

Introduction

I1 More than one hundred years ago, Georg Frobenius [26] proved his remarkable theorem affirming that, *for a prime p and a finite group G , if the quotient of the normalizer by the centralizer of any p -subgroup of G is a p -group then, up to a normal subgroup of order prime to p , G is a p -group.* Of course, it would be an anachronism to pretend that Frobenius, when doing this theorem, was thinking the *category* — noted \mathcal{F}_G in the sequel — where the objects are the p -subgroups of G and the morphisms are the group homomorphisms between them which are induced by the G -conjugation. Yet Frobenius' hypothesis is truly meaningful in this category.

I2 Fifty years ago, John Thompson [57] built his seminal proof of the nilpotency of the so-called *Frobenius kernel* of a *Frobenius group* G with arguments — at that time completely new — which might be rewritten in terms of \mathcal{F}_G ; indeed, some time later, following these kind of arguments, George Glauberman [27] proved that, under some — rather strong — hypothesis on G , the normalizer N of a suitable nontrivial p -subgroup of G *controls fusion* in G , which amounts to saying that *the inclusion $N \subset G$ induces an equivalence of categories $\mathcal{F}_N \cong \mathcal{F}_G$.*

I3 Thus, when about forty years ago we start working on finite groups, these kind of results pushed us to introduce the *Frobenius category* \mathcal{F}_G in the language of Claude Chevalley's seminar. At the beginning, it was essentially a language to guide our research, as for instance in our refinement [35, Ch. II and III] of the *Alperin Fusion Theorem* [1]. Moreover, we quickly realized that, in order to have a better insight into the structure of G — for instance, to follow Thompson's arguments in Chapter IV of the so-called *Odd paper* [25] — we had to consider suitable *extensions* determined by G of this category, namely what in [35, Ch. VI] we call “*localités à épimorphismes*”.

I4 The next step in the gestation of the ideas contained in this book came thirty years ago, when Michel Broué talked us about his will of extending to the *Brauer blocks* our Frobenius categories for groups. As a matter of fact, we found that Richard Brauer already had partially realized this project in [9]; indeed, in this paper, for what Brauer calls a *block of characters* of G , he considers a new *order relation* refining the inclusion between a suitable set of subgroups of a *defect group* of the block, already proving a suitable generalization of the *Alperin Fusion Theorem* in the new context.

15 Then, Michel Broué[†] reformulated the new inclusion introduced by Brauer in terms of *pairs* — which freed Broué from the choice of a *defect group* — formed by a p -subgroup P of G and a *block of characters* e (for short, a *block* in the sequel) of the centralizer $C_G(P)$ of P , and extended the new inclusion to *all* these pairs — ever since called *Brauer pairs*. At this point, we already had a *Frobenius category* for a block b of G , namely the category $\mathcal{F}_{(b,G)}$ where the objects are all the Brauer pairs (P, e) *containing* $(\{1\}, b)$ — with respect to the new inclusion — and where, again, the morphisms are the group homomorphisms between the p -subgroups induced by the G -conjugation, up to Brauer-Broué's inclusion.

16 The miracle happened again: the Frobenius Hypothesis on $\mathcal{F}_{(b,G)}$ — *all the automorphism groups of the objects are p -groups* — was meaningful too; in this situation — where b is called a *nilpotent block* — with Michel Broué we proved in [12] that there is a bijection between the set of ordinary *irreducible characters* of G in b and the set of ordinary *irreducible characters* of any defect group of b . This immediately implies that the *block algebra* over \mathbb{C} is *Morita equivalent* to the group \mathbb{C} -algebra of the defect group of b ; in other words, denoting by (P, e) a maximal Brauer pair containing $(\{1\}, b)$ and identifying b with the corresponding central idempotent, it implies that *the categories of $\mathbb{C}Gb$ - and $\mathbb{C}P$ -modules are equivalent*.

17 However, Broué guessed something more precise, namely that even over a complete discrete valuation ring \mathcal{O} , the categories of $\mathcal{O}Gb$ - and $\mathcal{O}P$ -modules were equivalent. In our joint work [12], we already proved that the centers $Z(\mathcal{O}Gb)$ and $Z(\mathcal{O}P)$ were isomorphic which, as Everett Dade pointed out to us, proved Broué's conjecture whenever P is Abelian. Soon after, following James Green's approach [29], we came to the idea of the *source algebra* of the block b — which is more than an \mathcal{O} -algebra: it is the \mathcal{O} -algebra $i(\mathcal{O}Gb)i$ endowed with the homomorphism mapping $u \in P$ on ui , for a primitive idempotent i of $(\mathcal{O}Gb)^P$ such that $\text{Br}_P(i) \neq 0$ (cf. 1.12) — and succeeded in determining the source algebra of a *nilpotent block* [36], which in particular proved Broué's conjecture.

18 As for the Frobenius Theorem mentioned above, it could be claimed that, strictly speaking, the category $\mathcal{F}_{(b,G)}$ is unnecessary to define a *nilpotent block* since we are just assuming that the quotient of the normalizer by the centralizer of any Brauer pair (Q, f) fulfilling $f\text{Br}_Q(b) \neq 0$ (cf. 1.13) is a p -group. Yet, the existence of the *hyperfocal subalgebra* in the source algebra of a block b — proved in [49] ten years ago — involves $\mathcal{F}_{(b,G)}$ more seriously.

