authors. The basic theme of this monograph is to generalize earlier results on rings and modules to homomorphism groups and to obtain new results on regularity in homomorphism groups that can then be specialized to rings and modules.

Chapter II contains basic properties of regular maps. Regular maps are associated with direct decompositions.

Corollary II.1.3 (Characterization of Regularity). $f \in \text{Hom}_R(A, M)$ is regular if and only if $\text{Ker}(f)$ is a summand A and $\text{Im}(f)$ is a summand M .

This characterization already establishes that groups and rings of matrices over fields are regular since matrices are linear transformations and kernels and images of linear transformations are direct summands.

Special quasi-inverses exist producing special decompositions.

Proposition II.1.6. Let $f \in \text{Hom}_R(A, M)$ be regular. Then there exists $g \in \text{Hom}_R(M, A)$ such that $f g f = f$ and $g f g = g$. If so, then

$$A = \text{Ker}(f) \oplus \text{Im}(g), \quad \text{and} \quad M = \text{Im}(f) \oplus \text{Ker}(g).$$

Given modules A_R and M_R , let $S = \text{End}(M_R)$ and $T = \text{End}(A_R)$. Then $\text{Hom}_R(A, M)$ is an S - T -bimodule and this is the setting in which we operate. Regular maps produce projective summands.

Theorem II.3.1. Let $0 \neq f \in \text{Hom}_R(A, M)$ be regular. Then the following statements hold.

- 1) Sf is a nonzero S -projective direct summand of ${}_S \text{Hom}_R(A, M)$ that is isomorphic to a cyclic left ideal of S that is a direct summand of S .
- 2) fT is a nonzero T -projective direct summand of $\text{Hom}_R(A, M)_T$ that is isomorphic to a cyclic right ideal of T that is a direct summand of T .

There exists a largest regular S - T -submodule of $H = \text{Hom}_R(A, M)$, denoted by $\text{Reg}(A, M)$. Here “largest” means that any other regular S - T -submodule of H is contained in $\text{Reg}(A, M)$.

Theorem II.4.3. $\text{Reg}(A, M) = \{f \in \text{Hom}_R(A, M) \mid SfT \text{ is regular}\}$ is the largest regular S - T -submodule of $\text{Hom}_R(A, M)$.

Previous authors studied regular rings and modules and modules over regular rings. We are now confronted with the more delicate problem of computing $\text{Reg}(A, M)$, and the corresponding specializations $\text{Reg}(M_R)$, $\text{Reg}(R)$, and $\text{Reg}(A, A) = \text{Reg}(\text{End}(A_R))$.

There are interesting results on the structure of $\text{Reg}(A, M)$.

Theorem II.4.6. Every finitely or countably generated S -submodule of $\text{Reg}(A, M)$ is a direct sum of cyclic S -projective submodules that are isomorphic to left ideals of S and these ideals are direct summands of S .

Every finitely generated S -submodule L of $\text{Reg}(A, M)$ is S -projective and a direct summand of ${}_S \text{Hom}_R(A, M)$.

The analogous results hold for $\text{Reg}(A, M)$ as a right T -module.

Proposition II.2.5. Every epimorphic image of ${}_S \text{Reg}(A, M)$ is S -flat, and every epimorphic image of $\text{Reg}(A, M)_T$ is T -flat.

Corollary II.4.8. Suppose that $\text{Reg}(A, M)$ contains no infinite direct sums of S -submodules. Then $\text{Reg}(A, M)$ is the direct sum of finitely many simple projective

S -modules, $\text{Hom}_R(A, M) = \text{Reg}(A, M) \oplus U$ for some S -submodule U and U contains no nonzero regular S - T -submodule. Every cyclic S -submodule of $\text{Reg}(A, M)$ is isomorphic to a left ideal of S that is a direct summand of S .

Corollary II.4.9. Suppose that $\text{Hom}_R(A, M)$ is regular. Then every finitely generated S -submodule L of $\text{Hom}_R(A, M)$ is S -projective and a direct summand of ${}_S \text{Hom}_R(A, M)$. Furthermore, L is the direct sum of finitely many cyclic S -projective submodules that are isomorphic to left ideals of S .

There is a connection between regular elements in $\text{Hom}_R(A, M)$ and regular elements in $\text{Hom}_R(M, A)$.

