# Chapter 2
# Proofs of Existence

## 7 The Pigeonhole Principle in Geometry

We start by offering the reader the opportunity to prove the *Pigeonhole Principle* itself. It is also known (in Europe) as the *Dirichlet Principle*, after the famous mathematician Peter Gustav Lejeune Dirichlet (1805–1859).

**Exercise 7.1.** (Pigeonhole Principle) If $kn + 1$ pigeons (where $k$ and $n$ are positive integers) sit on $n$ pigeonholes, then at least one of the holes has at least $k + 1$ pigeons on it.

Note that the principle guarantees the *existence* of a hole with lots of pigeons on it, but as often happens in mathematics, it gives us no way of finding this hole.

**Example 7.1.** Prove that among any five points located inside or on the boundary of a unit square there are two points at most $\frac{1}{\sqrt{2}}$ apart.

*Solution.* As you can see, *pigeons and pigeonholes are not given to us*. We have to invent them if we wish to use the Pigeonhole Principle.

Let us partition the given unit square into four quarter squares (Figure 7.1). These quarter squares will be our pigeonholes, and, of course, the five given points will serve as the pigeons.

Since $5 = 1 \times 4 + 1$, by the Pigeonhole Principle there is a pigeonhole that contains at least two pigeons. In other words, there is a quarter square that contains at least two given points $A$, $B$. The distance $|AB|$ is, of course, no greater than the diagonal $\frac{1}{\sqrt{2}}$ of the quarter square. ■

A. Soifer, *Geometric Etudes in Combinatorial Mathematics*,
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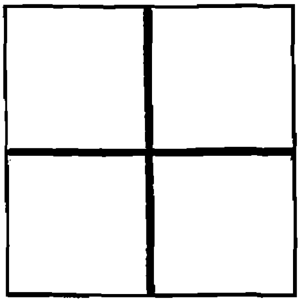

**Fig. 7.1**

**Example 7.2.** Prove that among any six points located in a $3 \times 4$ rectangle there are at least two points at most $\sqrt{5}$ apart.

*Solution. Sometimes we need to allow the pigeonholes to be different in size and shape*. In order to solve this problem, let us partition the $3 \times 4$ rectangle into five polygons that will be our pigeonholes (Figure 7.2).

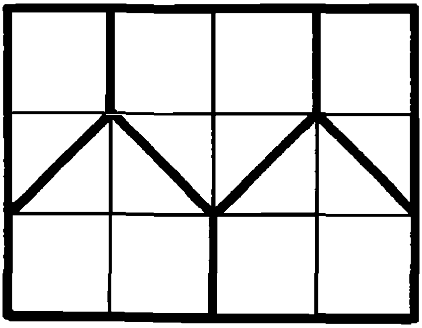

**Fig. 7.2** Example of using pigeonholes differing in size and shape

Six pigeons (the given points) sit on five pigeonholes. Therefore at least two pigeons sit on the same pigeonhole. As you can easily compute, the maximal distance between any two pigeons inside the same pigeonhole is $\sqrt{5}$. ■

**Exercise 7.2.** Prove that no matter how a plane is colored in two colors it must contain two points of the same color exactly one mile apart.

**Exercise 7.3.** (A. Soifer [S2], [S10]) Prove that among any nine points inside or on the boundary of a triangle of area 1, there are three points that form a triangle of area not exceeding $1/4$.

We can improve the result of the previous exercise:

**Exercise 7.4.** (A. Soifer [S2], [S10]) Prove that among any seven points inside or on the boundary of a triangle of area 1, there are three points that form a triangle of area not exceeding $1/4$.

The result of Exercise 7.4 can be improved, too; see the book [S2] or its expanded edition [S10].

**Exercise 7.5.** Suppose all vertices of a convex pentagon lie on the intersections of a grid. Prove that the pentagon (i.e., the interior plus the boundary) contains at least one more intersection of the grid.

**Exercise 7.6.** (A. Soifer and S. Slobodnik [SS], [S1], [S9]) Forty-one rooks are placed on a $10 \times 10$ chessboard. Prove that you can choose five of them that do not attack each other. (We say that one rook *attacks* another if they are in the same row or column of the chessboard.)

## *Solutions to Exercises*

**7.1.** Assume that there are no pigeonholes that contain $k+1$ pigeons. Then

$$\begin{array}{c}
\text{the 1st hole contains} \leq k \text{ pigeons} \\
\text{the 2nd hole contains} \leq k \text{ pigeons} \\
\\
\text{the } n\text{th hole contains} \leq k \text{ pigeons} \\
\hline
\text{the total number of pigeons} \leq k \times n
\end{array}$$

This contradicts the given fact that there are $kn+1$ pigeons. Therefore, there is a pigeonhole that contains at least $k+1$ pigeons. ■

**7.2.** Look at the vertices of an equilateral triangle with side one mile on the colored plane. Since its three vertices (pigeons) are painted in two colors (pigeonholes), we can choose two vertices painted in the same color.

This problem is the starting point of a celebrated open problem. The same statement (i.e., the *existence of two points of the same color distance 1 apart)* can be proven even if the plane is colored in three colors (try to prove it). It is also known that there is a coloring of the plane in seven colors that prevents the existence of two points of the same color 1 unit apart (try to show this too!).

*The question is still open for four, five, and six colors* after many years and numerous attempts to solve this problem.[1] ■

**7.3.** Midlines partition the given triangle into four congruent triangles of area $^1/_4$ (Figure 7.3).

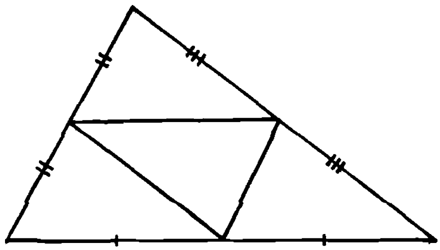

**Fig. 7.3**

These congruent triangles are our pigeonholes, and the given points are our pigeons. Now nine pigeons are sitting in four pigeonholes. Since $9 = 2 \times 4 + 1$, there is at least one pigeonhole containing at least three pigeons. ■

**7.4.** Since $7 = 2 \times 3 + 1$, *it would be nice to have three pigeonholes*— then at least one of them would have at least three pigeons! Let us draw only two midlines of the given triangle (Figure 7.4).