[†] A point of history. Before his stay at Chicago University where his joint paper with Jon Alperin [3] comes from, Michel Broué already had given a complete account of his reformulation in Chevalley's seminar.

I9 Indeed, it is well-known that the direct limit $\varinjlim \mathcal{F}_G$ is the maximal Abelian p -quotient of G [28, Ch. 7, Theorem 3.4]: pushing it further, from the Frobenius category \mathcal{F}_G it is possible to compute, inside a Sylow p -subgroup, its intersection — called a *hyperfocal subgroup* of G — with the kernel of the maximal p -quotient of G . Mimicking this computation in $\mathcal{F}_{(b,G)}$, we can define the *hyperfocal subgroup* $H_{(b,G)}$ of the block b of G ; the point is that *there exists an essentially unique P -stable subalgebra of the source algebra $i(kGb)i$ which intersects Pi in $H_{(b,G)}i$ and, together with Pi , generates the whole source algebra* [51, Theorems 14.7 and 15.10].

I10 However, the step which led us to seek for an abstract setting behind all these constructions was the discovery — twenty years ago — of the *localizer* of a *selfcentralizing* Brauer pair, together with the *localizing functor* over the category of *chains* of such Brauer pairs [44]. The *selfcentralizing* Brauer pairs (Q, f) are exactly those considered by Brauer in [9] and one of their possible definitions — justifying their name — is that $C_P(Q) \subset Q$ for any Brauer pair (P, e) containing (Q, f) ; in this case, it follows from [34] that there exists a suitable extension $L_G(Q, f)$ — the *localizer* of (Q, f) — of the quotient $N_G(Q, f)/C_G(Q)$ by the center $Z(Q)$. In the Frobenius category of a finite group G , these extensions were just the *automorphism group of the objects* in some of the “*localités à épimorphismes*” of G mentioned above.

I11 At that time there appeared the paper by Reinhard Knörr and Geoffrey Robinson [33] where they reformulated Jon Alperin’s Conjecture [2] in terms of an alternating sum over a set of *chains* of Brauer pairs — we are more precise below. Thus, quite naturally we considered the localizers $L_G(\mathfrak{q})$ of *chains* \mathfrak{q} — here *chain* stands for *totally ordered set throughout the inclusion* — of selfcentralizing Brauer pairs, and in [44] we prove that this correspondence can be extended to a *functor* from the suitable category of chains — where the morphisms are defined by the *set inclusion* and the *G -conjugation* — to the category of finite groups up to conjugation, namely to the *exterior quotient* $\widetilde{\mathfrak{Gr}}$ (cf. 1.3).

I12 Throughout all this work, it became clear that many arguments were *inner* arguments, in the sense that the blocks around it played no significant role, and we decided to look for a suitable abstract formulation. Actually, this was a reason to delay publication of [44] since we could hope to recover its contents in a more general setting, as we did. A key point of the endeavour to find such an axiomatic approach was the possibility to come back to Brauer’s point of view in [9], namely to the subgroups of a defect group; indeed, as for \mathcal{F}_G , if (P, e) is a maximal Brauer pair containing $(\{1\}, b)$, then $\mathcal{F}_{(b,G)}$ is equivalent to the full subcategory over the set $\mathfrak{B}_{(P,e)}$ of Brauer pairs (Q, f) contained in (P, e) and *the correspondence mapping (Q, f) on Q is an order-preserving bijection between $\mathfrak{B}_{(P,e)}$ and the set of subgroups of P* [1, Theorem 3.4].

I13 In autumn 1990, we began to work in this direction when doing a series of lectures [44] at the MSRI, and in spring 1991, invited by James Green at Warwick University, we already could give a first definition of an abstract *Frobenius category*[†]. The starting point was obvious: from a finite p -group P , we had to consider a subcategory of the category of finite groups \mathfrak{Gr} defined over the set of subgroups of P , the problem being to find suitable conditions tightening the situation enough to remain *near* the Frobenius categories of blocks of finite groups.

I14 That is to say, it was not difficult to imagine reasonable necessary conditions, yet they should allow us to go far toward mimicking the usual constructions in groups... till where? Although the definition did not change from spring 1991, we spent some time developing the machinery of normalizers, centralizers, quotients, the translation of our refinement of Alperin's Fusion Theorem... till we were able to state a reasonable criterion of *simplcity* — in the sense that, up to a normal subgroup of order prime to p , a finite group G is simple whenever the Frobenius category of G fulfills this abstract criterion (cf. 12.20). We wrote all this in a manuscript [46] which, in its first half, essentially covered chapters 2, 4, 5 and 12 below.

I15 In chapter 2, we state the conditions defining a *Frobenius P -category* — conditions admitting different equivalent forms — and show the existence of *normalizers* and *centralizers* of the subgroups of P , fulfilling the same conditions. The title of this chapter — *Frobenius P -categories: the first definition* — suggests that we have, at least, another definition; indeed, related with our effort for proving the existence of a *perfect locality* explained below, we found a quite different equivalent definition of a Frobenius P -category, stated in chapter 21. Although simpler from the formal point of view, this definition is farther from our main intuition and, for the moment, it takes second place.

