2

The Preliminaries

This chapter discusses features of static trade theory that are important components of the dynamic, multi-sector models developed in later chapters. Most of the notation used throughout the text is introduced and the style used to state the model's primitives, and to define and characterize equilibrium is presented.

The first section reviews key concepts and results from individual consumer and producer theory relevant to neoclassical trade theory. The exposition is simplified by assuming production technologies and preferences are differentiable and homothetic functions. Throughout the text we draw heavily upon the so called dual or indirect functions that characterize the constrained optimization behavior of individual agents. Readers interested in a more rigorous exposition of consumer theory should consult Cornes (1992) or Mas-Colell et al. (1995). A more rigorous treatment of producer theory can be found in Chambers (1988), and Fare and Grosskop (2004).

Using the concepts developed in Sections 1 and 2 introduces the Heckscher-Ohlin-Samuelson (HOS) model of a small open and competitive economy. The basic features of equilibrium and comparative statics as provided by the Stopler-Samuelson and Rybczynski theorems are discussed. Woodland (1982) provides an excellent characterization of this model. Section 3 considers, briefly, some further generalizations of the comparative statics of the HOS model. Section 4 concludes this chapter and presents a model of two traded goods, a home-good and three factors of production. A dynamic version of this model follows in later chapters.

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2.1 Microeconomic foundations

Throughout the text, the following notation denotes factor endowments, factor rental rates and output prices. Sectors are indexed by $j \in \{1, ..., M\}$, and denote the quantity of sector-j's output by the scalar Y_j. Corresponding output prices are denoted $\mathbf{p} = (p_1, ..., p_M) \in \mathbb{R}^M_{++}$, with the scalar p_j representing the per-unit price of sector-j output. We index factor endowments by $i \in \{1, ..., N\}$, and denote the economy's level of endowment i by the scalar V_i and the vector of factor endowments by $\mathbf{V} \equiv (V_1, ..., V_N) \in \mathbb{R}^N_{++}$. Corresponding factor rental rates are denoted $\mathbf{w} = (w_1, ..., w_N) \in \mathbb{R}^N_{++}$, with the scalar w_i representing the rental rate of factor V_i. For simplicity, outputs are often given a sector specific designation, such as agriculture, a, manufacturing, m, and the home-good, s. Likewise, endowments are often given designations like labor, L, capital, K, and land H.

2.1.1 Consumer preferences

The economy is composed of a large number of atomistic households. Each household faces the same vector of prices $\mathbf{p}$ and the same vector of factor rental rates $\mathbf{w}$. Let $\boldsymbol{\upsilon}^h = \left(\upsilon_1^h, ..., \upsilon_N^h\right) \in \mathbb{R}^N_{++}$ denote the level of factor endowments held by household-h, with υ_i^h representing the household's endowment of factor i. In most applications that follow we suppress the h superscript of $\boldsymbol{\upsilon}^h$ and υ^h, and use instead $\boldsymbol{\upsilon}$ and υ_i. Given factor rental rates $\mathbf{w}$, the household's income is given by $\mathbf{w} \cdot \boldsymbol{\upsilon}$, which is used to purchase q_j units of consumption good j at market price p_j, $j = 1, ..., M$. Then, the household's budget constraint is given by

$$\mathbf{w} \cdot \boldsymbol{\upsilon} \geq \mathbf{p} \cdot \mathbf{q}$$

where $\mathbf{q} = (q_1, ..., q_M) \in \mathbb{R}^M_{++}$. In other words, each household consumes a strictly positive level of each consumption good.

Consumer preferences over goods are represented by the utility function $u : \mathbb{R}^M_{++} \to \mathbb{R}_+$, defined as $u(\mathbf{q})$.

Assumption 1 $u(\mathbf{q})$ *satisfies the following properties:*

1. $u(\mathbf{q})$ is increasing and strictly concave in $\mathbf{q}$,

2. $u(\mathbf{q})$ is everywhere continuous, and everywhere twice differentiable,

3. $u(\mathbf{q})$ is homothetic.

Assumption 1.1 yields indifference curves that are convex, Assumption 1.2 ensures Marshallian demands are continuous functions, while Assumption 1.3 yields Marshallian demands that are separable in prices and income.