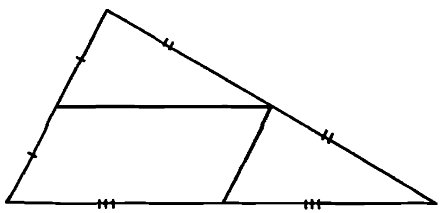

**Fig. 7.4**

[1] See new to this 2010 Springer edition Chapter 9 dedicated to this problem.

We get three pigeonholes; At least one of them must contain at least three pigeons. If one of the triangles contains three given points, we're done.

If the parallelogram contains three given points, then all we have left to prove is a simple lemma: *The maximum area of a triangle inscribed in a parallelogram of area 1/2 is equal to 1/4.* We leave the proof of this lemma to the reader. ■

**7.5.** Let us introduce the coordinate system on the grid (Figure 7.5).

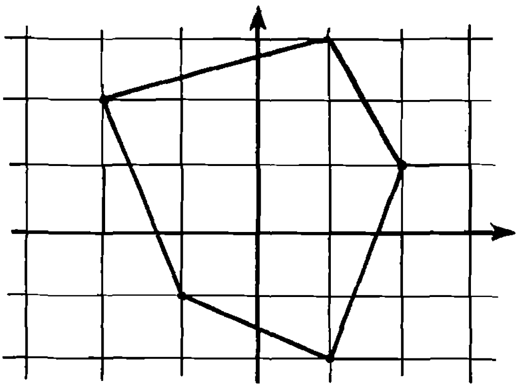

**Fig. 7.5** Cartesian coordinate system on a grid

Given a vertex $V$ of a pentagon with the coordinates $x, y$. We assign to $V$ the ordered pair of the remainders upon division of $x, y$ by 2. There are only four possible outcomes of this operation: (0,0), (0,1), (1,0), and (1,1). These are our pigeonholes. Since we have five pigeons (the vertices of the pentagon), by the Pigeonhole Principle there are two vertices, call them $M_1$ and $M_2$, in the same pigeonhole, i.e., their coordinates give the same pair of remainders. In order to complete the proof, all that is left to notice is that the midpoint of the segment $M_1M_2$ has integral coordinates and lies on the interior or boundary of the pentagon. ■

**7.6.** This solution was first published in [S1] and also appears in [S9]. Let's make a cylinder out of the chessboard by gluing together two opposite sides of the board. We color the cylinder diagonally in 10 colors (Figure 7.6).

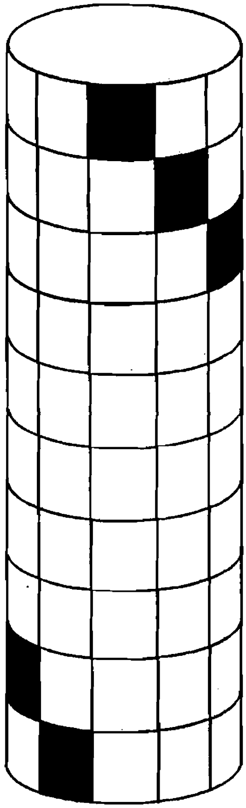

**Fig. 7.6** One out of the ten one-color diagonals is shown in black.

Now we have $41 = 4 \times 10 + 1$ pigeons (rooks) in 10 pigeonholes (one-color diagonals). Therefore, there is at least one hole containing at least 5 pigeons. But the 5 rooks located on the same one-color diagonal do not attack each other! ■

## 8 An Infinite Flock of Pigeons

What would happen if an infinite flock of pigeons were to land on a finite number of pigeonholes? The answer is clear: at least one of the holes would contain an infinite subflock! We will call this simple argument the *Infinite Pigeonhole Principle*. It will allow us to solve a number of important problems.

Let $M$ be an infinite *bounded subset* of the real line $R$. The word "bounded" means that there exists a positive number $m$ such that $|x| < m$ for every $x$ in $M$. A point $p$ of $R$ is said to be a *limit point* of the set $M$ if for every $\varepsilon > 0$ the segment $[p - \varepsilon, p + \varepsilon]$ contains infinitely many points of $M$.

Now we are ready for a classical result of mathematics.

**The Bolzano–Weierstrass Theorem 8.1.** Every infinite bounded subset $M$ of the real line $R$ has at least one limit point.

*Proof.* Imagine the points of $M$ as pigeons. So, $M$ is an infinite flock of pigeons. And what do we take as pigeonholes? Since $M$ is bounded, there is a positive integer $m$ such that $|x| < m$ for every $x$ from $M$. Now we take the segments

$$[-m, -m + 1], [-m + 1, -m + 2], \ldots, [-1, 0], [0, 1], \ldots, [m - 1, m] \tag{$*$}$$

as pigeonholes. Then, due to the infinite pigeonhole principle, at least one of them, say, [1, 2], contains infinitely many pigeons. Thus, we have infinitely many pigeons that in decimal form are equal to $1.a_1a_2a_3\ldots$, where $a_1, a_2, a_3, \ldots$ are digits.

The next step is to consider ten segments

$$[1.0, 1.1], [1.1, 1.2], \ldots, [1.8, 1.9], [1.9, 2.0] \tag{$**$}$$

as our new pigeonholes. Again we conclude that one of them, say, [1.4, 1.5], contains infinitely many pigeons. Thus, we have infinitely many pigeons of the form $1.4a_2a_3\ldots$

Now from ten segments

$$[1.40, 1.41], [1.41, 1.42], \ldots, [1.49, 1.50] \tag{$***$}$$

we choose one, say, [1.43, 1.44], that contains infinitely many pigeons, and so on.

We end up with a real number $x = 1.43\ldots$. We can easily show that $x$ is a limit point of the set $M$. Indeed, given a positive number $\varepsilon$, we choose an integer $m$, such that $\frac{1}{10^m} < \varepsilon$. Further, let $x_m = 1.43a_3a_4\ldots a_m$ be a *finite* decimal fraction that we obtain from $x$ by removing all decimal digits of $x$ except the first $m$ digits after the

decimal point. Then the segment $\left[x_m, x_m + \frac{1}{10^m}\right]$ contains infinitely many pigeons and is contained in the segment $[x - \varepsilon, x + \varepsilon]$.
This completes the proof. ■

Let $M$ be a nonempty bounded subset of the real line $R$. A point $\alpha$ of $R$ is said to be the *exact upper bound* of the set $M$ if it possesses the following properties:

i) there is no point $x$ in $M$ such that $x > a$;
ii) for every positive number ε, the segment $[a-\varepsilon, a]$ contains at least one point of the set $M$.

