2

Linear Random Vibration Systems

2.1 Introduction

By random vibration of a linear dynamic system we mean the vibration of a deterministic linear system exposed to random (stochastic) loads. Random processes are characterized by the fact that their behavior cannot be predicted in advance and therefore can be treated only in a statistical manner. An example of a *micro-stochastic process* is the "Brownian motion" of particles and molecules [218]. A macro-stochastic process example is the motion of the earth during an earthquake. During the launch of a spacecraft, it will be exposed to random loads of mechanical and acoustic nature. The random mechanical loads are the base acceleration excitation at the interface between the launch vehicle and the spacecraft. The random loads are caused by several sources, e.g. the interaction between the launch-vehicle structure and the engines, exhaust noise, combustion. Turbulent boundary layers will introduce random loads. In this chapter we review the theory of random vibrations of linear systems. For further study on the theory of random vibration see [16, 115, 136, 154].

2.2 Probability

The *cumulative probability function* $F(x)$, that $x(t) \leq X$, is (c.d.f.) given by

$$F(X) = \int_{-\infty}^{X} f(x)dx \tag{2.1}$$

where

- $f(x)$ is the *probability density function* (p.d.f.) with the following properties
- $f(x) \geq 0$
- $\int_{-\infty}^{\infty} f(x)dx = 1$

J. Wijker, *Random Vibrations in Spacecraft Structures Design*,
Solid Mechanics and Its Applications 165,

- $F(X+dx) - F(X) = \int_X^{X+dx} f(x)dx = f(X)dx, X \leq x(t) \leq X + dx$

The cumulative probability function has the following properties:

- $F(-\infty) = 0$
- $F(\infty) = 1$
- $0 \leq F(x) \leq 1$
- $f(x) = \frac{dF(x)}{dx}$

Examples of probability density functions are:

- The constant distribution $U(a,b)$; X is called equally distributed over the interval $[a,b]$, $X \sim G(a,b)$, $f(x) = \frac{1}{b-a}$, $a \leq x \leq b$, $f(x) = 0$ elsewhere.
- The normally distribution[1] $N(\mu,\sigma)$, $\sigma > 0$. X is normally distributed with the parameters μ and σ, $X \sim N(\mu,\sigma)$ when $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(x-\mu)^2}{2\sigma^2}}$.
- The log normal distribution $LN(\mu,\sigma)$, $\sigma > 0$. X is log normal distributed with the parameters μ and σ, $X \sim LN(\mu,\sigma)$, $x > 0$, when $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(ln(x)-\mu)^2}{2\sigma^2}}$.
- The Rayleigh distribution $R(\sigma)$, $\sigma > 0$. X is Rayleigh distributed with the parameter σ, $X \sim R(\sigma)$, $x > 0$, when $f(x) = (\frac{2x}{\sigma^2}) e^{\frac{-x^2}{\sigma^2}}$.

For an ergodic random process, the term $f(x)dx$ may be approximated by

$$f(x)dx \approx \lim_{T\to\infty} \frac{1}{T} \sum_i \delta t_i, \tag{2.2}$$

where the δt_i are the lingering periods of $x(t)$ between $\alpha \leq x \leq \beta$. This is illustrated in Fig. 2.1.

The mode is defined as the peak of the p.d.f. $f(x)$, and the *mean value* μ has an equal moment to the left and to the right of it

$$\int_{-\infty}^{\infty} (x-\mu) f(x)dx = 0. \tag{2.3}$$

This means that the *average value* (mean value, mathematical expectation) of x can be calculated from

$$E(x) = \mu = \frac{\int_{-\infty}^{\infty} x f(x)dx}{\int_{-\infty}^{\infty} f(x)dx} = \int_{-\infty}^{\infty} x f(x)dx. \tag{2.4}$$

The definition of the n-th moment about the mean value is as follows

$$\mu_n = \int_{-\infty}^{\infty} (x-\mu)^n f(x)dx. \tag{2.5}$$

[1] The normal distribution was discussed in 1733 by De Moivre. It was afterwards treated by Gauss and Laplace, and is often referred to as the Gauss or Gauss-Laplace distribution [41].

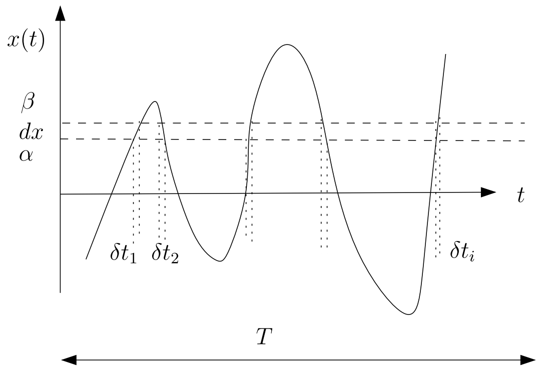

Fig. 2.1. Transient signal

The second moment is called the *variance* of a signal $x(t)$

$$\sigma^2 = \mu_2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx, \tag{2.6}$$

and σ is called the *standard deviation.*

Example. Suppose a sinusoidal signal $x(t) = A\sin\omega t$. Over one period T the signal $x(t)$ will cross a certain level twice when $X \leq x(t) \leq X + dx$, with a total time $2\delta t$. The p.d.f. can be estimated from $f(x)dx = \frac{2\delta t}{T} = \frac{\omega \delta t}{\pi}$ and with $\delta x = \omega A \cos\omega t \delta t$ the p.d.f. becomes $f(x) = \frac{1}{\pi A\cos\omega t} = \frac{1}{\pi A\sqrt{1-(\frac{x(t)}{A})^2}}$, $x < A$. The mean value is in accordance with (2.4)

$$E(x) = \mu = \int_{-\infty}^{\infty} xf(x)dx = 0,$$

and the variance σ^2 is in accordance with 2.6

$$\sigma^2 = \mu_2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \frac{A^2}{2}.$$

In general, within the framework of linear vibrations we may assume that the averaged (mean) value μ, of the response of a linear systems exposed to dynamic loads will be zero. So the second moment about the mean, the variance σ^2, is equal to the mean square value $E(x^2) = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx$.