I16 As announced in its title, the purpose of this book is not only to develop our abstract setting but to apply it to a better understanding of *Brauer blocks*. In chapter 3 we prove that the category $\mathcal{F}_{(b,G)}$ considered above (cf. I5) fulfills the conditions of a Frobenius P -category and, more generally, we illustrate in this case all the concepts introduced in 2.

I17 In chapter 4 we come back to the abstract Frobenius P -categories \mathcal{F} to introduce the *selfcentralizing* objects, mimicking the definition of selfcentralizing Brauer pairs mentioned above (cf. I10); indeed, these objects play the most important role in the structure of \mathcal{F} and, actually, the full subcategory \mathcal{F}^{sc} of \mathcal{F} over the set of them determines \mathcal{F} , as we prove in this chapter. But, in many arguments, the selfcentralizing Brauer pairs (Q, f) play a

[†] At present, there is some confusion with the terminology employed around this concept. Although we hope that this book will contribute to fix the original one, we mention some alternative names in footnotes and they can be found in *italics* in the Index.

role simply because f is a *nilpotent block* of $C_G(Q)$ (cf. I6), and this condition is preserved in a quotient of G by a central p -subgroup Z , whereas the self-centralizing condition need not. Analogously, we introduce the *nilcentralized* objects — we note \mathcal{F}^{nc} the full subcategory over the set of them; this is possible since we already have *centralizers* of the objects in \mathcal{F} and the *nilpotency* of a Frobenius P -category makes sense *via* the Frobenius Hypothesis above.

I18 Chapters 5 and 6 further illustrate the importance of selfcentralizing objects. In chapter 5 we develop in \mathcal{F} our refinement of the Alperin Fusion Theorem mentioned above, introducing the *essential* objects; actually, these objects already can be introduced in simpler structures called *divisible P -categories*, and then a suitable formulation of the Alperin Fusion Theorem becomes a necessary and “almost” sufficient condition to get a Frobenius P -category. In this case, the essential objects are selfcentralizing, proving again that \mathcal{F} is completely determined by the subcategory \mathcal{F}^{sc} . Moreover, in our approach we show that the Alperin Fusion Theorem concerns the *additive category* $\mathbb{Z}\mathcal{F}$ — where the morphisms between a pair of subgroups Q and R of P are the free \mathbb{Z} -modules over $\mathcal{F}(Q, R)$ and the composition is distributive — rather than \mathcal{F} itself, and it is somehow related to the *projective resolution* of the *trivial contravariant* functor over \mathcal{F} ; we believe that this relationship deserves more consideration.

I19 Chapter 6 exploits a remarkable feature of the *exterior quotient* $\tilde{\mathcal{F}}^{\text{sc}}$ of \mathcal{F}^{sc} — the quotient of \mathcal{F}^{sc} by the inner automorphisms of the corresponding subgroups of P (cf. 1.3) — namely the fact that in this category any morphism is an *epimorphism*. This leads to a canonical *partition* of the set of morphisms between two objects relative to a morphism with the same origin (cf. Proposition 6.7); then, the existence of these partitions implies the existence of a *direct product* in the *additive cover* $\mathbf{ac}(\tilde{\mathcal{F}}^{\text{sc}})$ of $\tilde{\mathcal{F}}^{\text{sc}}$ (cf. 6.2) — a construction introduced by Stefan Jackowski and James McClure in [32]. As a consequence, we get a *vanishing result for positive cohomology*, which is the key for the determination of the rank of the *Grothendieck groups associated with \mathcal{F}* in chapter 14.

I20 As in chapter 3, in chapter 7 we analyze the *nilcentralized* — and the *selfcentralizing* — objects mentioned above (cf. I17), inside the category $\mathcal{F}_{(b,G)}$ associated with a block b of G . As a matter of fact, any nilcentralized Brauer pair (Q, f) fulfilling $f\text{Br}_Q(b) \neq 0$ (cf. 1.13) appears associated with two meaningful *invariants*, namely with a central extension $\hat{\mathcal{F}}_{(b,G)}(Q)$ of the automorphism group

$$\mathcal{F}_{(b,G)}(Q) \cong N_G(Q, f)/C_G(Q) \quad \text{I20.1}$$

by k^* — where k is a fixed algebraically closed field of characteristic p — and with a *Dade $\bar{N}_P(Q)$ -algebra* S_Q — a simple k -algebra endowed with an action of $\bar{N}_P(Q) = N_P(Q)/Q$ which stabilizes a basis containing the unity element (cf. 1.20).

I21 As we explain below, from the point of view of Alperin's Conjecture [2] we were interested in the central extension $\hat{\mathcal{F}}_{(b,G)}(Q)$ rather than in the group $\mathcal{F}_{(b,G)}(Q)$ itself, and this raised a huge problem of coherence. Explicitly, for any *chain* \mathfrak{q} (cf. I11) of nilcentralized Brauer pairs fulfilling $f\mathrm{Br}_Q(b) \neq 0$ (cf. 1.12), we can define $\mathcal{F}_{(b,G)}(\mathfrak{q})$ as the stabilizer of \mathfrak{q} in $N_G(Q, f)/C_G(Q)$ for the maximal element (Q, f) in this chain, and then $\hat{\mathcal{F}}_{(b,G)}(\mathfrak{q})$ is the corresponding converse image. It was clear that the map sending \mathfrak{q} to $\mathcal{F}_{(b,G)}(\mathfrak{q})$ might be extended to a functor from the corresponding category of chains to \mathfrak{Gr} ; yet, from the point of view of Alperin's Conjecture, this functor was useless unless we were able to get a *lifting* of it sending \mathfrak{q} to $\hat{\mathcal{F}}_{(b,G)}(\mathfrak{q})$.