If $f \in \text{Hom}_R(A, M)$ and $g \in \text{Hom}_R(M, A)$ with $fgf = f$, then we call (f, g) a **regular pair**. Similarly, if $h \in \text{Hom}_R(M, A)$ and $k \in \text{Hom}_R(A, M)$ with $hkh = h$, then (h, k) is a regular pair. If (f, g) is a regular pair, then also (gfg, f) is a regular pair. We show that the so-called **transfer** $(f, g) \mapsto (gfg, f)$ produces all regular elements in $\text{Hom}_R(M, A)$ from those in $\text{Hom}_R(A, M)$.

We study “inherited regularity” when maps $\phi : A \rightarrow A'$ and $\mu : M \rightarrow M'$ are given and induce mappings on the homomorphism groups. The really important case of “inherited regularity” occurs when $A = A_1 \oplus \cdots \oplus A_m$ and $M = M_1 \oplus \cdots \oplus M_n$. We study the relationship between regular elements of $\text{Hom}_R(A, M)$ and those of its additive subgroups $\text{Hom}_R(A_i, M_j)$. It is convenient to identify $\text{Hom}_R(A, M)$, $\text{End}(A_R)$ and $\text{End}(M_R)$ with groups and rings of matrices.

The results for $m = n = 2$ are as follows.

Theorem II.6.14. Let $A = A_1 \oplus A_2$ and $M = M_1 \oplus M_2$ as before. We use the identifications

$$\text{Hom}_R(A, M) = \left\{ \begin{bmatrix} \xi_{11} & \xi_{21} \\ \xi_{12} & \xi_{22} \end{bmatrix}, \begin{array}{ll} \xi_{11} \in \text{Hom}_R(A_1, M_1), & \xi_{21} \in \text{Hom}_R(A_2, M_1), \\ \xi_{12} \in \text{Hom}_R(A_1, M_2), & \xi_{22} \in \text{Hom}_R(A_2, M_2). \end{array} \right\},$$

$$S = \text{End}_R(M) = \left\{ \begin{bmatrix} \mu_{11} & \mu_{21} \\ \mu_{12} & \mu_{22} \end{bmatrix} \mid \mu_{ij} \in \text{Hom}_R(M_i, M_j) \right\}$$

and

$$T = \text{End}_R(A) = \left\{ \begin{bmatrix} \alpha_{11} & \alpha_{21} \\ \alpha_{12} & \alpha_{22} \end{bmatrix} \mid \alpha_{ij} \in \text{Hom}_R(A_i, A_j) \right\}.$$

Then

$$\begin{bmatrix} \xi_{11} & \xi_{21} \\ \xi_{12} & \xi_{22} \end{bmatrix} \in \text{Reg}(A, M)$$

if and only if for all $\mu_{jt} \in \text{Hom}_R(M_j, M_t)$ and for all $\alpha_{si} \in \text{Hom}_R(A_s, A_i)$

$$\mu_{jt} \xi_{ij} \alpha_{si} \in \text{Reg}(A_s, M_t).$$

Theorem II.6.14 can be extended to arbitrary finite sums by induction. There is an interesting corollary.

Corollary II.6.18. Let $A = A_1 \oplus \cdots \oplus A_m$ and $M = M_1 \oplus \cdots \oplus M_n$. Then $\text{Hom}_R(A, M)$ is regular, i.e., $\text{Reg}(A, M) = \text{Hom}_R(A, M)$, if and only if

$$\forall i, j : \text{Hom}(A_i, M_j) = \text{Reg}(A_i, M_j).$$

In Chapter III we consider the case that either A or M is indecomposable and $\text{Reg}(A, M) \neq 0$. If this is the case much can be said about the structure of both modules.

Theorem III.1.1.

- 1) Suppose that there is $0 \neq f \in \text{Hom}_R(A, M)$ that is regular and μf is regular for all $\mu \in S = \text{End}(M_R)$. If M is directly indecomposable, then S is a division ring.
- 2) Suppose that there is $0 \neq f \in \text{Hom}_R(A, M)$ that is regular and $f\alpha$ is regular for all $\alpha \in T = \text{End}(A_R)$. If A is directly indecomposable, then T is a division ring.