Example. A random process x is randomly distributed between $0 \leq x \leq 1$ with a p.d.f.

$$f(x) = \begin{cases} 1 & 0 \leq x \leq 1, \\ 0 & x < 0, x > 1. \end{cases}$$

Calculate the mean value, the mean square value, the variance and the standard deviation of x:

- The mean value $E(x) = \mu = \int_{-\infty}^{\infty} x f(x) dx = \frac{1}{2}$
- The mean square value $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x) dx = \frac{1}{3}$
- The variance $\sigma^2 = E(x^2) - \mu^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x) dx = \frac{1}{12}$
- The standard deviation $\sigma = \sqrt{\frac{1}{12}} = 0.289$

The definition of a *cross probability function* or second order *probability distribution function* of two random processes $x(t)$ and $y(t)$ is given by

$$P(X,Y) = Prob[x(t) \leq X; y(t) \leq Y], \tag{2.7}$$

or, in terms of the specific probability density function

$$P(X,Y) = \int_{-\infty}^{X} \int_{-\infty}^{Y} f(x,y) dx dy. \tag{2.8}$$

Therefore, we can conclude that

$$Prob[X_1 \leq x(t) \leq X_2; Y_1 \leq y(t) \leq Y_2] = \int_{X_1}^{X_2} \int_{Y_1}^{Y_2} f(x,y) dx dy. \tag{2.9}$$

The specific probability density function $f(x,y)$ has the following properties

- $f(x,y) \geq 0$
- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) dx dy = 1.$

The probability density function of the first order can be obtained from the specific probability density function of the second order because

$$\begin{aligned} Prob[X_1 \leq x(t) \leq X_2; -\infty \leq y(t) \leq \infty] &= \int_{X_1}^{X_2} \left[\int_{-\infty}^{\infty} f(x,y) dy \right] dx \\ &= \int_{X_1}^{X_2} f(x) dx, \end{aligned} \tag{2.10}$$

where

$$f(x) = \int_{-\infty}^{\infty} f(x,y) dy. \tag{2.11}$$

In a similar manner it is found that

$$f(y) = \int_{-\infty}^{\infty} f(x,y) dx. \tag{2.12}$$

The probability density functions $f(x)$ and $f(y)$ are also called *marginal density functions*, [140].

If random variables $x(t)$ and $y(t)$ are statistically independent, then $f(x, y)$ satisfies

$$f(x, y) = f(x)f(y). \tag{2.13}$$

The mean value or *mathematical expectation* of a continuous function $g(x, y)$ is given by

$$E\{g(x, y)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)f(x, y)dxdy. \tag{2.14}$$

The mean values of $x(t)$ and $y(t)$ can be obtained as follows

$$E\{x(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xf(x, y)dxdy = \int_{-\infty}^{\infty} xf(x)dx, \tag{2.15}$$

$$E\{y(t)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} yf(x, y)dxdy = \int_{-\infty}^{\infty} yf(y)dy. \tag{2.16}$$

The *n-dimensional Gaussian probability density function* with the random variables $x_1(t)$, $x_2(t), \ldots, x_n(t)$ is given by, [149],

$$f(x_1, x_2, \ldots, x_n) = \frac{1}{\sigma_1\sigma_2\cdots\sigma_n\sqrt{(2\pi)^n\sigma}} e^{-\frac{1}{2\sigma}\sum_{k,l=1}^{n}\{\sigma_{kl}\frac{(x_k-m_k)(x_l-m_l)}{\sigma_k\sigma_l}\}}, \tag{2.17}$$

where

$$m_i = E\{x_i\}, \quad i = 1, 2, \ldots, n$$

represents the mean value, and

$$\sigma_i^2 = E\{(x_i(t) - m_i)^2\}, \quad i = 1, 2, \ldots, n$$

is the variance. In addition, the standard deviation σ is given by

$$\sigma = \begin{vmatrix} 1 & \varrho_{12} & \cdots & \varrho_{1n} \\ \varrho_{21} & 1 & \cdots & \varrho_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \varrho_{n1} & \varrho_{n2} & \cdots & 1 \end{vmatrix},$$

where

$$\varrho_{ij} = \frac{E\{(x_i - m_i)(x_j - m_j)\}}{\sigma_i\sigma_j}, \quad i, j = 1, 2, \ldots, n,$$

is the correlation coefficient of the two random variables x_i and x_j.

Characteristic Function

The *characteristic function* of a random variable x is defined as the Fourier transform of the probability density function [202]

Table 2.1. Characteristic functions

Distribution	$f(x)$	$E(x) = \mu_x$	σ_x^2	$M_x(\theta)$
$U(a,b)$	$\frac{1}{b-a}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	$\frac{1}{j\theta(b-a)}(e^{j\theta b} - e^{j\theta a})$
$N(\mu, \sigma^2)$	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$	μ	σ	$e^{j\mu\theta - \frac{\sigma^2\theta^2}{2}}$

$$M_x(\theta) = E\{e^{j\theta x}\} = \int_{-\infty}^{\infty} e^{j\theta x} f(x)dx. \tag{2.18}$$

Expanding the exponential term $e^{j\theta x}$ in power series will yield

$$M_x(\theta) = 1 + \sum_{n=1}^{\infty} \frac{(j\theta)^n}{n!} E\{x^n\}. \tag{2.19}$$

The moments of the random variable can be calculated from the characteristic function:

$$E\{x^n\} = \frac{1}{j^n} \frac{d^n M_x(\theta)}{d\theta^n}\bigg|_{\theta=0}. \tag{2.20}$$

The nth *cumulant function* can also be derived from the characteristic function

$$k_n(x) = \frac{1}{j^n} \frac{d^n \ln M_x(\theta)}{d\theta^n}\bigg|_{\theta=0}. \tag{2.21}$$

The first cumulant function is the same as the first moment, and the second and third cumulant functions are identical with the second and third central moments m_n

$$m_n = k_n(x) = \int_{-\infty}^{\infty} (x-\mu)^n f(x)dx,$$

where $m_1 = \mu$.

Table 2.1 shows two examples of the characteristic function.

Example. For a zero mean Gaussian random variable x, $\mu = 0$, the following expression can be derived $E\{x^4\} = 3(E\{x^2\})^2 = 3\sigma^4$. This can be proved using (2.20)

$$E\{x^4\} = \frac{1}{j^4} \frac{d^4 M_x(\theta)}{d\theta^4}\bigg|_{\theta=0,\mu=0} = 3\sigma^4.$$

A general recurrent expression for $E\{x^n\}$ is the subject of problem 2.5.