The *exact lower bound* of $M$ is defined similarly.

It is clear that if the exact upper bound $a$ of $M$ belongs to $M$, then $a$ is the *maximal* point of $M$. If not, then $a$ is a limit point of $M$. We are ready for another classical theorem of analysis.

**Theorem 8.2.** *Every nonempty bounded subset of R has the exact upper bound (and the exact lower bound).*

*Proof.* Let us use again the pigeonholes $(*)$ from the proof of Theorem 8.1. At least one of them contains a point of $M$. We take the *utmost right* pigeonhole that contains a point of $M$. Let it be the pigeonhole [1,2]. Then we consider ten pigeonholes $(**)$ and take the utmost right pigeonhole that contains a point of $M$. Let [1.4,1.5] be this pigeonhole. Then we consider the pigeonholes $(* * *)$ and so on. As a result, we obtain a real number $a = 1.43\ldots$. It can be easily shown (do) that $a$ is the exact upper bound of $M$. ■

Let now $M$ be an infinite *bounded point set* in the plane $R^2$. The word "bounded" means that there exists a square (or a disk) that contains $M$. Here and everywhere in this book by "*disk*" we mean "closed circular disk," i.e., a circle together with all of the points inside it. A point $p$ of $R^2$ is said to be a *limit point* of $M$ if each disk with center $p$ contains infinitely many points of $M$.

**Plane Sets Theorem 8.3.** Any infinite bounded subset $M$ of the plane $R^2$ has at least one limit point.

*Proof.* Let $m$ be a positive integer such that for every point $q = (x, y)$ of the set $M$, the coordinates $x$, $y$ satisfy the inequalities $|x| < m, |y| < m$. We divide the square with the vertices $(\pm m, \pm m)$ into unit squares (Figure 8.1). These unit squares are the pigeonholes, and

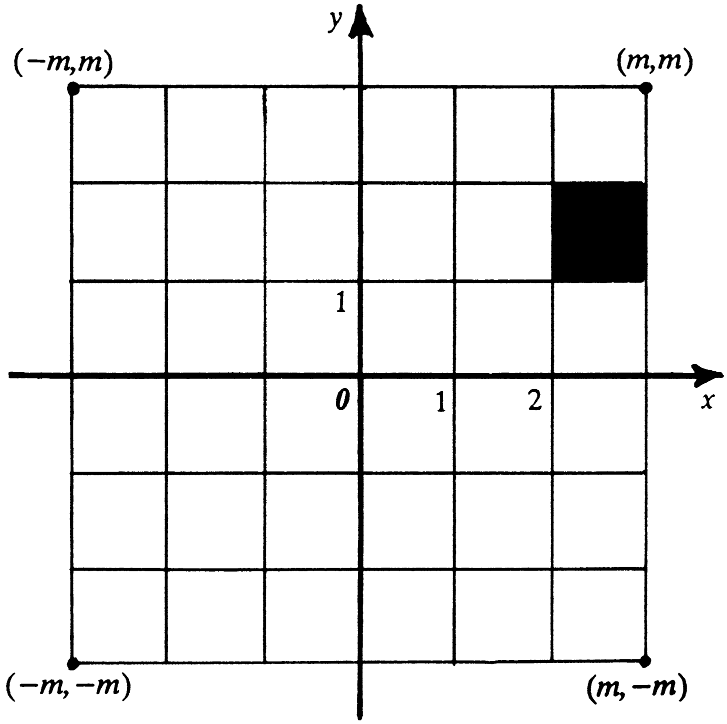


**Fig. 8.1**

all points of $M$ are the pigeons. By the Infinite Pigeonhole Principle, at least one pigeonhole $P$ contains infinitely many pigeons. Let, for example, the point (2,1) be the bottom left corner of $P$. Then each pigeon in this pigeonhole is of the form $q = (x, y)$ where $x = 2.a_1a_2a_3\ldots$ and $y = 1.b_1b_2b_3\ldots$.

We now divide the pigeonhole $P$ into 100 new pigeonholes, i.e., 100 small squares with the side length $\frac{1}{10}$. Again, there exists a pigeonhole $P_1$ that contains infinitely many pigeons. Let, for example, the point (2.7, 1.4) be the bottom left corner of $P_1$. Then each pigeon in $P_1$ is of the form $q = (x, y)$ where $x = 2.7a_2a_3\ldots$ and $y = 1.4b_2b_3\ldots$. Now we divide $P_1$ into 100 pigeonholes that are small squares with the side length $\frac{1}{100}$, and chose a pigeonhole $P_2$ with the bottom left corner, say (2.76, 1.43), containing infinitely many pigeons, and so on.

We end up with a pair of real numbers $q = (x, y)$ where $x = 2.76\ldots$, $y = 1.43\ldots$. It can be easily shown (do) that $q$ is a limit point of $M$. We are done. ■

A similar assertion is true for the space $R^3$ and for $n$-dimensional space $R^n$, where $n$ is an arbitrary positive integer.

The examples we discussed above allow us to introduce the notion of a compact set that is important in the last chapter of this book. A set $M$ in the plane $R^2$ (similarly in $R^n$ for arbitrary positive integer $n$) is said to be *closed* if it contains all its limit points. A closed bounded set is said to be *compact*.

The following proposition, known as the *Weierstrass Theorem*, indicates an important property of continuous functions on compact sets:

**The Weierstrass Theorem 8.4.** Each continuous function $f$ defined on a compact set $M$ (Figure 8.2) is bounded and reaches at at least one point $a$ the maximal value in $M$ (a similar statement is true for the minimal value).

We do not give here the exact definition of continuity (that is, "absence of breaks"), assuming that this notion has a clear visual meaning. Perhaps in one of the future volumes of our *Etudes* we will return to geometrical ideas in analysis and, in particular, to the idea of continuity.