Corollary III.2.3. Suppose that $0 \neq f \in \text{Reg}(A, M)$, and M is indecomposable. Then $\text{End}(M_R)$ is a division ring and either $A = K \oplus M_1 \oplus \cdots \oplus M_n$ where $M_i \cong M$, and $\text{Hom}_R(K, M) = 0$ or for every $i \in \mathbb{N}$ there is a decomposition $A = K_i \oplus M_1 \oplus \cdots \oplus M_i$ with $M_i \cong M$ and $K_i = K_{i+1} \oplus M_{i+1}$. In the latter case $\sum_{i=1}^{\infty} M_i = \bigoplus_{i=1}^{\infty} M_i$ and there is a homomorphism $\rho : A \rightarrow \prod_{i=1}^{\infty} M_i$ such that $\bigoplus_{i=1}^{\infty} M_i \subseteq \rho(A)$ and $\text{Ker}(\rho) = \bigcap_{i=1}^{\infty} K_i$.

There is a similar structure theorem when A is indecomposable (Corollary III.2.7).

In Chapter IV we specialize the general theory to the case $\text{Hom}_R(R, M) \cong M$. Then M is an S - R -bimodule where $S = \text{End}(M_R)$. An element $m \in M$ is **regular** with **quasi-inverse** $\varphi \in \text{Hom}_R(M, R)$ if $m = m\varphi(m)$. This means that $f \in \text{Hom}_R(R, M)$ is regular in $\text{Hom}_R(R, M)$ if and only if $f(1) \in M$ is regular in M .

Regular modules are special as the following example shows.

Theorem IV.1.2. Let $M \neq 0$ be a regular module over an integral domain R . Then R is a field. A vector space over a division ring is regular.

By means of the isomorphism $\text{Hom}_R(R, M) \cong M$ we can specialize our general results to immediately obtain numerous results on regularity in modules (Theorem IV.1.3, Theorem IV.1.4, Theorem IV.1.8, Theorem IV.3.1, Theorem IV.3.3, Theorem IV.3.4, Theorem IV.3.7).

In Chapter V we consider $H = \text{Hom}_R(A, M)$ as a bimodule but also as a one-sided module. Let $S = \text{End}(M_R)$ and $T = \text{End}(A_R)$. Then we have the bimodule ${}_S H_T$ and the one-sided modules ${}_S H$ and H_T and for each of these structures their concepts of regularity. Let $f \in H = \text{Hom}_R(A, M)$.

- f is **S -regular**, i.e., f is regular as an element of ${}_S H$, if and only if there exists $\sigma \in \text{Hom}_S(H, S)$ such that $(f\sigma)f = f$.
- f is **T -regular**, i.e., f is regular as an element of H_T , if and only if there exists $\tau \in \text{Hom}_T(H, T)$ such that $f(\tau f) = f$.

It turns out that the regularity of $f \in \text{Hom}_R(A, M)$ implies all other kinds of regularity.

Lemma V.2.1. Let $f \in \text{Hom}_R(A, M)$ be regular and let $g \in \text{Hom}_R(M, A)$ be such that $f g f = f$. Then the map

$$\sigma : {}_S \text{Hom}_R(A, M) \ni h \mapsto (h)\sigma = h \circ g \in {}_S \text{End}(M_R) = {}_S S$$

is a well-defined S -homomorphism and f is S -regular with quasi-inverse σ . The map

$$\tau : \text{Hom}_R(M, A)_T \ni h \mapsto \tau(h) = g \circ h \in \text{End}(A_R)_T = T_T$$

is a well-defined T -homomorphism and f is T -regular with quasi-inverse τ .

There are numerous immediate corollaries to our results on regularity for modules.

The existence of various largest regular submodules is an immediate consequence of Theorem II.4.3 and there are structure theorems that hold for the modules ${}_S H$ and H_T .

Theorem V.3.4. Let $H = \text{Hom}_R(A, M)$, let $S' = \text{End}(H_T)$ and suppose that N is a finitely generated S' -submodule of $\text{Reg}(H_T)$. Then N is an S' -projective direct summand of H . Furthermore, N is the direct sum of finitely many cyclic projective submodules, each of which is isomorphic to a left ideal of S' . The same is true for submodules of $\text{Reg}(H_T)$ considered as a right T -module.

In Chapter VI we introduce generalizations of regularity such as U -regularity and semiregularity. Semiregularity for modules was introduced by W.K. Nicholson ([26]). Zelmanowitz studied the class of regular modules, and Nicholson was interested in the wider class of semiregular modules. We list without proof some of his more striking results. Semiregularity is then defined for Hom and there are three different possibilities: semiregular, S -semiregular and T -semiregular. A sample result is as follows.

Theorem VI.4.2. If $f \in \text{Hom}_R(A, M)$ is semiregular, then Sf lies over an S -projective direct summand of ${}_S \text{Hom}_R(A, M)$.

