Adv. Math. Econ. 12, 23–69 (2009)

Advances in
MATHEMATICAL
ECONOMICS

Convergences in a dual space with applications to Fatou lemma

Charles Castaing[1] **and Mohammed Saadoune**[2]

[1] Département de Mathématiques, Université Montpellier II, Place E. Bataillon, 34095 Montpellier cedex, France
(e-mail: castaing.charles@numericable.fr)

[2] Département de Mathématiques, Université Ibn Zohr, Lot. Addakhla, B.P. 8106, Agadir, Marocco
(e-mail: mohammed.saadoune@gmail.com)

Received: July 27, 2008
Revised: September 9, 2008

JEL classification: C01, C02, C62

Mathematics Subject Classification (2000): 29B20, 28A20

Abstract. We present new convergence results and new versions of Fatou lemma in Mathematical Economics based on various tightness conditions and the existence of scalarly integrable selections theorems for the (sequential)-weak-star upper limit of a sequence of measurable multifunctions taking values in the dual E' of a separable Banach space E. Existence of conditional expectation of weakly-star closed random sets in a non norm separable dual space is also provided.

Key words: Biting lemma, conditional expectation, Fatou lemma, Komlós convergence, sequential weak upper limit, Tightness

1. Introduction

Motivated by the study of Fatou lemma in Mathematical Economics, we present several types of convergence for multifunctions taking on convex weakly-star compact values in the topological dual E' of a separable Banach space E with specific applications to Fatou lemma in several dimensions. There has been a great deal of research on Fatou lemma when the multifunctions take values in the primal space E. See, e.g., [1, 3, 4, 6–9, 14, 18, 19, 22, 23, 25, 28] for references. For the case of a dual space, we

DOI: 10.1007/978-4-431-92935-2_2,

mention the recent contributions of Benabdellah and Castaing [6], Cornet and Martins da Rocha [18], Balder and Sambucini [5], Castaing Raynaud de Fitte and Valadier [13], Castaing, Hess and Saadoune [12], Castaing and Saadoune [15, 16]. In the present paper we provide, under new tightness conditions, several new convergence results for multifunctions with values in E' with sharp localizations of the limits and several new variants of Fatou lemma in the dual space E' via the integrability of the sequential-weak-star upper limit of a sequence of measurable multifunctions taking values in this space [12]. We also provide the existence of Conditional Expectations (CE) for weakly-star closed random sets in E'. See, e.g., [10, 11, 20, 21] for the problems of convergence of CE in Banach spaces. Since L^1-boundedness assumption is relaxed here, we obtain several significant generalizations in this study.

2. Notations and preliminaries

Throughout this paper the triple $(\Omega, \mathcal{F}, \mu)$ is a complete probability space, E is a separable Banach space and $D := (x_p)_{p\in\mathbb{N}}$ is a fixed dense sequence in the closed unit ball $\overline{B}_E$. We denote by E'_s (resp. E'_b) (resp. E'_{m^*}) the topological dual E' endowed with the topology $\sigma(E', E)$ of pointwise convergence, alias w^* topology (resp. topology of the norm) (resp. the topology $m^* = \sigma(E', H)$, where H is the linear space of E generated by D, that is the Hausdorff locally convex topology defined by the sequence of semi-norms

$$p_k(x') = \max\{| < x', x_p > | : p \leq k\}, \quad x' \in E', (k \geq 1)).$$

Recall that the topology m^* is metrizable, for instance, by the metric

$$d_{E'_{m^*}}(x'_1, x'_2) := \sum_{p=1}^{p=+\infty} \frac{1}{2^p} |\langle x_p, x'_1\rangle - \langle x_p, x'_2\rangle|, \quad x'_1, x'_2 \in E'.$$

We assume from now that $d_{E'_{m^*}}$ is held fixed. Further, we have $m^* \subseteq w^* \subseteq s^*$. When E is infinite dimensional these inclusions are strict. On the other hand, the restrictions of m^* and w^* to any bounded subset of E' coincide and $\mathcal{B}(E'_s) = \mathcal{B}(E'_{m^*})$ [12, Proposition 5.1], but the consideration of $\mathcal{B}(E'_b)$ is irrelevant here. Noting that E' is the countable union of closed balls, we deduce that the space E'_{w^*} is Suslin, as well as the metrizable topological space E'_{m^*}. A $2^{E'_s}$ valued multifunction $X : \Omega \Rightarrow E'_s$ is measurable if its graph belongs to $\mathcal{F} \otimes \mathcal{B}(E'_s)$. Given a measurable multifunction $X : \Omega \Rightarrow E'_s$ and a Borel set $G \in \mathcal{B}(E'_s)$, the set

$$X^- G = \{\omega \in \Omega : X(\omega) \cap G \neq \emptyset\}$$

is measurable, that is $X^{-}G \in \mathcal{F}$. In view of the completeness hypothesis on the probability space, this is a consequence of the Projection Theorem (see, e.g., Theorem III.23 of [17]) and of the equality

$$X^{-}G = \mathrm{proj}_{\Omega}\{Gr(X) \cap (\Omega \times G)\}.$$

In particular, if X is measurable, the *domain* of X, defined by

$$\mathrm{dom}\, X = \{\omega \in \Omega : X(\omega) \neq \emptyset\}$$

is measurable, because $\mathrm{dom}\, X = X^{-}E$. Further if $u : \Omega \to E'_s$ is a scalarly measurable mapping, that is, for every $x \in E$, the scalar function $\omega \mapsto \langle x, u(\omega)\rangle$ is $\mathcal{F}$-measurable, then the function $f : (\omega, x') \mapsto ||x' - u(\omega)||_{E'_b}$ is $\mathcal{F} \otimes \mathcal{B}(E'_s)$-measurable, and for every fixed $\omega \in \Omega$, $f(\omega, .)$ is lower semicontinuous on E'_s, shortly, f is a normal integrand, indeed, we have

$$||x' - u(\omega)||_{E'_b} = \sup_{p \in \mathbb{N}} \langle x_p, x' - u(\omega)\rangle.$$

As each function $(\omega, x') \mapsto \langle x_p, x' - u(\omega)\rangle$ is $\mathcal{F} \otimes \mathcal{B}(E'_s)$-measurable and continuous on E'_s for each $\omega \in \Omega$, it follows that f is a normal integrand. Consequently, the graph of u belongs to $\mathcal{F} \otimes \mathcal{B}(E'_s)$. Besides these facts, let us mention that the function distance $d_{E'_b}(x', y') = ||x' - y'||_{E'_b}$ is lower semicontinuous on $E'_s \times E'_s$, being the supremum of continuous functions. If X is a measurable multifunction, the distance function $\omega \mapsto d_{E'_b}(x', X(\omega))$ is measurable, by using the lower semicontinuity of the function $d_{E'_b}(x', .)$ on E'_s and measurable projection theorem [17, Theorem III.23], and recalling that E'_s is a Suslin space. A mapping $u : \Omega \rightrightarrows E'_s$ is said to be scalarly integrable, alias Gelfand integrable, if, for every $x \in E$, the scalar function $\omega \mapsto \langle x, u(\omega)\rangle$ is integrable. We denoted by $G^1_{E'}[E](\Omega, \mathcal{F}, \mu)$ ($G^1_{E'}[E](\mu)$ for short) the space of all scalarly integrable (classes of) mappings $u : \Omega \Rightarrow E'_s$. The subspace of $G^1_{E'}[E](\mu)$ of all mappings u such that the function $|u| : \omega \mapsto ||u(\omega)||_{E'_b}$ is integrable is denoted by $L^1_{E'}[E](\mu)$. The measurability of $|u|$ follows easily from the above considerations and holds even E is an arbitrary Banach space, we refer to [6] for details. For any $2^{E'_s}$ valued multifunction $X : \Omega \Rightarrow E'_s$, we denote by $G\text{-}\mathcal{S}^1_X$ (resp. $\mathcal{S}^1_X$) the set of all $G^1_{E'}[E](\mu)$-selections (resp. $L^1_{E'}[E](\mu)$-selections) of X. The G-integral (resp. integral) of X over a set $A \in \mathcal{F}$ is defined by

$$G\text{-}\int_A X\, d\mu := \left\{ \int_A f\, d\mu : f \in G\text{-}\mathcal{S}^1_X \right\}$$

and

$$\int_A X\,d\mu := \left\{ \int_A f\,d\mu : f \in \mathcal{S}^1_X \right\}$$

respectively. Let (X_n) be a sequence of measurable multifunctions taking values in E'_s. The sequential weak* upper limit w^*-$ls\ X_n$ of (X_n) is defined by

$$w^*\text{-}ls\ X_n = \left\{ x' \in E' : x' = \sigma(E', E)\text{-}\lim_{j\to\infty} x'_{n_j};\ x'_{n_j} \in X_{n_j} \right\}.$$

By $cwk(E'_s)$ we denote the set of all nonempty $\sigma(E', E)$-compact convex subsets of E'_s. A multifunction $X : \Omega \Rightarrow E'_s$ is scalarly measurable if, for every $x \in E$, the function $\omega \to \delta^*(x, X(\omega))$ is measurable. Let us recall that any scalarly measurable $cwk(E'_s)$-valued multifunction, X, is measurable. Indeed, let $(e_k)_{k\in\mathbb{N}}$ be a sequence in E which separates the points of E', then we have $x \in X(\omega)$ iff, $\langle e_k, x\rangle \le \delta^*(e_k, X(\omega))$ for all $k \in \mathbb{N}$. Further, we denote by $\mathcal{G}^1_{cwk(E'_s)}(\Omega, \mathcal{F}, \mu)$ (shortly $\mathcal{G}^1_{cwk(E'_s)}(\mu)$) the space of all *scalarly integrable* $cwk(E'_s)$-valued multifunctions $X : \Omega \to cwk(E'_s)$, that is, for every $x \in E$, the function $\omega \to \delta^*(x, X(\omega))$ is integrable. By $\mathcal{L}^1_{cwk(E'_s)}(\Omega, \mathcal{F}, \mu)$ (shortly $\mathcal{L}^1_{cwk(E'_s)}(\mu)$) we denote the subspace of $\mathcal{G}^1_{cwk(E'_s)}(\mu)$ of all *integrably bounded* multifunctions X that is the function $|X| : \omega \to |X(\omega)|$ is integrable, here $|X(\omega)| := \sup_{y\in X(\omega)} ||y||_{E'_b}$, by the above consideration, it is easy to see that $|X|$ is measurable. A sequence (X_n) in $\mathcal{L}^1_{cwk(E'_s)}(\mu)$ is *bounded* (resp. *uniformly integrable*) if $(|X_n|)$ is bounded (resp. uniformly integrable) in $\mathcal{L}^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$. In the sequel, we apply the usual convention $|\emptyset| = 0$ and $1_\emptyset = 0$. Our main purpose is to introduce some new types of *tightness condition* so-called *Mazur tightness condition* for sequences in the spaces $\mathcal{L}^1_{cwk(E'_s)}(\mu)$ and $\mathcal{G}^1_{cwk(E'_s)}(\mu)$. These considerations led to several new results of convergence in this space with application to Fatou lemma. In §3 we show the relationships between these tightness conditions. In §4 we present our main result of convergence for a Mazur-tight sequence in the space $L^1_{E'}[E](\mu)$ of scalarly integrable and mean norm bounded E'-valued mappings. In §5, some new convergence results are developed for the space $G^1_{E'}[E](\mu)$ of scalarly integrable E'-valued mappings. These results allow to obtain new types of convergence for the spaces $\mathcal{L}^1_{cwk(E'_s)}(\mu)$ and $\mathcal{G}^1_{cwk(E'_s)}(\mu)$ that we present in §6. We also provide here several variants of Fatou Lemma. The existence of Conditional Expectations of $\sigma(E', E)$ closed random sets in the dual is also stated in §7. Here the L^1-bounded condition is no longer required by contrast of most results given in the literature.

3. Mazur tightness condition

In this section new tightness properties of Mazur type are introduced and examined for sequences in the space $\mathcal{L}^0_{cwk(E'_s)}(\mu)$ of all measurable $cwk(E'_s)$-valued multifunctions. Actually, they find their origin in [12, 15, 16]. First, let us recall the following tightness definition [15]:

Definition 3.1. *A sequence (X_n) in $\mathcal{L}^0_{cwk(E'_s)}(\mu)$ is $cwk(E'_s)$-tight (resp. compactly $cwk(E'_s)$-tight) if there is a scalarly measurable (resp. scalarly measurable and integrably bounded) $cwk(E'_s)$-valued multifunction $\Gamma_\varepsilon : \Omega \Rightarrow E'$ such that*

$$\inf_n \mu(\{\omega \in \Omega : X_n(\omega) \subset \Gamma_\varepsilon(\omega)\}) \geq 1 - \varepsilon.$$

The measurability of $\{\omega \in \Omega : X_n(\omega) \subset \Gamma_\varepsilon(\omega)\}$ is immediate using the considerations developed in the beginning of this section.

Definition 3.2. *A sequence (X_n) in $\mathcal{L}^0_{cwk(E'_s)}(\mu)$ is said to be L^0-* lim sup-*Mazur tight, (resp. L^1-* lim sup-*Mazur tight) if, for every subsequence (Y_i) of (X_n), there exists a sequence (r_n) in $L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ with $r_n \in co\{|Y_i|(.) : i \geq n\}$ such that* $\limsup_n r_n < \infty$ *(resp.* $\limsup_n r_n \in L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$*).*

Similarly, we introduce a weaker notion of the above Mazur tightness, namely, L^0- lim inf-*Mazur tightness* and L^1- lim inf-*Mazur tightness*:

Definition 3.3. *A sequence (X_n) in $\mathcal{L}^0_{cwk(E'_s)}(\mu)$ is said to be L^0-* lim inf-*Mazur tight, (resp. L^1-* lim inf-*Mazur tight) if, for every subsequence (Y_n) of (X_n), there exists a sequence (r_n) in $L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ with $r_n \in co\{|Y_i|(.) : i \geq n\}$ such that for every sequence (s_n) in $L^1_{\mathbb{R}}(\mu)$ such that $s_n \in co\{r_i : i \geq n\}$, one has* $\liminf s_n < \infty$ *(resp.* $\liminf_n s_n \in L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$*).*

These new notions are denoted respectively: L^ℓ- lim sup-MT and L^ℓ- lim inf-MT, $\ell = 0, 1$. A sequence (X_n) in $\mathcal{L}^0_{cwk(E'_s)}(\mu)$ is said to be scalarly L^ℓ- lim sup-MT (resp. L^ℓ- lim inf-MT), $\ell = 0, 1$ if, for each $x \in E$, the sequence $(\delta^*(x, X_n))$ is L^ℓ- lim sup-MT (resp. L^ℓ- lim inf-MT). From Proposition 2.1 in [15] we derive directly a useful characterization of L^ℓ- lim sup-Mazur tightness condition which may be regarded as an extension of the Biting lemma (see, e.g., [13, 24]), since every $\mathcal{L}^1_{cwk(E'_s)}(\mu)$-bounded sequence is obviously L^1- lim sup-MT. This result will be used extensively in all this work.

Proposition 3.4. *Let (X_n) be a sequence in $\mathcal{L}^0_{cwk(E'_s)}(\mu)$ and let $\ell = 0, 1$. Then the following are equivalent:*

(1) (X_n) is L^ℓ- $\limsup$-MT.
(2) Given any subsequence (Y_n) of (X_n), there exist a subsequence (Z_n) of (Y_n), φ_∞ in $L^\ell_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ and an increasing sequence (C_k) in $\mathcal{F}$ with $\lim_{k\to\infty} \mu(C_k) = 1$ such that for every $k \in \mathbb{N}$, $1_{C_k}\varphi_\infty \in L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ and

$$\forall A \in \mathcal{F}, \quad \lim_{n\to\infty} \int_{A\cap C_k} |Z_n|\, d\mu = \int_{A\cap C_k} \varphi_\infty\, d\mu.$$

As a consequence of this proposition we deduce the following useful property.

Proposition 3.5. *Let (X^1_n) and (X^2_n) be two sequences in $\mathcal{L}^0_{cwk(E'_s)}(\mu)$. If (X^1_n) and (X^2_n) are L^ℓ- $\limsup$-MT, ($\ell = 0, 1$), then the sequence $(|X^1_n| + |X^2_n|)$ is L^ℓ- $\limsup$-MT. Consequently, $(X^1_n + X^2_n)$ is L^ℓ- $\limsup$-MT.*

Proof. Let (φ_n) be any subsequence of $(|X^1_n| + |X^2_n|)$. Each φ_n is of the form $\varphi_n := |Y^1_n| + |Y^2_n|$, where (Y^1_n) and (Y^2_n) are two subsequences of (X^1_n) and (X^2_n) respectively. Applying Proposition 3.4 successively to the sequences (X^1_n) and (X^2_n) we can find a subsequence (ψ_n) of (φ_n) with $\psi_n := |Z^1_n| + |Z^2_n|$, where (Z^1_n) and (Z^2_n) are two subsequences of (Y^1_n) and (Y^2_n) respectively, two functions φ^1_∞, φ^2_∞ in $L^\ell_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ and two increasing sequences (C^1_k) and (C^2_k) in $\mathcal{F}$ with $\lim_{k\to\infty} \mu(C^1_k) = 1$ and $\lim_{k\to\infty} \mu(C^2_k) = 1$ such that for every $k \in \mathbb{N}$, the functions $1_{C^1_k}\varphi^1_\infty$ and $1_{C^2_k}\varphi^2_\infty$ are integrable and

$$\lim_{n\to\infty} \int_{A\cap C^1_k} |Z^1_n|\, d\mu = \int_{A\cap C^1_k} \varphi^1_\infty\, d\mu,$$
$$\lim_{n\to\infty} \int_{A\cap C^2_k} |Z^2_n|\, d\mu = \int_{A\cap C^2_k} \varphi^2_\infty\, d\mu,$$

for all $A \in \mathcal{F}$. Then taking $C_k := C^1_k \cap C^2_k$ and $\varphi_\infty := \varphi^1_\infty + \varphi^2_\infty$ we get

$$\lim_{n\to\infty} \int_{A\cap C_k} \psi_n\, d\mu = \int_{A\cap C_k} \varphi^1_\infty\, d\mu + \int_{A\cap C_k} \varphi^1_\infty\, d\mu = \int_{A\cap C_k} \varphi_\infty\, d\mu,$$

for all $A \in \mathcal{F}$. Since $1_{C_k}\varphi_\infty \in L^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$, applying again Proposition 3.4 we deduce that $(|X^1_n| + |X^2_n|)$ is L^ℓ- $\limsup$-MT. □

In the following proposition, the L^1- $\liminf$-Mazur tightness and the L^0- $\limsup$-Mazur tightness conditions together are connected to the L^1- $\limsup$-Mazur tightness:

Proposition 3.6. *Let (X_n) be a sequence in $\mathcal{L}^0_{cwk(E'_s)}(\mu)$. Then the following two statement (a) and (b) are equivalent:*

(a) (X_n) is L^1- $\limsup$-MT.
(b) (X_n) is L^1- $\liminf$-MT and L^0- $\limsup$-MT.

