Chapter 2
Contexts, Resources, and Reform

In the previous chapter, I outlined possibilities for teaching mathematical reasoning that involves learners communicating their thinking to their teacher and their peers and teachers taking learners' mathematical reasoning seriously to develop and transform it. This is in line with the visions of reform mathematics in a number of countries. However, international evidence suggests that very few teachers embrace reforms and those who do, experience significant challenges in their teaching. The challenges that I outlined at the end of the previous chapter are daunting in any context even in the most well resourced contexts. However, in South Africa and many other countries, resources in most schools are severely limited, adding to teachers' difficulties in enacting reforms.

At the same time, resources are not the only influence on reform teaching and the ways in which they exert an influence are not always obvious. In a review of recent international and South African studies, Fleisch (2007) shows that the studies are inconclusive on the effects of resources such as teacher qualifications, class size, and learning materials on learner achievement. This suggests that there are ways in which resources that do matter are most likely mediated by other variables. In this chapter, I discuss some responses to reform pedagogy across a range of contexts and discuss ways in which contextual constraints and resources may or may not be implicated in enactments of reform teaching. This discussion, together with a more general discussion of the resources available for teaching and learning in South Africa, serves to situate the description of the different contexts of the teachers in this study and the resources available to them as they worked to teach mathematical reasoning in their classrooms.

Responses to Reforms

A strong impetus for reform curricula in many countries is the need to redress inequalities in mathematics education. Internationally, success in mathematics is distributed according to race and socio-economic status (Association for Mathematics Education of South Africa 2000; Department of Education 2001;

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Moses and Cobb 2001; Secada 1992). While many of the reasons for this maldistribution originate outside of the classroom, there are arguments that classroom practices can begin to work towards equity (Boaler 2002). Allowing different ways of knowing mathematics to be available in the classroom may afford success for a wider range of learners (Boaler 2002; Boaler and Greeno 2000). Allowing learners to express their ideas in the classroom can lend to diverse ways of thinking, and help to teach learners that everyone's thinking can contribute to the development of mathematical knowledge (Lampert 2001).

There has been much debate as to whether current mathematics reforms can be a mechanism for ensuring more equitable participation and achievement in mathematics (see Brodie 2006, for a summary of these debates). Empirical evidence in well-resourced countries is beginning to show that reforms do mitigate achievement gaps between marginalised and other learners and also enable learners to develop more motivated and positive identities as mathematics learners (Boaler 1997; Boaler and Greeno 2000; Hayes et al. 2006; Kitchen et al. 2007; Schoenfeld 2002). However, the evidence also shows that implementation of reform curricula is not widespread and in fact it is likely that implementation of reforms is inequitably distributed (Kitchen et al. 2007), so that poorer learners are less-likely to experience reform curricula and pedagogy. Particularly in African contexts, issues of resources, including big classes and few materials, teacher confidence and knowledge, and support for teachers, can be major barriers to developing new ways of teaching (Tabulawa 1998; Tatto 1999). If reforms are successful in promoting equity and if they are not taken up in less-resourced contexts, then the existing division between rich and poor are likely to be exacerbated.

There is also growing evidence that teaching in reform-oriented ways is an extremely challenging task for teachers (Sherin 2002; Nathan and Knuth 2003) and that successful reform teachers are rare, even in well resourced schools in the United States where the reforms have been in place for 10 years longer than in South Africa. Among the 18 teachers in their study, Fraivillig et al. (1999) considered only six teachers to be "skillful" in eliciting and supporting learner thinking, while only one was successful in eliciting, supporting, and extending learner thinking. Hufferd-Ackles et al. (2004) described the development of reform practices through four levels. Of the four teachers they worked with, only one teacher's trajectory took her and her learners through all four levels. These studies were conducted in well-resourced classrooms and so suggest that while resources may be important, they are not the only challenge for teachers in working with learners' reasoning.

Studies of and by teachers who are successful in developing discussion and collaboration around learners' reasoning, identify a number of challenges in such work. These include: supporting learners to make contributions that are productive of further reasoning (Heaton 2000; Staples 2004); respecting and valuing all learners' thinking while working with the diversity of their mathematical ideas (Lampert 2001); respecting the integrity of learners' errors while trying to transform them and teach the appropriate mathematics (Chazan and Ball 1999); seeing beyond one's own long-held and taken-for-granted mathematical assumptions in order to hear and work with learners' reasoning (Chazan 2000; Heaton 2000); maintaining a "common ground", which enables all learners to follow the conversation and its mathematical purpose and to contribute appropriately (Staples 2004); and generating mathematical practices such as making connections, generalizing, and justifying (Boaler and Humphreys 2005). The above research shows that the pedagogical demands of mathematical conversations can be daunting and that we need to understand more about the practices involved in generating and sustaining these conversations (Brodie 2007b).