The cumulant functions can be calculated using (2.21)

$$k_1 = \mu = 0, \qquad k_2 = \sigma^2, \qquad k_n = 0, \quad n > 2.$$

Problems

2.1. The simply supported beam AB shown in Fig. 2.2 is carrying a load of 1000 N that may be placed anywhere along the span of the beam. This

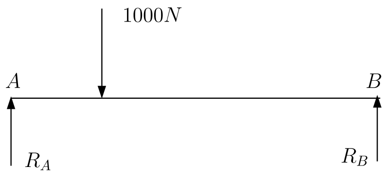

Fig. 2.2. Simply supported beam AB

problem is taken from [5]. The reaction force at support A, R_A, can be any value between 0 and 1000 N depending on the position of the load on the beam. What is the probability density function of the reaction force R_A? Calculate the probability that

- $Prob(100 \leq R_A \leq 200)$,
- $Prob(R_A \geq 600)$.

Answers: $f(x) = 1/1000$, $0 \leq x \leq 1000$, 0.10, 0.40

2.2. A random variable X is uniformly distributed over the interval $(a, b, a < b)$ and otherwise zero.

- Define the probability density function $f(x)$ such that $\int_{-\infty}^{\infty} f(x)dx = 1$.
- Calculate $E(X)$.
- Calculate $E(X^2)$.
- Calculate the variance $Var(X)$.
- Calculate the standard deviation σ_X, and
- Calculate the distribution function $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$, $a < x < b$.

Answers: $f(x) = \frac{1}{b-a}$, $E(X) = \frac{a+b}{2}$, $E(X^2) = \frac{a^2+b^2+ab}{3}$, $Var(X) = \frac{(b-a)^2}{12}$, $\sigma_X = \frac{(b-a)}{\sqrt{12}}$, and $F(x) = \frac{x-a}{b-a}$.

2.3. A continuous random variable X is said to have *gamma distribution* if the probability density function of X is

$$f(x, \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-\frac{x}{\beta}}, & x \geq 0; \\ 0, & \text{otherwise}, \end{cases}$$

where the parameters α and β satisfy $\alpha > 0$, $\beta > 0$. Show that the mean and variance of such a random variable X satisfy

$$E(X) = \alpha\beta, \qquad Var(X) = \alpha\beta^2.$$

The gamma function is defined by

$$\Gamma(k) = \int_0^\infty e^{-u} u^{k-1} du.$$

Show that

$$\Gamma(k) = (k-1)!$$

for integer k.

2.4. Each front tire on a particular type of vehicle is supposed to be filled to a pressure of 2.5 Bar. Suppose the actual pressure in each tire is a random variable, X for the right tire and Y for the left tire, with joint p.d.f.

$$f(x,y) = \begin{cases} K(x^2+y^2), & 2.0 \le x \le 3.0,\ 2.0 \le y \le 3.0; \\ 0, & \text{otherwise.} \end{cases}$$

- What is the value of K?
- What is the probability that both tires are underinflated?
- What is the probability that the difference in air pressure between the two tires is at most 0.2 Bar?
- Are X and Y independent random variables?

2.5. This problem is taken from [112]. Let X be a Gaussian random variable with a characteristic function

$$M_x(\theta) = e^{j\mu\theta - \frac{\sigma^2\theta^2}{2}}.$$

Show that

$$E\{X^n\} = \mu E\{X^{n-1}\} + (n-1)\sigma^2 E\{X^{n-2}\}.$$

2.6. The gamma probability density function is defined by

$$f(x) = \frac{\lambda(\lambda x)^{k-1} e^{-\lambda x}}{\Gamma(k)},$$

where λ and k are distribution parameters. The function $\Gamma(k)$ is the gamma function, which is given by

$$\Gamma(k) = \int_0^\infty e^{-u} u^{k-1} du.$$

Show that the mean and the variance are as follows:

$$\mu_x = \frac{k}{\lambda},$$
$$\sigma_x^2 = \frac{k}{\lambda^2}.$$

2.3 Random Process

A random process is random in time. The probability can be described with the aid of probabilistic theory of random processes [146]. The mean and the

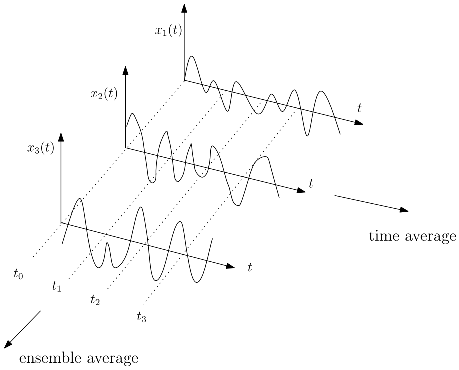

Fig. 2.3. Time history of a random process

mean square values are of great importance for random processes. We can make a distinction between *ensemble average* and *time average*. In this section we review the properties of random processes. The ensemble average $E\{\}$ of a collection of sampled records $x_j(t_i)$, $j = 1, 2, \ldots, n$ at certain times t_i is defined as

$$E\{x(t_i\} = \frac{1}{n}\sum_{j=1}^{n} x_j(t_i), \quad i = 0, 1, 2, \ldots . \tag{2.22}$$

This is illustrated in Fig. 2.3.

A random or stochastic process $x(t)$ is said to be stationary in the strict[2] sense if the set of finite dimensional joint probability distributions of the process is invariant under a linear translation $t \to t + a$.

$$\begin{aligned} F_x(x_1, t_1) &= F_x(x_1, t_1 + a), \\ F_x(x_1, t_1; x_2, t_2) &= F_x(x_1, t_1 + a; x_2, t_2 + a), \\ &\vdots \\ F_x(x_1, t_1; x_2, t_2; \ldots ; x_n, t_n) &= F_x(x_1, t_1 + a, x_2; t_2 + a; \ldots ; x_n, t_n + a). \end{aligned} \tag{2.23}$$

If (2.23) holds only for $n = 1$ and $n = 2$ the process is stationary in the weak sense or simply weakly stationary [203].

The time average (temporal mean) value of a record $x(t)$, over a very long sampling time T, is given by,

$$\langle x \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)dt. \tag{2.24}$$

An *ergodic process* is a stationary process in which ensemble and time averages are constant and equal to one another $E\{x\} = \langle x \rangle$.

In Table 2.2 a qualification of random processes is shown.

[2] Also mentioned strictly stationary process or strongly stationary process.

Table 2.2. Qualification of random process

Random process	Stationary	Ergodic
	Stationary	Non ergodic
	Non stationary	Non ergodic

For our purposes we will assume that all random processes are stationary and ergodic.