Proof. The implication $(a) \Rightarrow (b)$ is trivial. To prove $(b) \Rightarrow (a)$ let (Y_n) be any subsequence of (X_n) satisfying (b). Using the L^0- $\limsup$-Mazur tightness and Proposition 3.4, we find a measurable function $\varphi_\infty : \Omega \mapsto \mathbb{R}^+$, a subsequence of (Y_n) still denoted (Y_n) and a sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$ such that, for every $k \in \mathbb{N}$, $1_{C_k}\varphi_\infty \in L^1_{\mathbb{R}}(\mu)$, and the following holds:

$$\forall A \in \mathcal{F}, \quad \lim_{n\to\infty} \int_{A\cap C_k} |Y_n|\, d\mu = \int_{A\cap C_k} \varphi_\infty\, d\mu.$$

Let (r_n) be a sequence of measurable functions as in the definition of the $L^1 - \liminf$-Mazur tightness. From the preceding equality it follows that

$$\forall A \in \mathcal{F}, \quad \lim_{n\to\infty} \int_{A\cap C_k} r_n\, d\mu = \int_{A\cap C_k} \varphi_\infty\, d\mu.$$

Invoking Mazur's Theorem and appealing to a diagonal procedure (see, e.g., Lemma 3.1 in [14]), one can construct a sequence (s_n) with $s_n \in co\{r_i \ : \ i \geq n\}$ such that, for every k, $(1_{C_k} s_n)$ converges a.e. to $1_{C_k}\varphi_\infty$. Since $\mu(C_k) \mapsto 1$, (s_n) converges a.e. to φ_∞, which, in view of condition (b), shows that $\varphi_\infty \in L^1_{\mathbb{R}}(\mu)$. Returning to Proposition 3.4, we deduce that (X_n) satisfies (a). □

According to the two following results,the notion of L^0- $\liminf$-Mazur tightness (resp. L^1- $\liminf$-Mazur tightness) is in some sense stronger than $cwk(E'_s)$-tightness (resp. compactly $cwk(E'_s)$-tightness). The first one is a variant of Proposition 3.2 in [16] dealing with primal space E.

Proposition 3.7. *Suppose that E is a separable Banach space and (X_n) is a sequence in $\mathcal{L}^0_{cwk(E'_s)}(\mu)$ satisfying the condition L^0- $\limsup$-MT. Then (X_n) admits a $cwk(E'_s)$-tight subsequence.*

Proof. By Proposition 3.4, there exist a subsequence (Y_n) of (X_n) and a $\mathcal{F}$-measurable partition (C_k) of Ω such that for every $k \in \mathbb{N}$ the sequence $(|Y_n|_{|C_k})$ is bounded in the space $L^1_{\mathbb{R}}(C_k, C_k \cap \mathcal{F}, \mu_{|C_k})$. Hence $(|Y_n|_{|C_k})$ is $cwk(E'_s)$-tight with respect to the measure space $(C_k, C_k \cap \mathcal{F}, \mu_{|C_k})$, making use of the Markov inequality. Therefore, for every $\epsilon > 0$, there is a scalarly measurable $cwk(E'_s)$-valued multifunction $\Gamma_{k,\epsilon}$ such that

$$\sup_n \mu(C_k \setminus \{\omega \in C_k : Y_{n|C_k})(\omega) \subset \Gamma_{k,\epsilon}(\omega)\}) \leq \epsilon\mu(C_k). \qquad (*)$$

Now define the multifunction Γ_ϵ on Ω by

$$\Gamma_\epsilon = 1_{C_1}\, \Gamma_{1,\epsilon} + \sum_{k\geq 2} 1_{C_k}\, \Gamma_{k,\epsilon}$$

Then, since

$$\{\omega \in \Omega : Y_n(\omega) \subset \Gamma_\epsilon(\omega)\} = \bigcup_k \{\omega \in C_k : Y_{n|C_k}(\omega) \subset \Gamma_{k,\epsilon}(\omega)\},$$

(*) entails

$$\begin{aligned}\mu(\{\omega \in \Omega : Y_n(\omega)) \subset \Gamma_\epsilon(\omega)\} &= \mu(\bigcup_k \{\omega \in C_k : Y_{n|C_k}(\omega) \subset \Gamma_{k,\epsilon}(\omega)\}) \\ &= \sum_k \mu(\{\omega \in C_k : Y_{n|C_k}(\omega) \subset \Gamma_{k,\epsilon}(\omega)\}) \\ &\geq \sum_k \mu(C_k)(1-\epsilon) = 1-\epsilon.\end{aligned}$$

Thus the sequence (Y_n) is $cwk(E'_s)$-tight. □

It is worthy to mention that the L^1- lim sup-Mazur tightness condition does not imply that $(|X_n|)$ is bounded in $\mathcal{L}^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$. Indeed, it suffices to consider the space $\mathcal{L}^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ where $\Omega = [0, 1]$ endowed with the Lebesgue measure and f_n is given by $f_n(\omega) := n^2 1_{[0,1/n]}(\omega),\ \omega \in \Omega$. Then, (f_n) is not bounded in $\mathcal{L}^1_{\mathbb{R}}(\Omega, \mathcal{F}, \mu)$ but satisfies (*), because it converges a.e. to 0. However, we have the following result

Proposition 3.8. *Suppose that E is a separable Banach space and (X_n) is a sequence in $\mathcal{L}^0_{cwk(E'_s)}(\mu)$ satisfying the condition L^1- lim sup-MT. Then (X_n) admits a $cwk(E'_s)$-compactly tight subsequence.*

Proof. Applying Proposition 3.4, to the sequence $(|X_n(.)|)$ provides a subsequence (Y_n) of (X_n) a function $\varphi \in L^1_{\mathbb{R}^+}(\mu)$ and an increasing sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$ such that for every $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_{C_k} |Y_n| d\mu = \int_{C_k} \varphi d\mu,$$

for all $A \in \mathcal{F}$. Let $\epsilon > 0$ and choose $p_\epsilon \geq 1$ such that $\frac{1}{p_\epsilon}\int_\Omega \varphi d\mu \leq \epsilon$. Applying the Lemma 3.9 below we get

$$\limsup_n \mu(\{\omega \in \Omega : |Y_n|(\omega) > p_\epsilon\}) \le \frac{1}{p_\epsilon}\int_\Omega \varphi d\mu \le \epsilon.$$

Hence, there exists $N_\epsilon \in \mathbb{N}$, such that

$$\sup_{n > N_\epsilon} \mu(\{\omega \in \Omega : |Y_n|(\omega) > p_\epsilon\}) \le \epsilon.$$

Since the functions $|Y_1|, \ldots, |Y_{N_\epsilon}|$ are integrable, one can find $\rho_\epsilon > p_\epsilon$ such that

$$\sup_{n \le N_\epsilon} \mu(\{\omega \in \Omega : |Y_n|(\omega) > \rho_\epsilon\}) \le \epsilon.$$

Whence we get

$$\sup_{n \in \mathbb{N}} \mu(\{\omega \in \Omega : |Y_n|(\omega) > \rho_\epsilon\}) \le \epsilon.$$

This shows that (Y_n) is compactly $cwk(E'_s)$-tight, indeed, it suffices to take $\Gamma_\epsilon := \overline{B}_{E'_b}(0, \rho_\epsilon)$. □

Lemma 3.9. *Let (φ_n) be sequence in $L^1_{\mathbb{R}^+}$ which biting converges to an integrable function φ. Then*

$$\forall p \ge 1, \quad \limsup_n \mu(\{\omega \in \Omega : \varphi_n(\omega) > p\}) \le \frac{1}{p}\int_\Omega \varphi d\mu.$$

Proof. There exists an increasing sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$ such that for every $k \in \mathbb{N}$,

$$\lim_{n\to\infty} \int_{C_k \cap A} \varphi_n d\mu = \int_{C_k \cap A} \varphi d\mu,$$

for all $A \in \mathcal{F}$. Using the Markov inequality we get

$$\limsup_n \mu(\{\omega \in C_k : \varphi_n(\omega) > p\}) \le \frac{1}{p}\lim_n \int_{C_k} \varphi_n d\mu = \frac{1}{p}\int_{C_k} \varphi d\mu$$

whence

$$\begin{aligned}\limsup_n \mu(\{\omega \in \Omega : \varphi_n(\omega) > p\}) \le \frac{1}{p}\int_{C_k} \varphi d\mu \\ + \limsup_n \mu(\{\omega \in \Omega \setminus C_k : \varphi_n(\omega) > p\}) \\ \le \frac{1}{p}\int_{C_k} \varphi d\mu + \mu(\Omega \setminus C_k).\end{aligned}$$

Letting $k \to \infty$ we get

$$\limsup_n \mu(\{\omega \in \Omega : \varphi_n(\omega) > p\}) \le \frac{1}{p}\int_\Omega \varphi d\mu.$$

□

We are ready to present general convergence results in $\mathcal{L}^1_{cwk(E'_s)}(\mu)$ with localization of the limit which can be applied to various places in the study of Fatou lemma in Mathematical Economics. A key ingredient in our proofs relies to integrability of the weak* sequential limit of a sequence of measurable multifunctions taking values in E' [12] and the Mazur-tightness conditions introduced above. Further these results constitute a sharp continuation of a similar study initiated in [14] dealing convergences and Fatou lemma in the space $\mathcal{L}^1_{cwk(E)}(\mu)$ of scalarly integrable and integrably bounded multifunctions with convex weakly compact values in the primal space E. For this purpose, we introduce the following convergences in $\mathcal{G}^1_{cwk(E'_s)}(\mu)$. A sequence (X_n) in $\mathcal{G}^1_{cwk(E'_s)}(\mu)$ *weakly Komlós converges* to X_∞, if

$$\forall x \in E, \quad \frac{1}{n}\sum_{i=1}^{n} \delta^*(x, Y_i(\omega)) \to \delta^*(x, X_\infty(\omega)), \quad \text{a.e.} \quad \omega \in \Omega,$$

(X_n) $d_{E'_{m^*}}$*-Wijsman Komlós converges* to X_∞, if

$$\forall x' \in E', \quad \lim_n d_{E'_{m^*}}(x', \frac{1}{n}\sum_{i=1}^{n} Y_i) = d_{E'_{m^*}}(x', X_\infty) \quad a.e.$$

for every subsequence (Y_n) of (X_n), here the negligible set depends only on the subsequence under consideration. (X_n) *weakly biting converges* to X_∞, if there exist a sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$ such that

$$\forall k \geq 1, \quad \forall v \in L^\infty_E(\Omega, \mathcal{F}, \mu), \quad \lim_{n\to\infty} \int_{C_k} \delta^*(v, X_n) d\mu = \int_{C_k} \delta^*(v, X_\infty) d\mu.$$

This study will be achieved from its single-valued specialization, namely we will deal first with the spaces $L^1_{E'}[E](\mu)$ and $G^1_{E'}[E](\mu)$.

4. Convergences in $L^1_{E'}[E](\mu)$

The main result in this section is concerned with the following with application to Fatou lemma.

Theorem 4.1. *Let E is a separable Banach space. Let (f_n) be a sequence in $L^1_{E'}[E](\mu)$ satisfying the L^1-$\limsup$-MT condition. Then there exist a function $f_\infty \in L^1_{E'}[E](\mu)$, a subsequence (g_n) of (f_n) and an increasing sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$ such that the following holds:*

(1) $(1_{C_k}\|g_n\|_{E'_b})$ is uniformly integrable in $L^1_{\mathbb{R}}(\mu)$ for each k.

(2) (g_n) *weakly biting converges to* f_∞.
(3) (g_n) *weakly Komlós converges to* f_∞.
(4) $f_\infty(\omega) \in w^*\text{-cl}\, co\,[w^*\text{-}ls\; g_n(\omega)]$ *a.e.*
(5) If μ *is nonatomic then*

$$\forall A \in \mathcal{F}, \quad \int_A f_\infty \, d\mu \in w^*\text{-cl}\left(\int_A w^*\text{-}ls\; g_n \, d\mu\right).$$

The proof of Theorem 4.1 involves the three following lemmas. The first one, Lemma 4.2, is an adaptation of Lemma 4.1 in [16] in the framework of a dual space. Its proof follows the same lines but needs a careful look and involves a sequential $\sigma\left(L^1_{E'}[E], L^\infty_E\right)$-compactness result. The second one, Lemma 4.3, is derived from Proposition 3.5 in [15]. The third one, Lemma 4.4 transforms Theorem 5.6 (jjj) [16] into a general result on integration of multifunctions, in particular, it yields an extension of Ljapunov's theorem for the sequential weak* upper limit of a sequence of measurable multifunctions with values in E'.

Lemma 4.2. *Let* $\Delta : \Omega \rightrightarrows E'$ *be a nonempty valued measurable and integrably bounded multifunction. Then*

$$\mathcal{S}^1{}_{w^*-cl(\Delta)} \subset \text{sequ.}\,\sigma(L^1_{E'}[E], L^\infty_E)\text{-cl}(\mathcal{S}^1_\Delta). \tag{$\dagger$}$$

Consequently,

$$\int_A w^*\text{-cl}(\Delta) d\mu \subset \text{seq}\, w^*\text{-cl}\left(\int_A \Delta d\mu\right) = w^*\text{-cl}\left(\int_A \Delta d\mu\right). \tag{$\dagger\dagger$}$$

Proof. Since the multifunction Δ has a $\mathcal{F} \otimes \mathcal{B}(E'_s)$-measurable graph and E'_s is a Suslin space, invoking Theorem III.22, [17], one can find a sequence $(\sigma_n)_{n\geq 1}$ of scalarly measurable selectors of Δ such that for every $\omega \in \Omega$, $w^* - cl(\Delta(\omega)) = w^* - cl(\{\sigma_n(\omega)\}_{n\geq 1})$. Since Δ is integrably bounded, the functions σ_n are necessary $L^1_{E'}[E]$-integrable and one has

$$d_{E'_{m^*}} - cl(\Delta(\omega)) = d_{E'_{m^*}} - cl(\{\sigma_n(\omega)\}_{n\geq 1}), \tag{4.2.1}$$

because the restriction of the w^*-topology to the closed ball $|\Delta(\omega)|\overline{B}_{E'}$ of E' is metrizable by the metric $d_{E'_{m^*}}$. Now take σ in $\mathcal{S}^1{}_{w^*-cl(\Delta)}$. For each $q \geq 1$, let us define the sets

$$A^q_n := \left\{\omega \in \Omega : \; d_{E'_{m^*}}(\sigma(\omega), \sigma_n(\omega)) < \frac{1}{q}\right\} \quad (n \geq 1),$$

$$\Omega^q_1 := A^q_1, \quad \Omega^q_n := A^q_n \setminus \cup_{i<n} A^q_i \text{ for } n > 1$$

and the function

$$\varsigma_q := \sum_{n=1}^{+\infty} 1_{\Omega_n^q} \sigma_n.$$

Since the functions $\omega \to d_{E'_{m^*}}(\sigma(\omega), \sigma_n(\omega))$ are $\mathcal{F}$-measurable, $A_n^q \in \mathcal{F}$ for all n. Further, from (4.2.1) it follows that $\cup_n A_n^q = \Omega$ a.e. Then $(\Omega_n^q)_n$ constitutes a sequence of pairwise disjoint members of $\mathcal{F}$ which satisfies $\cup_n \Omega_n^q = \Omega$ a.e. So ς_q is a scalarly measurable selector of Δ. As Δ is integrably bounded, we conclude that $\varsigma_q \in \mathcal{S}_\Delta^1$. Furthermore, we have

$$d_{E'_{m^*}}(\sigma(\omega), \varsigma_q(\omega)) < \frac{1}{q}, \quad \forall \omega \in \Omega.$$

By integrating we get

$$\int_\Omega d_{E'_{m^*}}(\sigma, \varsigma_q)\, d\mu \leq \frac{1}{q}.$$

Letting $q \to +\infty$, this inequality entails

$$\forall p \in \mathbb{N}^*, \ \ \forall A \in \mathcal{F}, \quad \langle x_p, \int_A \sigma\, d\mu \rangle = \lim_{q \to \infty} \langle x_p, \int_A \varsigma_q\, d\mu \rangle. \tag{4.2.2}$$

On the other hand, since the sequence (ς_q) is mean norm bounded in the space $L^1_{E'}[E](\mu)$, there exists, by Theorem 6.5.9 [13], a subsequence of (ς_q) still denoted in the same way and a function $\sigma' \in L^1_{E'}[E](\mu)$ such that

$$\forall v \in L_E^\infty(\Omega, \mathcal{F}, \mu), \quad \lim_{q \to \infty} \int_\Omega \langle v, \varsigma_q \rangle d\mu = \int_\Omega \langle v, \sigma' \rangle d\mu. \tag{4.2.3}$$

From (4.2.2) and (4.2.3) it follows that $\sigma' = \sigma$ a.e. Hence

$$\sigma \in \text{sequ.}\, \sigma(L^1_{E'}[E], L_E^\infty)\text{-cl}(\mathcal{S}_\Delta^1),$$

which proves (†). To prove (††) let a be an arbitrary element of $\int_A w^*\text{-cl}(\Delta) d\mu$. Then there exists $f \in L^1_{E'}[E](\mu)$ such that $a = \int_\Omega f\, d\mu$. Since, by (†), $f \in \text{seq}\, \sigma(L^1_{E'}[E], L^\infty E)\text{-cl}(\mathcal{S}_\Delta^1)$, there is a sequence (f_n) in $\mathcal{S}_\Delta^1$ which $\sigma(L^1_{E'}[E], L_E^\infty)$ converges to f so that, for every $A \in \mathcal{F}$, $w^*\text{-}\lim_n \int_A f_n d\mu = \int_A f d\mu$, whence $\int_A f d\mu \in \text{seq w}^*\text{-cl}\left(\int_A \Delta d\mu\right)$. □

Lemma 4.3. *Assume that μ is nonatomic, and $\Gamma : \Omega \rightrightarrows E'$ is a $\sigma(E', E)$ compact valued measurable multifunction satisfying $\Gamma(\omega) \subset \Phi(\omega), \forall \omega \in \Omega$ where $\Phi : \Omega \rightrightarrows E'$ is a scalarly integrable $cwk(E'_s)$-valued multifunction. Then*

$$\forall A \in \mathcal{F}, \ G\text{-}\int_A w^*\text{-cl}\, co \Gamma d\mu = w^*\text{-cl}\left(G\text{-}\int_A \Gamma d\mu \right).$$