Research on the new curriculum in South Africa has shown that teachers who are enthusiastic about and express support for the new curriculum struggle to enact many of the ideas in their classrooms. These studies show that many classrooms remain teacher-centred, and teachers engage with learners' ideas in superficial ways, if they do so at all (Chisholm et al. 2000; Taylor and Vinjevold 1999). Other studies show some hybrid practices beginning to develop. Jansen (1999) found that while most teachers were not implementing the new curriculum, or were doing so very superficially, some of the more experienced and confident teachers were able to move between old and new practices and negotiate for themselves and with their learners what it means to implement the new curriculum. Brodie et al. (2002) found that many teachers set up tasks and group work situations where learners engaged with the tasks. However, when learners expressed their thinking, teachers struggled to support and engage with their reasoning to take them further and to develop them mathematically (Brodie 1999; Brodie et al. 2002). This finding was confirmed in more recent studies, where teachers selected tasks that could elicit mathematical reasoning, but did not engage learners' reasoning in classroom interaction (Jina and Brodie 2008; Modau and Brodie 2008; Stein et al. 1996; Stein et al. 2000).

There are a number of possible explanations for South African teachers' difficulties with the new curriculum. One claim, prominent at the moment, is that teachers do not know enough conceptual mathematics to teach in ways required by the new curriculum (Taylor and Vinjevold 1999). Other explanations are that teacher development around the new curriculum has been inadequate and that appropriate curriculum materials are not available (Chisholm et al. 2000; Taylor and Vinjevold 1999). A third possibility is that teachers are able to implement some aspects of the new curriculum, for example, higher level tasks, relatively easily, whereas other aspects, in particular, interaction with learners, are particularly difficult (Brodie et al. 2007). Slonimsky and Brodie (2006) argue that new curriculum practices require that teachers coordinate a complex set of contextual and knowledge constraints and that such coordination takes a long time to develop. All the above explanations acknowledge that resources are only part of the problem.

One of the aims of this project was to explore possibilities for developing learners' mathematical reasoning in a range of South African contexts and thereby begin to develop a deeper understanding of how resources are implicated in such practices. The next section presents some background on educational resources and achievement in South Africa and the following section describes the differently resourced contexts in which the five teachers in the project worked. In the discussion of resources, I include the material resources of the schools, the human resources, in particular teacher and learner knowledge, and finally the resources that the teachers chose to work with in the study – the tasks that they developed to engage their learners in reasoning mathematically.

The South African Context

As with all aspects of life in South Africa, the education system is characterized by large disparities between rich and poor, and most of our schools and learners are of very low socio-economic status. Most teachers in South Africa teach big classes in very poorly resourced schools. The latest national data shows that of the 25,145 operational schools in South Africa, 11.5% of schools did not have water, 16% had no electricity, 5.2% did not have ablution facilities, 80% did not have libraries and 67% did not have computers for teaching and learning (Department of Education 2007). The average learner–teacher ratio in schools was 32:1 and the average learner-classroom ratio was 38:1 (Department of Education 2008). These averages hide wide disparities between provinces and schools. A research study conducted in Gauteng and Limpopo (the richest and poorest provinces in South Africa respectively), observed averages of 35 students per class in secondary schools, with the three rural secondary schools averaging 60 learners per class and with some classes having as many as 120 learners (Adler and Reed 2002). There were limited resources such as overhead projectors and those resources that did exist, for example chalkboards, were often in poor condition.

While outrageous in any terms, this lack of resources is particularly significant for teachers attempting to work with their learners' mathematical reasoning. It is difficult to attend to learners' ideas when there are 50 or more learners in the class and few material resources. Moreover, many learners, having experienced poorly resourced education, often have weak mathematical knowledge (Fleisch 2007), and may be reluctant to participate in lessons. When they do participate, they may express barely coherent, or very problematic ideas, and teachers may not be able to engage with these ideas (Brodie 2000). The fact that most teaching and learning takes place in English, which is not the main language of most teachers and learners, also makes participation more difficult for learners and development of learner thinking more difficult for teachers.

Learners' weak mathematical knowledge is apparent in the annual results of the school leaving examinations[1], which are taken by approximately 500,000 Grade 12 learners each year, of which about 300,000 take the mathematics examinations. In 2005, 55% of about 303,000 learners passed mathematics and in 2006, 52% of about 317,500 learners passed mathematics. In 2005, about 44,000 learners took the examination at the Higher Grade level, which is required for entry into scientific fields at university, and 59% passed while in 2006, about 47,000 took the examination at this level and 53% passed (Department of Education 2008). The inequities of the system become apparent when we see that of 40,000 Higher Grade candidates in 2001, only 20,000 were "African",[2] and of these about 3,000 passed. Thus,

[1] South Africa has school learners examinations at the end of Grade 12. These examinations are high stakes and determine students' eligibility for further study and job opportunities.