For a stationary and ergodic random process $x(t)$ there are the following relations for the mean value:

$$\mu_x = \langle x \rangle = E(x) = \int_{-\infty}^{\infty} x f(x) dx = \lim_{T\to\infty} \frac{1}{T} \int_0^T x(t) dt, \tag{2.25}$$

and for the mean square value

$$\langle x^2 \rangle = E\{x^2\} = \int_{-\infty}^{\infty} x^2 f(x) dx = \lim_{T\to\infty} \frac{1}{T} \int_0^T x^2(t) dt = \sigma_x^2 + \mu_x^2. \tag{2.26}$$

The variance of stationary random process $x(t)$ is given by

$$\sigma_x^2 = E\{(x(t) - \mu_x)^2\} = E\{x^2\} - 2\mu_x E\{x\} + \mu_x^2 = E\{x^2\} - \mu_x^2. \tag{2.27}$$

This explains (2.26).

$$\begin{aligned} \sigma_x^2 &= E\{(x(t) - \mu_x)^2\} \\ &= \lim_{T\to\infty} \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} E\{x(t_1)x(t_2)\} dt_1 dt_2 - \mu_x^2 \\ &= \lim_{T\to\infty} \frac{1}{4T^2} \int_{-T}^{T} \int_{-T}^{T} R_{xx}(t_2 - t_1) dt_1 dt_2 - \mu_x^2, \end{aligned} \tag{2.28}$$

where $R_{xx}(t_2 - t_1)$ is the auto correlation function, which will be discussed later in the next section. The autocorrelation function describes the correlation of the random process $x(t)$ at different points t_1 and t_2 in time.

To put this result in a simpler form, consider the change of variables according to $\tau_1 = t_2 + t_1$ and $\tau_2 = t_2 - t_1$. The Jacobian of this transformation is

$$\left| \frac{\partial(t_1, t_2)}{\partial(\tau_1, \tau_2)} \right| = \frac{1}{2}. \tag{2.29}$$

In terms of the new variables (2.28) becomes,

$$\sigma_x^2 = \lim_{T\to\infty} \frac{1}{4T^2} \int_0^T \int_0^{T-\tau_2} \frac{1}{2} R_{xx}(\tau_2) d\tau_1 d\tau_2 - \mu_x^2, \tag{2.30}$$

where the domain of integration is a square shown in Fig. 2.4. It is seen that the integrand is an even function of τ_2 and is not a function of τ_1. Hence, the

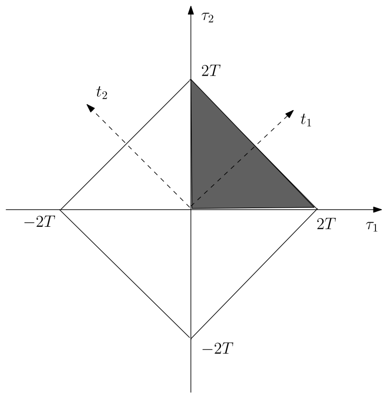

Fig. 2.4. Domain of integration

value of the integral is four times the value of the integral over the shaded area. Thus

$$\begin{aligned}\sigma_x^2 &= \lim_{T\to\infty} \frac{1}{2T^2} \int_0^{2T} \int_0^{2T-\tau_2} R_{xx}(\tau_2) d\tau_1 d\tau_2 - \mu_x^2 \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^{2T} R_{xx}(\tau_2) \left(1 - \frac{\tau_2}{2T}\right) d\tau_2 - \mu_x^2 \\ &= \lim_{T\to\infty} \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau_2}{2T}\right) [R_{xx}(\tau_2) - \mu_x^2] d\tau_2. \end{aligned} \tag{2.31}$$

The random variable $x(t)$ is ergodic in the mean if and only if [188]

$$\lim_{T\to\infty} \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau}{2T}\right) [R_{xx}(\tau) - \mu_x^2] d\tau = 0. \tag{2.32}$$

Example. A random signal $x(t)$ with zero mean has the following correlation function

$$R(\tau) = e^{-\lambda|\tau|}.$$

Show that this signal is ergodic in the mean using (2.32).

$$\begin{aligned}&\lim_{T\to\infty} \frac{1}{T} \int_0^{2T} \left(1 - \frac{\tau}{2T}\right) [R(\tau)] d\tau \\ &\quad = \lim_{T\to\infty} \frac{1}{\lambda T} \left(1 - \frac{1 - e^{-\lambda T}}{2\lambda T}\right) = 0.\end{aligned}$$

Normally, all vibration testing and analysis is carried out under the assumption that the random vibration is Gaussian. The primary reasons for this assumption are twofold [143]:

1. The Gaussian process is one of the few processes which have been mathematically defined, and
2. Many physical processes have been found to be at least approximately Gaussian (central limit theorem[3]).

2.3.1 Power Spectral Density

The *autocorrelation function* (*auto variance function*) of a stationary and an ergodic random process $x(t)$ is illustrated in Fig. 2.5. It expresses the correlation of a function with itself (auto) at points separated by various times τ. The autocorrelation function is defined by

$$\begin{aligned} R_{xx}(\tau) = E\{x(t)x(t+\tau)\} &= \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt \\ &= \lim_{T\to\infty} \frac{1}{T}\int_{0}^{T} x(t)x(t+\tau)dt, \end{aligned} \tag{2.33}$$

with the following properties:

- $\lim_{\tau\to\infty} R_{xx}(\tau) = \mu_x^2$, $x(t)$ and $x(t+\tau)$ become independent [84]
- $R_{xx}(\tau)$ is a real function
- $R_{xx}(\tau)$ is a symmetric function, $R_{xx}(\tau) = R_{xx}(-\tau)$, $R_{xx}(-\tau) = E\{x(t-\tau)x(t)\}$
- $R_{xx}(0) = E(x^2) = \lim_{T\to\infty} \frac{1}{T}\int_0^T x^2(t)dt = \sigma_x^2 + \mu_x^2$
- $R_{xx}(0) \geq |R_{xx}(\tau)|$, which can be proven with the relation $\lim_{T\to\infty} \frac{1}{T}\int_0^T [x(t) \pm x(t+\tau)]^2 dt = E\{[x(t) \pm x(t+\tau)]^2\} \geq 0$, $E\{[x(t)]^2\} + E\{[x(t+\tau)]^2\} \pm 2E\{x(t)x(t+\tau)\} \geq 0$, thus, $2R(0) \pm 2R(\tau) \geq 0$, finally, $R(0) \geq |R(\tau)|$. It should be emphasized, however, that the equality may hold, [209].