Proof. It is obvious that $G\text{-}\mathcal{S}^1_\Gamma \subset G\text{-}\mathcal{S}^1_{w^*\text{-}cl\, co\Gamma} \subset G\text{-}\mathcal{S}^1_\Phi$. By [17, Theorem V-13], $G\text{-}\mathcal{S}^1_{w^*\text{-}cl\, co\Gamma}$ is convex and $\sigma(G^1_{E'}[E], L^\infty \otimes E)$ compact. Arguing as in the $L^1_E(\mu)$ case [26, Lemma 2 and Theorem 3], it is not difficult to see that $G\text{-}\mathcal{S}^1_\Gamma$ is dense in $\mathcal{S}^1_{w^*\text{-}cl\, co\Gamma}$ with respect to the $\sigma(G^1_{E'}[E], L^\infty \otimes E)$ topology. Since $f \mapsto \int_\Omega f d\mu$ from $G^1_{E'}[E](\mu)$ into E' is continuous with respect to the $\sigma(L^1_{E'}[E], L^\infty \otimes E)$, the conclusion follows. □

Lemma 4.4. *Assume in addition that μ is nonatomic and let $(\Delta_q)_{q\geq 1}$ be a sequence of measurable multifunctions from Ω to $\sigma(E', E)$- compact subsets of E'. Suppose that Δ_q is integrably bounded, for all q, and $\mathcal{S}^1_{\cup_{q\geq 1}\Delta_q} \neq \emptyset$. Then for all $A \in \mathcal{F}$, the following equalities hold:*
(a) $w^*\text{-}cl \int_A \cup_q w^*\text{-}clco\Delta_q\, d\mu = w^*\text{-}cl \int_A \cup_q \Delta_q\, d\mu.$
(b) $w^*\text{-}cl(\int_A co \cup_q \Delta_q\, d\mu) = w^*\text{-}cl(\int_A \cup_q \Delta_q\, d\mu).$

Proof. Let σ be a fixed element of $\mathcal{S}^1_{\cup_{q\geq 1}\Delta_q}$ and set

$$\Lambda_q := \cup_{i=1}^{i=q}(1_{dom\Delta_i}\Delta_i + 1_{\Omega\setminus dom\Delta_i}\sigma).$$

Then (Λ_q) is increasing, $dom\Lambda_q = \Omega$, for all q, and $\cup_q\Lambda_q = \cup_q\Delta_q$. Next, for each $q \in \mathbb{N}$ and each $F \in \mathcal{F}$, define the following multifunction:

$$\Lambda_{F,q} := 1_F w^*\text{-}cl\, co\Lambda_q + 1_{\Omega\setminus F}\sigma.$$

We claim that

$$\mathrm{seq}\,\sigma(L^1_{E'}[E], L^\infty_E)\text{-cl}\,[\mathcal{S}^1{}_{\cup_{q\in\mathbb{N}} w^*\text{-}cl\, co\, \Lambda_q}] = \mathrm{seq}\,\sigma(L^1_{E'}[E], L^\infty_E)\text{-cl}\,[\bigcup_{q\in\mathbb{N}}\bigcup_{F\in\mathcal{F}}\mathcal{S}^1_{\Lambda_{F,q}}].$$

Here $\mathrm{seq}\,\sigma(L^1_{E'}[E], L^\infty_E)\text{-cl}(A)$ denotes the sequential $\sigma(L^1_{E'}[E], L^\infty_E)$-closure of a set $A \subset L^1_{E'}[E](\mu)$. Since, for each $q \in \mathbb{N}$ and each $F \in \mathcal{F}$, $\Lambda_{F,q} \subset \cup_{q\in\mathbb{N}} w^*\text{-}cl\, co\Lambda_q$, it suffices to prove the inclusion

$$\mathcal{S}^1_{\cup_{q\geq 1} w^*\text{-}cl\, co\, \Lambda_q} \subset \mathrm{seq}\,\sigma(L^1_{E'}[E], L^\infty_E)\text{-cl}\left(\bigcup_{q\in\mathbb{N}}\bigcup_{F\in\mathcal{F}}\mathcal{S}^1_{\Lambda_{F,q}}\right). \quad (4.4.1)$$

To show this, take $s \in \mathcal{S}^1_{\cup_{q\geq 1} w^*\text{-}cl\, co\Lambda_q}$ and for each $q \in \mathbb{N}$, define a set $F_q \in \mathcal{F}$ and an $L^1_{E'}[E](\mu)$ selector, s_q, of $\Lambda_{F_q,q}$ as follows:

$$F_q := \{\omega \in \Omega : \ s(\omega) \in w^*\text{-}cl\, co\Lambda_q(\omega)\} \quad \text{and} \quad s_q := 1_{F_q}s + 1_{\Omega\setminus F_q}\sigma.$$

Then we have

$$\int_\Omega |s - s_q|\, d\mu \leq \int_{\Omega\setminus F_q} |s| d\mu + \int_{\Omega\setminus F_q} |\sigma| d\mu.$$

Since $\lim_{q\to\infty}\mu(\Omega\setminus F_q)=0$, the preceding estimation implies that

$$s\in \operatorname{seq}\sigma(L^1_{E'}[E],L^\infty_E)\text{-cl}\left(\bigcup_{q\in\mathbb{N}}\bigcup_{F\in\mathcal{F}}\mathcal{S}^1_{\Lambda_{F,q}}\right).$$

Thus the desired inclusion follows.

Using (4.4.1) it is immediate that

$$\forall A\in\mathcal{F},\ \int_A \cup_{q\ge 1} w^*\text{-}cl\, co\Lambda_q\, d\mu := \left\{\int_A f\,d\mu : f\in \mathcal{S}^1_{\cup_{q\ge1} w^*\text{-}cl\, co\Lambda_q}\right\}$$

$$\subset \left\{\int_A f\,d\mu : f\in \text{sequ.}\,\sigma\left(L^1_{E'}[E],L^\infty_E\right)\text{-cl}\left(\bigcup_{q\in\mathbb{N}}\bigcup_{F\in\mathcal{F}}\mathcal{S}^1_{\Lambda_{F,q}}\right)\right\} \tag{4.4.2}$$

$$\subset w^*\text{-}cl\left(\left\{\int_A f\,d\mu : f\in \bigcup_{q\in\mathbb{N}}\bigcup_{F\in\mathcal{F}}\mathcal{S}^1_{\Lambda_{F,q}}\right\}\right)= w^*\text{-}cl\left(\bigcup_{q\in\mathbb{N}}\bigcup_{F\in\mathcal{F}}\int_A \Lambda_{F,q}\,d\mu\right).$$

On the other hand, since Λ_q is measurable, w^* compact valued and integrably bounded for every $q\in\mathbb{N}$ and every $F\in\mathcal{F}$, it follows from Lemma 4.3 that

$$\forall q\in\mathbb{N},\quad \forall A\in\mathcal{F},\quad \int_A w^*\text{-}cl\, co\Lambda_q\, d\mu = w^*\text{-}cl\left(\int_A \Lambda_q\, d\mu\right).$$

Consequently $\forall q\in\mathbb{N}$, $\forall F\in\mathcal{F}$, $\forall A\in\mathcal{F}$,

$$\begin{aligned}\int_A \Lambda_{F,q}\,d\mu &= \int_A 1_F\, w^*\text{-}cl\, co\Lambda_q\, d\mu + \int_A 1_{\Omega\setminus F}\sigma\, d\mu\\ &= w^*\text{-}cl\left(\int_A 1_F\Lambda_q\, d\mu\right) + \int_A 1_{\Omega\setminus F}\sigma\, d\mu\\ &= w^*\text{-}cl\left(\int_A 1_F\Lambda_q + 1_{\Omega\setminus F}\sigma\, d\mu\right).\end{aligned}$$

This yields $\forall A\in\mathcal{F}$,

$$w^*\text{-}cl\left(\bigcup_{q\in\mathbb{N}}\bigcup_{F\in\mathcal{F}}\int_A \Lambda_{F,q}\,d\mu\right) = w^*\text{-}cl\left(\bigcup_{q\in\mathbb{N}}\bigcup_{F\in\mathcal{F}}\int_A 1_F\Lambda_q + 1_{\Omega\setminus F}\sigma\, d\mu\right). \tag{4.4.3}$$

Since $1_F\Lambda_q + 1_{\Omega\setminus F}\sigma \subset \cup_{q\in\mathbb{N}}\Delta_q$, for all $q\in\mathbb{N}$ and all $F\in\mathcal{F}$, from (4.4.2) and (4.4.3) we deduce

$$\forall A\in\mathcal{F},\quad w^*\text{-}cl\left(\int_A \cup_{q\ge1} w^*\text{-}cl\, co\Delta_q\, d\mu\right)\subset\ w^*\text{-}cl\left(\int_A \cup_{q\in\mathbb{N}}\Delta_q\, d\mu\right)$$

and the equality (a) follows. Whereas (b) is a consequence of the preceding inclusion and the fact that

$$co \cup_{q\in\mathbb{N}} \Delta_q \subset \cup_{q\geq 1} w^*\text{-}cl\, co\Delta_q.$$

□

Before going further let us give a useful application of the preceding lemma to the sequential weak* upper limit of a sequence of measurable multifunctions.

Proposition 4.5. *Assume that μ is nonatomic and (X_n) is a sequence of measurable multifunctions with values in E'. If $\mathcal{S}^1_{w^*\text{-}ls\, X_n} \neq \emptyset$, then the following equalities hold*

$$w^*\text{-}cl \int_\Omega \cup_q w^*\text{-}cl co w^*\text{-}ls\, (X_n \cap \overline{B}_{E'}(0,q))\, d\mu = w^*\text{-}cl \int_\Omega w^*\text{-}ls\, X_n\, d\mu.$$

$$\forall A \in \mathcal{F}, \quad w^*\text{-}cl \left(\int_A co w^*\text{-}ls\, X_n\, d\mu \right) = w^*\text{-}cl \left(\int_A w^*\text{-}ls\, X_n\, d\mu \right)$$

*Moreover, the set w^*-*cl$(\int_A w^*\text{-}ls\, X_n\, d\mu)$ *is w^*-closed and* convex.

In particular, if $X_n = \Gamma$, for all n, where $\Gamma : \Omega \rightrightarrows E'$ is a measurable multifunction such that $\mathcal{S}^1_\Gamma \neq \emptyset$, then

$$\forall A \in \mathcal{F}, \quad w^*\text{-}cl \left(\int_A co\Gamma \right) = w^*\text{-}cl \left(\int_A \Gamma\, d\mu \right).$$

*Consequently the set w^*-*cl$\left(\int_A \Gamma\, d\mu\right)$ *is w^*-closed and* convex

Proof. Take $\Delta_q := w^*\text{-}ls(X_n \cap q\overline{B}_{E'})$. Then Δ_q is integrably bounded and, since $q\overline{B}_{E'}$ is compact metrizable with respect to the weak* topology, it is not difficult to see that Δ_q is w^*-compact valued and measurable (see, e.g., Theorem 5.4 in [12]). Furthermore, since a w^*-convergent sequence is bounded in E', we have for all $\omega \in \Omega$

$$w^*\text{-}ls\, X_n = \bigcup_{q\in\mathbb{N}} \Delta_q.$$

Then, in view of the condition $\mathcal{S}^1_{w^*\text{-}ls\, X_n} \neq \emptyset$, the multifunction $\bigcup_{q\in\mathbb{N}} \Delta_q$ admits at least one $L^1_{E'}[E](\mu)$-selection. Consequently, it is possible to apply Lemma 4.4 to the sequence (Δ_q), which entails the desired equalities. □

The next corollary shows that for a measurable multifunction having at least one $L^1_{E'}[E](\mu)$-integrable selector the integral is dense in the G-integral with respect to the w^*-topology.

Corollary 4.6. *Assume that μ is nonatomic, and $\Gamma : \Omega \Rightarrow E'$ is a measurable multifunction. If $\mathcal{S}^1_\Gamma \neq \emptyset$, then the following equality holds*

$$\forall A \in \mathcal{F},\ \ w^*\text{cl}\left(\int_A \Gamma d\mu\right) = w^*\text{-cl}\left(G\text{-}\int_A \Gamma d\mu\right).$$

Proof. It suffices to prove the inclusion

$$G\text{-}\int_A \Gamma d\mu \subset w^*\text{cl}\left(\int_A \Gamma d\mu\right).$$

Let a be an arbitrary element of $G\text{-}\int_A \Gamma\, d\mu$. Then there exists a function $f \in G\text{-}\mathcal{S}^1_\Gamma$ such that $a = \int_A f\, d\mu$. For each $p \in \mathbb{N}$, define the measurable set

$$M_p := \{\omega \in \Omega : \|f(\omega) - \sigma(\omega)\|_{E'_b} \leq p\},$$

where σ is a fixed $L^1_{E'}[E](\mu)$-integrable selector of Γ. Then, since $0 \in \Gamma - \sigma$ a.e., one has

$$\forall x \in E, \quad \int_A 1_{M_p}(\Gamma - \sigma)\, d\mu \subset \int_A \Gamma - \sigma\, d\mu.$$

Therefore

$$\begin{aligned}
\langle x, \int_A f - \sigma\, d\mu\rangle &= \int_A \langle x, f - \sigma\rangle\, d\mu = \lim_{p\to\infty} \int_A \langle x, 1_{M_p}(f - \sigma)\rangle\, d\mu \\
&= \lim_{p\to\infty} \langle x, \int_A 1_{M_p}(f - \sigma)\, d\mu\rangle \\
&\leq \lim_{p\to\infty} \delta^*(x, \int_A 1_{M_p}(\Gamma - \sigma\, d\mu)) \\
&\leq \delta^*(x, w^*\text{-cl}\,(\int_A \Gamma - \sigma\, d\mu)),
\end{aligned}$$

for every $x \in E$ and for every $A \in \mathcal{F}$. Moreover, by Proposition 4.5, the set $w^*\text{-cl}\,(\int_A \Gamma\, d\mu)$ is convex w^*-closed and so is $w^*\text{-cl}\,\left(\int_A \Gamma - \sigma\, d\mu\right)$. Consequently,

$$\int_A f - \sigma\, d\mu \in w^*\text{-cl}\left(\int_A \Gamma - \sigma\, d\mu\right),$$

which is equivalent to

$$a \in w^*\text{-cl}\left(\int_A \Gamma\, d\mu\right).$$

This finishes the proof. □

Proof of Theorem 4.1. By Theorem 2.3 in [15] and its proof, there exist an increasing sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$, a subsequence (g_n) of (f_n) and $f_\infty \in L^1_{E'}[E](\Omega, \mathcal{F}, \mu)$ satisfying (1), (2) and (3). Now let us prove the localization properties (4) and (5). We shall proceed in two steps.
Step 1. In view of Proposition 3.7, we may suppose, for simplicity, that (g_n) is compactly $cwk(E'_s)$-tight. Consequently, we can construct a non decreasing sequence, $(\Gamma_q)_q$, of $cwk(E'_s)$-valued scalarly measurable integrably bounded multifunctions (in the special case we consider here, we may take for Γ_q the closed ball $\overline{B}_{E'}(0, \rho_q)$ of center 0 and with radius ρ_q) such that

$$\forall n, \quad \mu(\Omega \setminus A_{n,q}) \leq \frac{1}{q}, \tag{4.1.1}$$

where

$$A_{n,q} := \{\omega \in \Omega : g_n(\omega) \in \Gamma_q(\omega)\}.$$

Now, from the condition L^1- lim sup-MT and Theorem 5.7 in [16] it follows that the multifunction $w^*\text{-}ls\, g_n$ admits a $L^1_{E'}[E](\mu)$ selector σ. Let (e'_m) be a fixed dense sequence in $\bar{B}_{E'}$ for the Mackey topology and define

$$g^m_{n,q} = 1_{A_{n,q}}(g_n - \ell\sigma - e'_m), \quad (q, m \in \mathbb{N}), \quad (\ell = 0, 1).$$

It is obvious that the sequence $(g^m_{n,q})_n$ satisfies the condition L^1- lim sup-MT so that we may apply again Theorem 2.3 in [15]. Thus, using a standard diagonal procedure, it is possible to find a subsequence (not relabeled) of $(g^m_{n,q})$ and $f^m_{\infty,q} \in L^1_{E'}[E](\Omega, \mathcal{F}, \mu)$ such that

$$\forall e \in E, \quad \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \langle e, h_i(\omega)\rangle \to \langle e, f^m_{\infty,q}(\omega)\rangle, \quad \text{a.e.} \quad \omega \in \Omega, \tag{4.1.2}$$

for every subsequence (h_n) of $(g^m_{n,q})$ with

$$f^m_{\infty,q}(\omega) \in w^*\text{-}cl\, co \left(\bigcap_{p=1} w^*\text{-}cl\{g^m_{i,q}(\omega) : i \geq p\} \right), \quad \text{a.e.} \quad \omega \in \Omega.$$