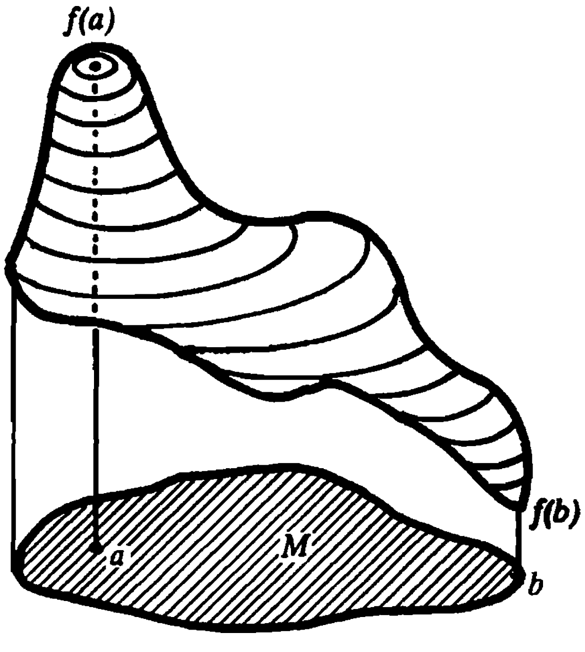


**Fig. 8.2** Queen Dido's Puzzle

The following proposition, known as the *Intermediate Value Theorem,* expresses another important property of continuous functions.

**Intermediate Value Theorem 8.5.** Let $M$ be a connected set (that is, $M$ consists of "one piece") and $f$ be a continuous function on $M$. If $a$ and $b$ are two points of $M$ and $y$ is a number such that $f(a) < y < f(b)$, then there exists a point $c$ of $M$ for which $f(c) = y$ (Figure 8.3).

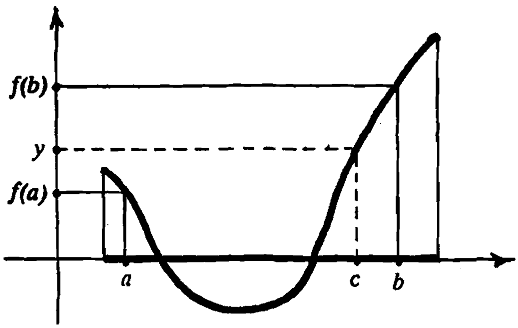


**Fig. 8.3**

**Exercise 8.1.** A compact figure $F$ of area $S$ and a line $L$ are given in the plane. Prove that there exists a line parallel to $L$ that divides $F$ into two parts, each of area $\frac{1}{2}S$ (Figure 8.4).

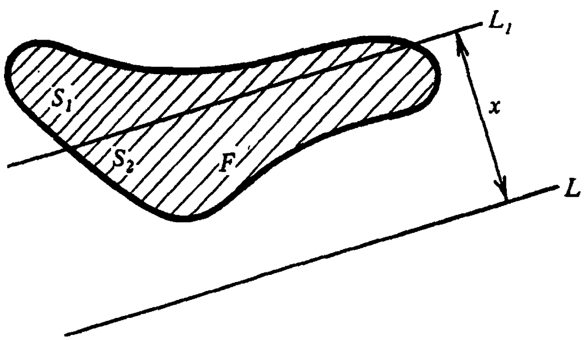


**Fig. 8.4**

**Exercise 8.2.** A compact figure $F$ of area $S$ and a point $p$ are given in the plane. Prove that there exists a line through $p$ that divides $F$ into two parts, each of area $\frac{1}{2}S$.

**Exercise 8.3.** The Russian mathematician Pavel Uryson noticed that for every compact figure $F$ in the plane there exists a square circumscribed about $F$. Can you prove it?

The above exercises and similar problems can be found in [S2], [S10] and the fine article [T].

Let $M_1, M_2, \ldots$ be arbitrary figures in the plane (or in $R^n$). The *intersection* $M_1 \cap M_2 \cap \ldots$ of these figures is the set of all points that belong to *all* the figures $M_1, M_2, \ldots$. If there is at least one point common to all the figures $M_1, M_2, \ldots$, then we say that the intersection of these figures is *nonempty.*

**Example 8.4.** Given a *decreasing* sequence $M_1, M_2, \ldots$ that is, $M_j$ is contained in $M_i$ if $j > i$ of compact figures in the plane (or in $R^n$). Prove that the intersection of all figures $M_i$ (that is, the set of points that belong to all the figures) is a nonempty compact figure.

*Proof.* The intersection of closed sets is always closed. The intersection of bounded figures is bounded. So, it remains to prove that the intersection $I$ of a decreasing sequence of nonempty compact figures $M_i$ is nonempty. Assume the opposite, i.e., that is empty. We take a point $a_1$ of the figure $M_1$. Then $a_1$ does not belong to $I$ (since $I$ is empty). So there exists an index $i$ such that $a_1$ does not belong to $M_i$. Without loss of generality we can assume that $a_1$ does not belong to $M_2$ (if necessary, we throw away the figures $M_2, \ldots, M_{i-1}$ and rename $M_i$ as $M_2$). Now we choose a point $a_2$ of $M_2$. Then $a_1$ is distinct from $a_2$. Again, $a_2$ does not belong to $I$ and, consequently, $a_2$ does not belong to a figure $M_j$ for some index $j$. Without loss of generality we assume that $a_2$ does not belong to $M_3$. Continuing, we obtain a sequence $a_1, a_2, \ldots$ of distinct points such that $a_i$ belongs to $M_i$ but not to $M_{i+1}, M_{i+2}, \ldots$. Moreover, the set $\{a_1, a_2, a_3, \ldots\}$ is bounded (it is contained in $M_1$).

According to the Bolzano-Weierstrass Theorem there is a limit point $b$ of the set $\{a_1, a_2, a_3, \ldots\}$. Since all the points $a_1, a_2, a_3, \ldots$, except $a_1, a_2, a_3, \ldots, a_{i-1}$, belong to $M_i$, $b$ also belongs to $M_i$ (recall that $M_i$ is closed, that is, it contains all its limit points). Thus, each figure $M_i$ ($i = 1, 2, \ldots$) contains $b$, contradicting the assumption that the intersection of all these figures is empty. ■

Let us note that the theorem we just proved is not true for noncompact figures (even closed). For example, let us denote the half-plane determined in a coordinate system $(x, y)$ by the inequality $x \geq i$ (where $i = 1, 2, \ldots$) by $M_i$. We obtain a decreasing sequence of closed noncompact figures $M_1, M_2, \ldots$, such that their intersection is empty (see Figure 18.14). We are done.