[2] The apartheid system of racial classification was four-tiered and funding for education was directly linked to these tiers. Although many white South Africans refer to themselves as African, in this context "African" refers to black South Africans who are not "coloured" or of Indian descent, and whose schools and colleges received least funding under apartheid. Black Africans make up about 80% of the South African population.

while about 85% of white Higher Grade candidates passed, only 15% of black African candidates did. Figures for 2004 are that 7,236 African learners passed out of a total of 40,000 candidates[3] and of these, 2,406 African learners passed with a "C" grade or higher, which is 0.5% of the total number who wrote mathematics and 6% of those who wrote Higher Grade mathematics (Centre for Development and Enterprise 2007).

Research in the lower grades shows that learners begin to struggle with mathematics as early as Grade 3. Contrary to other developing countries, South African learners are almost all in school. Fleisch et al. (2008) show that in 2007, more than 95% of children of compulsory school age attended an educational institution and that this reflects an improvement since 2001 in each age cohort between 7 and 15 years of age. However, Motala and Dieltiens (2008) raise questions as to what these learners actually learn in school, suggesting that about 60% are disengaged and disaffected and learn very little.

Reviewing the research in mathematics, Taylor et al. (2003) conclude that "studies conducted in South Africa from 1998 to 2002 suggest that learners' scores are far below what is expected at all levels of the schooling system, both in relation to other countries (including other developing countries) and in relation to the expectation of the South African curriculum." Many Grade 3 learners struggle with basic skills such as adding and subtracting two-digit numbers that require "carrying" or "borrowing". South Africa also performs poorly in international comparison studies. In the third TIMSS study, South Africa came last out of 41 countries at all three grade levels tested, doing significantly worse than countries with similar GDPs, for example Latvia and Lithuania (Howie and Hughes 1998). In the 2003 TIMSS study, South African Grade 8 learners again came last out of 46 countries, and more significantly did worse than countries with lower Human Development Indices, including Ghana et al. (Reddy 2006). In the Monitoring Learner Assessment Study (MLA), which compared Grade 4 learners across 12 African countries, South Africa had a mean score of 30% for numeracy, which was the lowest of all 12 countries (Taylor et al. 2003). These average scores hide the large disparities between black, low socio-economic status learners and wealthier, white learners, but they serve to show the extent of the "crisis" in mathematics learning in South Africa, which is quantitatively different from many other countries. While we acknowledge critiques and limitations of such comparative studies (for example, Keitel 2000; Reddy 2006), they do serve as an indication of some of the challenges that our education system faces and the difficult conditions under which many teachers work.

The "crisis" also extends to the availability and quality of mathematics teachers in South Africa. Many South African teachers, because they were under-served by apartheid education, have relatively weak knowledge of mathematics and how best to teach it (Taylor and Vinjevold 1999). This situation does not look set to change in the near future. Given the limited numbers of students who graduate from school with strong mathematical knowledge, the pool for potentially well-qualified

[3] The number of African candidates who wrote is not available.

teachers is small. Students who do well in mathematics and science usually have a range of more attractive career options in other fields. Knowledgeable teachers of mathematics are often recruited by industry with far better salaries and working conditions. There is a lack of detailed data about mathematics teachers in South Africa (Centre for Development and Enterprise 2007) but the following give only a part of the picture over the past 10 years. In 1997, only 50% of teachers of mathematics had specialized in mathematics in their training (Department of Education 2001); in 2004, a survey of 1,766 secondary schools (out of a total of about 5,600) showed that there were 1,734 qualified mathematics teachers at these schools and of those only 1,362 were actually teaching the subject (Centre for Development and Enterprise 2007); and in 2006, 16 universities graduated a total of about 550 mathematics teachers (Centre for Development and Enterprise 2007).

I have taken some time to review these statistics because they provide a background for understanding the debates about the new curriculum in South Africa, and the contexts of the schools in this study. It is imperative to provide access to mathematics for large numbers of low socio-economic status learners. The government's response has been the development of policies that encourage the transformation of the curriculum and pedagogy in South African schools. The curriculum was developed in consultation with local and international experts and draws on the international research that suggests that reform pedagogy can reduce inequality in mathematics achievement. However, the international experience of teacher difficulties with reform curricula, in addition to the particular challenges of the South African situation suggest that we cannot assume that the new curriculum will reduce inequality – it may even increase it. This was a concern for all of the teachers in this project, and so we wanted to examine how teachers worked with learners' mathematical reasoning in a range of South African classrooms.

Five Schools: Contexts and Resources

Race and Socio-Economic Status

Fifteen years after the end of apartheid, although there have been some shifts, schools still largely reflect historical divisions of race and class. Table 2.1 gives a description of each of the five teachers' schools in terms of race and socio-economic status[4].

[4] Historically race and class have been closely related in South Africa. Although this is changing for some small sections of the population, to a large extent this trend still exists and is largely reflected in this sample. In order to determine socio-economic status, we used the location of the school, the school fees, which public schools in South Africa are allowed to charge, and the teachers' knowledge of the typical occupations of the parents.