As

$$\sup_n \|g^m_{n,q}(\omega)\|_{E'_b} \leq |\Gamma_q|(\omega) + \ell\|\sigma(\omega)\|_{E'_b} + 1 < \infty$$

and the restriction of the w^*-topology to the closed ball $(|\Gamma_q|(\omega) + \ell\|\sigma(\omega)\|_{E'_b} + 1)\overline{B}_{E'}$ is metrisable, it follows that

$$w^*\text{-}ls\, g^m_{n,q}(\omega) = \bigcap_{p=1} w^*\text{-}cl(\{g^m_{i,q}(\omega) : i \geq p\}),$$

for all $q \in \mathbb{N}$, for all $m \in \mathbb{N}$ and for all $\omega \in \Omega$. Hence

$$f^m_{\infty,q}(\omega) \in w^* - cl\, co\, [w^*\text{-}ls\; g^m_{n,q}(\omega)], \quad \text{a.e.} \quad \omega \in \Omega \tag{4.1.3}$$

for all $q \in \mathbb{N}$ and for all $m \in \mathbb{N}$. Next, putting

$$L_q := \bigcup_{i=1}^{i=q} 1_{D_i} w^*\text{-}ls\,(g_n \cap \Gamma_i) + 1_{\Omega \setminus D_i}\sigma, \quad \text{where} \quad D_i := \text{dom}\, w^*\text{-}ls\,(g_n \cap \Gamma_i),$$

$$\phi(\omega) := \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \|g_i(\omega)\|_{E'_b},$$

and

$$F^\ell_{m,\sigma}(\omega) := \overline{B}_{E'}(f_\infty(\omega) - \ell\sigma(\omega), \phi(\omega) + \ell\|\sigma(\omega)\|_{E'_b} + \|e'_m\|_{E'_b}), \quad (\ell = 0, 1),$$

we claim that

$$f_\infty(\omega) - \ell\sigma(\omega) \in w^* - cl(\cup_{q\geq 1} w^* - clco[(L_q(\omega) - \ell\sigma(\omega)) \cup \{e'_m\}] \cap F^\ell_{m,\sigma}(\omega)) \tag{4.1.4}$$

a.e., for all $m \in \mathbb{N}$. Indeed, it follows from (3) and (4.1.2) that

$$\begin{aligned}
&\|(f_\infty(\omega) - \ell\sigma(\omega) - e'_m) - f^m_{\infty,q}(\omega)\|_{E'_b} \\
&= \sup_{e\in \overline{B}_E} |\langle e, f_\infty(\omega) - \ell\sigma(\omega) - e'_m\rangle - \langle e, f^m_{\infty,q}(\omega)\rangle| \\
&= \sup_{e\in \overline{B}_E} \lim_{n\to\infty} \frac{1}{n} |\sum_{i=1}^{n} \langle e, g_i(\omega) - \ell\sigma(\omega) - e'_m\rangle - \langle e, g^m_{i,q}(\omega)\rangle| \\
&= \sup_{e\in \overline{B}_E} \lim_{n\to\infty} \frac{1}{n} |\sum_{i=1}^{n} \langle e, 1_{A^c_{i,q}}(g_i(\omega) - \ell\sigma(\omega) - e'_m)\rangle| \\
&\leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} 1_{\Omega\setminus A_{i,q}} \|g_i(\omega) - \ell\sigma(\omega) - e'_m\|_{E'_b} := \phi^m_q(\omega)
\end{aligned} \tag{4.1.5}$$

a.e. Using Fatou lemma and (4.1.1) we get

$$\begin{aligned}
\lim_{q\to\infty} \int_{C_k} \phi^m_q(\omega) d\mu &\leq \lim_{q\to\infty} \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \int_{C_k \cap (\Omega\setminus A_{i,q})} \|g_i(\omega) - \ell\sigma(\omega) - e'_m\|_{E'_b}\, d\mu \\
&\leq \lim_{q\to\infty} \sup_n \int_{C_k\cap(\Omega\setminus A_{n,q})} \|g_n(\omega) - \ell\sigma(\omega) - e'_m\|_{E'_b}\, d\mu = 0,
\end{aligned}$$

for every $k \in \mathbb{N}$. Hence the sequence $(\phi^m_q(\omega))_q$ converges to 0 in the Banach space $L^1_{\mathbb{R}}(C_k)$ when $q \to \infty$. By extracting subsequences, we may assume that $(\phi^m_q)_q$ converges to 0 a.e. on each C_k. From (4.1.3) and (4.1.5) we have

$$f^m_{\infty,q}(\omega) \in \overline{co}[w^*\text{-}lsg^m_{n,q}(\omega)] \cap \overline{B}_{E'}(f_\infty(\omega) - \ell\sigma(\omega) - e'_m, \phi^m_q(\omega)))$$
$$\subset \overline{co}[(L_q(\omega) - \ell\sigma(\omega) - e'_m) \cup \{0\}] \cap (F^\ell_{m,\sigma}(\omega) - e'_m)\ a.e,$$

because

$$\overline{co}[w^*\text{-}lsg^m_{n,q}(\omega)] \subset \overline{co}[w^*\text{-}ls((g_n(\omega) - \ell\sigma(\omega) - e'_m) \cap (\Gamma_q(\omega) - \ell\sigma(\omega) - e'_m)) \cup \{0\}]$$
$$\subset \overline{co}[(L_q(\omega) - \ell\sigma(\omega) - e'_m) \cup \{0\}],$$

and

$$\phi^m_q(\omega) \le \phi(\omega) + \ell\|\sigma(\omega)\|_{E'_b} + \|e'_m\|_{E'_b}.$$

We deduce that

$$d_{E'_b}(f_\infty(\omega) - \ell\sigma(\omega) - e'_m, w^* - clco[(L_q(\omega) - \ell\sigma(\omega) - e'_m) \cup \{0\}] \cap (F^\ell_{m,\sigma}(\omega) - e'_m)$$
$$\le \|f_\infty(\omega) - \ell\sigma(\omega) - e'_m - f^m_{\infty,q}(\omega)\|_{E'_b} \le \phi^m_q(\omega).$$

As $(\phi^m_q)_q$ converges to 0 a.e. on each C_k and $\mu(C_k) \to 1$, it follows that

$$f_\infty(\omega) - \ell\sigma(\omega) - e'_m$$
$$\in w^* - cl\big(\cup_{q\ge1} w^* - clco[(L_q(\omega) - \ell\sigma(\omega) - e'_m) \cup \{0\}] \cap (F^\ell_{m,\sigma}(\omega) - e'_m)\big)$$

a.e., which is equivalent to (4.1.4). Now, to prove (4) we repeat an argument in the proof of Theorem 8 in [2]. We assert that, for every subset C in E',

$$w^*cl\ co\ C = \bigcap_m w^*cl\ co[C \cup \{e'_m\}]. \tag{‡}$$

Indeed, assume that C is nonempty. If $x' \notin w^*cl\ co\ C$, there is $e \in E$ and $r \in \mathbb{R}$ such that

$$\delta^*(e, C) < r < \langle e, x'\rangle.$$

Taking e'_m in $\{y \in E' : \delta^*(e, C) < \langle e, y\rangle < r\}$ we get

$$w^*\text{-}cl\ co[C \cup \{e'_m\}] \subset \{y \in E' : \langle e, y\rangle \le r\}.$$

Hence $x \notin w^*\text{-}cl\ co[C \cup \{e'_m\}]$.

Applying (‡) in our case we get

$$\begin{aligned} &w^* - cl(\cup_{q\ge1} w^* - cl\ co\ L_q(\omega)) \\ &= \bigcap_{m\ge1} w^*cl\ co[(\cup_{q\ge1} w^*cl\ co\ L_q(\omega)) \cup \{e'_m\}] \\ &= \bigcap_m w^* - cl(\cup_{q\ge1} w^* - clco[L_q(\omega) \cup \{e'_m\}]), \end{aligned} \tag{4.1.6}$$

for all $\omega \in \Omega$, where the last equality follows from the fact that the sequence (L_q) is increasing. Consequently, (4) follows from (4.1.4) for $\ell = 0$, and (4.1.6). It remains to prove (5).

Step 2. Writing

$$F^1_{m,\sigma}(\omega) = f_\infty(\omega) - \sigma(\omega) + (\phi(\omega) + \|\sigma(\omega)\|_{E'_b} + \|e'_m\|_{E'_b})\overline{B}_{E'}(0,1),$$

we deduce easily that $F^1_{m,\sigma} \in \mathcal{L}^1_{cwk(E'_s)}(\mu)$. On the other hand, it is not difficult to see that L_q is measurable (see, e.g., Theorem 5.4 in [12]) and so is the multifunction $w^*\text{-cl}\, co[L_q - \sigma \cup \{e'_m\}]$. From these facts and (4.1.4) for $\ell = 1$, it follows that the multifunction $\cup_{q\geq 1} w^*\text{-cl}\, co[L_q - \sigma \cup \{e'_m\}] \cap F^1_{m,\sigma}$ satisfies the conditions of Lemma 4.2. Then

$$\begin{aligned}
&\int_A w^* - cl[\cup_{q\geq 1} w^*\text{-cl}\, co[L_q - \sigma \cup \{e'_m\}] \cap F^1_{m,\sigma}]\, d\mu\\
&\subset w^*\text{-cl}\left(\int_A \cup_{q\geq 1} w^*\text{-cl}\, co[L_q - \sigma \cup \{e'_m\}] \cap F^1_{m,\sigma}\, d\mu\right)\\
&\subset w^*\text{-cl}\left(\int_A \cup_{q\geq 1} w^*\text{-cl}\, co[L_q - \sigma \cup \{e'_m\}]\, d\mu\right), \qquad (4.1.7)
\end{aligned}$$

for every m. Now an appeal to Lemma 4.4 shows that

$$\begin{aligned}
&w^*\text{-cl}\left(\int_A \cup_{q\geq 1} w^*\text{-cl}\, co[L_q - \sigma \cup \{e'_m\}]\right) d\mu\\
&= w^*\text{-cl}\left(\int_A \cup_{q\geq 1} L_q - \sigma \cup \{e'_m\}\, d\mu\right). \qquad (4.1.8)
\end{aligned}$$

Thus using (4.1.4) for $\ell = 1$, (4.1.7) and (4.1.8) we get

$$\begin{aligned}
\int_A f_\infty - \sigma\, d\mu &\in \int_A \bigcap_m w^* - cl[\cup_{q\geq 1}(w^*\text{-cl}\, co[L_q - \sigma \cup \{e'_m\}] \cap F^1_{m,\sigma})]\, d\mu\\
&\subset \bigcap_m \int_A w^* - cl[\cup_{q\geq 1}\left(w^*\text{-cl}\, co[L_q - \sigma \cup \{e'_m\}] \cap F^1_{m,\sigma}\right)]\, d\mu\\
&\subset \bigcap_m w^*\text{-cl}\left(\int_A \cup_{q\geq 1} L_q - \sigma \cup \{e'_m\}\, d\mu\right)\\
&\subset \bigcap_m w^*\text{-cl}\left(\int_A w^*\text{-}ls\, g_n - \sigma \cup \{e'_m\}\, d\mu\right),
\end{aligned}$$

where the last inclusion follows from the fact that, $\cup_{q\geq 1} L_q \subset w^*\text{-}ls\, g_n$. On the other hand, noting that the multifunction $w^*\text{-}ls\, g_n$ is measurable, (by Theorem 5.5 in [12]) and $0 \in w^*\text{-}ls\, g_n - \sigma$, we can prove easily that

$$\int_A w^*\text{-}ls\, g_n - \sigma \cup \{e'_m\}\, d\mu \subset \int_A w^*\text{-}ls\, g_n - \sigma\, d\mu + \int_A \{e'_m\} \cup \{0\}\, d\mu,$$

which implies

$$\begin{aligned} & w^*\text{-cl}\left(\int_A w^*\text{-}ls\, g_n - \sigma \cup \{e'_m\}\, d\mu\right) \\ & \subset w^*\text{-cl}\left(\int_A w^*\text{-}ls\, g_n - \sigma\, d\mu\right) + \int_A \{e'_m\} \cup \{0\}\, d\mu, \end{aligned}$$

since the set $\int_A \{e'_m\} \cup \{0\}\, d\mu$ is compact. Hence, there exists

$$a \in w^*\text{-cl}\left(\int_A w^*\text{-}ls\, g_n - \sigma\, d\mu\right) \quad \text{and} \quad b_m \in \int_A \{e'_m\} \cup \{0\}\, d\mu$$

such that

$$\int_A f_\infty - \sigma\, d\mu = a + b_m,$$

for every m. But, taking a subsequence (e'_{m_k}) of (e'_m) which w^*-converges to 0 one has $\lim_{k\to\infty}\langle x, b_{m_k}\rangle = 0$. Indeed, choose s_m in $\mathcal{S}^1_{\{e'_m\}\cup\{0\}}$ such that $\int_A s_m\, d\mu = b_m$. Then we have

$$\begin{aligned} |\langle x, b_m\rangle| &= |\langle x, \int_A s_m\, d\mu\rangle| = |\int_A \langle x, s_m\rangle\, d\mu| \\ &\le \int_A |\langle x, s_m\rangle|\, d\mu \le \int_A |\langle x, e'_m\rangle|\, d\mu = \mu(A)|\langle x, e'_m\rangle|. \end{aligned}$$

Hence

$$\int_A f_\infty - \sigma\, d\mu \in w^*\text{-cl} \int_A w^*\text{-}ls\, g_n - \sigma\, d\mu.$$

Equivalently

$$\int_A f_\infty\, d\mu \in w^*\text{-cl}\left(\int_A w^*\text{-}\, ls\, g_n\, d\mu\right).$$

□

From Theorem 4.1, we get easily the following version of Fatou lemma.

Corollary 4.7. *Let E be a separable Banach space. Let (f_n) be sequence in the space $L^1_{E'}[E](\mu)$ such that:*

(i) (f_n) satisfies the condition L^1- lim sup-MT.
(ii) For every $x \in E$, the sequence $(\langle x, f_n\rangle)$ is uniformly integrable.
(iii) There exists $b \in E'$ such that $b = w^*\text{-}\lim_{n\to\infty} \int_\Omega f_n d\mu$.

Then there exists $f_\infty \in L^1_{E'}[E](\mu)$ *such that:*

(j) $b = \int_\Omega f_\infty d\mu$.
(jj) For almost all $\omega \in \Omega$ *one has* $f_\infty(\omega) \in w^*\text{-}cl\, co\,[w^*\text{-}\, ls f_n(\omega)]$.
(jjj) In particular, if μ *is nonatomic, then*

$$\int_\Omega f_\infty d\mu \in w^*\text{-}cl\left(\int_\Omega w^*\text{-}ls\ f_n\, d\mu\right).$$

We end this section by the following version of Fatou which is an analog of Corollary 4.4 [16] in the framework of $L^1_{E'}[E](\mu)$ space.

Corollary 4.8. *Let E is a separable Banach space. Let* (f_n) *be a sequence in* $L^1_{E'}[E](\Omega, \mathcal{F}, \mu)$ *satisfying the condition* L^1*-* lim sup*-MT. If* μ *is nonatomic, then the following inclusion holds*

$$w^*\text{-}\,ls \int_\Omega f_n\, d\mu \subset w^*\text{-cl} \int_\Omega w^*\text{-}\,ls\ f_n\, d\mu - C^*,$$

where C is the cone of all $x \in E$ *for which* $(max[0, \langle -x, f_n\rangle])$ *is uniformly integrable and* C^* *is the polar cone of* C.

Proof. Let b be an arbitrary element of $w^*\text{-}\,ls \int_\Omega f_n\, d\mu$. Then there exist a subsequence of (f_n) (not relabeled) such that $b = w^* - \lim_n \int_\Omega f_n\, d\mu$. An appeal to Theorem 4.1 produces a function $f_\infty \in L^1_{E'}[E](\mu)$, a subsequence (g_n) of (f_n) and an increasing sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$ for which (1)–(5) hold.

Let $\varepsilon > 0$ and $x \in C$ be given. Pick $k_0 \geq 1$ such that

$$\int_{C_{k_0}} \langle x', f_\infty\rangle\, d\mu \geq \int_\Omega \langle x', f_\infty\rangle\, d\mu - \epsilon$$

and that

$$\limsup_n \int_{\Omega\setminus C_{k_0}} \langle x, g_n\rangle^-\, d\mu \leq \varepsilon.$$

These two inequalities combined with (1) and a routine computation give

$$\begin{aligned}\langle x, b\rangle &\geq \lim_n \int_{C_{k_0}} \langle x, g_n\rangle\, d\mu - \limsup_n \int_{\Omega\setminus C_{k_0}} \langle x, g_n\rangle^-\, d\mu \\ &\geq \lim_n \int_{C_{k_0}} \langle x, g_n\rangle\, d\mu - \epsilon \\ &= \int_{C_{k_0}} \langle x, f_\infty\rangle\, d\mu - \epsilon \\ &\geq \int_\Omega \langle x, f_\infty\rangle\, d\mu - 2\epsilon.\end{aligned}$$

Thus $\langle x, b\rangle \geq \int_\Omega \langle x, f_\infty\rangle \, d\mu$. As μ is nonatomic, by (5), we conclude that

$$b \in \int_\Omega f_\infty \, d\mu - C^* \in w^*\text{-}cl\left(\int_\Omega w^*\text{-}ls\ f_n d\mu\right) - C^*.$$

$\square$

5. Convergences in $G^1_{E'}[E](\mu)$

In this section we proceed to a new convergence result in the space $G^1_{E'}[E](\mu)$ of scalarly integrable mappings $f : \Omega \to E'$ and its applications to Fatou lemma. Since $|f| \notin L^1_{\mathbb{R}}(\mu)$, this study involves both the L^0- lim sup-Mazur and the scalar L^1- lim inf-Mazur tightness conditions by contrast with the L^1- lim sup-Mazur tightness condition occurring in the space $L^1_{E'}[E](\mu)$. Before going further, we need the following $G^1_{E'}[E]$-extension of Lemma 4.4.

Lemma 5.1. *Assume that μ is nonatomic and let $(\Delta_q)_{q\geq 1}$ be a sequence of measurable multifunctions from Ω to $\sigma(E', E)$- compact subsets of E'. Suppose that $1_{dom\Delta_q}\Delta_q$ is scalarly integrable, for all q,and $G\text{-}\mathcal{S}^1_{\cup_{q\geq 1}\Delta_q} \neq \emptyset$. Then:*

(a) $w^*\text{-}cl\,(G - \int_\Omega \cup_q w^*\text{-}cl co\Delta_q \, d\mu) = w^*\text{-}cl\,(G - \int_\Omega \cup_q \Delta_q \, d\mu)$.

(b) $\forall A \in \mathcal{F}, \quad w^*\text{-}cl\,(G - \int_A co \cup_q \Delta_q \, d\mu) = w^*\text{-}cl\,(G - \int_A \cup_q \Delta_q \, d\mu)$.

Proof. Let σ be a fixed element of $G\text{-}\mathcal{S}^1_{\cup_{q\geq 1}\Delta_q}$ and set

$$\Lambda_q := \cup_{i=1}^{i=q}(1_{dom\Delta_i}\Delta_i + 1_{\Omega\setminus dom\Delta_i}\sigma).$$

Then (Λ_q) is increasing, $dom\Lambda_q = \Omega$, for all q, and $\cup_q\Lambda_q = \cup_q\Delta_q$. Next, for each $q \in \mathbb{N}$ and each $F \in \mathcal{F}$, define the following multifunction:

$$\Lambda_{F,q} := 1_F w^*\text{-}cl\ co\Lambda_q + 1_{\Omega\setminus F}\sigma.$$

We claim that

$$\text{seq}\, w^*\text{-cl}\left(G\text{-}\int_A \cup_{q\in\mathbb{N}} w^*\text{-}cl\ co\ \Lambda_q \, d\mu\right) = \text{seq}\, w^*\text{-cl}\left(G\text{-}\int_A \cup_{q\in\mathbb{N}} \cup_{F\in\mathcal{F}} \Lambda_{F,q} \, d\mu\right).$$

Here seq w^*-cl(Y) denotes the sequential w^*-closure of a set $Y \subset E'$. Since, for each $q \in \mathbb{N}$ and each $F \in \mathcal{F}$, $\Lambda_{F,q} \subset \cup_{q\in\mathbb{N}} w^*\text{-}cl\ co\Lambda_q$, it suffices to prove the inclusion

$$G\text{-}\int_A \cup_{q\in\mathbb{N}} w^*\text{-}cl\ co\ \Lambda_q \, d\mu \subset \text{seq}\, w^*\text{-cl}\left(\bigcup_{q\in\mathbb{N}}\bigcup_{F\in\mathcal{F}} G\text{-}\int_A \Lambda_{F,q} \, d\mu\right). \tag{5.1.1}$$

To show this, take $s \in G\text{-}\mathcal{S}^1_{\cup_{q\geq 1} w^*\text{-}cl\, co\Delta_q}$ and for each $q \in \mathbb{N}$, define a set $F_q \in \mathcal{F}$ and an $G^1_{E'}[E](\mu)$ selector, s_q, of $\Lambda_{F_q,q}$ as follows:

$$F_q := \{\omega \in \Omega : \ s(\omega) \in w^*\text{-}cl\, co\Lambda_q(\omega)\} \quad \text{and} \quad s_q := 1_{Fq} s + 1_{\Omega\setminus F_q}\sigma.$$

Then we have

$$\forall x \in E, \quad \int_\Omega |\langle x, s - s_q\rangle|\, d\mu \leq \int_{\Omega\setminus F_q} |\langle x, s\rangle| d\mu + \int_{\Omega\setminus F_q} |\langle x, \sigma\rangle| d\mu.$$

Since $\lim_{q\to\infty} \mu(\Omega \setminus F_q) = 0$, the preceding estimation implies that

$$s \in \text{seq}\, w^*\text{-cl}\, (G\text{-}\int_A \cup_{q\in\mathbb{N}} \cup_{F\in\mathcal{F}} \Lambda_{F,q}\, d\mu).$$

Thus the desired inclusion follows.