Now let us introduce a new notion. Let $M$ be a figure in the plane, and $\varepsilon$ a positive number. A point $x$ is said to be $\varepsilon$-*close* to the figure $M$, if there exists a point $y$ of $M$ such that the distance $|xy|$ between these points does not exceed $\varepsilon$. The set $E_\varepsilon(M)$ of all points $x$ that are $\varepsilon$-close to $M$ is called the $\varepsilon$-*extension* of the figure $M$ (Figure 8.5). In other words, the $\varepsilon$-extension of $M$ is the union of all disks of radius $\varepsilon$ with centers at the points of the figure $M$. It can be easily shown that if $M$ is compact, then its $\varepsilon$-extension is also compact.

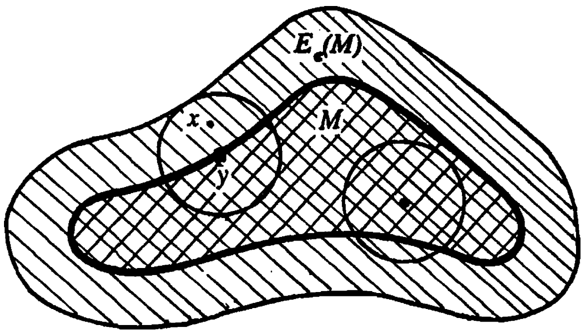


**Fig. 8.5**

The German mathematician Felix Hausdorff introduced a notion of *distance between two compact figures*. Here is his definition: let $M_1$ and $M_2$ be two distinct compact figures in the plane. The *distance d* $(M_1, M_2)$ between $M_1$ and $M_2$ is the least positive number $\varepsilon$ such that $M_1$ is contained in the $\varepsilon$-extension of $M_2$ and $M_2$ is contained in the $\varepsilon$-extension of $M_1$.

For example, if $M_1$ and $M_2$ are two equilateral triangles with parallel sides and common centroid (Figure 8.6), then the distance $d(M_1, M_2)$ is equal to $h\sqrt{3}$, where $h$ is the distance between the cor-

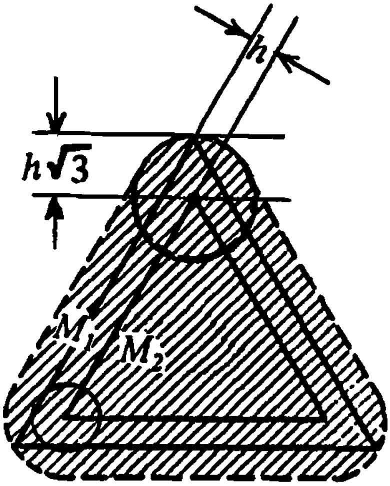


**Fig. 8.6**

responding parallel sides of the triangles. Indeed, the $h$-extension of the smaller triangle does not contain the vertices of the larger one.

Now let $M_1, M_2, \ldots$ be a sequence of compact figures in the plane and $M$ be one more compact figure. The figure $M$ is said to be the *limit* of the sequence $M_1, M_2, \ldots$ if the distance $d(M, M_k)$ approaches 0 as $k$ increases without bound. In this case we say that the sequence $M_1, M_2, \ldots$ *converges* to $M$.

*A* sequence $M_1, M_2, \ldots$ of compact figures in the plane is called *bounded* if there exists a square that contains *all* the figures $M_1, M_2, \ldots$. The following theorem has important applications in mathematics. It is, in a sense, the *Bolzano-Weierstrass Theorem for Compact Figures*.

**Bolzano-Weierstrass Theorem for Compact Figures 8.6.** Any bounded sequence of compact figures in the plane (or more generally, in the space $R^n$) has a convergent subsequence.

*Proof.* Have you forgotten the pigeonholes? Let $M_1, M_2, \ldots$ be a bounded sequence of compact figures, and $m$ be a positive integer such that all the figures $M_1, M_2, \ldots$ are contained in the square with vertices $(\pm m, \pm m)$. We divide this square into unit squares (Figure 8.1) and presume that each unit square contains its boundary. Now each union of any number of unit squares we call a pigeonhole.

So, there is a finite number of pigeonholes (more precisely, there are $4m^2$ unit squares and, consequently, $2^{4m^2}$ pigeonholes, including the empty pigeonhole, that is, the "union" of the empty set of squares). Certainly, our compact sets $M_1, M_2, \ldots$ are the pigeons! But in what sense is a pigeon sitting on a pigeonhole?

Let $F$ be a compact figure contained in the square with the vertices $(\pm m, \pm m)$. We denote the union of the unit squares that have at least one common point with $F$ by $C_1(F)$. The figure $C_1(F)$ is said to be the *container* of $F$.

Now we say that the pigeon $M_k$ is sitting on a pigeonhole if this pigeonhole is the container of $M_k$. So, each pigeon is sitting on *only one* pigeonhole. The pigeon $M_k$ and the pigeonhole on which it is sitting are drawn in Figure 8.7.

Thus, we have a finite number of pigeonholes and infinitely many pigeons. Therefore, by the Infinite Pigeonhole Principle, there exists a pigeonhole $P_1$ that contains infinitely many pigeons. We pick one of