I22 Since the eighties, we knew how to prove the existence of a lifting

$$\hat{\mathcal{F}}_{(b,G)}(\mathfrak{q}) \longrightarrow \hat{\mathcal{F}}_{(b,G)}(\mathfrak{r}) \quad \text{I22.1}$$

for *any* morphism $\mathfrak{r} \rightarrow \mathfrak{q}$ between chains (cf. Theorem 7.16), and this was already announced in [44]; it is a consequence of a Splitting Theorem that Everett Dade announced in 1979 at the Santa Cruz Conference [20] and never published (cf. Theorem 9.21 below or [40]). But the huge problem is to prove that it is possible to do *coherent* choices in order to get a functor, a question that we only have solved when preparing this book.

I23 In the meanwhile, we tentatively have followed two strategies. In one of them, we tried to improve our proof in [40] of Dade's Splitting Theorem, showing that it was possible to make a choice once for ever — a *polarization* — in the set of *equivalence* classes of Dade P -algebras for any finite p -group P , which then we could apply to our problem. Although what we obtained in this direction did not solve the problem, we explain our result in chapter 9; in particular, it includes a proof of Dade's Splitting Theorem somewhat different and more detailed than in [40]. Previously, in chapter 8 we recall the main facts on Dade P -algebras we need in the sequel.

I24 Still since the eighties, we already knew that the existence of a $\mathcal{F}_{(b,G)}(\mathfrak{q})$ -stable Dade P -algebra S *gluing* together all the Dade $\bar{N}_P(Q)$ -algebras S_Q — namely, a Dade P -algebra S such that $\mathrm{Res}_Q^P(S)$ and $\mathrm{Res}_\varphi(S)$ are *equivalent* for any nilcentralized Brauer pair (Q, f) contained in (P, e) and any $\mathcal{F}_{(b,G)}$ -morphism φ from (Q, f) to (P, e) , and that S_Q is *equivalent* to the *Brauer quotient* $S(Q)$ (see 11.6) — easily would solve our problem. Indeed, replacing kGb by the P -algebra $S^\circ \otimes_k \mathrm{Res}_P^G(kGb)$, we could find inside the *same* extensions $\hat{\mathcal{F}}_{(b,G)}(\mathfrak{q})$ and $\hat{\mathcal{F}}_{(b,G)}(\mathfrak{r})$ — up to *noncanonical* isomorphisms — and then this P -algebra provided coherent *extension group homomorphisms*

$$\hat{\mathcal{F}}_{(b,G)}(\mathfrak{q}) \longrightarrow \hat{\mathcal{F}}_{(b,G)}(\mathfrak{r}) \quad \text{I24.1}$$

(see Proposition 11.8 and Theorem 11.10 for the precise arguments).

I25 Thus, the other strategy was to look for such a Dade P -algebra. One way to find S was from a conjectural *gluing* result on Dade P -algebras, which is indeed true when P is Abelian [45]; but presently, after the complete classification of the Dade P -algebras in [17] and [6], such a general result seems impossible. In chapter 10 we state a result in this direction which should be combined with a suitable *polarization*... that does not exist. Another way to find S could be from the P -algebra $\text{Res}_P^G(kGb)$ itself and, although we have no construction to propose, we cannot close this possibility.

I26 Actually, the existence of such a Dade P -algebra S also would prove the existence of a suitable k^* -extension $\hat{\mathcal{F}}^{\text{nc}}$ of \mathcal{F}^{nc} — a question raised by Markus Linckelmann in the 2002 Durham Symposium — which naturally appears inside $S^\circ \otimes_k \text{Res}_P^G(kGb)$. It is this fact that gives a solution to our problem: from the Dade $\bar{N}_P(Q)$ -algebras S_Q themselves, we are able to construct k^* -extensions of suitable subcategories of \mathcal{F}^{nc} and the point is that these k^* -extensions are quite independent of our choice of Dade algebras; although we do not get a complete k^* -extension $\hat{\mathcal{F}}^{\text{nc}}$, we get a complete coherent choice for the *extension group homomorphisms* I24.1. All this is explained in chapter 11.

I27 In chapters 12 and 13, we pursue the development of the abstract Frobenius P -categories. In chapter 12 we discuss the existence of *quotients* of a Frobenius P -category \mathcal{F} by suitable subgroups of P . If $\mathcal{F} = \mathcal{F}_G$ is the Frobenius category of a finite group G , any Sylow p -subgroup of a normal subgroup of G naturally determines one of those quotients of \mathcal{F} ; although the converse is not true, using the Classification of Finite Simple Groups, it is not difficult to check that, provided P is not Abelian, there are not so many exceptions. But the main purpose of this chapter is to determine the p' -*quotients* of \mathcal{F} — the quotients reduced to a finite group of order prime to p — since we need them to state the *simplicity* criterion (cf. 12.20) and to talk about *solvability* in Frobenius P -categories. As a matter of fact, all the p' -quotients of \mathcal{F} have a *kernel* which is also a Frobenius P -category, and there is a smallest such Frobenius P -category (cf. Corollary 12.17).