Table 2.1 Demographics of schools

Teacher	Learners' race[a]	Learners' socio-economic status	School fees (per year)	No. of teachers/ no. of learners
Mr. Nkomo	Black	Working class	R200	1,650/42 (39:1)
Mr. Mogale	Black	Working class	R200	1,700/46 (37:1)
Mr. Daniels	All races	Middle and lower-middle class	R4,000	1,600/60 (27:1)
Mr. Peters	Black and "coloured"	Working class	R400	1,250/36 (35:1)
Ms. King	White, with a few learners of other races	Middle and upper-middle class	R40,000 (private)	850/65 (13:1)

[a]We use apartheid terminology to describe race, which is standard practice in South Africa to indicate shifts or lack thereof in historical racial divisions

Under apartheid Mr. Nkomo's and Mr. Mogale's schools served only black learners, Mr. Peters' school served only "coloured" learners, and Mr. Daniels' and Ms. King's schools served only white learners. Ms. King's school is a private, boys-only school, with some boys living at the school, and all the others are public, co-educational non-residential schools. The racial profile of the teachers matched those of the learners. Since schools began to integrate in the early 1990s, Mr. Peters' school has black and "coloured" learners; Mr. Daniels' school is racially diverse with learners from all four racial "groups"; and Ms. King's school has a few black, "coloured", and Indian learners, but is still predominantly white. Teacher diversity across the schools has occurred much more slowly. All the teachers in Mr. Nkomo's and Mr. Mogale's schools are black, most of the teachers in Mr. Peters' school are "coloured", with some black teachers, almost all of the teachers in Ms. King's school are white, with a few teachers of other races. Mr. Daniels' school has made the most progress in integrating teachers across race. Although many teachers are still white, there are a number of "coloured", Indian and black teachers at the school. Mr. Daniels' himself is "coloured", and moved to this school from a "coloured" school about 4 years ago.

School Resources

All of the five schools are known in their areas as good schools. They all regularly achieved pass rates of 65% and above in the Grade 12 examinations, well above average nationally, and a little above average for Gauteng province in which they are located (Motala and Perry 2002). They are all functional most of the time, in contrast to many other schools in South Africa (Christie and Potterton 1997; Taylor et al. 2003; Taylor and Vinjevold 1999). This means that school starts on time, most learners are present, absentees are noted, learners move between classes relatively quickly, teachers come to class and teach, learners return to classes after breaks, there are regular teacher meetings, and there are administrative staff and administrative computer systems. According to the teachers, the principals are supportive of efforts to improve teaching and learning in their schools. All the principals supported the teachers studying further and all were eager for this research to take place in their schools. So, while the five schools represent some diversity, they do not capture the full diversity of South African schools. The study is limited to an urban area, in the wealthiest province in South Africa, and well-functioning schools. However, as the first study of teaching mathematical reasoning in South African high schools, it was important and appropriate to limit our study in this way.

All of the five schools have fences or walls and control access to the schools[5]. Ms. King's school is located on a beautiful, large, peaceful property, situated next to a busy business district. There are dormitories for the learners who live on campus, houses for the teachers who live on campus, a church, plenty of sports fields and a dam with guinea fowl. It is easy to forget that you are in the middle of a big city while at the school. Mr. Daniels' school is located in a residential suburb, on a hill with a beautiful view of neighbouring suburbs. It has a number of sports fields and has a "green" feel to it. Mr. Nkomo's, Mr. Mogale's, and Mr. Peters' schools are located in residential areas which are poorer and less "green". There is little open space in Mr. Peters' school and some dusty fields in the other two schools. Mr. Peters' school is an area that is well known for gang activity and violence. Sometimes learners come to school with knives and there have been some incidents with guns. During the time of the study, a teacher was robbed at gunpoint of his laptop and cellphone in the school grounds, an incident that caused considerable disruption to teaching and learning for about a week. Learners also are subject to attack when they leave the school, especially many black learners who come from other areas and have to walk through "coloured" gang territory in order to get home. Table 2.2 indicates other resources available at each of the teacher's schools.

Table 2.2 Resources available at the schools

Teacher	Staffroom	Library	Computer room	Photocopying
Mr. Nkomo	Yes	Old books: mainly: textbooks:	Non functional	Yes
Mr. Mogale	Yes			Yes
Mr. Daniels	Tea and coffee	Well equipped	Yes	Yes
Mr. Peters	Yes	No	Non functional	Yes
Ms. King	Tea, coffee, computers	Well equipped	Yes	Yes

[5] Christie and Potterton (1997) argue that this contributes to the functionality of schools.

Classroom Resources

Each teacher chose one Grade 10 or 11 class in which we would conduct the research. Table 2.3 gives an overview of the classes that comprised the research sample.