In Chapter VII we consider additional substructures of $H = \text{Hom}_R(A, M)$ and study relations between them.

The **singular submodule** of ${}_S H_T$ is by definition the S - T -submodule $\Delta(A, M) = \{f \in \text{Hom}_R(A, M) \mid \text{Ker}(f) \text{ is large in } A\}$. The **cosingular submodule** of H is the S - T -submodule $\nabla(A, M) = \{f \in \text{Hom}_R(A, M) \mid \text{Im}(f) \text{ is small in } M\}$, and

there are three different concepts of radical in H : the radical of ${}_S H$ as an S -module, and similarly the radical of H_T as a T -module, and finally the most important radical for which there are two equivalent definitions: $\text{Rad}(\text{Hom}_R(A, M)) = \text{Rad}(A, M) = \{f \in H \mid f \text{Hom}_R(M, A) \subseteq \text{Rad}(S)\} = \{f \in H \mid \text{Hom}_R(M, A)f \subseteq \text{Rad}(T)\}$.

Proposition VII.2.4. If A or M is large restricted or injective, then $\Delta(A, M) \subseteq \text{Rad}(A, M)$.

Corollary VII.3.1.

- 1) If R_R or M_R is injective, then $\Delta(M_R) \subseteq \text{Rad}(M_R)$.
- 2) If R_R is injective, then $\Delta(R_R) \subseteq \text{Rad}(R)$.

There is a correspondence between ideals of endomorphism rings and bi-submodules of Hom .

Theorem VII.6.3. Let A and M be right R -modules, $S = \text{End}(M_R)$, and $T = \text{End}(A_R)$. Let $\mathcal{L}(S)$ denote the lattice of all two-sided ideals of S and $\mathcal{L}(H)$ the lattice of all S - T -submodules of H . Then

$$\text{Mdl} : \mathcal{L}(S) \rightarrow \mathcal{L}(H) : \text{Mdl}(I) = \{f \in H \mid f \text{Hom}_R(M, A) \subseteq I\}$$

and

$$\text{Idl} : \mathcal{L}(H) \rightarrow \mathcal{L}(S) : \text{Idl}(K) = \sum_{g \in \text{Hom}_R(M, A)} Kg$$

are inclusion preserving maps with

$$\text{Mdl}(\text{Idl}(\text{Mdl}(I))) = \text{Mdl}(I) \text{ and } \text{Idl}(\text{Mdl}(\text{Idl}(K))) = \text{Idl}(K).$$

In Chapter VIII we deal with regularity in homomorphism groups of abelian groups.

The theory of abelian groups is a highly developed subject so that conclusive answers are possible using the available tools.

We wish to compute the maximum regular bi-submodule of $\text{Hom}(A, M)$. A first step consists in checking when $\mathbb{Z}f$ is regular for a regular homomorphism $f \in \text{Hom}(A, M)$.

Proposition VIII.2.9. Let A be a group and let $f \in \text{Hom}(A, M)$. Then $\mathbb{Z}f = \{nf \mid n \in \mathbb{Z}\}$ is a regular subgroup of $\text{Hom}(A, M)$ if and only if $\text{Ker}(f)$ is a direct summand of A , $\text{Im}(f)$ is a direct summand of M , and for all $n \in \mathbb{Z}$,

$$\text{Im}(f) = n \text{Im}(f) \oplus \text{Im}(f)[n].$$

We can settle the torsion-free and torsion cases.

Theorem VIII.2.13. Let A and M be torsion-free abelian groups. Then $\text{Reg}(A, M) = 0$ unless A and M are both divisible. If both A and M are divisible, then $\text{Reg}(A, M) = \text{Hom}(A, M)$.

Theorem VIII.2.14. Let A and M be p -primary groups.

- 1) Suppose that M is not reduced. Then $\text{Reg}(A, M) = 0$.
- 2) Suppose that M is reduced. There are decompositions $A = A_1 \oplus A_2$ and $M = M_1 \oplus M_2$ such that A_1 and M_1 are elementary, and A_2 and M_2 have no direct summands of order p . Then
 - i) $\text{Reg}(A, M) = 0$ if $M_2 \neq 0$,
 - ii) $\text{Reg}(A, M) = 0$ if A_2 is not divisible,
 - iii) $\text{Reg}(A, M) = \text{Hom}(A, M)$ if $M_2 = 0$ and A_2 is divisible, i.e., if A is the direct sum of an elementary group and a divisible group and M is elementary.