I28 In chapter 13 we discuss the p -*quotients* of \mathcal{F} ; as implicitly mentioned in I9 above, it is still possible to define the *hyperfocal subgroup* $H_{\mathcal{F}}$ of P in \mathcal{F} ; once again, $P/H_{\mathcal{F}}$ is indeed the maximal p -*quotient* of \mathcal{F} and there exists a suitable Frobenius $H_{\mathcal{F}}$ -category \mathcal{F}^{\flat} — called the *hyperfocal* subcategory of \mathcal{F} . Then, the contents of these two chapters allow us to state a definition of *solvability* for Frobenius P -categories, and one main point is that a *solvable Frobenius P -category is necessarily the Frobenius category of a p -solvable finite group*, which responds positively to our requirement of remaining near the Frobenius categories of finite groups (cf. I13). This result does not appear till chapter 19 since its proof needs the existence of the *localizer* proved in chapter 18.

I29 Coming back to the Frobenius category $\mathcal{F}_{(b,G)}$ of a block (b,G) , we already know from chapter 11 that there is a functor from the *proper category of* $(\mathcal{F}_{(b,G)})^{\text{nc}}$ -chains (cf. A2.8) to the category $k^*\text{-}\mathfrak{Gr}$ of the *central* k^* -extensions of finite groups sending a $(\mathcal{F}_{(b,G)})^{\text{nc}}$ -chain \mathfrak{q} to the k^* -extension $\hat{\mathcal{F}}_{(b,G)}(\mathfrak{q})$ above (cf. I21). On the other hand, the *modular Grothendieck group* evidently defines a *contravariant* functor from $k^*\text{-}\mathfrak{Gr}$ to the category of free \mathbb{Z} -modules. A main fact stated in this book is that *the inverse limit of the composition of both functors* — called the *Grothendieck group* of $\mathcal{F}_{(b,G)}$ (cf. 14.3) — is a free \mathbb{Z} -module tightly related with Alperin's Conjecture.

I30 Let us be more explicit. Recall that, following Alperin [2], a *weight* of the block b is a pair (Q, χ) formed by a p -subgroup Q of G and by an irreducible character χ of $N_G(Q)$, associated with $\text{Br}_Q(b)$ (cf. 1.13), which comes from a modular irreducible character $\bar{\chi}$ in a block of $\bar{N}_G(Q) = N_G(Q)/Q$ of *defect zero* (cf. 1.17). Then, Alperin's Conjecture affirms that *the number of G -conjugacy classes of weights of the block b coincides with the number $|\text{Irr}_k(G, b)|$ of modular irreducible characters in b .*

I31 But, for a weight (Q, χ) , an $\bar{N}_G(Q)$ -module M affording $\bar{\chi}$ is simple and projective (cf. 1.17), and therefore its restriction to $\bar{C}_G(Q)$ is semisimple and projective too, determining a set of blocks \bar{f} of $\bar{C}_G(Q)$ of *defect zero* (cf. 1.17). That is to say, lifting \bar{f} to a block f in $Z(kC_G(Q))$, (Q, f) is a *self-centralizing Brauer pair* (cf. I10 and Corollary 7.3) and, up to G -conjugation, may assume it is contained in (P, e) ; moreover, it is not difficult to prove that *the set of weights determining the same selfcentralizing Brauer pair (Q, f) bijectively correspond with the set of isomorphism classes of $k_*\hat{\mathcal{F}}_{(b,G)}(Q)$ -modules, both simple and projective* [53, Theorem 3.7]. Consequently, denoting by $\text{IrPr}_k(\hat{\mathcal{F}}_{(b,G)}(Q))$ the corresponding set of modular characters, Alperin's Conjecture affirms

$$|\text{Irr}_k(G, b)| = \sum_Q |\text{IrPr}_k(\hat{\mathcal{F}}_{(b,G)}(Q))| \quad \text{I31.1}$$

where Q runs over a set of representatives for the $\mathcal{F}_{(b,G)}$ -isomorphism classes of $\mathcal{F}_{(b,G)}$ -selfcentralizing objects.

I32 At this point, the arguments of Reinhard Knörr and Geoffrey Robinson in [33] prove, in the language above, that Alperin's Conjecture is equivalent to the equality

$$|\text{Irr}_k(G, b)| = \sum_{\mathfrak{q}} (-1)^{|\mathfrak{q}|-1} |\text{Irr}_k(\hat{\mathcal{F}}_{(b,G)}(\mathfrak{q}))| \quad \text{I32.1}$$

where \mathfrak{q} runs over a set of representatives for the $\mathcal{F}_{(b,G)}$ -isomorphism classes of $(\mathcal{F}_{(b,G)})^{\text{sc}}$ -chains (cf. I11). But, in chapter 14 we prove that *the \mathbb{Z} -rank of the Grothendieck group of $\mathcal{F}_{(b,G)}$ coincides with the right-hand member of this*

equality — actually, the statement makes sense for any Frobenius P -category \mathcal{F} endowed with an analogous functor from the category of nilcentralized \mathcal{F} -chains to $k^*\text{-}\mathfrak{Gr}$, and we prove that equality in this general context. Thus, Alperin's Conjecture is equivalent to the assertion that *the \mathbb{Z} -ranks of the Grothendieck groups of the block (b, G) and the category $\mathcal{F}_{(b, G)}$ coincide.*