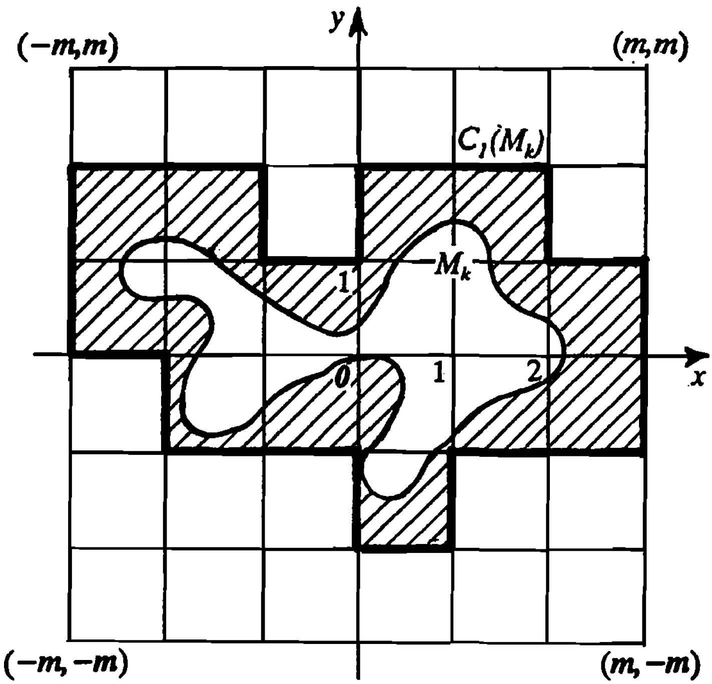


**Fig. 8.7**

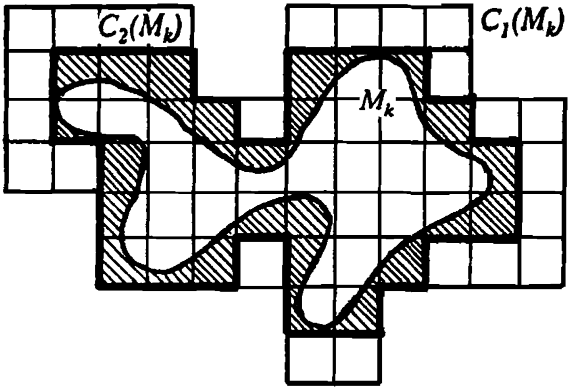


**Fig. 8.8**

these pigeons and denote it by $N_1$. Obviously, $C_1(N_1) = P_1$. It is not difficult to show that if $F$ is a compact figure with $C_1(F) = P_1$, then

$$d(N_1, F) \leq \sqrt{2},$$

since the length of the diagonal of the unit square is equal to $\sqrt{2}$.

We now divide each unit square contained in $P_1$ into four squares of side $\frac{1}{2}$ (Figure 8.8). If $M_k$ is a pigeon whose container coincides with $P_1$, then we denote the union of all small squares (with side length $1/2$) that have at least one common point with $M_k$ by $C_2(M_k)$. The figure $C_2(M_k)$ is the new pigeonhole on which the pigeon $M_k$ is sitting. Again, we have a finite number of pigeonholes whereas the remaining flock of pigeons (for which $C_2(M_k) = P_1$) is infinite. Consequently, there exists a pigeonhole $P_2$ that contains infinitely many pigeons.

Out of these infinitely many pigeons sitting on $P_2$ we pick one, call it $N_2$, such that the index of $N_2$ in the original sequence $M_1, M_2, \ldots$ is greater than the index of $N_1$. Of course, $C_2(N_2) = P_2$. As in the first step, if $F$ is a compact figure with $C_2(F) = P_2$, then

$$d(N_2, F) \leq \frac{\sqrt{2}}{2}.$$

We then divide each square of $P_2$ into four squares with side length $\frac{1}{4}$, and so on.

We end up with an infinite sequence of "narrow" pigeonholes $P_1, P_2, \ldots$, and an infinite sequence $N_1, N_2, \ldots$ of distinct pigeons that is a subsequence of the given sequence $M_1, M_2, \ldots$ of compact figures. According to the construction, for each pigeon $N_k$ of the created subflock, we have $C_1(N_k) = P_1$; for each $N_k$ with $k \geq 2$, we have $C_2(N_k) = P_2$, and so on. This means that

$$d(N_1, N_k) \leq \sqrt{2} \text{ for all } k > 1,$$
$$d(N_2, N_k) \leq \frac{\sqrt{2}}{2} \text{ for all } k > 2,$$

and generally,

$$d(N_j, N_k) \leq \frac{\sqrt{2}}{2^{j-1}} \text{ for all } k > j.$$

Finally, we denote the intersection of all pigeonholes $P_1, P_2, \ldots$ by $N$. Then $N$ is a nonempty compact figure (see Example 8.4). It can be easily shown that $C_1(N) = P_1, C_2(N) = P_2, \ldots$. Consequently, $d(N_j, N) \leq \frac{\sqrt{2}}{2^{j-1}}$. This means that the sequence $N_1, N_2, \ldots$ converges to $N$. This completes the proof. ■

**Example 8.5.** Mathematicians of ancient Greece knew that *among all figures of given perimeter P the circle has maximal area*. They knew it but could not prove it! A nice proof was suggested in the nineteenth century by the Swiss mathematician Jacob Steiner. But his proof had a hole. We discuss here the wonderful idea of Steiner and fill in the hole in his proof. This will serve as a good example of an application of the above Theorem 8.4. The problem of determining the figure $F$ of maximal area with the given perimeter $P$ is called the *isoperimetric problem*. In this case, the figure F is called *extremal*.

Let $F$ be a figure with the given perimeter $P$. If it is not convex, then we slide a rubber loop over it (Figure 8.9). We obtain a figure

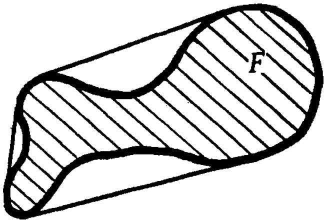


**Fig. 8.9**

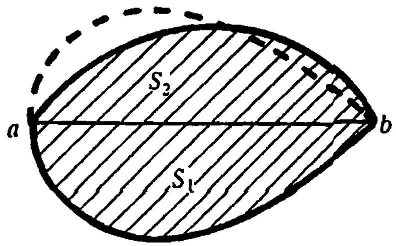


**Fig. 8.10**

of a larger area and smaller perimeter (we'll discuss convex figures in greater detail in Chapter 4). So, if a figure is not convex, then it is not extremal, that is, it cannot be a solution of the isoperimetric problem.

Let $F$ be a convex figure. We call a chord $[a, b]$ of $F$ a *cross-cut* if it divides the boundary of $F$ into two arcs of equal length $\frac{1}{2}P$. Steiner noticed that if $F$ is extremal, then every cross-cut divides its area into two *equal* parts. Indeed, if *ab* is a cross-cut and the areas of parts $S_1$ above and $S_2$ below it are unequal and $|S_1| > |S_2|$ (Figure 8.10), then by replacing $S_2$ with the symmetric image of $S_1$, we obtain a figure with the same perimeter $P$ and a *greater* area, contradicting the extremality of $F$.