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[3] Let $x_1, x_2, \ldots, x_n$ be a sequence of independent random variables with the means $\mu_1, \mu_2, \ldots, \mu_n$ and the variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$. Let S_n be the sum of the sequence

$$S_n = \sum_{i=1}^{n} x_i, \qquad \mu_{sn} = \sum_{i=1}^{n} \mu_i, \qquad \sigma_{sn}^2 = \sum_{i=1}^{n} \sigma_i^2.$$

As $n \to \infty$ the normalized variable z_n, with mean $\mu_z = 0$ and $\sigma_z = 1$ is given by

$$z_n = \frac{s_n - \mu_{sn}}{\sigma_{sn}}.$$

The variable z_n has the following normalized distribution

$$f_{sn}(z) = \frac{1}{2\pi} e^{-\frac{1}{2}z^2}.$$

For any individual distribution of x_i, the distribution of the sum converges to a normalized Gaussian distribution.

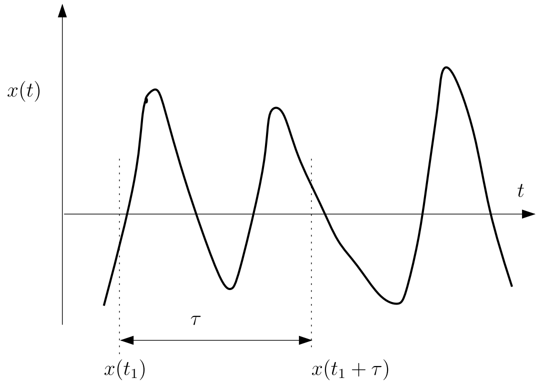

Fig. 2.5. Autocorrelation

- The correlation between $x(t)$ and $\dot{x}(t)$ is $R_{x\dot{x}}(\tau) + R_{\dot{x}x}(\tau) = 0$, because $\frac{\partial R_{xx}(\tau)}{\partial t} = 0$, and therefore $R_{x\dot{x}}(0) = -R_{\dot{x}x}(0) = 0$
- If $x(t) = \alpha y(t) + \beta z(t)$ then $R_{xx}(\tau) = \alpha^2 R_{yy}(\tau) + \alpha\beta R_{yz}(\tau) + \alpha\beta R_{zy}(\tau) + \beta^2 R_{zz}(\tau) = \alpha^2 R_{yy}(\tau) + 2\alpha\beta R_{yz}(\tau) + \beta^2 R_{zz}(\tau)$
- The Fourier transform requirement is satisfied for the autocorrelation function when $\int_{-\infty}^{\infty} |R_{xx}(\tau)| d\tau < \infty$.
- The *normalized correlation coefficient*is defined as $r(\tau) = \frac{R_{xx}(\tau)}{R_{xx}(0)}$. The normalized correlation coefficient of many real physical stochastic processes can be approximated by the formula [181] $e^{-\alpha|\tau|}(\cos\gamma\tau + \frac{\alpha}{\gamma}\sin\gamma|\tau|)$, where α and γ are constants.
- The *correlation time* τ_c is defined as $\tau_c = \frac{1}{R_{xx}(0)}\int_{-0}^{\infty} |R_{xx}(\tau)| d\tau = \int_{-0}^{\infty} |r(\tau)| d\tau$.

The *cross-correlation function* $R_{xy}(\tau)$ is defined as

$$R_{xy}(\tau) = E\{x(t)y(t+\tau)\} = \lim_{T\to\infty} \frac{1}{T}\int_0^T x(t)y(t+\tau)dt. \tag{2.34}$$

It can be proven that $|R_{xy}(0)| = \frac{1}{2}[R_{xx}(0) + R_{yy}(0)]$, and $|R_{xy}(0)|^2 \leq R_{xx}(0)R_{yy}(0)$.

Example. Calculate the autocorrelation function of the function $x(t) = A\sin\omega t$. In accordance with (2.33) the autocorrelation function becomes $R_{xx}(\tau) = \frac{1}{T}\int_0^T x(t)x(t+\tau)dt = \frac{\omega A^2}{2\pi}\int_0^{\frac{2\pi}{\omega}} \sin\omega t \sin\omega(t+\tau)dt = \frac{A^2}{2}\cos\omega\tau$. The mean square value of $x(t)$ can be easily calculated $E\{x^2\} = R_{xx}(0) = \frac{A^2}{2}$.

The *covariance function* $C_{xx}(\tau)$ is defined as

$$C_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_0^T \{x(t) - \mu_x\}\{x(t+\tau) - \mu_x\}dt, \tag{2.35}$$

and that function is related to the autocorrelation function as follows

$$R_{xx}(\tau) = C_{xx}(\tau) + \mu_x^2. \tag{2.36}$$

And if the average value of $x(t)$ is $\mu_x = 0$, then we see that

$$R_{xx}(\tau) = C_{xx}(\tau). \tag{2.37}$$

It can be proved that

$$|C_{xx}(\tau)| \leq \sigma_x^2. \tag{2.38}$$

The *Fourier transform* of a function $x(t)$ is defined in [89, 145] as

$$F\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt, \tag{2.39}$$

and the inverse of the Fourier transform

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega, \tag{2.40}$$

assuming that $\int_{-\infty}^{\infty} |x(t)|dt < \infty$, and any discontinuities are finite.

Example. Calculate the Fourier transform of a rectangular pulse:

$$f(t) = \begin{cases} A & |t| \leq T, \\ 0 & |t| > T. \end{cases}$$

From (2.39) we have

$$F(\omega) = \int_{-T}^{T} Ae^{-j\omega t}dt = \left[-\frac{A}{j\omega}e^{-j\omega t}\right]_T^T = \frac{2A}{\omega}\sin\omega T.$$

The Fourier transform of the autocorrelation function $R_{xx}(\tau)$ is called the *power spectral density function* $S_{xx}(\omega)$ (also called auto *spectral density*)

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau = 2\int_{0}^{\infty} R_{xx}(\tau)\cos\omega\tau d\tau, \tag{2.41}$$

and

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}d\omega = \frac{1}{\pi}\int_{0}^{\infty} S_{xx}(\omega)\cos\omega\tau d\omega. \tag{2.42}$$

Use has been made of Euler's identity, namely $e^{j\omega t} = \cos\omega t + j\sin\omega t$.

Table 2.3 contains the spectral densities $S(\omega)$ for various correlation functions $R(\tau)$.