I33 Everyone understands that the sentence “*an alternating sum of ranks which coincides with the rank of an inverse limit*” necessarily suggests the possible existence of a *differential complex* with a unique nonzero *cohomology group* at degree zero. But, it has to be noticed that our *contravariant* functor mapping an $(\mathcal{F}_{(b, G)})^{\text{sc}}$ -chain \mathbf{q} on the *Grothendieck group* of $\hat{\mathcal{F}}_{(b, G)}(\mathbf{q})$ does *not* come from a *contravariant* functor defined over the category $(\hat{\mathcal{F}}_{(b, G)})^{\text{sc}}$; in other words, we are not dealing with the usual *cohomology groups* of $(\hat{\mathcal{F}}_{(b, G)})^{\text{sc}}$. Moreover, our sum runs over a set of representatives for a set of suitable $\mathcal{F}_{(b, G)}$ -isomorphism classes and this fact has to be integrated in our hypothetical differential complex.

I34 All these remarks forced us to enlarge the usual construction of the cohomology groups of a category \mathfrak{C} in order to include our situation; the new cohomology groups we consider need not fulfill the *long exact sequence* condition (cf. A3.11.4), but they are useful for our purposes. We explain our point of view — which possibly has been already employed in other situations — in the Appendix. We have adopted the language of the 2-categories since, when constructing the cohomology groups of \mathfrak{C} , we dislike expressions such as “*consider a sequence of n \mathfrak{C} -morphisms which can be composed*”, that we replace by “*consider a functor from Δ_n to \mathfrak{C}* ”; but, if the *simplex* Δ_n becomes a *category* then the *simplicial category* Δ becomes a 2-category...

I35 Let us come back to our discussion on Alperin's Conjecture. Of course, two free \mathbb{Z} -modules with the same \mathbb{Z} -rank are isomorphic, and therefore Alperin's Conjecture is also equivalent to the assertion that *the Grothendieck groups of the block (b, G) and the category $\mathcal{F}_{(b, G)}$ are isomorphic*. But, the stabilizer $\text{Out}(G)_b$ of b in the group of *outer automorphisms* of G has a natural action over both Grothendieck groups and then an obvious question arises: *is there an $\text{Out}(G)_b$ -stable isomorphism between the Grothendieck groups of the block (b, G) and the category $\mathcal{F}_{(b, G)}$* ? Actually, even a positive answer for a suitable scalar extension of the Grothendieck groups would be welcome.

I36 A good indication towards a positive answer to this question is that, up to a suitable scalar extension, both Grothendieck groups have the *same* behaviour throughout the restriction to the normal subgroups, provided we can “follow” the subcategory $(\mathcal{F}_{(b, G)})^{\text{nc}}$ in the normal subgroups; we expose our reduction results in chapter 15. Here there appears a significant difference between our method and the method which consists of restricting any irreducible character in the block b individually; indeed, in the second one,

we are forced to apply the so-called *Clifford Theory* [31, Ch. V] which involves the unknown *Clifford extensions* of the stabilizers, which makes any tentative induction enormously difficult.

I37 In chapter 16 we develop a strategy toward reducing a possible positive answer to the question above, to a positive *verification* of the same question “around” the noncommutative simple groups; by “around” we mean that, for any noncommutative simple group S , we have to consider the central k^* -extensions of suitable subgroups of $\text{Aut}(S)$ containing S . Our strategy itself already needs the Classification of Simple Groups since it quotes some facts which are only known from this classification, as for instance the *solvability* of $\text{Out}(S)$. The precise result is stated in Theorem 16.45.

I38 The last part of this book deals again with an abstract Frobenius P -category \mathcal{F} . In the second half of our manuscript [46], we investigated to what extent we still got *localizers* together with the *localizing functor* (cf. I10) in our abstract setting — the main purpose and the crucial test in building it. The answer had been “almost” positive — as we explain in 18.5, it remained to prove that some 1-*cocycle* was a 1-*coboundary*, which now is done — and, since it was not reasonable to foresee a finite group as a possible direct limit of the *localizing functor*, we considered the possibility of the existence of a *topological space* as, roughly speaking, a direct limit of the functor defined by the *classifying spaces* of the *localizers*.

I39 In 1994, we proposed this idea to Dave Benson, who already had constructed a *topological space* [5] from a configuration considered by Ron Solomon when discussing finite simple groups with the same Sylow 2-subgroups P as the third Conway’s group [56]; actually, as it could be expected, Solomon’s configuration is nothing but a Frobenius P -category [13]. After a while, Benson raised the question of the existence of an *extension* \mathcal{L}^{sc} of the full subcategory \mathcal{F}^{sc} mimicking a suitable “*localité à épimorphismes*” (cf. I3), namely having the *localizer* as the automorphism group of any selfcentralizing object.

I40 After Benson’s publication [5], Carles Broto, Ran Levi and Bob Oliver became interested in our manuscript [46][†] in order to prove that the topological space coming from the category \mathcal{L}^{sc} guessed by Benson had good enough properties to be a “*classifying space*” of \mathcal{F} ; in [13] they proved the “good properties” of \mathcal{L}^{sc} but did not succeed in proving its existence and its uniqueness, just giving some sufficient conditions.