Two indirect functions emerge from the consumer's problem: the indirect utility function and the expenditure function. The *indirect utility function* gives the household's maximum attainable utility given income $\mathbf{w} \cdot \boldsymbol{\upsilon}$, defined as

$$\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \boldsymbol{\upsilon}) \equiv \max_{\mathbf{q}} \{u(\mathbf{q}) : \mathbf{w} \cdot \boldsymbol{\upsilon} \geq \mathbf{p} \cdot \mathbf{q}\}$$

The indirect utility function inherits the following properties from the direct utility function (see Cornes, pp. 67–70):

V1. Homogeneous of degree zero in $\mathbf{p}$ and $\mathbf{w} \cdot \boldsymbol{\upsilon}$; $\mathcal{V}(\theta\mathbf{p}, \theta\mathbf{w} \cdot \boldsymbol{\upsilon}) = \mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \boldsymbol{\upsilon})$, $\theta > 0$,

V2. $\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \boldsymbol{\upsilon})$ is convex in $\mathbf{p}$,

V3. $\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \boldsymbol{\upsilon})$ is continuous and differentiable in $\mathbf{p}$ and $\mathbf{w} \cdot \boldsymbol{\upsilon}$,

V4. $\mathcal{V}(\mathbf{p}, \mathbf{w} \cdot \boldsymbol{\upsilon}) = v(\mathbf{p})\, \mathbf{w} \cdot \boldsymbol{\upsilon}$: separable in $\mathbf{p}$ and $\mathbf{w} \cdot \boldsymbol{\upsilon}$,

By V4, the marginal utility of an additional unit of income is $v(\mathbf{p})$.

V5. Given differentiability, Marshallian demands follow from Roy's identity,

$$q^j(\mathbf{p})(\mathbf{w} \cdot \boldsymbol{\upsilon}) = -\frac{v_{p_j}(\mathbf{p})}{v(\mathbf{p})} \mathbf{w} \cdot \boldsymbol{\upsilon} \tag{2.1}$$

where, throughout the text, the subscript on a function indicates a partial derivative, e.g., $v_{p_j} = \partial v(\mathbf{p}) / \partial p_j$ and $v_{p_1 p_2} = \partial^2 v(\mathbf{p}) / \partial p_1 \partial p_2$.

Since consumers face the same prices and have identical preferences, the "community" indirect utility function is given by

$$\mathcal{V} = v(\mathbf{p})(\mathbf{w} \cdot \mathbf{V})$$

while the total domestic Marshallian demand for good j is

$$Q_j = q^j(\mathbf{p})(\mathbf{w} \cdot \mathbf{V}), \ \forall j \in \{1, \cdots, M\} \tag{2.2}$$

These functions are the simple aggregation of individual consumer welfare and demands. It also follows from V1 that (2.2) is homogeneous of degree minus one in prices $\mathbf{p}$ and of degree one in income.

The *expenditure function* gives the minimum cost of achieving utility level $q \in \mathbb{R}$ at given prices $\mathbf{p}$, and is defined as

$$E(\mathbf{p}, q) \equiv \min_{\mathbf{q}} \{\mathbf{p} \cdot \mathbf{q} : q \leq u(\mathbf{q})\}$$

The expenditure function inherits from $u(\cdot)$, the following properties:

E1. $E(\mathbf{p}, q) > 0$ for any $\mathbf{p}$ and $q > 0$,

E2. $E(\mathbf{p}, q)$ is non-decreasing in $\mathbf{p}$ and q,

E3. $E(\mathbf{p}, q)$ is concave and continuous in $\mathbf{p}$,

E4. $E(\lambda\mathbf{p}, q) = \lambda E(\mathbf{p}, q)$, $\lambda > 0$*:* homogeneous of degree 1 in $\mathbf{p}$,

E5. $E(\mathbf{p}, q) = \mathcal{E}(\mathbf{p})\, q$*:* separable in $\mathbf{p}$ and q,

E6. Shephard's lemma: If $E(\mathbf{p}, q)$ is differentiable in $\mathbf{p}$, then

$$q_j = E_{p_j}(\mathbf{p}, q) = \mathcal{E}_{p_j}(\mathbf{p})\, q, \ j = 1, ..., M$$

E1 says purchasing a strictly positive consumption bundle is costly. **E**2 says, all else equal, (i) if the price of a consumption good increases, then the cost of achieving the same level of utility increases, or (ii) increasing utility requires an increase in expenditures. By **E**3, the expenditure function is continuous and yields downward sloping Hicksian demand functions. Condition **E**4 implies demand functions are homogeneous of degree zero in $\mathbf{p}$. **E**5 results from Assumption 1.3 and implies demand functions are separable in $\mathbf{p}$ and q (see Chambers, 1988, Chapter 2). Later, we interpret the quantity q to be a composite consumption good, the unit cost of which is $\mathcal{E}(\mathbf{p})$.

2.1.2 Production technologies

Each sector j is composed of a large number of identical, atomistic firms. Each firm faces the same vector of input and output prices. Let y_j be the output of each firm in sector j and let $\boldsymbol{\upsilon}^j \equiv \left(\upsilon_1^j, ..., \upsilon_N^j\right) \in \mathbb{R}_+^N$ represent the vector of productive factors used by that firm, where υ_i^j is the level of factor i used by the sector j firm. Represent the technology of a sector j firm by the production function $f^j : \mathbb{R}_+^N \to \mathbb{R}_+$, defined as $y_j = f^j(\boldsymbol{\upsilon}^j)$.