On the other hand, since, for each $q \in \mathbb{N}$, the multifunction Λ_q satisfies all the conditions of Lemma 4.3. we have

$$\forall q \in \mathbb{N}, \ \ \forall A \in \mathcal{F}, \quad G\text{-}\int_A w^*\text{-}cl\, co\Lambda_q d\mu = w^*\text{-}cl\, (G\text{-}\int_A \Lambda_q d\mu).$$

Consequently $\forall q \in \mathbb{N}, \forall F \in \mathcal{F}, \forall A \in \mathcal{F}$,

$$\begin{aligned} G\text{-}\int_A \Lambda_{F,q} d\mu &= G\text{-}\int_A 1_F w^*\text{-}cl\, co\Lambda_q d\mu + \int_A 1_{\Omega\setminus F}\sigma d\mu \\ &= w^*\text{-}cl\, (G\text{-}\int_A 1_F \Lambda_q d\mu) + \int_A 1_{\Omega\setminus F}\sigma d\mu \\ &= w^*\text{-}cl\, (G\text{-}\int_A (1_F\Lambda_q + 1_{\Omega\setminus F}\sigma) d\mu). \end{aligned}$$

This yields $\forall A \in \mathcal{F}$,

$$w^*\text{-}cl\, (\bigcup_{q\in\mathbb{N}} \bigcup_{F\in\mathcal{F}} G\text{-}\int_A \Lambda_{F,q}\, d\mu) = w^*\text{-}cl\, (\bigcup_{q\in\mathbb{N}} \bigcup_{F\in\mathcal{F}} G\text{-}\int_A 1_F\Lambda_q + 1_{\Omega\setminus F}\sigma d\mu). \tag{5.1.2}$$

Since $1_F\Lambda_q + 1_{\Omega\setminus F}\sigma \subset \cup_{q\in\mathbb{N}}\Delta_q$, for all $q \in \mathbb{N}$ and all $F \in \mathcal{F}$, from (5.1.1) and (5.1.2) we deduce

$$\forall A \in \mathcal{F}, \ w^*\text{-}cl\, (G\text{-}\int_A \cup_{q\geq 1} w^*\text{-}cl\, co\Delta_q\, d\mu) \subset w^*\text{-}cl\, (G\text{-}\int_A \cup_{q\in\mathbb{N}}\Delta_q d\mu).$$

Hence the equality (a) follows. Finally, the equality (b) is a consequence of the preceding inclusion and the fact that

$$co\, \cup_{q\in\mathbb{N}} \Delta_q \subset \cup_{q\geq 1} w^*\text{-}cl\, co\Delta_q.$$

□

The following result is a reformulation of Proposition 4.5 for the multi-valued Gelfand integral. Its proof is similar using Lemma 5.1.

Proposition 5.2. *Let (X_n) be a sequence of measurable multifunctions with values in E'. If μ is nonatomic and if the set $G\text{-}\mathcal{S}^1_{w^*\text{-}ls\, X_n}$ is nonempty, then for all $A \in \mathcal{F}$, the equalities*

$$w^*\text{-}cl\left(G\text{-}\int_A \cup_q w^*\text{-}cl co w^*\text{-}ls\,(X_n \cap \overline{B}_{E'}(0,q))\,d\mu\right) = w^*\text{-}cl\left(G\text{-}\int_A w^*\text{-}ls\,X_n\,d\mu\right),$$

$$w^*\text{-}cl\left(G\text{-}\int_A co w^*\text{-}ls\,X_n\,d\mu\right) = w^*\text{-}cl\left(G\text{-}\int_A w^*\text{-}ls\,X_n\,d\mu\right),$$

hold. Moreover, the set $w^*\text{-}\mathrm{cl}\left(G\text{-}\int_A w^*\text{-}ls\,X_n\,d\mu\right)$ is w^*-closed and convex.

In particular, if $X_n = \Gamma$, for all n, where $\Gamma : \Omega \rightrightarrows E'$ is a measurable multifunction such that $G\text{-}\mathcal{S}^1_\Gamma \neq \emptyset$, then

$$\forall A \in \mathcal{F}, \quad w^*\text{-}cl\left(G\text{-}\int_A co\Gamma\right) = w^*\text{-}cl\left(G\text{-}\int_A \Gamma\,d\mu\right).$$

Consequently the set $w^*\text{-}\mathrm{cl}\left(G\text{-}\int_A \Gamma\,d\mu\right)$ is w^*-closed and convex

Theorem 5.3. *Let E is a separable Banach space. Let (f_n) be a in $G^1_{E'}[E](\mu)$ satisfying the following conditions:*

(i) (f_n) is L^0- lim sup*-MT.*
(ii) (f_n) is scalarly L^1- lim inf*-MT.*

Then there exist a function $f_\infty \in G^1_{E'}[E](\mu)$, a subsequence (g_n) of (f_n) and a sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$ such that the following hold:

(1) $\forall k \in \mathbb{N}$, $\forall n \geq k$, $1_{C_k} g_n \in L^1_{E'}[E](\mu)$, $1_{C_k} f_\infty \in L^1_{E'}[E](\mu)$.
(2) $(1_{C_k}\|g_n\|_{E'_b})_{n\geq k}$ is uniformly integrable in $L^1_{\mathbb{R}}(\mu)$ for each k.
(3)

$$\forall v \in L^\infty_E(\Omega,\mathcal{F},\mu), \qquad \lim_{n\to\infty, n\geq k} \int_{C_k} \langle v, g_n\rangle d\mu = \int_{C_k} \langle v, f_\infty\rangle d\mu.$$

(4) (g_n) weakly Komlós converges to f_∞.
(5) $f_\infty(\omega) \in w^*\text{-}\mathrm{cl}\,co\,[w^*\text{-}ls\,g_n(\omega)]$ a.e.
(6) If μ is nonatomic, then $\forall A \in \mathcal{F}$,

$(\in_1)$ $$\int_A 1_{C_k} f_\infty\,d\mu \in w^*\text{-}\mathrm{cl}\left(\int_A 1_{C_k} w^*\text{-}ls\,g_n\,d\mu\right).$$

$(\in_2)$
$$\int_A f_\infty\,d\mu \in w^*\text{-}\mathrm{cl}\left(G\text{-}\int_A w^*\text{-}ls\,f_n\,d\mu\right) \quad \text{provided that} \quad G\text{-}\mathcal{S}^1_{w^*\text{-}ls\,f_n} \neq \emptyset.$$

Proof. On account of the L^0-lim sup-MT tightness condition (i) and Proposition 3.4, we provide a subsequence (g_n) of (f_n), a measurable function $\varphi_\infty : \Omega \mapsto \mathbb{R}^+$, and a sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$ such that

$$\lim_{n\to\infty} \int_{A\cap C_k} |g_n|\, d\mu = \int_{A\cap C_k} \varphi_\infty\, d\mu < \infty,$$

for all $k \in \mathbb{N}$ and for all $A \in \mathcal{F}$. In view of this equality, we observe that, for each $k \in \mathbb{N}$, every subsequence (h_n) of (g_n) admits a subsequence (h_n^k) with $\int_{C_k} |h_n^k|\, d\mu < \infty$, for all $n \in \mathbb{N}$, such that $(1_{C_k} h_n^k)$ is uniformly integrable. Using this fact and applying Theorem 4.1 to $(1_{C_k} g_n)$ via a standard diagonal procedure, it is possible to find a subsequence of (g_n) (not relabeled) and a function $f_\infty^k \in L^1_{E'}[E](\mu)$ such that

$$(1_{C_k} g_n)_{n\geq k} \quad \text{is uniformly integrable in} \quad L^1_{E'}[E](\mu),$$

$$(1_{C_k} g_n) \quad \text{weakly Komlós converges to} \quad f_\infty^k,$$

$$\forall k \in \mathbb{N},\ \ \forall v \in L^\infty_E(\Omega, \mathcal{F}, \mu), \qquad \lim_{n\to\infty, n\geq k} \int_\Omega \langle v, 1_{C_k} g_n\rangle d\mu = \int_\Omega \langle v, f_\infty^k\rangle d\mu,$$

$$f_\infty^k(\omega) \in w^*\text{-cl}\, co\, [w^*\text{-}ls\ g_n(\omega)] \quad \text{a.e} \quad \omega \in C_k.$$

Furthermore, if μ is nonatomic, then

$$\forall A \in \mathcal{F}, \quad \int_A f_\infty^k\, d\mu \in w^*\text{-cl}\left(\int_A w^*\text{-}ls\ 1_{C_k} g_n\, d\mu\right).$$

Put

$$C_1' := C_1 \quad \text{and} \quad C_k' := C_k \setminus C_{k-1} \quad \text{for} \quad k > 1,$$

and

$$f_\infty := \sum_{k=1}^{k=\infty} 1_{C_k'} f_\infty^k.$$

Since $\frac{1}{n}\sum_{i=1}^n g_i$ w^*-converges to f_∞^k a.e. on each C_k and $(C_k) \uparrow$, it follows that

$$\forall k, \forall j \leq k, \quad f_\infty^j = f_\infty^k \quad \text{a.e on} \quad C_j,$$

and then

$$\forall k, \quad f_\infty = f_\infty^k \quad \text{a.e on} \quad C_k.$$

Consequently we get

$$(g_n) \quad \text{weakly Komlós converges to} \quad f_\infty,$$

$$\forall k \in \mathbb{N},\ \ \forall v \in L^\infty_E(\Omega, \mathcal{F}, \mu), \qquad \lim_{n\to\infty, n\geq k} \int_{C_k} \langle v, g_n\rangle d\mu = \int_{C_k} \langle v, f_\infty\rangle d\mu,$$

$$f_\infty(\omega) \in w^*\text{-cl}\, co\, [w^*\text{-} ls\, g_n(\omega)] \quad \text{a.e.}$$

and, if μ is nonatomic, we have

$$\forall A \in \mathcal{F}, \quad \int_{A\cap C_k} f_\infty\, d\mu = \int_{A\cap C_k} f_\infty^k\, d\mu \in w^*\text{-cl}\left(\int_{A\cap C_k} w^*\text{-}ls\; 1_{C_k} g_n\, d\mu\right)$$
$$= w^*\text{-cl}\left(\int_A w^*\text{-}ls\; 1_{C_k} g_n\, d\mu\right).$$

thus proving (1), (2), (3), (4), (5) and (6)-($\in_1$). Next, let us show that f_∞ is scalarly integrable. Fix x in E. By conditions (i), (ii) and Proposition 3.6, the sequence $(\langle x, g_n\rangle)$ is L^1- lim sup-MT. So, applying Proposition 3.4 to the sequence $(\langle x, g_n\rangle)$, provides a function $\varphi_\infty^x \in L^1_{\mathbb{R}^+}(\mu)$, a subsequence of (g_n) still denoted (g_n) and an increasing sequence (C_k^x) in $\mathcal{F}$ with $\lim_k \mu(C_k^x) = 1$ such that, for every $k \in \mathbb{N}$, the following holds:

$$\forall A \in \mathcal{F}, \quad \lim_{n\to\infty} \int_{A\cap C_k^x} |\langle x, g_n\rangle|\, d\mu = \int_{A\cap C_k^x} \varphi_\infty^x\, d\mu.$$

Using successively this equality, conclusion (4) and the classical Fatou lemma we get

$$\int_{C_k^x} |\langle x, f_\infty\rangle|\, d\mu = \int_{C_k^x} \lim_{n\to\infty} |\frac{1}{n}\sum_{i=1}^n \langle x, g_i\rangle|\, d\mu \le \int_{C_k^x} \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n |\langle x, g_i\rangle| d\mu$$
$$\le \liminf_{n\to\infty} \frac{1}{n}\sum_{i=1}^n \int_{C_k^x} |\langle x, g_i\rangle| d\mu$$
$$= \lim_{n\to\infty} \int_{C_k^x} |\langle x, g_n\rangle|\, d\mu = \int_{C_k^x} \varphi_\infty^x\, d\mu$$

for all $k \in \mathbb{N}$. Whence

$$\int_\Omega |\langle x, f_\infty\rangle|\, d\mu = \lim_{k\to\infty} \int_{C_k^x} |\langle x, f_\infty\rangle|\, d\mu \le \lim_{k\to\infty} \int_{C_k^x} \varphi_\infty^x\, d\mu = \int_\Omega \varphi_\infty^x\, d\mu < \infty,$$

proving the required integrability property. Finally, let us prove the second inclusion of (6). Let $\sigma \in G\text{-}\mathcal{S}^1_{w^*\text{-}ls\, f_n}$. Then from ($\in_1$) and the inclusion $0 \in w^*\text{-}ls\, f_n - \sigma$, it follows that

$$\int_{A\cap C_k} (f_\infty - \sigma)\, d\mu \in w^*\text{-cl}\left(G - \int_A 1_{C_k}\left(w^*\text{-}ls\, f_n - \sigma\right)\, d\mu\right)$$
$$\subset w^*\text{-}cl\left(G - \int_A w^*\text{-}ls\, f_n - \sigma\, d\mu\right),$$

for every $k \in \mathbb{N}$ and every $A \in \mathcal{F}$. Consequently, since f_∞ and σ are scalarly integrable, one has $\forall x \in E, \forall A \in \mathcal{F}$,

$$\begin{aligned}\left\langle x, \int_A f_\infty - \sigma \, d\mu \right\rangle &= \int_A \langle x, f_\infty - \sigma \rangle \, d\mu = \lim_{k\to\infty} \int_A \left\langle x, 1_{C_k}(f_\infty - \sigma)\right\rangle d\mu \\ &= \lim_{k\to\infty} \left\langle x, \int_A 1_{C_k}(f_\infty - \sigma)\, d\mu \right\rangle \\ &\le \delta^* \left(x, w^*\text{-cl}\left(G - \int_A w^*\text{-}ls\ f_n - \sigma\, d\mu \right)\right).\end{aligned}$$

Moreover, by Proposition 5.2, the set w^*-cl$(G - \int_A w^*\text{-}ls\ f_n\, d\mu)$ is w^*-closed convex and so is w^*-cl$(G - \int_A w^*\text{-}ls\ f_n - \sigma\, d\mu)$. Hence we get

$$\forall A \in \mathcal{F}, \quad \int_A f_\infty - \sigma\, d\mu \in w^*\text{-cl}\left(G - \int_A w^*\text{-}ls\ f_n - \sigma\, d\mu \right).$$

Thus

$$\int_A f_\infty\, d\mu \in w^*\text{-cl}\left(G - \int_A w^*\text{-}ls\ f_n\, d\mu \right).$$

This completes the proof. □

The following result is a direct consequence of Theorem 5.3, Corollary 4.6 and Theorem 5.8 in [12].

Corollary 5.4. *Let E is a separable Banach space. Let (f_n) be a sequence in $G^1_{E'}[E](\mu)$ satisfying the following two conditions*

(i) (f_n) *is* L^0- lim sup-*MT.*
(ii) (f_n) *is scalarly* L^1- lim inf-*MT.*
(iii) $\liminf |f_n| \in L^1_{\mathbb{R}}(\mu)$.

Then there exist a function $f_\infty \in G^1_{E'}[E](\mu)$, a subsequence (g_n) of (f_n) and a sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$ satisfying (1)–(5) of Theorem 5.3 and, if μ is nonatomic, then

$$(6') \qquad \forall A \in \mathcal{F}, \quad \int_A f_\infty\, d\mu \in w^*\text{-cl}\left(\int_A w^*\text{-}ls\ f_n\, d\mu \right).$$

Theorem 5.3 extends Theorem 4.1 to the space $G^1_{E'}[E](\mu)$, by the way, we get the following $G^1_{E'}[E](\mu)$-extension of Corollary 4.8.

Corollary 5.5. *Suppose that μ is nonatomic, E is a separable Banach space and (f_n) is a sequence in $G^1_{E'}[E](\mu)$ satisfying the following two conditions*

(i) (f_n) *is* L^0- lim sup-*MT.*
(ii) (f_n) *is scalarly* L^1- lim inf-*MT.*
(iii) $\liminf |f_n| \in L^1_{\mathbb{R}}(\mu)$.

Then the following inclusion holds

$$w^*\text{-}ls \int_\Omega f_n \, d\mu \subset w^*\text{-cl}\left(\int_\Omega w^*\text{-}ls\ f_n \, d\mu\right) - C^*,$$

where C is the cone of all $x \in E$ for which $(max[0, \langle -x, f_n\rangle])$ is uniformly integrable and C^* is the polar cone of C.

The proof is the same as that of Corollary 4.8 using Corollary 5.4 and is omitted. □

6. Convergences in $\mathcal{L}^1_{cwk(E'_s)}(\mu)$ and $\mathcal{G}^1_{cwk(E'_s)}(\mu)$

Our main result in this section is concerned with new convergence results in the space $\mathcal{L}^1_{cwk(E'_s)}(\mu)$ and $\mathcal{G}^1_{cwk(E'_s)}(\mu)$.