The power spectral density function $S_{xx}(\omega)$ quantifies the distribution of power of signal $x(t)$ with respect to the frequency. In the expression of the

Table 2.3. Correlation function versus spectral density [181, 203]

$R(\tau)$	$S(\omega)$
$C\delta(\tau)$	$C =$ constant
$\sum_{k=1}^{n} C_k \delta^{(k)}(\tau)$	$\sum_{k=1}^{n} C_k (j\omega)^k$
$Ce^{-\alpha\|\tau\|}$	$\frac{2\alpha C}{\alpha^2+\omega^2}$
$Ce^{-\alpha\|\tau\|}\cos\beta\tau$	$\alpha C(\frac{1}{\alpha^2+(\beta+\omega)^2} + \frac{1}{\alpha^2+(\beta-\omega)^2})$
$Ce^{-\alpha\|\tau\|}(\cos\beta\tau - \frac{\alpha}{\beta}\sin\beta\|\tau\|)$	$\frac{C\omega^2}{(\omega^2-\alpha^2-\beta^2)^2+4\alpha^2\omega^2}$
$Ce^{-\alpha\|\tau\|}(\cos\beta\tau + \frac{\alpha}{\beta}\sin\beta\|\tau\|)$	$\frac{4C(\alpha^2+\omega^2)}{(\omega^2-\alpha^2-\beta^2)^2+4\alpha^2\omega^2}$
$Ce^{-(\alpha\tau)^2}\cos\beta\tau$	$\frac{C\sqrt{\pi}}{2\alpha}[e^{\{-\frac{(\omega+\beta)^2}{4\alpha^2}\}} + e^{-\frac{(\omega-\beta)^2}{4\alpha^2}}]$
$\sum_{k=0}^{n} C_k \cos\frac{k\pi\tau}{T}$ for $\|\tau\| \le T$ 0 for $\|\tau\| > T$	$2T\sum_{k=0}^{n}(-1)^k C_k \frac{\omega T\sin\omega T}{(\omega T)^2-(k\pi)^2}$
$C(1-\frac{\|\tau\|}{T})$ for $\|\tau\| \le T$ 0 for $\|\tau\| > T$	$CT(\frac{\sin\frac{\omega T}{2}}{\frac{\omega T}{2}})^2$
$-C\sum_{j=1}^{n}\frac{e^{s_j\|\tau\|}}{s_j}\frac{B(s_j)B(-s_j)}{\prod_{k=1,k\neq j}^{n}(s_k^2-s_j^2)}$ $\Re\{s_j\} < 0$	$C\frac{B(j\omega)B(-j\omega)}{A(j\omega)A(-j\omega)}$ $B(s) = b_0 s^m + b_1 s^{m-1} + \cdots + b_m$ $A(s) = a_0 s^n + a_1 s^{n-1} + \cdots + a_n = \prod_{j=1}^{n}(s-s_j)$ $n > m$, s_j are roots of $A(s) = 0$

power spectral density, **spectral** indicates a measure of the frequency content, and the **power** is the quantity to which the various frequency components contributes in the mean square value of the variable $x(t)$. **Density** tells us that the frequencies are not discrete but continuously distributed, so we cannot speak of the contribution of a single frequency ω but only of the contribution of a band of frequencies between ω and $\omega + d\omega$.

Both the autocorrelation function $R_{xx}(\tau)$ and the power spectral density function $S_{xx}(\omega)$ are symmetric functions about $\tau = 0$ and $\omega = 0$.

The pair of (2.41) and (2.42) is called the Wiener-Khintchine (in German Wiener-Chintschin [81]) relationship. It is evident that for processes monotonically decreasing the integral (2.42) exits. Therefore $S_{xx}(\omega)$ is for large ω of the following order of magnitude

$$S_{xx}(\omega) \sim O\left(\frac{1}{\omega^{1+\varepsilon}}\right), \quad \text{where } \varepsilon > 0. \tag{2.43}$$

If the range of frequency ω, in which the spectral density does not vanish, is much smaller than a certain frequency ω_o belonging to this range, this process is called a *narrow band process* (Fig. 2.6). Thus, a narrow band process is that one that satisfies the condition $\frac{\Delta}{\omega_o} \ll 1$, where Δ is the band width at the half power points. Otherwise the process is called a *wide band process.*

Example. The correlation function $R_{xx}(\tau)$ of a random binary wave is given by:

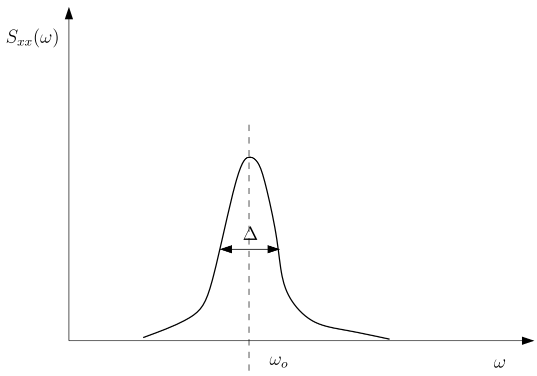

Fig. 2.6. Narrow band process

$$R_{xx}(\tau) = \sigma^2 \begin{cases} 1 - \frac{|t|}{\epsilon} & |t| \leq \epsilon, \\ 0 & |t| > \epsilon. \end{cases}$$

The power spectral density function $S_{xx}(\omega)$ becomes

$$S_{xx}(\omega) = 2\int_0^\infty R_{xx}(\tau)\cos\omega\tau d\tau = 2\int_0^\epsilon R_{xx}(\tau)\cos\omega\tau d\tau = \frac{4\sigma^2}{\epsilon\omega^2}\sin^2\left(\frac{\omega\epsilon}{2}\right).$$

The *total energy* E of the signal $x(t)$ is given by [89]

$$E = \int_{-\infty}^{\infty} \{x(t)\}^2 dt. \tag{2.44}$$

The (average) *power* P of the signal $x(t)$ is given by [89]

$$P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \{x(t)\}^2 dt = R_{xx}(0). \tag{2.45}$$

Using (2.40) we can rewrite (2.44) as

$$E = \int_{-\infty}^{\infty} \{x(t)\}^2 dt = \int_{-\infty}^{\infty} x(t)dt\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega\right]. \tag{2.46}$$

By changing the order of the integration (2.46) becomes

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\int_{-\infty}^{\infty} x(t)e^{j\omega t}dt\right]d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)d\omega, \tag{2.47}$$

hence,

$$\begin{aligned} E = \int_{-\infty}^{\infty} \{x(t)\}^2 dt &= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)X^*(\omega)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\omega)|^2 d\omega. \end{aligned} \tag{2.48}$$

The resulting equation (2.48) is called *Parseval's theorem* [89], with $|X(\omega)|^2$ the *energy spectral density* (ESD).[4] The ESD is an even function. Parseval's theorem describes how the energy in the signal is distributed along the frequency axis by the function $|X(\omega)|^2$.