As usual the general case of a mixed group is much more difficult. If $f \in \text{Reg}(A, M)$, then pf must be regular for every prime p . This fact alone has far-reaching consequences.

Theorem VIII.3.6. Let A and M be reduced abelian groups and assume that $\text{Reg}(A, M) \neq 0$. Then there exists a non-void set of primes $\mathbb{P}(A, M)$ such that

$$\forall p \in \mathbb{P}(A, M), \quad A = pA \oplus A[p], \quad A[p] \neq 0,$$

and

$$\forall p \in \mathbb{P}(A, M), \quad M = M[p] \oplus M', \quad M[p] \neq 0.$$

The equations $A = pA \oplus A[p]$ clearly are very restrictive.

Corollary VIII.3.10. Suppose that A is a reduced abelian group such that, for every prime $p \in \mathbb{P}$, there is a decomposition $A = pA \oplus A[p]$. Then $A \cong G$ where G is a group with $\bigoplus_{p \in \mathbb{P}} A[p] \subseteq G \subseteq \prod_{p \in \mathbb{P}} A[p]$ such that $G / \bigoplus_{p \in \mathbb{P}} A[p]$ is divisible.

If $\text{Reg}(A, M) \neq 0$ we are dealing with groups of the kind in Corollary 3.10 in the best of situations. These groups appear to be very concrete and accessible yet numerous studies have shown that they are very elusive. Let $\mathbf{t}(G)$ denote the maximal torsion subgroup of G .

For each $p \in \mathbb{P}$, let T_p be a zero or nonzero elementary p -group, set $P = \prod_{p \in \mathbb{P}} T_p$, $T = \bigoplus_{p \in \mathbb{P}} T_p = \mathbf{t}(P)$, and let $\mathbb{P}(T) = \{p \in \mathbb{P} \mid T_p \neq 0\}$. A group A is a $\mathcal{G}^*(T)$ -group if $T \subseteq A \subseteq P$ such that A/T is (torsion-free) and $\mathbb{P}(T)$ -divisible. A $\mathcal{G}^*(T)$ -group A will be called **slim** if T_p is cyclic or zero for every $p \in \mathbb{P}$.

Theorem VIII.3.16. Suppose that $A \in \mathcal{G}^*(T_1)$ and $M \in \mathcal{G}^*(T_2)$. Then $\mathbf{t}(\text{Hom}(A, M))$ is a regular bi-submodule of $\text{Hom}(A, M)$.

There are $\mathcal{G}^*(T)$ -groups A, M such that $\text{Hom}_R(A, M)$ contains non-regular homomorphisms.

The situation is somewhat easier if $A = M$, which is the study of regularity in endomorphism rings of mixed groups. Despite numerous attempts ([29], [12], [30],

[13], [24]), it is not known which mixed abelian groups have regular endomorphism rings but there is one conclusive result.

Theorem VIII.4.2. ([13, Theorem 4.1]) Let A be a slim $\mathcal{G}^*(T)$ -group such that A/T has finite rank and is divisible. Then $\text{End}(A)$ is regular.

The definition of regularity makes sense in any category and this is the topic of Chapter IX. In preadditive categories $\text{Reg}(A, M)$ exists as it does in module categories.

Theorem IX.1.3. Let \mathcal{C} be a preadditive category, A, M objects of \mathcal{C} , $S = \mathcal{C}(M, M)$ and $T = \mathcal{C}(A, A)$. Then $\text{Reg}(A, M) = \{f \in \mathcal{C}(A, M) \mid SfT \text{ is regular}\}$ is the largest regular S - T -submodule of $\mathcal{C}(A, M)$.

Preadditive categories are looked at in some detail, in particular, kernels, cokernels, images and coimages of morphisms in a category are needed for further discussions of regularity.

Let A be an object in a category \mathcal{C} and $e = e^2 \in \mathcal{C}(A, A)$. Then the **idempotent e splits in \mathcal{C}** if there exists an object M and mappings $\iota \in \mathcal{C}(M, A)$, $\pi \in \mathcal{C}(A, M)$ such that $\iota\pi = e$ and $\pi\iota = 1_M$. We say that **idempotents split in \mathcal{C}** if all idempotents in \mathcal{C} are splitting. The splitting of idempotents means that every idempotent determines a direct decomposition. By $A \in_{\mathcal{C}} A_1 \oplus A_2$ we mean that there exists a set of structural maps (insertions and projections) that make A the biproduct of A_1 and A_2 .

If idempotents split in a preadditive category \mathcal{C} , then we have the familiar decompositions associated with regular maps.