Steiner fixed a cross-cut $[a, b]$ of the extremal figure $F$ and showed that for any boundary point $c$ of $F$ distinct from $a$ and $b$, the angle *acb* must be equal to 90°. Indeed, if the angle *abc* were not equal to 90° (Figure 8.11), then we could install a hinge in the point $c$ and rotate one of the shaded areas about $c$ until the angle *acb* is equal to 90° (Figure 8.11). While doing so, we certainly would not change the shaded areas, but would increase the area of the triangle *acb* (prove it). Now we can replace the lower half of $F$ with a symmetric image of the upper half. Thus, we get a figure of the same perimeter as $F$ but of a greater area.

But if for every boundary point $c$ of $F$ (except $a$ and $b$), the angle *acb* is equal to 90°, then $F$ would have to be a disk.

May we now conclude that the disk of perimeter $P$ has the maximal area among all figures of perimeter $P$? In order to answer this question, let us consider the following hypothetical situation.

A customer came to a clock shop and asked to have his antique clock with a very complicated mechanism repaired. The owner of the shop asked his experts for help, and they replied that nobody

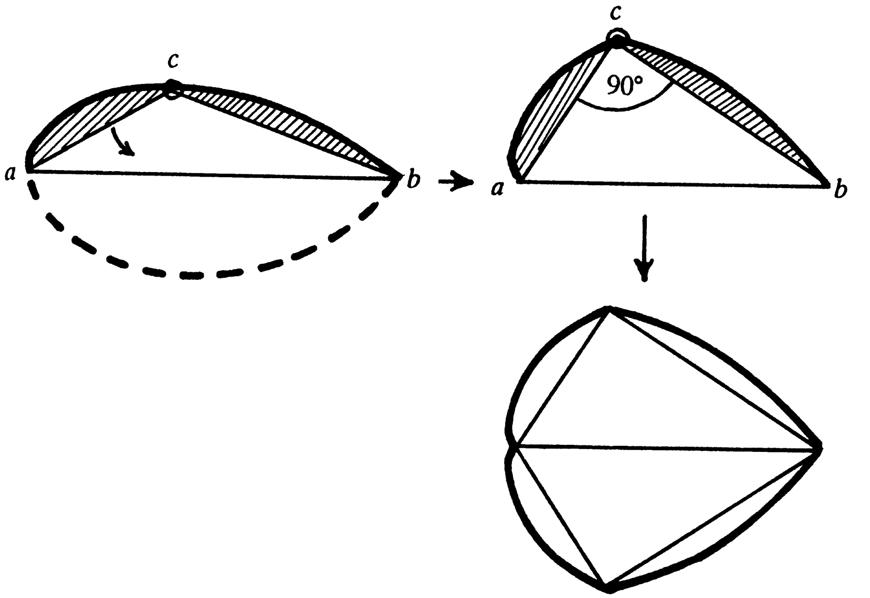


**Fig. 8.11**

except the foreman Smith could repair the clock. Smith was ill so the customer was asked to come next week. However, the next week it became clear that the foreman Smith also was unable to repair that clock. So "nobody except Smith can" does not mean yet that "Smith can."

Similarly, Steiner's reasoning above shows that no figure except the disk can be extremal. Does it mean that the disk is surely the extremal figure? Not at all! Perhaps no figure except the disk is extremal, and the disk is not extremal either. If we had a guarantee of the *existence* of an extremal figure, then (knowing that no figure except the disk is extremal) we could conclude that the disk *really is* the extremal figure. The lack of such a guarantee of the existence was the hole in Steiner's reasoning.

And how can we get a guarantee for the existence of an extremal figure? Theorem 8.4 gives such a guarantee! Indeed, let us denote the exact upper bound of areas of all figures with perimeter $P$ by $S$. Then for every positive integer $k$ there exists a figure $M_k$ of perimeter $P$ whose area is greater than $S - \frac{1}{k}$. We can place all the figures $M_1, M_2, \ldots$ in a bounded piece of the plane (say, in a circle of radius $P$). We obtain the bounded sequence $M_1, M_2, \ldots$ of compact figures in the plane. By Theorem 8.1 there exists a convergent subsequence. Now it is not difficult to prove that the limit figure $N$ of this subsequence has area $S$, that is, the figure $N$ is extremal. This is a way to fill in the hole in Steiner's reasoning. Thus, the disk *is* the only extremal figure. ■

**Exercise 8.4.** According to a legend, Queen Dido permitted a town to be built by the sea "bounded by an ox's skin." The skin was cut into thin strips and then the strips were tied into a long ribbon. But how were they to bind the region of maximal area with the help of this ribbon (Figure 8.12)? Can you help the people in this legend?

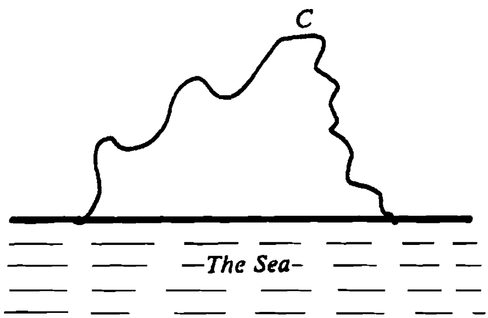


**Fig. 8.12** Queen Dido's Puzzle

**Exercise 8.5.** (B. Grünbaum; same as Exercise 6.9) Prove that every tiling of an infinite strip by copies of a polyominal tile $T$ contains a bounded part that can be used to tile a cylinder.