The above table shows the disparities in class size among the teachers, in relation to the socio-economic status of learners at the schools. The schools of lower socio-economic status tend to have larger classes. The only class that does not fit the trend is Mr. Nkomo's, where mathematics classes are smaller in Grade 11 and 12. There is a difference of almost 20 learners between Mr. Peters' and Ms. King's classes. The levels of the classes relate both to tracking practices at the schools and the level of examination that learners are being prepared for. Some schools teach standard and higher grade learners in the same class (Mr. Nkomo's school in this study) while some differentiate them (Mr Mogale's and Mr. Peters' schools). Some schools track even further beyond this (Mr. Mogale's and Ms. King's schools).

Ms. King's classroom is part of a newly built wing of the school, is carpeted and has air-conditioning. There is a big table and chair for each learner, which can be arranged for work in groups. There are whiteboards and pens, a teacher's desk with a computer, cupboards and tables for storing paper and worksheets, notice boards filled with math posters, an overhead projector and screen, and a television set which can be used for presentations from a computer. Each learner has a textbook, which is purchased by the learner. Ms. King has a range of texts and resources, including international texts, from which she and her colleagues develop and share worksheets[6]. Learners have access to a computer lab and so they are given projects to do, either using the mathematical software or using the internet as a research tool, for example a project on the history of mathematics. Mr. Daniels' classroom has small tables and chairs, which are arranged in groups of four. The classroom is in good repair, and there are notice boards with a few math posters that Mr. Daniels has obtained. There is a teacher's desk and one cupboard that overflows with supplies, worksheets, learners' work, and other documents (there is no other storage space).

Table 2.3 Description of research classes

Teacher	Grade	Class size	Tracking/level of class
Mr. Nkomo	11	28	Untracked: mixed standard and higher grade[a]
Mr. Mogale	11	43	Tracked: higher grade (top class)
Mr. Daniels	11	35	Tracked: standard grade
Mr. Peters	10	45	Untracked
Ms. King	10	27	Tracked: second highest class of seven

[a]At the time of the study, the Grade 12 mathematics examination could be taken at two levels, standard and higher grade. Success on the higher grade (or exceptionally good marks on the standard grade) granted access to scientific fields at university

[6]Two mathematics teachers at this school developed a very strong curriculum of investigations in the 1980's and Ms. King and her colleagues still use some of this (McLachlan & Ryan, 1994).

There is a chalkboard and chalk and an overhead projector and screen, although electricity is not always available and so it does not always work. The school issues a textbook to each learner, although Mr. Daniels and his colleagues work predominantly from worksheets, which they develop and share.

Mr. Nkomo and Mr. Mogale's classrooms are similar. Tables are shared between two learners, with enough space for both. These are put together in pairs to form groups of four learners. There is a chalkboard and chalk, but no teacher's desk and no overhead projector and screen. Mr. Nkomo's classroom has a cupboard that stores cleaning supplies, but he has an office nearby where he keeps texts, worksheets, and learners' work. Mr. Mogale keeps his work in the staffroom. Mr. Nkomo's classroom is in good repair, although there is graffiti on the cupboard and notice boards. He occasionally puts posters and learners' work on the notice boards but has to be careful because they often disappear. Electricity is regularly available. In Mr. Mogale's classroom, some windows and the door are broken. There is no regular supply of electricity but on darker days (when it rains) lights can be provided with a starter. Both Mr. Nkomo's and Mr. Mogale's learners are issued a textbook, although they work from worksheets most of the time. Mr. Peters' classroom has an "old style" desk with adjoined chair for each learner, which means that they cannot easily be arranged in groups of more than two. There is a teacher's desk and cupboard, a chalkboard and chalk, no overhead projector and screen and no electricity, so on rainy days the classroom is dark. Some windows are broken. The school has some textbooks but there are not enough for each learner and so they are not issued and Mr. Peters works from worksheets that he develops.

I have spent some time describing the schools and the classrooms. This serves three purposes. First, it gives the readers a picture of the contexts so that they might better understand our later analyses of teaching and learning in these classrooms. Second, it shows that the five schools in which we worked are generally better resourced than most South African schools, although not as well resourced as many classrooms in "developed" countries. Third, it shows the differences in resources and socio-economic profiles across the five schools, which will allow me to make some claims in relation to equity and teaching and learning mathematical reasoning. The five classrooms create a matrix design across grades and socio-economic status as described in Table 2.4. I will show later in the chapter how this design is both reinforced and complicated by learners' knowledge in relation to their socio-economic status and by the tasks chosen by the teachers.