Assumption 2 *$f^j(\boldsymbol{\upsilon}^j)$ satisfies the following properties:*

1. $f^j(\mathbf{0}) = 0$, and $f^j(\boldsymbol{\upsilon}^j) > 0$ for any $\boldsymbol{\upsilon}^j \gg \mathbf{0}^N$,

2. $f^j(\boldsymbol{\upsilon}^j)$ is linearly homothetic in $\boldsymbol{\upsilon}^j$,

3. $f^j(\boldsymbol{\upsilon}^j)$ is non-decreasing and strictly concave in $\boldsymbol{\upsilon}^j$,

4. $f^j(\boldsymbol{\upsilon}^j)$ is everywhere continuous and everywhere twice differentiable in $\boldsymbol{\upsilon}^j$.

Here $\mathbf{0}^N \in \mathbb{R}_+^N$ is a vector of N zeros and the notation $\boldsymbol{\upsilon}^j \gg \mathbf{0}$ means at least one element of $\boldsymbol{\upsilon}^j$ is strictly positive. Assumption 2.1 ensures it is not possible to produce a positive level of output with no input, and ensures there are no fixed costs. Assumption 2.2 says individual firm technologies satisfy constant returns to scale (CRS). An important implication of Assumption 2.2 is, when all firms face the same output and input prices, sectoral production levels and input demands are simple linear aggregations of individual firm choices. Another implication is the corresponding cost function is separable in input prices and output levels. Assumption 2.3 ensures the production technology is well-behaved and yields the familiar convex isoquants: it imposes diminishing marginal returns on individual input use. Assumptions 2.1 and 2.3 also ensure the existence of a cost and aggregate value-added (GDP) function defined later. Finally, Assumption 2.4 allows the use of differential calculus to derive corresponding cost and GDP functions.

Two indirect functions associated with the producer's problem are the cost function and the sectoral value-added function. The *cost function* is defined as:

$$c^j\left(\mathbf{w}, y_j\right) \equiv \min_{\boldsymbol{v}^j} \left\{\mathbf{w} \cdot \boldsymbol{v}^j : y_j \leq f^j\left(\boldsymbol{v}^j\right)\right\},\ j = 1, 2, ..., M$$

and is the firm's analog of the household's expenditure function. It inherits from Assumption 2, the following properties:

C1. $c^j\left(\mathbf{w}, y_j\right) > 0$ for any $\mathbf{w}$ and $y_j > 0$,

C2. $c^j\left(\mathbf{w}, y_j\right)$ is non-decreasing in $\mathbf{w}$ and y_j,

C3. $c^j\left(\mathbf{w}, y_j\right)$ is concave and continuous in $\mathbf{w}$,

C4. $c^j\left(\theta\mathbf{w}, y_j\right) = \theta c^j\left(\mathbf{w}, y_j\right)$: homogeneous of degree one in $\mathbf{w}$,

C5. $c^j\left(\mathbf{w}, y_j\right) = C^j\left(\mathbf{w}\right) y_j$: separable in $\mathbf{w}$ and y_j,

where $C^j\left(\mathbf{w}\right)$ is the unit cost of producing output j. Finally, we have

C6. Shephard's lemma: If $c^j\left(\mathbf{w}, y_j\right)$ is differentiable in $\mathbf{w}$, then

$$v_i^j = C_{w_i}^j\left(\mathbf{w}\right) y_j,\ i = 1, ..., N,$$

where $C_{w_i}^j\left(\cdot\right)$ is the derived unit demand for input i from sector j.

C1 says producing a strictly positive level of output is costly. **C**2 says, all else equal, if the price of an input increases production cost increases, or increasing output increases production costs. By **C**3, the cost function is continuous and yields conditional input demand functions that are decreasing in own prices. Condition **C**4 implies input demand functions are homogeneous of degree zero in $\mathbf{w}$. **C**5 results from Assumption 2.2 and implies constant marginal and average costs. Furthermore, given **C**5, the output supply and input demand functions are both separable in $\mathbf{w}$ and y_j (see Chambers, 1988, Chapter 2).

Since all firms in a sector employ the same technology and face the same output and input prices, characterizing the aggregate technology for the sector is straightforward. Let $\mathbf{V}^j \equiv \left(V_1^j, ..., V_N^j\right) \in \mathbb{R}_+^N$ denote the vector of factors employed in producing output Y_j, where V_i^j is the aggregate level of factor i used by sector j firms. While the total number of firms in a sector are indeterminate, their identical nature implies if each firm produces a share, Υ_j^o, of total sectoral output j, then the firm also employs the same $\bar{\Upsilon}_j$ share of factor inputs, i.e.,

$$\begin{aligned} y_j^o &= \Upsilon_j^o Y_j, \\ v_i^o &= \Upsilon_j^o V_i^j, \ \forall i \in \{1, \cdot \cdot \cdot, N\} \end{aligned}$$