Theorem 6.1. *Let (X_n) be a sequence in $\mathcal{L}^1_{cwk(E'_s)}(\mu)$ satisfying the condition L^1-$\limsup$-MT. Then there exist a subsequence (X'_n) of (X_n), $X_\infty \in \mathcal{L}^1_{cwk(E'_s)}(\mu)$ and a sequence (D_k) in $\mathcal{F}$ with $\lim_k \mu(D_k) = 1$ such that the following hold:*

(1) $(|X'_n|)$ is uniformly integrable in $L^1_{\mathbb{R}}(D_k)(\mu)$ on each D_k.
(2) (X'_n) weakly Komlós converges to X_∞.
(3) (X'_n) d_{m^*}-Wijsman Komlós converges a.e. to X_∞.
(4) (X'_n) weakly biting converges to X_∞.

$$\forall k \geq 1, \forall v \in L^\infty_E(\Omega, \mathcal{F}, \mu), \quad \lim_{n\to\infty} \int_{D_k} \delta^*(v, X'_n) d\mu = \int_{D_k} \delta^*(v, X_\infty) d\mu,$$

(5) $X_\infty(\omega) \subset w^*\text{-cl}\, co\, [w^*\text{-}ls\ X_n(\omega)]$ a.e.
(6) If μ is nonatomic then

$$\forall A \in \mathcal{F}, \quad \int_A X_\infty \, d\mu \subset w^*\text{-cl}\left(\int_A w^*\text{-}ls\ X'_n \, d\mu\right).$$

Proof. We will use several arguments of the proof of Theorem 4.1 with appropriate modifications.
Step 1. On account of the condition L^1-$\limsup$-M, Theorem 2.1 in [15] and Proposition 3.4, there exist a subsequence (X'_n) of (X_n) a function $\varphi \in L^1_{\mathbb{R}}(\mu)$ and an increasing sequence (D_k) in $\mathcal{F}$ with $\lim_k \mu(D_k) = 1$ such that

$$(|X'_n|) \quad \text{is uniformly integrable in} \quad L^1_{\mathbb{R}}(D_k)(\mu) \quad \text{on each} D_k, \tag{6.1.1}$$

$$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |Y_i|(\omega) = \varphi(\omega) \quad \text{exists a.e.,} \quad \text{and} \tag{6.1.2}$$

$$\lim_{n\to\infty} \delta^*\left(x_p, \frac{1}{n}\sum_{i=1}^{n} Y_i\right) \quad \text{exists a.e.} \tag{6.1.3}$$

for every subsequence (Y_n) of (X'_n). On the other hand, in view of Proposition 3.8, we may suppose, for simplicity, that (X'_n) is compactly $cwk(E'_s)$-tight. Consequently, we can construct a non decreasing sequence, (K_q), in $\mathcal{L}^1_{cwk(E'_s)}(\mu)$ such that

$$\forall n, \quad \mu(\Omega \setminus \{\omega \in \Omega : X'_n(\omega) \subset K_q(\omega)\}) \leq \frac{1}{q}. \tag{6.1.4}$$

Next, for every $p \in \mathbb{N}$, pick a maximal $L^1_{E'}[E](\mu)$-selection $v_{n,p}$ of X'_n, that is

$$\delta^*(x_p, X'_n) = \langle x_p, v_{n,p}\rangle.$$

It is clear that $(v_{n,p})$ is also L^1- lim sup-MT and we have

$$\forall n,\ \mu(\Omega \setminus \{\omega \in \Omega : v_{n,p}(\omega) \subset K_q(\omega)\}) \leq \mu(\Omega \setminus \{\omega \in \Omega : X'_n(\omega) \subset K_q(\omega)\}) \leq \frac{1}{q}.$$

As in the proof of Theorem 4.1, let us set

$$Q_q := \bigcup_{i=1}^{i=q} 1_{B_i}\, w^*\text{-}ls\,(X'_n \cap K_i) + 1_{\Omega\setminus B_i}\tau,$$

where $B_i :=$ dom w^*-$ls\,(X'_n \cap Q_i)$ and τ a fixed $L^1_{E'_s}(\mu)$-integrable selector of w^*-$ls X'_n$ (such a function is ensured by Theorem 5.7 in [16]). Repeating mutandis the arguments of the proof of Theorem 4.1 for the sequence $(v_{n,p})_n$ instead of (f_n) but replacing (C_k), Γ_q, L_q, ϕ and σ respectively by (D_k), K_q, Q_q, φ and τ, we can always find a subsequence of $(v_{n,p})$ (not relabeled) and $v_{\infty,p} \in L^1_{E'}[E](\mu)$ for each $p \in \mathbb{N}$, such that

$$(v_{n,p}) \quad \text{weakly Komlós converges to} \quad v_{\infty,p} \in L^1_{E'}[E](\mu) \quad \text{for each} \quad p \in \mathbb{N} \tag{6.1.5}$$

$$v_{\infty,p}(\omega) - \ell\tau(\omega) \in w^* - cl[\cup_{q\geq 1}(w^*\text{-cl}\, co[Q_q(\omega) - \ell\tau(\omega) \cup \{e'_m\}] \cap G^{\ell}_{m,\tau}(\omega))], \tag{6.1.6}$$

where

$$G^{\ell}_{m,\tau}(\omega) := \overline{B}_{E'}(v_{\infty,p}(\omega) - \ell\tau(\omega), \varphi(\omega) + \ell\|\tau(\omega)\|_{E'_b} + \|e'_m\|_{E'_b}), \quad (\ell = 0, 1).$$

Step 2. Let us prove (2) and (3). To do this consider the multifunctions

$$S_n = \frac{1}{n}\sum_{i=1}^{n} X'_i \quad \text{and} \quad X_\infty = w^*\text{-}li\ \frac{1}{n}\sum_{i=1}^{n} X'_i,$$

where w^*-$li\ C_n$ is the sequential weak* lower limit of a sequence (C_n) in $2^{E'}$ defined by

$$w^*\text{-}li\ C_n = \{x' \in E' : x' = \sigma(E', E)\text{-}\lim_{n\to\infty} x'_n;\ x'_n \in C_n\}.$$

Then, by (6.1.2), (S_n) is pointwise bounded a.e., and X_∞ is $cwk(E'_s)$-valued, the w^*-closedness of X_∞ follows easily from the fact that the restriction of the weak* topology to bounded sets is metrizable. Moreover, $v_{\infty,p} \in \mathcal{S}^1_{X_\infty}$ and we have

$$\lim_{n\to+\infty} \delta^*(x_p, S_n(\omega)) = \lim_n \left\langle x_p, \frac{1}{n}\sum_{i=1}^{n} v_{i,p}\right\rangle = \langle x_p, v_{\infty,p}\rangle \le \delta^*(x_p, X_\infty(\omega)) \quad \text{a.e.}$$

On the other hand it easy to see that

$$\delta^*(x_p, X_\infty(\omega)) \le \lim_{n\to\infty} \delta^*(x_p, S_n(\omega)) \quad \text{a.e.}$$

Whence we get

$$\lim_{n\to\infty} \delta^*(x_p, S_n(\omega)) = \delta^*(x_p, X_\infty(\omega)) \quad \text{a.e.} \tag{6.1.7}$$

We will use an argument in [10, Lemma 3.2]. We have

$$\begin{aligned} |\delta^*(x, S_n) - \delta^*(x, X_\infty)| \le{}& \max\{\delta^*(x - x_p, S_n), \delta^*(x_p - x, S_n)\} \\ &+ |\delta^*(x_p, S_n) - \delta^*(x_p, X_\infty)| \\ &+ \max\{\delta^*(x - x_p, X_\infty), \delta^*(x_p - x, X_\infty) \end{aligned}$$

for all $x' \in E'$ and for all j. Now let $x \in \overline{B}_E$ and $\varepsilon > 0$. There is $x_p \in D$ such that $||x - x_p|| \le \varepsilon$. Then we have

$$|\delta^*(x, S_n) - \delta^*(x, X_\infty)| \le \varepsilon \sup_n |S_n| + |\delta^*(x_p, S_n) - \delta^*(x_p, X_\infty)| + \varepsilon\, |X_\infty|.$$

Thus, by (6.1.7) and the pointwise boundedness of (S_n), it follows

$$\forall x \in E, \quad \lim_{n\to\infty} \delta^*(x, S_n(\omega)) = \delta^*(x, X_\infty(\omega)) \quad \text{a.e.} \tag{6.1.8}$$

By this equality, X_∞ is scalarly measurable, and hence measurable, (see, e.g., Corollary 5.3, [12]). Furthermore, returning again to (6.1.2), we get

$$|X_\infty| \le \liminf_{n\to\infty} |S_n| \le \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} |X'_i|\, d\mu = \varphi,$$

hence $\int_\Omega |X_\infty|\, d\mu < \infty$.
Next, we claim that

$$\lim_n d_{E'_{m^*}}(x', S_n) = d_{E'_{m^*}}(x', X_\infty),$$

for all $x' \in E'$ and almost all $\omega \in \Omega$. Indeed, since the multifunctions S_n and X_∞ are $cwk(E'_s)$-valued and $p(x') := d_{E'_{m^*}}(0, x')$ is a m^*-continuous semi-norm, we can invoke Theorem II.18 in [17], which, together with (6.1.8), entail

$$\begin{aligned}
\liminf_n d_{E'_{m^*}}(x', S_n) &= \liminf_n \sup_{x\in U^0} [\langle x, x'\rangle - \delta^*(x, S_n)] \\
&\ge \sup_{x\in U^0} \lim_n [\langle x, x'\rangle - \delta^*(x, S_n)] \\
&= \sup_{x\in U^0} [\langle x, x'\rangle - \delta^*(x, X_\infty)] \\
&= d_{E'_{m^*}}(x', X_\infty)
\end{aligned}$$

for every $x' \in E'$ and for almost all $\omega \in \Omega$, where $U := \{x' \in E' : p(x') < 1\}$ and U^0 its polar. By definition of X_∞ we have

$$\limsup_n d_{E'_{m^*}}(x', S_n)) \le d_{E'_{m^*}}(x', X_\infty),$$

for every $x' \in E'$ and for almost all $\omega \in \Omega$. Hence

$$\lim_n d_{E'_{m^*}}(x', S_n)) = d_{E'_{m^*}}(x', X_\infty),$$

for every $x' \in E'$ and for almost all $\omega \in \Omega$.

Applying the results obtained above for the sequence (X'_n) to any other subsequence (Y_n) of (X'_n) gives $X'_\infty \in \mathcal{L}^1_{cwk(E^*_s)}(\mu)$ such that

$$\forall x \in E, \quad \lim_{n\to\infty} \delta^*\left(x, \frac{1}{n}\sum_{i=1}^{n} Y_i(\omega)\right) = \delta^*\left(x, X'_\infty(\omega)\right) \quad \text{a.e.}$$

$$\forall x' \in E', \quad \limsup_n d_{E'_{m^*}}\left(x', \frac{1}{n}\sum_{i=1}^{n} Y_i(\omega)\right) = d_{E'_{m^*}}\left(x', X'_\infty\right), \quad \text{a.e.}$$

Then returning to (6.1.3) and (6.1.8) we deduce that $X_\infty = X'_\infty$, thus completing the proof of (2) and (3).
Step 3. Using (6.1.1), conclusion (2) and Lebegue–Vitali theorem, we get

$$\forall k \geq 1, \ \ \forall v \in L^\infty_E(\Omega, \mathcal{F}, \mu), \ \ \lim_{n\to\infty} \int_{D_k} \delta^*(v, \frac{1}{n}\sum_{i=1}^n Y_i) d\mu = \int_{D_k} \delta^*(v, X_\infty) d\mu,$$

for every subsequence (Y_n) of (X'_n). This is equivalent to

$$\forall k \geq 1, \ \ \forall v \in L^\infty_E(\Omega, \mathcal{F}, \mu), \quad \lim_{n\to\infty} \int_{D_k} \delta^*(v, X'_i) d\mu = \int_{D_k} \delta^*(v, X_\infty) d\mu,$$

thus proving (4).
Step 4. To prove (5) and (6) let us set

$$r_1(\omega) := |X_\infty|(\omega) + \varphi(\omega) + \|e'_m\|_{E'_b} \quad \text{and}$$

$$r_2(\omega) := |X_\infty|(\omega) + 2\|\tau(\omega)\|_{E'_b} + \varphi(\omega) + \|e'_m\|_{E'_b}.$$

Since $v_{\infty,k} \in \mathcal{S}^1_{X_\infty}$, it follows from (6.1.6) for $\ell = 0$ that

$$\begin{aligned}
\delta^*(x_p, X_\infty(\omega)) &= \langle x_p, v_{\infty,j}(\omega)\rangle \\
&\leq \delta^*(x_p, w^* - cl\big[\cup_{q\geq 1} w^*\text{-cl}\, co[Q_q(\omega) \cup \{e'_m\}] \\
&\qquad \cap w^*\text{-cl}\, co \bigcup_{s\in\mathcal{S}^1_{X_\infty}} \overline{B}_{E'}(s(\omega), \varphi(\omega) + \|e'_m\|_{E'_b})\big]) \\
&\leq \delta^*\left(x_p, w^* - cl\big[\cup_{q\geq 1} w^*\text{-cl}\, co[Q_q(\omega) \cup \{e'_m\}] \cap \overline{B}_{E'}(0, r_1(\omega))\big]\right),
\end{aligned}$$

for every p and for every m. Since the multifunction

$$\omega \Rightarrow w^* - cl[\cup_{q\geq 1}(w^*\text{-cl}\, co[Q_q(\omega) \cup \{e'_m\}] \cap \overline{B}_{E'}(0, r_1(\omega)))]]$$

is $cwk(E'_s)$-valued, E'_s is Suslin and its dual is equal to E, by virtue of Proposition III-35 in [17], the preceding inequality entails

$$\begin{aligned}
X_\infty(\omega) &\subset w^* - cl\big[\cup_{q\geq 1} w^*\text{-cl}\, co[Q_q(\omega) \cup \{e'_m\}] \cap \overline{B}_{E'}(0, r_1(\omega))\big] \\
&\subset w^* - cl\, (\cup_{q\geq 1} w^*\text{-cl}\, co[Q_q(\omega) \cup \{e'_m\}]).
\end{aligned} \tag{6.1.9}$$

Similarly, using again (6.1.6) but this time for $\ell = 1$, we obtain

$$X_\infty(\omega) - \tau(\omega) \subset w^* - cl\big[\cup_{q\geq 1}(w^*\text{-cl}\, co[Q_q(\omega) - \tau(\omega) \cup \{e'_m\}] \cap \overline{B}_{E'}(0, r_2(\omega))\big]. \tag{6.1.10}$$

Inclusion (5) is a consequence of (6.1.9). Indeed, it suffices to proceed as in the Step 1 of the proof of Theorem 4.1, by using an argument in the proof of Theorem 8 in [2]. Finally, to prove (6) take $u \in \mathcal{S}^1_{X_\infty}$ and note that (6.1.10) entails

$$u(\omega)-\tau(\omega) \in w^*-cl \left[\cup_{q\geq 1} w^*\text{-cl}\, co[Q_q(\omega)-\tau(\omega)\cup\{e'_m\}]\cap\overline{B}_{E'}(0, r_2(\omega))\right].$$

Since the sequence $\left(w^*\text{-cl}\, co[Q_q(\omega) - \tau \cup \{e'_m\}] \cap \overline{B}_{E'}(0, r_2(\omega))\right)$ satisfies all the conditions of Lemma 4.4, repeating exactly the same arguments as in the Step 2 of the proof of Theorem 4.1, we deduce that

$$\int_A u\, d\mu \in w^*\text{-cl}\left(\int_A \cup_{q\geq 1} Q'_q\, d\mu\right) = w^*\text{-cl}\left(\int_A w^*\text{-}ls\, X'_n\, d\mu\right),$$

which yields the desired inclusion (6). □

From Theorem 6.1 we derive the following Biting lemma in $\mathcal{L}^1_{cwk(E'_s)}(\mu)$. We refer to [8, 9, 14] dealing with Biting lemma in $\mathcal{L}^1_{cwk(E)}(\mu)$.

Corollary 6.2. *Suppose that E is a separable Banach space, (X_n) is a bounded sequence in $\mathcal{L}^1_{cwk(E'_s)}(\mu)$. Then there exist a subsequence (X'_n) of (X_n) and $X_\infty \in \mathcal{L}^1_{cwk(E'_s)}(\mu)$ such that the following hold:*

(i) (X'_n) weakly biting converges to X_∞
(ii) $X_\infty(\omega) \subset w^*clco\,[w^*\text{-}ls\, X'_n(\omega)]$ a.e.

Our second main result presents a version of Theorem 5.3 for multifunctions in the space $\mathcal{G}^1_{cwk(E'_s)}(\mu)$. The L^1- lim sup-MT condition is replaced by the $L^0 - \limsup$ and the scalar $L^1 - \liminf$ Mazur tightness conditions

Theorem 6.3. *Let (X_n) be a sequence in $\mathcal{G}^1_{cwk(E'_s)}(\mu)$ satisfying the following conditions:*

(i) (X_n) is L^0- lim sup-MT.
(ii) (X_n) is scalarly L^1- lim inf-MT.

Then there exist a subsequence (X'_n) of (X_n), $X_\infty \in \mathcal{G}^1_{cwk(E'_s)}(\mu)$ and a sequence (D_k) in $\mathcal{F}$ with $\lim_k \mu(D_k) = 1$ such that the following hold:

(1) $\forall k \in \mathbb{N}$, $\forall n \geq k$, $1_{C_k} X'_n \in \mathcal{L}^1_{cwk(E'_s)}(\mu)$, $1_{C_k} X_\infty \in \mathcal{L}^1_{cwk(E'_s)}(\mu)$.
(2) $(1_{C_k}|X'_n|)_{n\geq k}$ is uniformly integrable in $L^1_{\mathbb{R}}(\mu)$ for each k.
(3) (X'_n) weakly Komlós converges to X_∞.
(4) (X'_n) d_{m^*}-Wijsman Komlós converges a.e. to X_∞.