Table 2.4 Variation across schools

	Socio-economic status	
Grade	High	Low
Grade 11	Mr. Daniels	Mr. Nkomo Mr. Mogale
Grade 10	Ms. King	Mr. Peters

Learner Knowledge

Learner knowledge was ascertained across the five classes through classroom observations and task-based interviews with learners. It was clear that Mr. Peters' learners had extremely weak knowledge, probably a few grade levels below Grade 10. Mr. Nkomo's learners were closer to grade level but showed some weaknesses, particularly in relation to Mr. Daniels' learners, who were around grade level. This is somewhat surprising, given that Mr. Nkomo's class was untracked and consisted of both higher and standard grade learners, while Mr. Daniels' class was a standard grade class only (see Table 2.2). The learners' knowledge in these cases reflects the socio-economic status of the schools. Further reflecting socio-economic status, Ms. Kings' Grade 10 learners were extremely strong and were the second highest class in a strongly tracked grade. In fact, their knowledge was stronger than both Mr. Nkomo's and Mr. Daniels' Grade 11 learners. Mr. Mogale's learners provide an interesting counterpoint to the SES/learner knowledge link. Although his is a low SES school, this class had been chosen in Grade 9 as the strongest learners in the grade, had been kept together as a class and had Mr. Mogale as their mathematics teacher since then. He had worked to build their mathematics knowledge and confidence over 3 years, informed by the principles of the new curriculum, which he had learned on in-service workshops.

In the task-based interviews, two or three learners were interviewed together and encouraged to help each other. High-achieving learners were chosen from Mr. Nkomo, Mr. Peters, and Mr. Daniels' classes while the learners from Mr. Mogale's and Ms. King's classes were close to average achievement.

Mr. Peters' learners showed very little facility in solving mathematics problems. Their solutions showed procedural errors in almost every step and suggested that they were looking for rules, rather than thinking about the meaning of what they were doing. Even the rules that they did remember, for example "what you do to the one side, you do to the other" were almost always applied incorrectly. Occasionally, with the easier problems, one learner used trial and error methods and obtained correct solutions but then did not know what to do with these, nor how to relate them to the mistaken rule-based calculations. Occasionally through prompting by the interviewer, this same learner was able to make some conceptual connections. The two learners did not manage to communicate with each other in ways that helped their problem solving; rather their conversations seemed to encourage even more mistakes and misunderstandings.

Mr. Nkomo's learners showed some facility with mathematical procedures and calculations and working algebraically without mistakes. However, they did not relate their calculations to the underlying mathematical meaning and when they were confronted with something slightly out of the ordinary, could not make sense of it. They did not use trial and error methods and were heavily dependent on the interviewer's help to solve most of the problems. They were able to work together and sometimes help each other with procedural issues.

The learners in Mr. Daniels' class showed procedural fluency and were able to talk conceptually about the mathematical solutions they were developing. They were able to reason with mathematical objects, although some of their reasoning was flawed and somewhat problematic from a mathematical standpoint. They spent useful time talking and explaining ideas to each other, correcting mistakes and resolving conflicting ideas, while checking that their procedures were correct and eventually coming to consensus on most solution methods. They were able to work with the interviewer's prompts and incorporate them into their own problem solving activities.

The learners in Ms. King's class were much more procedurally fluent with equations than both Mr. Nkomo's and Mr. Peters' learners, and even slightly more fluent than Mr. Daniels' learners even though they were a grade lower. They had been taught some Grade 11 concepts and procedures as "extras" in Grade 10 and were able to work with these as well, with some assistance from the interviewer. Conceptually, one learner struggled with some of the same issues that Mr. Daniels' Grade 11 learners struggled with, while the other was able to reason mathematically in a particularly perceptive way. The two boys were able to work together and help each other.

The learners in Mr. Mogale's class were procedurally fluent. They made occasional mistakes, which they noticed themselves because they continuously checked their work, looking for mistakes. They also estimated answers as a check on their procedures. They understood the meanings of the mathematical objects they worked with and reasoned mathematically with them. They went further than the learners in the other classes, in that they noticed links with other areas of mathematics and posed interesting questions about their observations. So they extended their thinking, creating new conjectures about the relationships between mathematical ideas.

These differences in the learners' knowledge were evident in the classroom interactions as well. These differences cannot be read as a comment on the particular teachers in this study (except possibly in the case of Mr. Mogale who had taught these learners for 3 years). It is clear that both the strengths and the weaknesses in the learners' knowledge comes from prior years of schooling and is a function of far more than only particular teachers. In four of the five classrooms, learners' knowledge is strongly associated with the racial and socio-economic profiles of the schools. This makes sense because schools in poorer areas usually have fewer resources, larger classes, and generally, less knowledgeable teachers (Fleisch 2007). The strong association of learner knowledge with race and socio-economic status in this sample modifies the matrix design in Table 2.4 slightly (see Table 2.5 below). Given that Mr. Mogale's learners provide a strong exception to the rule, being of low socio-economic status but with strong learner knowledge, we will be able to make some arguments which de-link learner knowledge and socio-economic status in subsequent chapters.