Hence, the sector level production function is a linear expansion of individual firm production functions. That is,

$$\Upsilon_j^o Y_j = f^j\left(\boldsymbol{v}^j\right) = f^j\left(\Upsilon_j^o \mathbf{V}^j\right)$$

which implies the sector level production function is

$$Y_j = f^j\left(\mathbf{V}^j\right)$$

To distinguish between firm level and aggregate sectoral production however, it is convenient to represent the aggregate technology for sector j by the production function $\mathcal{F}^j : \mathbb{R}_+^N \to \mathbb{R}_+$, defined as

$$Y_j = \mathcal{F}^j\left(\mathbf{V}^j\right) \tag{2.3}$$

Then, the corresponding sectoral cost function, denoted TC_j, is given by

$$TC_j = C^j\left(\mathbf{w}\right) Y_j \tag{2.4}$$

The economy-wide gross national product function is obtained by maximizing aggregate sectoral income subject to the technology (2.3) and the endowment constraints. In this case we have:

$$G(\mathbf{p},\mathbf{V}) \equiv \max_{\mathbf{V}^1,\ldots,\mathbf{V}^M} \left\{ \sum_{j=1}^{M} p_j \mathcal{F}^j\left(\mathbf{V}^j\right) : V_i \geq \sum_{j=1}^{M} V_i^j,\ i = 1,\ldots,M \right\} \tag{2.5}$$

Woodland (1982, p. 123) shows the function $G(\cdot)$ satisfies the following properties:

G1. $G(\mathbf{p},\mathbf{V}) \geq 0$ for all $\mathbf{p}$ and $\mathbf{V}$,

G2. $G(\lambda\mathbf{p},\mathbf{V}) = \lambda G(\mathbf{p},\mathbf{V}),\ \lambda > 0$: linearly homogeneous in $\mathbf{p}$,

G3. $G(\mathbf{p},\lambda\mathbf{V}) = \lambda G(\mathbf{p},\mathbf{V})\ \lambda > 0$: linearly homogeneous in $\mathbf{V}$,

G4. $G(\mathbf{p},\mathbf{V})$ is continuous, non-decreasing, and convex in $\mathbf{p}$,

G5. $G(\mathbf{p},\mathbf{V})$ is continuous, non-decreasing, and concave in $\mathbf{V}$,

G6. Hotelling's lemma. If $G(\cdot)$ is everywhere differentiable in $\mathbf{p}$ and $\mathbf{V}$, then

$$\begin{aligned} Y_j &= G_{p_j}(\mathbf{p},\mathbf{V}) \\ w_i &= G_{V_i}(\mathbf{p},\mathbf{V}) \end{aligned}$$

The major implications of conditions G1 – G6 are that the gradients of $G(\cdot)$ yield aggregate sectoral supply functions, $G_{p_j}(\mathbf{p},\mathbf{V})$, that are non-decreasing in own-price, homogeneous of degree zero in prices $\mathbf{p}$, and homogeneous of degree one in endowments $\mathbf{V}$. The inverse factor demand functions $G_{V_i}(\mathbf{p},\mathbf{V})$ are downward sloping in own factor levels, homogeneous of degree one in prices and homogeneous of degree zero in endowments. The Hessian matrix of $G(\mathbf{p},\mathbf{V})$ is positive semi-definite.[1]

[1] Young's theorem implies that the second derivative matrix of $G(\mathbf{p},\mathbf{V})$ is symmetric, $G_{p\mathbf{V}}(\cdot) = G_{\mathbf{V}p}(\cdot)$. Thus, an increase in w_i due to a unit increase in p_j is equal to the increase in Y_j due to an increase in $v_{i.}$ See Diewert (1973, 1974).

It is also convenient to specify a sectoral value-added function. For many of the models developed in this text, at least one productive sector is endowed with a factor specific to its production process. For example, we typically model land as a factor used only in producing agricultural products. Farmers can rent land in and out among themselves at some market determined land rental rate, but they do not rent land to producers in other sectors of the economy. Since each farmer's production function satisfies Assumption 2, there exists a corresponding sectoral agricultural production and cost function (2.3) and (2.4). However, in the case of the sectoral production function, the sector specific factor is pre-determined or fixed, and the sectoral level production function exhibits decreasing returns to scale in all the other factors employed in other sectors of the economy. This property gives rise to a sector level value-added function.