## *Solutions to Exercises*

**8.1.** Let us denote by $x$ the distance between the given line $L$ and the constructed line $L_1$ that cuts $F$ into two parts (Figure 8.4). We denote by $S_1$ and $S_2$ the areas of these parts. Then $S_1 - S_2$ is a continuous function of $x$ (we don't give a proof of this visually clear assertion because we did not introduce the exact definition of continuity). But when $x$ is small, $S_1 - S_2$ is equal to $S$ (Figure 8.4), and when $x$ is large $S_1 - S_2 = -S$. So, according to the Intermediate Value Theorem, there is a value $x = c$ for which $S_1 - S_2 = 0$, that is, $S_1 = S_2 = \frac{1}{2}S$. ■

**8.2.** Let $L$ be a directed line through $p$ that forms angle $\phi$ with a fixed initial ray $L_0$ (Figure 8.13). We denote the areas of the parts into which $L$ divides the figure $F$ by $S_1$ and $S_2$ ($S_1$ is to the left of $L$). Then $S_1 - S_2$ is a continuous function of $\phi$. But as $\phi$ runs through all the values from 0 to $\pi$, the areas $S_1$ and $S_2$ interchange their roles. So, if $S_1 - S_2$ is negative for $\phi = 0$, then it will be positive for $\phi = \pi$. Consequently, by the Intermediate Value Theorem, there exists a value $\phi = c$ for which $S_1 - S_2 = 0$, that is $S_1 = S_2 = \frac{1}{2}S$. ■

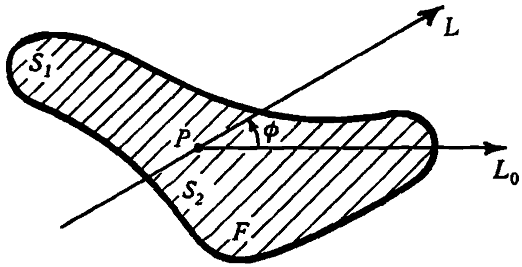


**Fig. 8.13**

**8.3.** We squeeze the figure $F$ between two parallel lines $L_1$ and $L_2$ that form angle $\phi$ with a fixed initial ray (Figure 8.14). Then we squeeze $F$ between two parallel lines $M_1$ and $M_2$ that form the angle $\phi + 90°$ with the initial ray (Figure 8.15). The four lines define a rectangle circumscribed about $F$. Let $a$ be the length of the side parallel

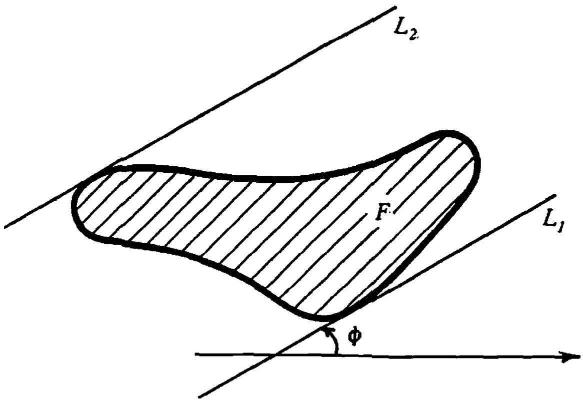


**Fig. 8.14**

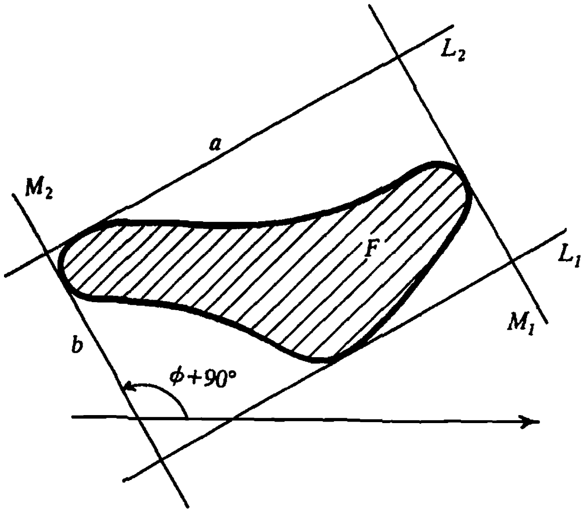


**Fig. 8.15**

to $L_1$ and $b$ be the length of the side parallel to $M_1$. Then $a - b$ is a continuous function of $\phi$. But when $\phi$ runs through the values from 0 to $\pi$, the lengths $a$ and $b$ interchange their roles. This means that if $a - b > 0$ for $\phi = 0$, then $a - b < 0$ for $\phi = \pi$. Consequently, there exists an angle $\phi = $ c for which $a - b = 0$, that is, the circumscribed rectangle turns into a square. ■

**8.4.** Let $C$ be a curve bounding the town (Figure 8.12), $q$ its length, and $C'$ the curve symmetric to $C$ with respect to the sea line $L$ (see Figure 8.16). Then the union of $C$ and $C'$ is a closed curve of length $2q$. We obtain the maximal area when this union is a circle; hence, $C$ is a semicircle (Figure 8.17). ■

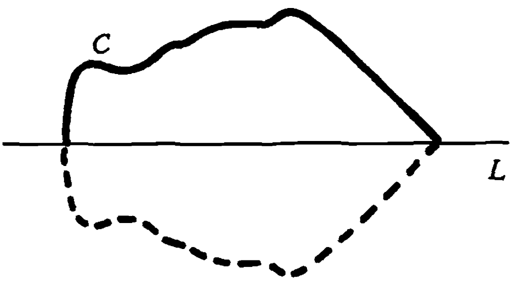


**Fig. 8.16**

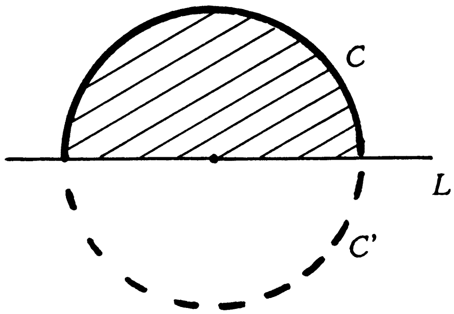


**Fig. 8.17**

**8.5.** Let $S$ be an infinite strip tiled by copies of tile $T$. There are finitely many shapes of step-lines (can you prove it?) cutting across $S$ along the boundaries of tiles (Figure 8.18 shows one such step-line cut).

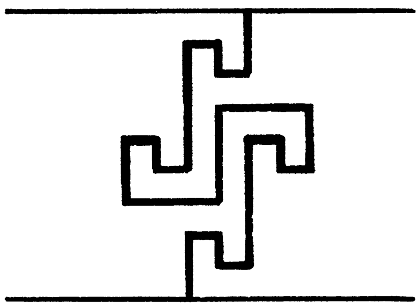

**Fig. 8.18**

On the other hand, there are infinitely many step-line cuts because $S$ is an infinite strip. By the Infinite Pigeonhole Principle, there is a cut that repeats at least twice. We get a region $F$ just like the one in Figure 6.5; you can glue a cylinder out of $F$. ■

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