Example. Consider the signal

$$x(t) = e^{-\alpha t}, \quad \alpha, t \geq 0.$$

Calculate the total energy E of the signal $x(t)$ using both sides of Parseval's theorem (2.48). Start with the left hand side (LHS) of (2.48), thus

$$E = \int_{-\infty}^{\infty} \{x(t)\}^2 dt = \int_{0}^{\infty} e^{-2\alpha t} dt = \frac{1}{2\alpha}.$$

The right hand side (RHS) of (2.48) is obtained. The Fourier transform of $x(t)$ is given by

$$X(\omega) = \frac{1}{\alpha + j\omega}.$$

The spectral density $|X(\omega)|^2$ becomes

$$|X(\omega)|^2 = \frac{1}{\alpha^2 + \omega^2}.$$

The total energy of the signal $x(t)$ is

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{\alpha^2 + \omega^2} d\omega = \frac{1}{2\pi} \left[\frac{\arctan \frac{\omega}{\alpha}}{\alpha} \right]_{-\infty}^{\infty} = \frac{1}{2\alpha}.$$

The definition of the energy of a signal relies on the *time domain* representation of the signal $x(t)$. Parseval's theorem gives a second way to compute the total energy based on the Fourier transform of the signal. That means the calculation of the total energy is done in the *frequency domain*. Parseval's theorem relates a time domain representation of the energy in a signal to the frequency domain description.

Equation (2.45), using Parseval's theorem, can be written as

$$P = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \{x(t)\}^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \lim_{T \to \infty} \frac{1}{2T} |X(\omega)|^2 d\omega, \tag{2.49}$$

where $\lim_{T \to \infty} \frac{1}{2T} |X(\omega)|^2$ is the *power spectral density*[5] (PSD) of $x(t)$. Parseval's theorem is a relation that states an equivalence between the power P of a signal computed in the time domain and that computed in the frequency domain.

---

[4] If $z = x + jy$ and $z^* = x - jy$ then $zz^* = x^2 + y^2 = |z|^2$.

[5] Also called autospectral density or autospectrum.

Equation (2.41) can be written, after multiplying by $e^{j\omega t}e^{-j\omega t}$, as follows

$$S_{xx}(\omega) = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T}\left[\int_{-T}^{T} x(t)x(t+\tau)dt\right] e^{j\omega t}e^{-j\omega(t+\tau)}d\tau$$
$$= \lim_{T\to\infty} \frac{1}{2T}|X(\omega)|^2. \qquad (2.50)$$

The average power P, using (2.49) and (2.42), can be expressed as

$$P = \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T}|X(\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega = R_{xx}(0), \qquad (2.51)$$

hence

$$R_{xx}(0) = E\{x^2\} = \langle x^2\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega. \qquad (2.52)$$

$S_{xx}(\omega)$ has the following properties:

- $S_{xx}(\omega) = S_{xx}(-\omega)$
- $S_{xx}(\omega) \geq 0$.

The *spectral moment* m_i of a stationary random process $X(t)$ is defined as [154]

$$m_i = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^i S_{xx}(\omega)d\omega. \qquad (2.53)$$

For a process $x(t)$ with $\mu_x = 0$ we may use the Wiener-Khintchine relations to find

$$\sigma_x^2 = m_0 = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)d\omega, \qquad (2.54)$$

and

$$\sigma_{\dot{x}}^2 = m_2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^2 S_{xx}(\omega)d\omega, \qquad (2.55)$$

and

$$\sigma_{\ddot{x}}^2 = m_4 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^4 S_{xx}(\omega)d\omega. \qquad (2.56)$$

A *normalized moment* with the dimension of circular frequency can be defined as

$$\gamma_n = \left(\frac{m_n}{m_0}\right)^{\frac{1}{n}}, \qquad (2.57)$$

where γ_1 is the central frequency and has a geometrical meaning of being the centroid of the spectral distribution $S_{xx}(\omega)$, and γ_2 has the geometrical meaning of the radius of gyration of $S_{xx}(\omega)$ about the origin. A variance parameter δ describing the dispersion of $S_{xx}(\omega)$ around the central frequency is defined as

$$\delta = \frac{\sqrt{(m_0 m_2 - m_1^2)}}{m_1} = \sqrt{\frac{m_0 m_2}{m_1^2} - 1}, \qquad (2.58)$$

where the range of δ is $0 \le \delta < \infty$. For a harmonic process, $m_1 = \omega_0 m_0$ and $m_2 = \omega_0^2 m_0$, $\delta = 0$, hence small values of δ indicate a narrow band process. Another bandwidth parameter γ is defined by

$$\gamma = \sqrt{1 - \frac{m_2^2}{m_0 m_4}} = \sqrt{1 - \alpha_2^2}, \tag{2.59}$$

where the *irregularity factor* α_2 is defined to be

$$\alpha_2 = \frac{m_2}{\sqrt{m_0 m_4}}. \tag{2.60}$$

The PSD function $S_{xx}(\omega)$ is *two-sided.* It is more practical to replace ω (rad/s) with f (Hz, cycles/s) and to replace the two-sided PSD function $S_{xx}(\omega)$ with a *one-sided* PSD function $W_{xx}(f)$ and then (2.52) becomes

$$R_{xx}(0) = E\{x^2\} = \langle x^2 \rangle = \frac{4\pi}{2\pi} \int_0^\infty S_{xx}(\omega) df = \int_0^\infty W_{xx}(f) df, \tag{2.61}$$

where $W_{xx}(f) = 2S_{xx}(\omega)$.

In the narrow frequency band Δf it is assumed that W_{xx} is constant and therefore

$$W_{xx} \Delta f = \lim_{T \to \infty} \frac{1}{T} \int_0^T x^2(t) dt, \tag{2.62}$$

and

$$W_{xx} = \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{x^2(t)}{\Delta f} dt = \frac{\langle x^2 \rangle}{\Delta f}, \tag{2.63}$$

where T is the averaging time, $x^2(t)$ the instantaneous square of the signal within Δf, $\langle x^2 \rangle$ the mean square value, and $\Delta f \to 0$.