More formally, divide the input vector $\mathbf{V}^j$ into two subvectors, a vector of variable inputs and a vector of sector specific inputs. Let the first ς_j factors be variable and the remaining $N - \varsigma_j$ factors be sector specific. Denote the vector of variable factors by $\mathbf{V}^j = \left(V_1^j, ..., V_{\varsigma_j}^j\right) \in \mathbb{R}_+^{\zeta_j}$, and denote the vector of sector specific factors by $\bar{\mathbf{V}}^j = \left(\bar{V}_{\zeta_j+1}^j, ..., \bar{V}_N^j\right) \in \mathbb{R}_+^{N-\zeta_j}$. With fixed factors $\bar{\mathbf{V}}^j$, the j^{th} sector value-added function can be defined as:

$$\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) \equiv$$
$$\max_{y_j, \mathbf{V}^j} \left\{p_j Y_j - \mathbf{w} \cdot \left(\mathbf{V}^j, \mathbf{0}^{N-\zeta_j}\right) : Y_j \geq \mathcal{F}^j\left(\mathbf{V}_\varsigma^j, \bar{\mathbf{V}}^j\right)\right\} \tag{2.6}$$

where $\mathbf{0}^{N-\zeta_j} \in \mathbb{R}_+^{N-\zeta_j}$ is a vector of zeros. Given Assumption 2, the sectoral value-added function properties include:

Π1. $\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) \geq 0$ for all $p_j, \mathbf{w}$, and $\bar{\mathbf{V}}^j$,

Π2. $\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right)$ is nondecreasing in p_j and nonincreasing in $\mathbf{w}$,

Π3. $\lambda\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) = \Pi^j\left(\lambda p_j, \lambda\mathbf{w}, \bar{\mathbf{V}}^j\right)$, $\lambda > 0$: linearly homogeneous in p_j and $\mathbf{w}$, Π4. $\lambda\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) = \Pi^j\left(p_j, \mathbf{w}, \lambda\bar{\mathbf{V}}^j\right)$, $\lambda > 0$: linearly homogeneous in $\bar{\mathbf{V}}^j$

Π5. $\Pi^j\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) = \pi^j\left(p_j, \mathbf{w}\right)\Phi\left(\bar{\mathbf{V}}^j\right)$: separable in fixed endowments,

Π6. Hotelling's lemma. If $\Pi^j(\cdot)$ is everywhere differentiable in $\mathbf{p}$, $\mathbf{w}$ and $\bar{\mathbf{V}}^j$, then sectoral supply Y_j and sectoral factor demand V_i^j are, respectively,

$$\begin{aligned} Y_j &= \Pi^j_{p_j}\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) \\ V_i^j &= -\Pi^j_{\tilde{w}_i}\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right) \end{aligned}$$

The factor rental rate (or shadow price) of the sector specific factors is given by

$$w_i^j = \Pi^j_{\bar{V}_i^j}\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right)$$

For the case of a single sector specific factor, say land, that is rented in or out among farmers, $\pi^j\left(p_j, \mathbf{w}\right)$ is the rental rate that clears the land rental market. Moreover, it can be shown that the output price gradient of the economy-wide GDP function yields the same level of supply as the corresponding output price gradient of the sector value-added function,

$$Y_j = G_{p_j}\left(\mathbf{p}, \mathbf{V}\right) = \Pi^j_{p_j}\left(p_j, \mathbf{w}, \bar{\mathbf{V}}^j\right)$$

where the gradients are evaluated at values $(\mathbf{p}, \mathbf{w}, \mathbf{V})$ yielding an equilibrium to the economy. This property is particularly useful for decomposing effects into direct and indirect. For instance, the direct effect of a price change on Y_j is $\partial\Pi^j_{p_j}/\partial p_j$ while the indirect effects are transmitted through factor markets and are given by $\left(\partial\Pi^j_{p_j}/\partial w_i\right)\left(\partial w_i/\partial p_j\right)$. Together, they equal the total effect which can be shown to equal $\partial G_{p_j}\left(\mathbf{p}, \mathbf{V}\right)/\partial p_j$.

2.2 The Heckscher-Ohlin-Samuelson model

The optimizing behavior of producers and consumers embodied in expressions (2.2) and (2.4) provide the building blocks for specifying the well known Heckscher-Ohlin-Samuelson (HOS) model. The economy is small, open and competitive, endowed with two factors, and produces two outputs. Denote the endowment vector by $\mathbf{V} = (L, K)$, and interpret K as units of physical capital and interpret L as units of labor. Neither endowment is traded internationally. A main feature of the model is that the number of traded goods equal the number of factors, $M = N$.

2.2.1 The behavior of households

The individual household[2] is endowed with resources $\boldsymbol{\upsilon} = (\ell, k) \in \mathbb{R}^2_{++}$, where ℓ and k denote labor and capital, respectively. The household provides the services of these resources to firms in return for wages, w, and capital rents, r, yielding income $w\ell + rk$.