Table 2.5 Variation across schools

Learner knowledge/SES Grade	Strong/high	Weak/low
Grade 11	Mr. Daniels Mr. Mogale (knowledge)	Mr. Nkomo Mr. Mogale (SES)
Grade 10	Ms. King	Mr. Peters

The Tasks

As part of the research design, the teachers worked together to plan tasks, which would engage learners in mathematical reasoning. The two Grade 10 teachers worked together and the three Grade 11 teachers worked together. They worked with drafts of new South African textbooks which were being developed in relation to the new curriculum (Jaffer and Johnson 2004; Johnson et al. 2006), as well as some of their own resources, in order to choose, modify, and develop tasks that they thought would be useful to elicit mathematical reasoning and would also fit in with their curriculum. They spent two sessions of 2½ h each, planning the tasks and how they might teach them.

The Grade 11 Tasks

The tasks developed by the Grade 11 teachers (see Appendix) aimed to get the learners to explore how horizontal and vertical shifts of a parabola on a Cartesian plane produce differences in the equations of the graphs. The task consisted of three activities. The first activity required the learners to trace a copy of the graph $y = x^2$ onto a transparency, shift the transparency three units to the right and four units to the left, and observe what happened to the values of corresponding points on the shifted graphs in relation to the original graph. The second activity showed the original and the two shifted graphs on the plane, with their equations: $y = (x - 3)^2$ and $y = (x + 4)^2$ and asked learners to compare and contrast the graphs, and then to focus on the more general question of how the value of p in $y = a(x - p)^2$ affects the graph. The third activity dealt with vertical shifts, with the graphs of $y = (x - 3)^2$, $\mathrm{y} = (x - 3)^2 + 2$ and $y = (x - 3)^2 - 3$. Again the learners were asked to compare and contrast the graph and then answer the more general questions of how the value of q in $y = a(x - p)^2 + q$ affects the graph.

An analysis of the task using Stein et al.'s (2000) framework (see Chap. 3 for more detail) shows that that it demands higher level thinking from learners, predominantly at level three – "procedures with connections". According to Stein and her colleagues' criteria, the activities suggest pathways, to follow, that are closely connected to underlying conceptual ideas, are represented in multiple ways to help learners build connections and develop meaning, and require learners to engage with conceptual ideas in order to complete the task successfully.

Two additional aspects of the task are important for the subsequent analysis. First, the task is inductive, in that it asks learners to explore particular examples of shifting graphs and then make generalizations based on these examples. It does not ask for any form of deductive proof or justification. Exploratory questions such as "Discuss with a partner how these graphs differ and are the same" and "What do you observe?" are relatively open and unconstrained, except by the graphs, in what could count as an acceptable response. Learners could comment on only one observation or on as many as they could find. They could comment with or without explicit justification.

Second, the task contains a number of possibilities for misconceptions to arise and become visible. A key, counter-intuitive idea that is entailed in the task is that when the graph shifts in the positive direction, the equation has a negative sign in the brackets, i.e., $y=(x-3)^2$ is the equation of the graph that shifts three units to the right. Similarly, when the graph shifts in the negative direction, the sign in the brackets becomes positive. Many learners in all three Grade 11 classrooms demonstrated the misconception that the sign in the brackets should follow the direction of shift of the graph, making for some interesting discussions and exploration of the links between equations and graphs. Other conceptual issues that learners struggled with were: what does it mean if a graph extends infinitely along one axis; what counts as corresponding points; and how to read variables such as p and q in an equation. How these misconceptions influenced the teaching of mathematical reasoning in these classes is discussed in subsequent chapters.

The Grade 10 Tasks

Ms. King and Mr. Peters began their planning by looking for tasks that would enable learners to engage in all five strands of mathematical proficiency identified by Kilpatrick et al. (2001): conceptual understanding; procedural fluency; strategic competence; adaptive reasoning and productive disposition (see Chap. 6 for more detail). Ms. King wanted to focus on the integrated development of all five strands among her learners while Mr. Peters wanted to focus on the adaptive reasoning strand and develop his learners' ability to justify their thinking. Given their slightly different foci, and because of Mr. Peters' concerns that the tasks might be too challenging for his learners, Ms. King and Mr. Peters used the same first task, but different subsequent tasks. I will first discuss Ms. King's tasks and then Mr. Peters' tasks (see Appendix).

Tasks 1 and 5 on Ms. King's worksheet are primarily deductive tasks in that they require the learners to evaluate conjectures as true or false and then justify their decision. Task 1 asks whether x^2+1 can equal zero and what the smallest value for x^2+1 is if x is a real number. Task 5 asks whether n^2-n+11 is a prime number, if n is a natural number. Learners might test the conjectures using specific examples. However, they do not need to, they could work on a general justification from the beginning. In the case of task 5, if they do test examples, the first 10 natural numbers will give prime numbers but 11 will not and so makes the point that inductive testing is not good enough because there can be counter-examples. In this case, the general argument is that for n^2-n+p, $n=p$ gives p^2 which is not a prime number. Tasks 1 and 5 make demands on learners who are at level four (the highest level) of Stein et al.'s (2000) framework, which they call "doing mathematics". The tasks require nonalgorithmic thinking, they do not suggest specific solution approaches, they require learners to integrate existing knowledge to form understandings of new relationships, they require learners to examine task constraints, and they require some self-regulation and self-monitoring of the learners' thinking processes.