[†] A point of history. In december 1999, coming back from Wuhan, we found an e-mail sent by Bob Oliver asking us for a copy of our manuscript. We personally gave him a copy on the basis of a possible collaboration. Only in October 2000, did we learn that Carles Broto and Ran Levi were not only interested but already deeply engaged in our manuscript.

I41 It has to be understood that the existence of a suitable extension of \mathcal{F}^{sc} , or even of \mathcal{F} , already supplies *localizers* and *localizing functors* — namely, the *automorphism groups* of the objects in such an extension and the *automorphism group functor* over the corresponding category of chains (cf. Proposition A2.10). Thus, in chapter 17 we systematically consider extensions of \mathcal{F} — called \mathcal{F} -localities since they generalize our point of view in [35] — which we would like to be *determined* by \mathcal{F} and P ; for some precise meaning of the word *determined*, this condition imposes a biggest possibility that we call *perfect \mathcal{F} -locality* and corresponds to the category expected by Benson. In particular, we prove that if \mathcal{F} holds a perfect \mathcal{F} -locality \mathcal{L} , then any quotient $\bar{\mathcal{F}}$ of \mathcal{F} as considered in chapter 12 holds a perfect $\bar{\mathcal{F}}$ -locality $\bar{\mathcal{L}}$ defined as a quotient of \mathcal{L} ; of course, had we the existence and uniqueness of a perfect \mathcal{F} -locality such a result would be redundant.

I42 In this case, chapter 18 on the *localizers* would be somewhat redundant too, since the *localizers* we announce would be nothing but the automorphism groups of the objects in this category, as we said above. But, the *localizer* of a selfcentralizing object admits a *direct* group-theoretical characterization, given in [46] and coming from [34], which deserves to be stated. Moreover, as we mention above, this result is useful to prove, in chapter 19, that a solvable Frobenius P -category is necessarily the Frobenius category of a p -solvable finite group. In chapter 18 we also prove the existence and the uniqueness of the *localizing functor* $\text{loc}_{\mathcal{F}}$ mentioned above (cf. I10), together with some kind of “universality” of it, which is quite useful in chapter 23.

I43 In chapter 20 we prove that the existence of a perfect \mathcal{F}^{sc} -locality \mathcal{L}^{sc} forces the existence of a perfect \mathcal{F} -locality \mathcal{L} by a direct *necessarily unique* construction of \mathcal{L} from \mathcal{L}^{sc} ; we obviously proceed by induction, but cannot avoid the distinction between the *fully centralized* subgroups of P (cf. 2.6) and the others, as we cannot avoid the distinction between normal and ordinary inclusions. All this generates a long proof even if it is nothing but routine. Does there exist a general result guaranteeing that some kind of properties of \mathcal{F}^{sc} can be extended to \mathcal{F} ?

I44 In chapter 21 we expose the second definition of a Frobenius P -category \mathcal{F} , which leads to the *basic \mathcal{F} -locality*. This equivalent definition comes from an original contribution of Broto, Levi and Oliver[†] to the behaviour of a Frobenius P -category \mathcal{F} , namely the existence of a suitable $P \times P$ -set Ω , where P acts freely on the left and on the right, which has some precise \mathcal{F} -stable property and $P \times P$ -orbits determined by \mathcal{F} [13, Proposition 5.5] — that we call \mathcal{F} -basic. Roughly speaking, Ω keeps some properties — which can be stated in terms of the Frobenius P -category \mathcal{F}_G — of the action of a Sylow p -subgroup P on a finite group G , by left and right multiplication.

[†] They credit Markus Linckelmann and Peter Webb for the original idea.

I45 But, the point is that we can define a *basic* $P \times P$ -set Ω independently of any Frobenius P -category, as a $P \times P$ -set with free actions on the left and on the right, fulfilling suitable extreme equalities — actually, the conditions are so simple that to give further details amounts to stating our definition here! Then, any *basic* $P \times P$ -set Ω supplies a Frobenius P -category \mathcal{F}^Ω and, by the Broto-Levi-Oliver result mentioned above, any Frobenius P -category comes from a *basic* $P \times P$ -set; of course, a Frobenius P -category \mathcal{F} may come from two different basic $P \times P$ -sets Ω and Ω' , but then we can construct a third basic $P \times P$ -set Ω'' containing both Ω and Ω' , and still fulfilling $\mathcal{F}^{\Omega''} = \mathcal{F}$.

I46 In particular, for any Frobenius P -category \mathcal{F} and any \mathcal{F} -*basic* $P \times P$ -set Ω — a basic $P \times P$ -set such that $\mathcal{F}^\Omega = \mathcal{F}$ — we consider the group G of permutations σ of Ω which centralize the action of P on the right; then, by the action on the left, P becomes a subgroup of G and, by the very definition of a basic $P \times P$ -set, for any subgroup Q of P we have

$$N_G(Q)/C_G(Q) \cong \mathcal{F}(Q) \quad \text{I46.1.}$$

The elementary but careful work we do in chapter 22 consists of determining all the centralizers $C_G(Q)$ and the inclusions between them.