Given prices (p_1, p_2), the household's budget constraint is

$$w\ell + rk \geq p_1 q_1 + p_2 q_2$$

Consumer preferences are given by the utility function $u(q_1, q_2)$ satisfying Assumption 1. Consequently, the consumer's optimization problem yields the indirect utility function

$$\mathrm{v}(p_1, p_2)(w\ell + rk) \equiv \max_{q_1, q_2} \{u(q_1, q_2) : w\ell + rk \geq p_1 q_1 + p_2 q_2\}$$

where $\mathrm{v}(p_1, p_2)(w\ell + rk)$ satisfies properties V1 – V4. The corresponding Marshallian demands are ,

$$q^j(p_1, p_2)(w\ell + rk) = -\frac{\mathrm{v}_{p_j}(p_1, p_2)}{\mathrm{v}(p_1, p_2)}(w\ell + rk), \;\; j = 1, 2$$

Since consumers face the same prices and have identical preferences, the community indirect utility function is

$$\mathcal{V} = \mathrm{v}(p_1, p_2)(wL + rK)$$

while aggregate domestic Marshallian demand for good j is

$$Q_j = q^j(p_1, p_2)(wL + rK), \;\; j = 1, 2 \tag{2.7}$$

[2] The term household is used instead of the consumer to reinforce the point that resource endowments are not owned by firms.

Given homothetic preferences, the community indirect utility function and Marshallian demands are simple linear aggregates of individual consumer welfare and demand. The marginal utility of income $v(p_1, p_2)$, and the good specific income effect $q^j(p_1, p_2)$ are common to all households.

2.2.2 The price taking firm

As in Section 2.1.2, both sectors are composed of a large number of identical, atomistic firms. All firms face the same input and output prices. Let y_j be the output of a firm in sector j and let $\boldsymbol{v}^j = (\ell_j, k_j) \in \mathbb{R}^2_{++}$ represent the level of labor ℓ_j and capital k_j employed by the firm. The technology for sector $j = 1, 2$ is represented by the production function $f^j : \mathbb{R}^2_{++} \to \mathbb{R}_+$, defined as $y_j = f^j(\ell_j, k_j)$, where $f^j(\cdot)$ satisfies Assumption 2. Recall from the previous discussion that production of either output requires a strictly positive level of capital and labor.

Inputs are chosen to maximize profits. Each firm can be viewed as maximizing profits in two steps. First, it chooses the input bundle (ℓ_j, k_j) that minimizes the cost of producing y_j units of output. The corresponding cost function is given by

$$C^j(w,r)\,y_j \equiv \min_{\ell_j, k_j} \left\{ w\ell_j + rk_j : y_j \leq f^j(\ell_j, k_j) \right\}, \; j = 1, 2$$

and satisfies conditions **C**1 – **C**6. In the second step, given the cost function $C^j(\cdot)\,y_j$, the firm solves the optimization problem

$$\Pi^j(p_j, w, r) \equiv \max_{y_j} \left\{ p_j y_j - C^j(\cdot)\,y_j \right\}$$

The optimal choice of y_j must satisfy the following complementary slackness condition

$$y_j \geq 0; \; p_j - C^j(\cdot) \leq 0; \text{ and } \left[p_j - C^j(\cdot) \right] y_j = 0$$

Hence, in a competitive equilibrium only zero profits are possible.

2.2.3 Characterization of equilibrium

Restricting our analysis to the case where both sectors are open, i.e. $Y_1, Y_2 > 0$, equilibrium is defined by a set of factor prices and output levels $(w, r, Y_1, Y_2) \in \mathbb{R}^4_{++}$ satisfying the following four conditions:
Firms earn zero profits in each output market,

$$C^1(w,r) - p_1 = 0 \tag{2.8}$$
$$C^2(w,r) - p_2 = 0 \tag{2.9}$$

Labor and capital markets clear,

$$\sum_{j=1}^{2} C^j_w(w,r) Y_j = L \tag{2.10}$$
$$\sum_{j=1}^{2} C^j_r(w,r) Y_j = K \tag{2.11}$$

Expressions (2.8) and (2.9) require that the marginal cost of production in sector j be equal to the per-unit output price for the sector. Expression (2.10) ensures the aggregate demand for labor from the two sectors is equal to the endowment of labor L. Likewise, expression (2.11) ensures the capital market clears.

In principle, since (2.8) and (2.9) consists of two equations in the unknowns w and r, the solution may be written as

$$w = W(p_1, p_2) \tag{2.12}$$
$$r = R(p_1, p_2) \tag{2.13}$$

Notice that endowments do not appear as arguments in these equations. This result obtains because the number of traded goods equals the number of endowed factors of production.

Substituting (2.12) and (2.13) into the factor market clearing conditions (2.10) and (2.11) yields two linear equations with input-output coefficients $C^j_i(W(p_1,p_2), R(p_1,p_2))$, $i, j = 1, 2$, and unknowns Y_1 and Y_2. The system is linear because the prices p_1, p_2 are exogenous in which case $C^j_i(\cdot)$ is a scalar value. Assuming both sectors produce at positive levels, denote the solution to the resulting system as

$$Y_j = Y^j\left(p_1, p_2, L, K\right), \; j = 1, 2 \tag{2.14}$$

When both sectors produce at positive levels, and the elasticity of factor substitution between L_j and K_j is the same for both technologies, then the solution w^*, r^* satisfying the zero profit conditions is unique. Furthermore, it follows from (2.5) that $W(\cdot)$ and $R(\cdot)$ are homogeneous of degree one in p_1 and p_2,[3] while the supply functions (2.14) are homogeneous of degree zero in prices, and of degree one in endowments L and K.