(5)

$$\forall k \geq 1, \ \ \forall v \in L^{\infty}_{E}(\Omega, \mathcal{F}, \mu), \quad \lim_{n\to\infty, n\geq k} \int_{D_k} \delta^*(v, X'_n)d\mu = \int_{D_k} \delta^*(v, X_\infty)d\mu.$$

In particular,

$$\forall x \in E, \quad \lim_{n\to\infty} \int_{C_k} \delta^*(x, X'_n)d\mu = \int_{C_k} \delta^*(x, X_\infty)d\mu.$$

(6) $X_\infty(\omega) \subset w^*\text{-cl}\, co\,[w^*\text{-}ls\, X'_n(\omega)]$ *a.e.*
(7) If μ *is nonatomic then* $\forall A \in \mathcal{F}$,

$$(\subset_1) \qquad \int_A 1_{D_k} X_\infty \, d\mu \subset w^*\text{-cl}\left(\int_A 1_{D_k} w^*\text{-}ls\, X'_n\, d\mu\right).$$

$$(\subset_2) \quad G\text{-}\int_A X_\infty\, d\mu \subset w^*\text{-cl}\,(G\text{-}\int_A w^*\text{-}ls\, X_n\, d\mu), \quad \textit{provided that} \quad G\text{-}\mathcal{S}^1_{w^*\text{-}ls\, X_n} \neq \emptyset.$$

Proof. Reasoning as in the beginning of the proof of Theorem 5.3 by using Proposition 3.6 and Theorem 6.1, we find a subsequence (X'_n) of (X_n), $X^k_\infty \in \mathcal{L}^1_{cwk(E'_s)}(\mu)$ and an increasing sequence (D_k) in $\mathcal{F}$ with $\lim_k \mu(D_k) = 1$ such that $\forall k \in \mathbb{N}$,

$$(1_{D_k}|X'_n|)_{n\geq k} \quad \text{is uniformly integrable in} \quad L^1_{E'}[E](\mu).$$

$$(1_{D_k}X'_n) \quad \text{weakly Komlós converges to} \quad X^k_\infty.$$

$$\forall v \in L^\infty_E(\Omega, \mathcal{F}, \mu), \quad \lim_{n\to\infty, n\geq k} \int_\Omega \delta^*(v, 1_{D_k}X'_n(\omega))d\mu = \int_\Omega \delta^*(v, X^k_\infty)d\mu.$$

$$X^k_\infty(\omega) \subset w^*\text{-cl}\, co\, w^*\text{-}ls\, X'_n(\omega) \quad \text{a.e. on each} \quad D_k.$$

Furthermore, if μ is nonatomic, then

$$\forall A \in \mathcal{F}, \quad \int_A X^k_\infty\, d\mu \subset w^*\text{-cl}\left(\int_A w^*\text{-}ls\, 1_{D_k}X'_n\, d\mu\right).$$

Put

$$D'_1 := D_1 \quad \text{and} \quad D'_k := D_k \setminus D_{k-1} \quad \text{for} \quad k > 1,$$

and

$$X_\infty := \sum_{k=1}^{k=\infty} 1_{D'_k} X^k_\infty.$$

Since $\frac{1}{n}\sum_{i=1}^n \delta^*(x, X'_i(\omega))$ converges to $\delta^*(x, X^k_\infty(\omega))$ for all $x \in E$ and for almost everywhere $\omega \in D_k$ and $(D_k)\uparrow$, it follows that

$$\forall k, \forall j \leq k, \quad \delta^*(x, X^j_\infty(\omega)) = \delta^*(x, X^k_\infty(\omega)),$$

for all $x \in E$ and for almost everywhere $\omega \in D_j$. Hence

$$\forall k, \forall j \leq k, \quad X_\infty^j = X_\infty^k \quad \text{a.e. on} \quad D_j,$$

which yields

$$\forall k, \quad X_\infty = X_\infty^k \quad \text{a.e. on} \quad D_k.$$

Consequently we get

$$(X'_n) \quad \text{weakly Komlós converges to} \quad X_\infty,$$

$$\forall k \in \mathbb{N}, \forall v \in L_E^\infty(\Omega, \mathcal{F}, \mu), \qquad \lim_{n\to\infty, n\geq k} \int_{D_k} \delta^*(v, X'_n) d\mu = \int_{D_k} \delta^*(v, X_\infty) d\mu,$$

$$X_\infty(\omega) \subset w^*\text{-cl}\, co\, [w^*\text{-} ls\, X'_n(\omega)] \quad \text{a.e.}$$

and, if μ is nonatomic, we have

$$\forall A \in \mathcal{F}, \quad \int_A 1_{D_k} X_\infty \, d\mu = \int_A 1_{D_k} X_\infty^k \, d\mu = \int_{A\cap D_k} X_\infty^k \, d\mu$$

$$\subset w^*\text{-cl}\,(\int_{A\cap D_k} w^*\text{-}ls\, 1_{D_k} X'_n \, d\mu) = w^*\text{-cl}\,(\int_A w^*\text{-}ls\, 1_{D_k} X'_n \, d\mu)$$

whence follow (1), (2), (3), (4), (5), (6) and (7)-($\subset_1$). Next, let us show that X_∞ is scalarly integrable. Fix x in E. By conditions (i), (ii) and Proposition 3.5, the sequence $(\delta^*(x, X'_n))$ is L^1- lim sup-MT. Applying Proposition 3.4, provides a function $\theta_\infty^x \in L^1_{\mathbb{R}^+}(\mu)$, a subsequence of (X'_n) still denoted (X'_n) and a sequence (B_k^x) in $\mathcal{F}$ with $\lim_k \mu(B_k^x) = 1$ such that, for every $k \in \mathbb{N}$, the following holds:

$$\forall A \in \mathcal{F}, \quad \lim_{n\to\infty} \int_{A\cap B_k^x} |\delta^*(x, X'_n)| \, d\mu = \int_{A\cap B_k^x} \theta_\infty^x \, d\mu.$$

This equality, conclusion (3) and the classical Fatou lemma entail

$$\begin{aligned}
\int_{B_k^x} |\delta^*(x, X_\infty)| \, d\mu &= \int_{B_k^x} \lim_{n\to\infty} |\frac{1}{n} \sum_{i=1}^n \delta^*(x, X'_i)| \, d\mu \\
&\leq \int_{B_k^x} \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^n |\delta^*(x, X'_i)| d\mu \\
&\leq \liminf_{n\to\infty} \frac{1}{n} \sum_{i=1}^n \int_{B_k^x} |\delta^*(x, X'_i)| d\mu \\
&= \lim_{n\to\infty} \int_{B_k^x} |\delta^*(x, X'_n)| \, d\mu = \int_{B_k^x} \theta_\infty^x \, d\mu
\end{aligned}$$

for all $k \in \mathbb{N}$. Whence

$$\int_{\Omega} |\delta^*(x, X_\infty)|\, d\mu = \lim_{k\to\infty} \int_{B_k^x} |\delta^*(x, X_\infty)|\, d\mu$$
$$\leq \lim_{k\to\infty} \int_{B_k^x} \theta_\infty^x, d\mu = \int_{\Omega} \theta_\infty^x, d\mu < \infty,$$

which shows the desired integrability property.

Finally, let us show the second inclusion of (7). Let $\sigma \in G\text{-}\mathcal{S}^1_{w^*\text{-}ls\, X_n}$, then from inclusion ($\subset_1$) we deduce

$$\forall x \in E,\ G\text{-}\int_A 1_{D_k}(X_\infty - \sigma)\, d\mu \subset w^*\text{-cl}\left(G\text{-}\int_A 1_{D_k}(w^*\text{-}ls\, X_n - \sigma)\, d\mu\right)$$
$$\subset w^*\text{-cl}\left(G\text{-}\int_A w^*\text{-}ls\, X_n - \sigma\, d\mu\right), \tag{6.3.1}$$

where the last inclusion follows the fact $0 \in w^*\text{-}ls\, X_n - \sigma$ a.e. Since $1_{D_k} X_\infty \in \mathcal{L}^1_{cwk(E_s')}(\mu)$, for all $k \in \mathbb{N}$, by Strassen formula (see again Theorem V-14, [17]), it follows

$$\forall k \in \mathbb{N}, \quad \int_A \delta^*(x, 1_{D_k}(X_\infty - \sigma))\, d\mu$$
$$= \int_A \delta^*(x, 1_{D_k} X_\infty)\, d\mu - \int_A \langle x, 1_{D_k}\sigma\rangle\, d\mu$$
$$= \delta^*(x, \int_A 1_{D_k} X_\infty\, d\mu) - \left\langle x, \int_A 1_{D_k}\sigma\, d\mu\right\rangle$$
$$= \delta^*(x, G\text{-}\int_A 1_{D_k}(X_\infty - \sigma)\, d\mu). \tag{6.3.2}$$

From (6.3.1), (6.3.2) and the fact that $\mu(D_k) \uparrow 1$ it follows

$$\forall x \in E, \quad \forall A \in \mathcal{F}, \quad \delta^*\left(x, G - \int_A X_\infty - \sigma\, d\mu\right)$$
$$\leq \int_A \delta^*(x, X_\infty - \sigma)\, d\mu$$
$$= \lim_{k\to\infty} \int_A \delta^*(x, 1_{D_k}(X_\infty - \sigma))\, d\mu$$
$$= \lim_{k\to\infty} \delta^*(x, G\text{-}\int_A 1_{D_k}(X_\infty - \sigma) d\mu)$$
$$\leq \delta^*(x, w^*\text{-cl}\,(G\text{-}\int_A w^*\text{-}ls\, X_n - \sigma\, d\mu)). \tag{6.3.3}$$

Moreover, by Proposition 5.2, the set $w^*\text{-cl}\,(G\text{-}\int_A w^*\text{-}ls\, X_n\, d\mu)$ is convex w^*-closed and so is $w^*\text{-cl}\,(G\text{-}\int_A w^*\text{-}ls\, X_n - \sigma\, d\mu)$. Therefore (6.3.3) entails

$$G\text{-}\int_A X_\infty - \sigma\, d\mu \subset w^*\text{-cl}\left(G\text{-}\int_A w^*\text{-}ls\, X_n - \sigma\, d\mu\right).$$

Equivalently

$$G\text{-}\int_A X_\infty\, d\mu \subset w^*\text{-cl}\left(G\text{-}\int_A w^*\text{-}ls\, X_n\, d\mu\right),$$

which is the desired inclusion ($\subset_2$). □

As a direct consequence of Theorem 6.3, Corollary 4.6 and Theorem 5.8 in [12] we have the following

Corollary 6.4. *Let (X_n) be a sequence in $\mathcal{G}^1_{cwk(E'_s)}(\mu)$ satisfying the following conditions:*

(i) (X_n) *is* L^0- lim sup-*MT*.
(ii) (X_n) *is scalarly* L^1- lim inf -*MT*.
(iii) $\liminf d_{E'_b}(0, X_n) \in L^1_{\mathbb{R}}(\mu)$.

Then there exist a multifunction $X_\infty \in \mathcal{G}^1_{cwk(E'_s)}(\mu)$, a subsequence (X'_n) of (X_n) and a sequence (D_k) in $\mathcal{F}$ with $\lim_k \mu(D_k) = 1$ *satisfying (1)–(6) of Theorem 6.3, and if μ is nonatomic*

$$(7') \qquad \forall A \in \mathcal{F}, \quad G\text{-}\int_A X_\infty\, d\mu \subset w^*\text{-cl}\left(\int_A w^*\text{-}ls\, X_n\, d\mu\right).$$

The following is an application of the preceding result to weak compactness in the space $\mathcal{G}^1_{cwk(E'_s)}(\mu)$

Corollary 6.5. *Let (X_n) be a sequence in $\mathcal{G}^1_{cwk(E'_s)}(\mu)$ satisfying the following conditions:*

(i) (X_n) *is* L^0- lim sup-*MT*.
(ii) $(\delta^*(x, X_n))$ *is uniformly integrable.*

Then there exist a subsequence (X'_n) of (X_n) and $X_\infty \in \mathcal{G}^1_{cwk(E^*_s)}(\mu)$ such that
$\forall A \in \mathcal{F}, \forall x \in E,$

$$\lim_{n\to\infty} \int_A \delta^*(x, X'_n)\, d\mu = \int_A \delta^*(x, X_\infty)\, d\mu$$

with $X_\infty(\omega) \subset w^*clco\,[w^*\text{-}\,ls\, X'_n(\omega)] \quad a.e.$

We finish this section by providing the following Fatou lemma which is a multivalued version of Corollary 5.5. Its proof is essentially based on Theorem 5.3 and Proposition 3.5.

Proposition 6.6. *Suppose that μ is nonatomic, E is a separable Banach space and (X_n) in $\mathcal{G}^1_{cwk(E'_s)}(\mu)$ satisfying the following conditions:*

(i) (X_n) *is* L^0- lim sup-*MT.*
(ii) (X_n) *is scalarly* L^1- lim inf -*MT.*
(iii) $\liminf d_{E'_b}(0, X_n) \in L^1_{\mathbb{R}}(\mu)$.

Then the following inclusion holds

$$w^*\text{-}ls\ G\text{-}\int_\Omega X_n\,d\mu \subset w^*\text{-cl}\left(\int_\Omega w^*\text{-}ls\ X_n\,d\mu\right) - C^*,$$

where C is the cone of all $x \in E$ for which $(max[0, \delta^*(-x, X_n)])$ is uniformly integrable and C^* is the polar cone of C.

Proof. Let b be an arbitrary element of $w^*\text{-}ls\ G\text{-}\int_\Omega X_n\,d\mu$. Then there exist a subsequence of (X_n) (not relabeled) and an associated sequence (f_n) of $G^1_{E'}[E](\mu)$-selectors such that $b = w^*\text{-}\lim_{n\to\infty}\int_\Omega f_n\,d\mu$. By (i) the sequence (f_n) is L^0- lim sup-MT. Furthermore, by (i), (ii) and Proposition 3.6, (X_n) is scalarly L^1- lim sup-MT. Using the inequality

$$\forall x \in E,\quad |\langle x, f_n(\omega)| \le |\delta^*(x, X_n(\omega)| + |\delta^*(-x, X_n(\omega)| \quad \text{a.e.},$$

and Proposition 3.5, we conclude that (f_n) is scalarly L^1- lim sup-MT. Consequently, according to Theorem 5.3, we find $f_\infty \in G^1_{E'}[E](\mu)$, a subsequence (g_n) of (f_n) and a sequence (C_k) in $\mathcal{F}$ with $\lim_k \mu(C_k) = 1$ such that

$$\forall k \in \mathbb{N},\ \ \forall x \in E,\ \ \forall A \in \mathcal{F}\quad \lim_{n\to\infty}\int_{A\cap C_k}\langle x, g_n\rangle d\mu = \int_{A\cap C_k}\langle x, f_\infty\rangle d\mu, \tag{6.6.1}$$

$$\forall A \in \mathcal{F},\quad \int_A 1_{C_k} f_\infty\,d\mu \in w^*\text{-cl}\left(\int_A 1_{C_k} w^*\text{-}ls\ g_n\,d\mu\right). \tag{6.6.2}$$

We claim that

$$\int_A f_\infty\,d\mu \in w^*\text{-cl}\left(\int_A w^*\text{-}ls\ X_n\,d\mu\right). \tag{6.6.3}$$

Indeed, since condition (iii) ensures that $\mathcal{S}^1_{w^*\text{-}ls\ X_n}$ is non empty, thanks to Theorem 5.8 in [12], we can choose σ in $\in \mathcal{S}^1_{w^*\text{-}ls\ X_n}$. Then from (6.6.2) and the inclusion $0 \in w^*\text{-}ls\ X_n - \sigma$ a.e., it follows that

$$\forall x \in E, \quad \int_A 1_{C_k}(f_\infty - \sigma)\, d\mu \in w^*\text{-cl}\left(\int_A 1_{C_k}(w^*\text{-}ls\, g_n - \sigma)\, d\mu\right)$$
$$\subset w^*\text{-cl}\left(\int_A w^*\text{-}ls\, X_n - \sigma\, d\mu\right).$$

This inclusion and the same arguments used in the proof of Theorem 5.3 - ($\in_2$) prove our claim. On the other hand, let C' be the cone of all $x \in E$ for which $(max[0, -\langle x, f_n\rangle])$ is uniformly integrable and C' its polar cone. Since, for each $n \in \mathbb{N}$, g_n is a selector of X_n, necessary $C \subset C'$. Using (6.6.1) and reasoning as in the proof of Corollary 4.8 we deduce

$$b \in \int_\Omega f_\infty\, d\mu - C' \subset \int_\Omega f_\infty\, d\mu - C^*. \tag{6.6.4}$$

Combining (6.6.3)) and (6.6.4) gives

$$b \in w^*\text{-cl}\left(\int_A w^*\text{-}ls\, X_n\, d\mu\right) - C^*.$$

□

7. Conditional expectation of weakly* closed convex random sets in the dual

We finish our paper by providing the existence of conditional expectation of w^*-closed random sets which led to Fatou lemma for conditional expectation in the space $L^1_{E'}[E](\mu)$ and $\mathcal{L}^1_{cwk(E'_s)}(\mu)$.

In the following, $\mathcal{B}$ is a complete sub σ-algebra of $\mathcal{F}$. For any subset $\mathcal{H}$ in $L^1_{E'}[E](\mathcal{B}, \mu)$, and for any $v \in L^\infty_E(\mathcal{B}, \mu)$ we set

$$\delta^*(v, \mathcal{H}) = \sup_{u \in \mathcal{H}} \langle v, u\rangle.$$

Definition 7.1. *We shall say that Γ is a $\mathcal{F}$-random (resp. $\mathcal{B}$-random) closed convex set in E'_s, if the multifunction $\Gamma : \Omega \Rightarrow E'_s$ is $\mathcal{F}$ (resp. $\mathcal{B}$) measurable, that is, the graph of Γ belongs to $\mathcal{F} \times \mathcal{B}(E'_s)$ (resp. $\mathcal{B} \times \mathcal{B}(E'_s)$).*

We begin to state the existence and uniqueness of conditional expectation of an integrably bounded $\mathcal{F}$-random closed convex set Γ in E'_s (that is, $|\Gamma| \in L^1_{\mathbb{R}}(\mathcal{F})$) .

Definition 7.2. *A $\mathcal{B}$-random closed convex set Σ in E'_s is called conditional expectation of Γ if:*
(i) There is $\beta \in L^1_{\mathbb{R}^+}(\mathcal{B})$ such that $\Sigma(\omega) \subset \beta(\omega)\overline{B}_{E'}$ a.e.
(ii) $\forall x \in E, \forall B \in \mathcal{B}$, $\int_B \delta^*(x, \Sigma(\omega))d\mu(\omega) = \int_B \delta^*(x, \Gamma(\omega))d\mu(\omega)$.

Since E'_s is Suslin and its dual is equal to E, by virtue of Theorem V.14 (ii) is equivalent to

$$\forall B \in \mathcal{B}, \quad \int_B \Sigma(\omega) d\mu(\omega) = \int_B \Gamma(\omega) d\mu(\omega),$$

which is equivalent to

$$\forall x \in E, \quad \delta^*(x, \Sigma(\omega)) = E^{\mathcal{B}} \delta^*(x, \Gamma(\omega)) \quad \text{a.e.}$$

here $E^{\mathcal{B}} f$ denotes the usual conditional expectation of an integrable function f. We provide an existence and uniqueness result of conditional expectation of an integrably bounded $\mathcal{F}$-random closed convex set in E'_s extending Theorem VIII.34 in [17] because here the strong dual E'_b of E is no longer separable. This need a careful look involving a sequentially compactness result in [13, Corollary 6.5.10], and some other techniques.