I47 Naturally, these centralizers contain full *symmetric groups* coming from the possible mutually isomorphic $P \times P$ -orbits of Ω , but fortunately the minimal normal subgroups in the centralizers containing these *symmetric groups* form a “*localité*” in the old sense of [35] and therefore they determine an \mathcal{F} -*locality*. Although this \mathcal{F} -locality — called the *basic* \mathcal{F} -locality \mathcal{L}^b — is far from being *perfect*, it is *canonically* associated with \mathcal{F} in the sense that it does not depend on the choice of Ω provided it is “big enough”.

I48 The “universal” property of the *localizing functor* proved in chapter 18 guarantees that, if a perfect \mathcal{F} -locality does exist, it should be related to the *basic* \mathcal{F} -locality \mathcal{L}^b — at least over the set of \mathcal{F} -selfcentralizing subgroups of P — because of the rich structure of the *additive cover* $\mathbf{ac}(\tilde{\mathcal{F}}^{\text{sc}})$ of the *exterior* quotient $\tilde{\mathcal{F}}^{\text{sc}}$ of \mathcal{F}^{sc} developed in chapter 6. The problem with the relationship between the *basic* and the possible *perfect* \mathcal{F}^{sc} -localities is the *thickness* of the first one — in the sense that the *kernel* of the canonical functor $\mathcal{L}^{b, \text{sc}} \rightarrow \mathcal{F}^{\text{sc}}$ is too big. In chapter 23 we start by showing that the full subcategory $\mathcal{L}^{b, \text{sc}}$ of \mathcal{L}^b over the set of \mathcal{F} -selfcentralizing subgroups of P admits a quotient — the *polycentral* \mathcal{F}^{sc} -locality $\mathcal{L}^{c, \text{sc}}$ — already *narrowing* $\mathcal{L}^{b, \text{sc}}$.

I49 As a matter of fact, the kernel of the corresponding canonical functor $\mathcal{L}^{c, \text{sc}} \rightarrow \mathcal{F}^{\text{sc}}$ admits a quite general description — we discuss it in our Appendix from A2.14 to A2.17 — in terms of *representations* and *semidirect products*. In our situation in chapter 23, this formulation leads to a

vanishing result for *stable* positive cohomology groups — a type of nonstandard cohomology groups introduced in our Appendix (cf. I34 and A3.8) — *narrowing* twice the *polycentral* \mathcal{F}^{sc} -locality $\mathcal{L}^{\text{c,sc}}$ till we reach the *reduced* \mathcal{F}^{sc} -locality $\mathcal{L}^{\text{r,sc}}$. Moreover, in chapter 23 we explain our difficulties with finding a perfect \mathcal{F}^{sc} -sublocality of $\mathcal{L}^{\text{r,sc}}$ and give a (strongly!) sufficient condition to overcome them.

I50 Finally, in order to prove in chapter 24 that any perfect \mathcal{F}^{sc} -locality \mathcal{L}^{sc} is contained in the *reduced* \mathcal{F}^{sc} -locality, we have to exhibit a suitable *basic* $P \times P$ -set Ω . Where to find such a $P \times P$ -set? The answer comes from the fact that any morphism in \mathcal{L}^{sc} is an *epimorphism* — actually, it is a *monomorphism* too — and, in particular, all the arguments on $\mathbf{ac}(\tilde{\mathcal{F}}^{\text{sc}})$ in chapter 6 can be repeated in $\mathbf{ac}(\mathcal{L}^{\text{sc}})$; namely, the category $\mathbf{ac}(\mathcal{L}^{\text{sc}})$ admits a *direct product*, allowing us to consider *the direct product of P by P* which, being an $\mathbf{ac}(\mathcal{L}^{\text{sc}})$ -object, involves some finite set Ω (cf. 6.2): this is the set we are looking for.

I51 A last remark. The reader may ask himself whether or not it is possible to define an *ordinary Grothendieck group* for the Frobenius category $\mathcal{F}_{(b,G)}$ of a block (b, G) ; namely, to carry out an analogous construction with the Grothendieck groups obtained from the categories of representations over a field of characteristic zero, opening the possibility of dealing with Dade's Conjecture [21]. Firstly note that, without any extra effort, the direct sum of suitable scalar extensions of the *Grothendieck group* of the Frobenius categories $\mathcal{F}_{(g, C_G(u))}$ when (g, u) runs over a set of representatives for the set of G -conjugacy classes of *Brauer* (b, G) -elements [11] provides a satisfactory definition for the ordinary Grothendieck group of $\mathcal{F}_{(b,G)}$.

I52 Does it coincide with the *inverse limit* of the composition of the k^* -localizing functor $\widehat{\mathbf{loc}}_{\mathcal{F}_{(b,G)}}$ with the corresponding *ordinary Grothendieck group* functor? — here $\widehat{\mathbf{loc}}_{\mathcal{F}_{(b,G)}}$ denotes the *pull-back* of the *localizing functor* $\mathbf{loc}_{\mathcal{F}_{(b,G)}}$ (cf. I42) and the functor $\hat{\mathcal{F}}_{(b,G)}(\bullet)$ in I29 above. But, even if the answer was in the affirmative, in order to deal with Dade's Conjecture some extra idea would be necessary to fit the *defect* of ordinary irreducible characters considered in [21] inside the functorial framework.

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Frobenius Categories versus Brauer Blocks

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