The factor rental rate equations (2.12), (2.13) and the supply functions (2.14) can each be used to determine the gross domestic product function

$$\begin{aligned} G\left(p_1, p_2, L, K\right) &= W\left(p_1, p_2\right) L + R\left(p_1, p_2\right) K \qquad (2.15) \\ &= p_1 Y^1\left(p_1, p_2, L, K\right) + p_2 Y^2\left(p_1, p_2, L, K\right) \end{aligned}$$

Equation (2.15) indicates that in this model, GDP measured via the cost of production or via the value of output, yields the same result. As noted in the previous section, we can also derive the aggregate GDP function by maximizing aggregate revenue given technology (2.3) and the endowment constraints. In this case we have

$$G\left(p_1, p_2, L, K\right) \equiv$$
$$\max_{L_1, L_2, K_1, K_2} \left\{ \sum_{j=1}^{2} p_j \mathcal{F}^j\left(L_j, K_j\right) : L \geq \sum_{j=1}^{2} L_j, \; K \geq \sum_{j=1}^{2} K_j \right\} \tag{2.16}$$

where $G(\cdot)$ satisfies conditions G1 – G4. If expression (2.16) is continuously differentiable, the following envelope properties apply

$$\begin{aligned} W\left(p_1, p_2\right) &= G_L\left(p_1, p_2, L, K\right) \\ R\left(p_1, p_2\right) &= G_K\left(p_1, p_2, L, K\right) \end{aligned}$$

[3] This result follows intuitively from the fact that $C^j(w, r) Y_j \equiv W(p_1, p_2) L_1 + R(p_1, p_2) K_1$ where the right hand expression is obviously homgenous of degree one in w and r.

and

$$Y^j (p_1, p_2, L, K) = G_{p_j} (p_1, p_2, L, K) \tag{2.17}$$

Given the factor rental rate and supply functions, the excess demand for good j is expressed as

$$XD_j (p_j) \equiv$$

$$Y^j (p_1, p_2, L, K) - q^j (p_1, p_2) G (p_1, p_2, L, K) \quad \begin{array}{ll} > 0 & \text{export} \\ < 0 & \text{import} \end{array}$$

Since households spend all factor income on goods, together with (2.15), Walras' law requires the value of exports to equal the value of the economy's imports,

$$\sum_{j=1}^{2} p_j XD_j (p_j) = 0$$

2.2.4 Comparative statics

The Stopler-Samuelson and the Rybczynski theorems summarize the key comparative static results of the HOS model. Stopler-Samuelson establishes the relationship between the change in the price of output and the change in factor rental rates, while Rybczynski establishes the relationship between a change in factor endowments and the change in output. These theorems apply to the two good and two endowment case, and tend to break down for more general cases. Nevertheless, the basic insights they provide can be extended, in part, to cases where the number of traded goods and number of factor endowments exceed two, as well as when the number of traded goods are greater or less than the number of factors.

The Stopler-Samuelson theorem

The theorem states, if there is an increase in the relative price of a good, then the factor used intensively in the production of that good will experience an increase in real income, while the other factor will suffer a loss in real income. In other words, an increase in the price of good j will lead to an increase (decrease) in the rental rate of the factor used intensively (extensively) in its production. Furthermore, the theorem also states, the increase (decrease) in the factor rental rate will be in greater proportion than the change in the relative price of sector j's output.

The theorem is proven in two steps. The first step shows that an increase in the price of good j causes the rental rate of the factor used intensively in its production to increase. The second step shows the percent increase in the rental rate is greater than the corresponding increase in output price. We next provide a definition of relative factor intensity. Let $j, j^* = 1, 2$, $j \neq j^*$.

Definition 1 *Sector j is capital intensive if the ratio of the profit maximizing level of K_j to L_j employed in producing good j is greater than the corresponding profit maximizing level of K_{j^*} to L_{j^*} employed in producing good j^**

$$\frac{K_j}{L_j} - \frac{K_{j^*}}{L_{j^*}} > 0 \Rightarrow S_{Kj} > S_{Kj^*}$$

where S_{Kj} denotes the share of factor K_j in the total cost of producing output j.[4]