Tasks 2 and 3 require learners to work with the definition and meaning of a function, and with function notation. They make level three demands in Stein and her colleagues' hierarchy, suggesting solution methods that connect to underlying meanings and requiring multiple representations. Task 4 gives learners practice with function notation, which Ms. King thought was important in helping learners develop both conceptual understanding of and procedural fluency with functions. This task can be enacted at either level two or three of Stein's hierarchy, depending on how individual learners approach it. The task can be approached using the procedures of substitution and simplification of algebraic expressions without thinking much more deeply about the notion of a function. Alternatively, the task might support learners to make connections with what they have done before, and come to a more fluent and better understanding of functions. Because making connections is not explicitly asked for in the task, this task would be considered to be a level two task – "procedures without connections".

Mr. Peters worked with the same first task as Ms. King, although he excluded the second part: what is the smallest value for x^2+1. His first task read: Consider the following conjecture: "x^2+1 can never be zero". Prove whether this statement is true or false if $x \in R$. During the planning process, Mr. Peters expressed concern about how his learners would approach this task because of their weak knowledge. He worried about moving on with the same tasks as Ms. King, expecting that his learners would need more time working with the first task and that he would need to give them additional guidance. He developed a second task (Task 1B) where he scaffolded the learners' substituting into various single-term expressions and working with the sign of the expression. After two lessons where learners struggled with Task 1, he decided that this task (1B) would not help, as learners tended to focus on the sign rather than the value of the expression. So while teaching and monitoring learners' responses, he developed a second task that he hoped would address these difficulties, because the sign in front of each expression is not the sole determining factor of the sign of the expression (Task 2).

Both Tasks 1 and 2 are primarily deductive and can be approached by using a combination of inductive and deductive methods. Mr. Peters hoped that both tasks would encourage the learners to use a combined inductive–deductive approach, through substituting, testing, and justifying conjectures. In Task 2, he had the more specific goals of developing procedural fluency in substituting into the expressions, conceptual understanding that the expressions represent a range of values, strategic competence in that learners should not read off whether the expression was positive or negative from superficial aspects of the expressions and adaptive reasoning in that they justified their answers. Task 1 would count as "doing mathematics" in Stein and her colleagues' hierarchy, while Task 2, as Mr Peters intended it to be solved, would count as "procedures with connections". In Task 2 there is a specified solution method (not in the task as such but Mr. Peters made it clear in class), which is intended to help learners make connections with underlying meanings and concepts.

The above discussion of the tasks has shown differences in the ways they support learners to make connections between procedures and meanings; integrate the various strands of mathematical proficiency; and how they constrain what might count as an acceptable solution. In subsequent chapters, we will show how the tasks afforded and constrained the contributions that learners made in the classroom and how the teachers dealt with these. For the purposes of this chapter, the discussion of tasks fills out the matrix design of the study in Tables 2.4 and 2.5. Given that the tasks used in each grade were the same or similar, comparisons within and across grades are made easier. The matrix design in Table 2.6 enables some extrication of the variables of task, learner knowledge, and socio-economic status in relation to the possibilities for teaching mathematical reasoning in differently resourced classrooms.

All grade 11 teachers used the same tasks, which were inductive and which supported procedures with connections to meaning. Mr. Daniels' learners had strong mathematical knowledge and were of high SES while Mr. Nkomo's learners had weak mathematical knowledge and were of low SES. Mr. Mogale's learners provide a contrast to the others in that they were of low SES but had strong mathematical knowledge. The two grade 10 teachers used similar tasks, which were mainly deductive and which varied from "doing mathematics", through procedures with connections to procedures without connections. Ms. King's learners were of high SES and had very strong mathematical knowledge while Mr. Peters' learners were of low SES and had weak mathematical knowledge. These similarities and differences among the teachers enable comparisons in relation to tasks, school context, and learner knowledge, which we pursue in Part 3 of this book. In Part 2, we look at the individual case studies of each of the teachers, which provide in-depth descriptions of their teaching of mathematical reasoning.

Table 2.6 Variation across teachers in tasks, learner knowledge and SES

Learner knowledge/SES Tasks	Stronger/higher	Weaker/lower
Grade 11 Inductive Procedures with connections	Mr. Daniels Mr Mogale (knowledge)	Mr. Nkomo Mr Mogale (SES)
Grade 10 Deductive (with some inductive) Procedures with and without connections, doing mathematics	Ms. King	Mr. Peters

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