Theorem IX.2.7. Let \mathcal{C} be a preadditive category in which idempotents split and suppose that $f \in \mathcal{C}(A, M)$ is regular and $fgf = f$ for $g \in \mathcal{C}(M, A)$. Then the following statements hold.

- 1) $e = fg \in \mathcal{C}(M, M)$ is an idempotent.
- 2) $d = gf \in \mathcal{C}(A, A)$ is an idempotent.
- 3) There are structural maps $\iota_{A_i} : A_i \rightarrow A$, and $\pi_{A_i} : A \rightarrow A_i$ such that $d = \iota_{A_1}\pi_{A_1}$, $\pi_{A_1}\iota_{A_1} = 1_{A_1}$, $1_A - d = \iota_{A_2}\pi_{A_2}$, $\pi_{A_2}\iota_{A_2} = 1_{A_2}$, and $A \in_{\mathcal{C}} A_1 \oplus A_2$.
- 4) There are structural maps $\iota_{M_i} : M_i \rightarrow M$, and $\pi_{M_i} : M \rightarrow M_i$ such that $e = \iota_{M_1}\pi_{M_1}$, $\pi_{M_1}\iota_{M_1} = 1_{M_1}$, $1_M - e = \iota_{M_2}\pi_{M_2}$, $\pi_{M_2}\iota_{M_2} = 1_{M_2}$, and $M \in_{\mathcal{C}} M_1 \oplus M_2$.
- 5) $\pi_{M_1}f\iota_{A_1} : A_1 \rightarrow M_1$ is an isomorphism with inverse $\pi_{A_1}g\iota_{M_1}$.
- 6) $\iota_{A_2} \in \text{Ker}(f)$ and $\iota_{M_2} \in \text{Ker}(fg)$.
- 7) $\iota_{M_1} \in \text{Im}(f)$ and $\iota_{A_1} \in \text{Im}(gf)$.

In the preadditive category \mathcal{A} of torsion-free abelian groups of finite rank there is only one group whose endomorphism ring is a division ring, namely \mathbb{Q} ,

therefore regularity is not very interesting. Also there are highly non-unique direct decompositions in this category. To remedy this problem the **quasi-isomorphism category** $\mathbb{Q}\mathcal{A}$ was introduced by Bjarni Jonsson ([16], [17]). The objects of $\mathbb{Q}\mathcal{A}$ are the same as those of \mathcal{A} , namely all torsion-free abelian groups of finite rank, which is the same as all additive subgroups of finite-dimensional \mathbb{Q} -vector spaces. However, the morphism groups are now

$$\mathbb{Q}\mathrm{Hom}(A, M) = \{rf \mid r \in \mathbb{Q}, f \in \mathrm{Hom}(A, M)\} \subseteq \mathrm{Hom}(\mathbb{Q}A, \mathbb{Q}M)$$

where $\mathbb{Q}A$ and $\mathbb{Q}M$ are the vector subspaces spanned by A and M in the \mathbb{Q} -vector spaces in which they are contained as additive subgroups, and $\mathrm{Hom}(\mathbb{Q}A, \mathbb{Q}M)$ is just the groups of linear transformations. The category $\mathbb{Q}\mathcal{A}$ is a Krull-Schmidt category, i.e., every object is “uniquely” the direct sum of indecomposable objects. Also this category contains an abundance of groups whose endomorphism rings are division rings. The regularity picture in the category $\mathbb{Q}\mathcal{A}$ is transparent but there are no conclusive simple theorems because the indecomposable objects are too numerous, complex, and unknown.

Finally, we look at regularity in the category of (non-commutative) groups where semidirect products appear.

Theorem IX.5.2. Let G and H be groups and $f \in \mathrm{Hom}(G, H)$.

- 1) Assume that f is regular. Let $g \in \mathrm{Hom}(H, G)$ such that $f g f = f$. Then $e = g f \in \mathrm{End}(G)$ is an idempotent, $d = f g \in \mathrm{End}(H)$ is an idempotent and

$$G = \mathrm{Ker}(f) \rtimes \mathrm{Im}(e), \quad \mathrm{Im}(e) \cong \mathrm{Im}(f), \quad \text{and} \quad H = \mathrm{Ker}(d) \rtimes \mathrm{Im}(f).$$

- 2) Suppose that $G = \mathrm{Ker}(f) \rtimes K$ and $H = N \rtimes \mathrm{Im}(f)$. Then f is regular.



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