With use of (2.41) $W_{xx}(f)$ becomes as follows

$$W_{xx}(f) = 4 \int_0^\infty R_{xx}(\tau) \cos \omega\tau d\tau, \tag{2.64}$$

and (2.50) can be written as

$$W_{xx}(f) = \lim_{T \to \infty} \frac{1}{T} |X(\omega)|^2. \tag{2.65}$$

We will apply (2.64) and (2.65) later in this book to estimate the PSD values numerically.

The definition of the PSD function of $x(t)$ in relation with the Wiener-Khintchine relations and the mean square value will be recapitulated hereafter [142], because several definitions exist.

First Definition of Auto Spectral Density Function

- $S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2T}|X(\omega)|^2$

Wiener-Khintchine theorem $R_{xx}(\tau) \leftrightarrow S_{xx}(\omega)$

- $S_{xx}(\omega) = \int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau = 2\int_0^{\infty} R_{xx}(\tau)\cos(\omega\tau)d\tau$
- $R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}d\omega = \frac{1}{\pi}\int_0^{\infty} S_{xx}(\omega)\cos(\omega\tau)d\omega$

Relation between spectral density function and average energy (mean square value, variance)

- $\int_0^{\infty} S_{xx}(\omega)d\omega = \pi R_{xx}(0) = \pi\, Var\{x(t)\} = \pi\sigma_x^2$

Second Definition of Auto Spectral Density Function

- $S_{xx}(\omega) = \lim_{T\to\infty} \frac{1}{2\pi T}|X(\omega)|^2$

Wiener-Khintchine theorem $R_{xx}(\tau) \leftrightarrow S_{xx}(\omega)$

- $S_{xx}(\omega) = \frac{1}{\pi}\int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau = \frac{2}{\pi}\int_0^{\infty} R_{xx}(\tau)\cos(\omega\tau)d\tau$
- $R_{xx}(\tau) = \frac{1}{2}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}d\omega = \int_0^{\infty} S_{xx}(\omega)\cos(\omega\tau)d\omega$

Relation between spectral density function and average energy (mean square value, variance)

- $\int_0^{\infty} S_{xx}(\omega)d\omega = R_{xx}(0) = Var\{x(t)\} = \sigma_x^2$

Third Definition of Auto Spectral Density Function

- $W_{xx}(f) = \lim_{T\to\infty} \frac{1}{T}|X(2\pi f)|^2$

Wiener-Khintchine theorem $R_{xx}(\tau) \leftrightarrow W_{xx}(f)$

- $W_{xx}(f) = 2\int_{-\infty}^{\infty} R_{xx}(\tau)e^{-j\omega\tau}d\tau = 4\int_0^{\infty} R_{xx}(\tau)\cos(\omega\tau)d\tau$
- $R_{xx}(\tau) = \int_0^{\infty} W_{xx}(f)e^{j2\pi f\tau}d\omega = \int_0^{\infty} W_{xx}(f)\cos(2\pi f\tau)df$

Relation between spectral density function and average energy (mean square value, variance)

- $\int_0^{\infty} W_{xx}(f)d\omega = R_{xx}(0) = Var\{x(t)\} = \sigma_x^2$

White Noise

White noise contains equal amounts of energy at all frequencies. If the power spectral density function of a signal $x(t)$ is constant over the complete frequency range, $W_{xx}(f) = W_0$, $0 \le x(t) \le \infty$ we talk about white noise.

The power spectral density function $S_{xx}(\omega) = \frac{W_0}{2}, -\infty \le \omega \le \infty$. The autocorrelation function $R_{xx}(\tau)$ can be calculated as follows

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}d\omega = \frac{W_0}{4\pi}\int_{-\infty}^{\infty} e^{j\omega\tau}d\omega = \frac{W_0}{2}\delta(\tau),$$

where $\int_{-\infty}^{\infty} \delta(\tau)d\tau = 1$, and $\delta(\tau)$ is the *Dirac delta function*. A random process with a constant (white noise by analogy to white light in optics) PSD function between two frequencies (band-limited) is considered. Calculate the associated autocorrelation function $R_{xx}(\tau)$ if

- $W(f) = W_0,\ f_1 \leq f \leq f_2$
- $W(f) = 0,\ f < f_1$ and $f > f_2$

The autocorrelation function $R_{xx}(\tau)$ becomes

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega\tau}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\cos\omega\tau d\omega,$$

and

$$R_{xx}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)\cos\omega\tau d\omega = \frac{1}{\pi}\int_{0}^{\infty} S_{xx}(\omega)\cos\omega\tau d\omega.$$

Hence

$$R_{xx}(\tau) = \frac{W_0}{2\pi}\int_{2\pi f_1}^{2\pi f_2} \cos\omega\tau d\omega = \frac{W_0}{2\pi\tau}[\sin 2\pi f_2\tau - \sin 2\pi f_1\tau], \tag{2.66}$$

so that

$$R_{xx}(0) = \lim_{\tau\to 0}\frac{W_0}{2\pi\tau}[\sin 2\pi f_2\tau - \sin 2\pi f_1\tau] = W_0[f_2 - f_1].$$

Assume a very narrow bandwidth $[f_2 - f_1] = \Delta f$. Then (2.66) becomes

$$R_{xx}(\tau) = \frac{W_0}{2\pi\tau}\big[\sin 2\pi(f_1 + \Delta f)\tau - \sin 2\pi f_1\tau\big].$$

Using Taylor series

$$f(x + \Delta x) = f(x) + \frac{f'(x)}{1!}\Delta x + \frac{f''(x)}{2!}\Delta x^2 + \cdots, \tag{2.67}$$

for a sinus expansion of $\sin(f + \Delta f)$ we obtain

$$\sin\{2\pi(f_1 + \Delta f)\tau\} \approx \sin(2\pi f_1\tau) + \frac{\cos(2\pi f_1\tau)}{1!}2\pi\Delta f\tau. \tag{2.68}$$

The autocorrelation function $R_{xx}(\tau)$, with (2.67, 2.68), can now be calculated

$$R_{xx}(\tau) = \frac{W_0}{2\pi\tau}\big[\cos\big\{2\pi(f_1\tau)\big\}2\pi\Delta f\tau\big] = W_0\Delta f\cos(2\pi f_1\tau), \tag{2.69}$$

and

$$R_{xx}(0) = W_0\Delta f. \tag{2.70}$$

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