Theorem 7.3. *Under the foregoing hypotheses there exists a unique (for equality a.e.) conditional expectation of* Γ, Σ. *Moreover* Σ *has the properties:*

(a) $\Sigma(\omega) \subset E^{\mathcal{B}}(|\Gamma])(\omega)\overline{B}_{E'}$ *a.e.*
(b) The integral functionals

$$I_\Sigma : v \mapsto \int_\Omega \delta^*(v(\omega), \Sigma(\omega)) d\mu(\omega) \quad \text{and} \quad I_\Gamma : v \mapsto \int_\Omega \delta^*(v(\omega), \Gamma(\omega)) d\mu(\omega)$$

are continuous on the closed unit ball $\overline{B}_{L^\infty_E(\mathcal{B})}$ *of* $L^\infty_E(\Omega, \mathcal{B}, \mu)$ *endowed with the topology of convergence in measure and coincide on the subset of all simple functions* $v = \sum_{i=1}^n 1_{B_i} x_i$, *with the disjoint* $B_i \in \mathcal{B}$, $x_i \in E$.
(c) $\mathcal{S}^1_\Sigma(\mathcal{B})$ *is sequentially* $\sigma(L^1_{E'}[E](\mathcal{B}), L^\infty_E(\mathcal{B}))$ *compact (here* $\mathcal{S}^1_\Sigma(\mathcal{B})$ *denotes the set of all* $L^1_{E'}[E](\Omega, \mathcal{B}, \mu)$ *selections of* Σ*) and satisfies the inclusion*

$$E^{\mathcal{B}} \mathcal{S}^1_\Gamma(\mathcal{F}) \subset \mathcal{S}^1_\Sigma(\mathcal{B}).$$

(d) Furthermore one has

$$\delta^*(v, E^{\mathcal{B}} \mathcal{S}^1_\Gamma(\mathcal{F}) = \delta^*(v, \mathcal{S}^1_\Sigma(\mathcal{B}))$$

for all $v \in L^\infty_E(\mathcal{B})$.

Proof. Step 1 To prove the existence of Σ we apply Theorem V.17 in [17] by recalling that E'_s is a e.l.c Suslin space and E is its dual. Then we take in this theorem $\Lambda = L^1_{\mathbb{R}}(\Omega, \mathcal{B}, \mu)$ and $\Lambda^* = L^\infty_{\mathbb{R}}(\Omega, \mathcal{B}, \mu)$. We put $M(f) = \int f \Gamma d\mu$ for $f \in \Lambda^*$. Since $E' = \cup_n n\overline{B}_{E'}$, the mapping M mets conditions (i)–(iv) of Theorem V.17 in [17]. So there exist a $\mathcal{B}$-measurable convex $\sigma(E', E)$ compact valued, scalarly integrable multifunction Σ such that, $\forall f \in \Lambda^*, M(f) = \int f \Sigma d\mu$. Taking $f = 1_B (B \in \mathcal{B})$, we obtain $\int_B \Sigma d\mu = \int_B \Gamma d\mu$. The uniqueness follows easily as in the proof of Theorem VIII.34 in [17]. Indeed, let Σ_1 and Σ_2 be two convex $\mathcal{B}$-measurable convex $\sigma(E', E)$ compact valued, scalarly integrable multifunction such that

$$\forall f \in \Lambda^*, \quad M(f) = \int f \Sigma_1 d\mu = \int f \Sigma_2 d\mu.$$

By Strassen Theorem V.14 in [17], we have, for every $x \in E$, $\delta^*(x, \Sigma_1(\omega)) = \delta^*(x, \Sigma_2(\omega))$ a.e. By Proposition III.35, we deduce that $\Sigma_1(\omega) = \Sigma_2(\omega)$ a.e.

We will denote $E^{\mathcal{B}}\Gamma = \Sigma$ the unique $\mathcal{B}$-measurable convex $\sigma(E', E)$ compact valued, scalarly integrable multifunction Σ which verifies

$$\forall f \in \Lambda^*, \quad M(f) = \int f \Sigma d\mu.$$

Taking $f = 1_B (B \in \mathcal{B})$, we obtain

$$\int_B E^{\mathcal{B}}\Gamma d\mu = \int_B \Gamma d\mu.$$

Now we provide the properties of the conditional expectation $E^{\mathcal{B}}\Gamma$. It is worthy to mention that, when $\Gamma = u \in L^1_{E'}[E](\mathcal{F})$, then the EB of u, $E^{\mathcal{B}}u$, belongs to $L^1_{E'}[E](\mathcal{B})$ and satisfies

$$\forall f \in L^\infty_{\mathbb{R}}(\Omega, \mathcal{B}, \mu), \quad \int f u \, d\mu = \int f E^{\mathcal{B}} u \, d\mu.$$

Step 2 (a) For $x \in E$, one has

$$\begin{aligned} \delta^*(x, \Sigma(\omega)) &= E^{\mathcal{B}}(\delta^*(x, \Gamma(.))(\omega) \le E^{\mathcal{B}}(||x||.|\Gamma|)(\omega) \\ &= ||x||E^{\mathcal{B}}(|\Gamma|)(\omega) = E^{\mathcal{B}}(|\Gamma|)(\omega)\delta^*(x, \overline{B}_{E'}). \end{aligned}$$

for a.e. $\omega \in \Omega$. Again by [17, Proposition III.35], we have $\Sigma(\omega) \subset E^{\mathcal{B}}(|\Gamma|)(\omega)\overline{B}_{E'}$.
(b) It is clear that the formula

$$\int \delta^*(v(\omega), \Sigma(\omega)) d\mu(\omega) = \int \delta^*(v(\omega), \Gamma(\omega)) d\mu(\omega)$$

holds if $v = \sum_{i=1}^n 1_{B_i} x_i$, with the disjoint $B_i \in \mathcal{B}, x_i \in E$. Now we claim that the integral functionals

$$I_\Sigma : v \mapsto \int_\Omega \delta^*(v(\omega), \Sigma(\omega)) d\mu(\omega) \quad \text{and} \quad I_\Gamma : v \mapsto \int_\Omega \delta^*(v(\omega), \Gamma(\omega)) d\mu(\omega)$$

are continuous on the closed unit ball $\overline{B}_{L^\infty_E(\mathcal{B})}$ of $L^\infty_E(\Omega, \mathcal{B}, \mu)$ endowed with the topology of convergence in measure. Indeed we have for $v, w \in \overline{B}_{L^\infty_E(\mathcal{B})}$ the estimate

$$\begin{aligned}
&\left| \int_\Omega \delta^*(v(\omega), \Sigma(\omega)) d\mu(\omega) - \int_\Omega \delta^*(w(\omega), \Sigma(\omega)) d\mu(\omega) \right| \\
&\le \int_\Omega |\delta^*(v(\omega), \Sigma(\omega)) - \delta^*(w(\omega), \Sigma(\omega))| d\mu(\omega) \\
&\le \int_\Omega \max(\delta^*(v(\omega) - w(\omega), \Sigma(\omega)), \delta^*(w(\omega) - v(\omega), \Sigma(\omega))) d\mu(\omega) \\
&\le 2 \int ||v(\omega) - w(\omega)|| E^{\mathcal{B}}(|\Gamma|)(\omega) d\mu(\omega)
\end{aligned}$$

and similarly

$$\begin{aligned}
&\left| \int \delta^*(v(\omega), \Gamma(\omega)) d\mu(\omega) - \int \delta^*(w(\omega), \Gamma(\omega)) d\mu(\omega) \right| \\
&\le \int |\delta^*(v(\omega), \Gamma(\omega)) - \delta^*(w(\omega), \Gamma(\omega))| d\mu(\omega) \\
&\le \int_\Omega \max(\delta^*(v(\omega) - w(\omega), \Gamma(\omega)), \delta^*(w(\omega) - v(\omega), \Gamma(\omega))) d\mu(\omega) \\
&\le 2 \int ||v(\omega) - w(\omega)|| |\Gamma|(\omega) d\mu(\omega).
\end{aligned}$$

So (b) follows. If E is reflexive, one can see that I_Σ and I_Γ are Mackey continuous since the topology of convergence in measure on $\overline{B}_{L^\infty_E(\mathcal{B})}$ coincides with the Mackey convergence $\tau(L^\infty_E, L^1_{E'})$.
(c) The sequential $\sigma(L^1_{E'}[E](\mathcal{B}), L^\infty_E(\mathcal{B}))$ compactness of $\mathcal{S}^1_\Sigma(\mathcal{B})$ follows from Step 1 and Corollary 6.5.10 in [13]. Let $u \in \mathcal{S}^1_\Gamma(\mathcal{F})$ and $x \in E$. Then

$$\langle x, E^{\mathcal{B}} u \rangle \le E^{\mathcal{B}}[(\delta^*(x, \Gamma(.))] = \delta^*(x, \Sigma(.)) \quad \text{a.e.}$$

So again by [17, Proposition III.35], $E^{\mathcal{B}} u \in \mathcal{S}^1_\Sigma(\mathcal{B})$ and hence

$$E^{\mathcal{B}} \mathcal{S}^1_\Gamma(\mathcal{F}) \subset \mathcal{S}^1_\Sigma(\mathcal{B}). \tag{*}$$

(d) By (*), it is immediate that

$$\delta^*(v, E^{\mathcal{B}} \mathcal{S}^1_\Gamma(\mathcal{F})) \le \delta^*(v, \mathcal{S}^1_\Sigma(\mathcal{B}))$$

for all $v \in L^\infty_E(\mathcal{B})$. Let us prove the converse inequality

$$\delta^*(v, E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F})) \geq \delta^*(v, \mathcal{S}^1_\Sigma(\mathcal{B})). \tag{**}$$

Let $v \in L^\infty_E(\mathcal{B})$. Let u be a maximal $\mathcal{F}$ measurable selection of Γ associated with v, that is

$$\langle v(\omega), u(\omega)\rangle = \delta^*(v(\omega), \Gamma(\omega)), \quad \forall \omega \in \Omega.$$

See [17, Theorem III.22]. Then it is obvious that $u \in L^1_{E'}[E](\Omega, \mathcal{F}, \mu)$. Furthermore, by applying the equality of conditional expectation given in Step 2(b)

$$\int \langle v, u\rangle \, d\mu = \int \langle v, E^{\mathcal{B}}u\rangle \, d\mu.$$

One has

$$\begin{aligned}
&\delta^*(v, E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F})) \geq \langle v, E^{\mathcal{B}}u\rangle = \int \langle v, E^{\mathcal{B}}u\rangle \, d\mu \\
&= \int \langle v, u\rangle \, d\mu = \langle v, u\rangle = \int_\Omega \delta^*(v(\omega), \Gamma(\omega)) \, d\mu(\omega) \\
&= \int_\Omega \delta^*(v(\omega), \Sigma(\omega)) \, d\mu(\omega) \quad \text{(by (b) and approximation)} \\
&\geq \langle v, u_1\rangle \quad \text{for any} \quad u_1 \in \mathcal{S}^1_\Sigma(\mathcal{B}).
\end{aligned}$$

Finally

$$\delta^*(v, E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F})) = \delta^*(v, \mathcal{S}^1_\Sigma(\mathcal{B})) \tag{$***$}$$

for all $v \in L^\infty_E(\mathcal{B})$. □

Remarks. When E is a reflexive separable Banach space and $\Gamma \in \mathcal{L}^1_{cwk(E')}(\Omega, \mathcal{F}, \mu)$, then conditional expectation $E^{\mathcal{B}}\Gamma$ of Γ belongs to $\mathcal{L}^1_{cwk(E')}(\Omega, \mathcal{B}, \mu)$ and satisfies

$$\delta^*(v, E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F})) = \delta^*(v, \mathcal{S}^1_\Sigma(\mathcal{B}))$$

for all $v \in L^\infty_E(\mathcal{B})$ so that

$$E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F}) = \mathcal{S}^1_\Sigma(\mathcal{B})$$

because $E^{\mathcal{B}}\mathcal{S}^1_\Gamma(\mathcal{F})$ and $\mathcal{S}^1_\Sigma(\mathcal{B})$ are convex $\sigma(L^1_{E'}(\mathcal{B}), L^\infty_E(\mathcal{B}))$ compact, meanwhile the existence and uniqueness of EB met in Theorem 7.3 are unusual because the dual space is not strongly separable. See also [27] dealing with CE of Random Sets in the dual of a separable Fréchet space via the regular conditional probability.

To end the paper we provide the existence and uniqueness of conditional expectation of a closed convex $\mathcal{F}$ random set in E'_s in the line of [17, Theorem VIII.35].

Theorem 7.4. *Let Γ be a closed convex $\mathcal{F}$-random set in E'_s which admits an selection $u_0 \in L^1_{E'}[E](\mathcal{F})$. For every n and very ω, let*

$$\Gamma_n(\omega) := \Gamma(\omega) \bigcap_n (u_0(\omega) + n\overline{B}_{E'}),$$

$$\Sigma(\omega) = w^* cl \left[\bigcup_n E^{\mathcal{B}}(\Gamma_n)(\omega) \right].$$

Then: (a) Σ which is a.e. convex, is a unique (for the equality a.e.) $\mathcal{B}$ closed convex random set in E'_s with

$$\forall v \in L^\infty_E(\mathcal{B}), \quad \int_\Omega \delta^*(v(\omega), \Sigma(\omega)) d\mu(\omega) = \int_\Omega \delta^*(v(\omega), \Gamma(\omega)) d\mu(\omega).$$

(b) Σ is the smallest (for inclusion a.e.) of the $\mathcal{B}$ closed convex random set Θ such that

$$E^{\mathcal{B}} \mathcal{S}^1_\Gamma(\mathcal{F}) \subset \mathcal{S}^1_\Theta(\mathcal{B}).$$

We shall denote $E^{\mathcal{B}}\Gamma = \Sigma$ and says that Σ is the conditional expectation of Γ.

Proof. The proof is the same as in [17, Theorem VIII.35], using Theorem 7.3, the monotone convergence theorem and measurable projection theorem which ensures the uniqueness. Here the measurability of Σ is ensured thanks to Corollary 5.3 in [12]; at this point, let us mention that Γ admits a $L^1_{E'}[E](\mathcal{F})$ selection iff $d(0, \Gamma)$ is μ-integrable (see Lemma 5.6 in [12]). $\square$

The results obtained in this section led to Fatou lemma for conditional expectation of weak-star random sets in a dual space.

References

1. Artstein, Z.: A note on Fatou lemma's in several dimensions. J. Math. Econ. **6**, 277–282 (1979)
2. Amrani, A., Castaing, C., Valadier, M.: Méthodes de troncatures appliquées à des problèmes de convergences faible ou forte dans L^1. Arch. Rational Mech. Anal **117**, 167–191 (1992)
3. Balder, E.J.: A Fatou lemma for Gelfand integrals by means of Young measure theory. Positivity **6**, 317–329 (2002)
4. Balder, E.J., Hess, Ch.: Two generalizations of Komlós theorem with lower closure-type aplications. J. Convex Anal. **3**, 25–44 (1996)

5. Balder, E.J., Sambucini, A.R.: Fatou's lemma for multifunctions with unbounded values in a dual space. J. Convex Anal. **12**, 383–395 (2005)
6. Benabdellah, H., Castaing, C.: Weak compactness and convergences in $L^1_{E'}[E]$. Adv. Math. Econ. **3**, 1–44 (2001)
7. Castaing, C.: Quelques résultats de convergences des suites adaptées. Sém. Anal. Conv. Exp **2**, 2.1–2.24 (1987); Acta Mathematica Vietnamica **14**(1), 51–66 (1989)
8. Castaing, C.: Méthodes de compacité et de decomposition, applications: minimisation, convergences des martingales, lemme de Fatou multivoque. Ann. Math. Pura Apli. **164**, 51–75 (1993)
9. Castaing, C., Clauzure, P.: Lemme de Fatou multivoque, Ann. Math. Pura Appl. **XXXIX**, 303–320 (1991)
10. Castaing, C., Ezzaki, F.: Some convergence results for multivalued martingales in the limit. Séminaire d Analyse Convexe Montpellier 1990, Exposé No 1
11. Castaing, C., Ezzaki, F., Hess, Ch.: Convergence of conditional expectations for unbounded closed convex random sets. Stud. Math. **124**(2), 133–148 (1997)
12. Castaing, C., Hess, Ch., Saadoune, M.: Tightness conditions and integrability of the sequential upper limit of a sequence of multifunctions. Adv. Math. Econ. **11**, 1–34 (2008)
13. Castaing, C., Raynaud de Fitte, P., Valadier, M.: Young measures on topological spaces. With applications in control theory and probability theory. Dordrecht: Kluwer 2004
14. Castaing, C., Saadoune, M.: Dunford–Pettis-types theorem and convergences in set-valued integration. J. Nonlinear Convex Anal. **1**, 37–71 (1999)
15. Castaing, C., Saadoune, M.: Komlós type convergence with applications. Adv. Math. Econ. **10**, 1–26 (2007)
16. Castaing, C., Saadoune, M.: On various versions of Fatou lemma. Submitted (2008)
17. Castaing, C., Valadier, M.: Convex analysis and measurable multifunctions. Lectures Notes in Mathematics, vol. 580. Berlin: Springer 1977
18. Cornet, B., Martins da Rocha, V.F.: Fatou's lemma for unbounded Gelfand integrable mapping. Preprint 109, CERNSEM, Université Paris 1 (2002)
19. Cornet, B., Médecin, J.P.: Fatou's lemma for Gelfand integrable mappings. Positivity **6**, 297–315 (2002)
20. Hiai, F.: Convergence of conditional expectations and strong laws of large numbers for multivalued random variables. TAMS **291**(2), (613–627) 1985
21. Hiai, F., Umegaki, H.: Integrals, Conditional expectations, and martingales of multivalued functions. J. Multivariate Anal. **7**, 149–182 (1977)
22. Hildenbrand, W.: Core and equilibria of a large economy. Princeton: Princeton University Press 1974
23. Khan, M.A., Majumdar, M.: Weak sequential convergence in $L_1(\mu, X)$ and an approximate version of Fatou's lemma. J. Math. Anal. Appl. **114**, 569–573 (1986)
24. Saadoune, M., Valadier, M.: Extraction of a good subsequence from a bounded sequence of integrable functions. J. Convex Anal. **2**(1/2), 345–357 (1995)
25. Schmeidler, D.: Fatou's lemma in several dimensions, Proc. Am. Math. Soc. **24**, 300–306 (1970)

26. Valadier, M.: Semi-continuité de fonctionnnelles integrales, Séminaire d Analyse Convexe, Montpellier 1977, Exposé No 2
27. Valadier, M.: On conditional expectation of random sets. Annali di Mathematica pura ed applicata (IV) **CXXVI**, 81–91 (1980)
28. Yannelis, N.C.: Fatou's lemma in infinite-dimensional space. Proc. Am. Math. Soc. **102** 303–310 (1988)

Springer
http://www.springer.com/978-4-431-92934-5

Advances in Mathematical Economics Volume12
Kusuoka, S.; Maruyama, T. (Eds.)
2009, V, 170 p., Hardcover
ISBN: 978-4-431-92934-5