Establishing the Stopler-Samuelson theorem only requires manipulating the zero profit conditions (2.8) and (2.9). Totally differentiating expressions (2.8) and (2.9), and manipulating the resulting expressions yields

$$C_w^j(\cdot)Y_j w\tilde{w} + C_r^j(\cdot)Y_j r\tilde{r} = p_j Y_j \tilde{p}_j, \ j = 1, 2 \tag{2.18}$$

where "$\tilde{\ }$" denotes proportional changes, e.g., $\tilde{w} = dw/w$ and $\tilde{p}_j = dp_j/p_j$. The zero profit conditions require total revenue $p_j Y_j$ be exactly equal to total cost TC_j. Dividing expression (2.18) by TC_j yields

$$\frac{wC_w^j(\cdot)Y_j}{TC_j}\tilde{w} + \frac{rC_r^j(\cdot)Y_j}{TC_j}\tilde{r} = \frac{p_j Y_j}{TC_j}\tilde{p}_j, \ j = 1, 2$$

[4] If technology is constant returns to scale Cobb-Douglas, then profit maximization implies $\alpha_{ij} = S_{ij} = w_i v_{ij}/TC_j$ where α_{ij} is the input elasticity of the i-th factor employed in the production of the j-th output.

or

$$S_{Lj}\tilde{w} + S_{Kj}\tilde{r} = \tilde{p}_j, \qquad j = 1, 2 \tag{2.19}$$

where $S_{Lj} = wC_w^j(\cdot)Y_j/TC_j$ is the (factor) cost share of labor in producing output j and $S_{Kj} = rC_r^j(\cdot)Y_j/TC_j$ is the cost (factor) share of capital in producing that output. Given zero profit, $p_jY_j = TC_j$.

Expression (2.19) is a set of two equations expressed in terms of factor shares and proportional factor rental rates $\tilde{w}$ and $\tilde{r}$. Solving this system for the proportional change in factor rental rates yields the following two equations

$$\tilde{w} = \frac{S_{K2}\tilde{p}_1 - S_{K1}\tilde{p}_2}{D_s} \tag{2.20}$$

$$\tilde{r} = \frac{-S_{L2}\tilde{p}_1 + S_{L1}\tilde{p}_2}{D_s}, \tag{2.21}$$

where

$$D_s \equiv S_{L1}S_{K2} - S_{L2}S_{K1} = S_{L1}S_{L2}\left(\frac{S_{K2}}{S_{L2}} - \frac{S_{K1}}{S_{L1}}\right)$$
$$= S_{L1}S_{L2}\frac{r}{w}\left(\frac{K_2}{L_2} - \frac{K_1}{L_1}\right)$$

By Equations (2.20) and (2.21), the sign of $\partial\tilde{w}/\partial\tilde{p}_j$ and $\partial\tilde{r}/\partial\tilde{p}_j$ each depend on the sign of D_s. If sector 2 is capital intensive, then by definition $S_{K2} > S_{K1}$, implying $K_2/L_2 > K_1/L_1$ and D_s is positive. It follows that $\partial\tilde{w}/\partial\tilde{p}_1 > 0$, while $\partial\tilde{r}/\partial\tilde{p}_1 < 0$, and conversely for a change in $\tilde{p}_2$. On the other hand, if sector 2 is labor intensive, then $S_{K2} < S_{K1}$ and D_s is negative. In this case, $\partial\tilde{w}/\partial\tilde{p}_1 < 0$, while $\partial\tilde{r}/\partial\tilde{p}_1 > 0$, and conversely for a change in $\tilde{p}_2$. Hence, an increase in the price of output j causes the rental rate of the factor used intensively in its production to rise, while the rental rate of the factor used intensively in producing output j^* falls.

Next, observe that the factor rental equations (2.12) and (2.13) are homogeneous of degree one in prices. Using Euler's theorem, and expressing the expressions in elasticity terms yields

$$\varepsilon_{p_1}^i + \varepsilon_{p_2}^i = 1, \; i = w, r \tag{2.22}$$

where $\varepsilon^i_{p_j}$ is the price elasticity of input i with respect to the price of output j. For example, the elasticity of w with respect to p_1 is equal to $\varepsilon^w_{p_1} \equiv W_{p_1}(\cdot)(p_1/w)$. Again, if sector 2 is capital intensive, then a change in the price of good 1 leads to an increase in the wage rate and a decrease in the rate of return to capital. It follows from (2.22) that one of the elasticities is negative, and consequently one elasticity in each equation must be greater than one, i.e., $\partial\tilde{w}/\partial\tilde{p}_1 > 0$ and $\partial\tilde{r}/\partial\tilde{p}_1 < 0$, implying $\varepsilon^w_{p_2} < 0$ and $\varepsilon^r_{p_1} > 1$. Thus, if sector 2 is capital intensive, $\tilde{p}_1 > 0$ implies w will increase in greater proportion than the increase in p_1. The capital rental rate r declines.

Rybczynski Theorem

The Rybczynski Theorem establishes that if the endowment of a factor increases, then the industry which uses that factor relatively intensively will (a) expand, and (b) expand more than proportionately to the percentage increase in the endowment – the other industry will contract (Woodland, 1982, p. 83). The factor market clearing conditions (2.10) and (2.11) are used to show (a) and (b).

To establish the result of the Rybczynski theorem, first substitute the factor rental rate equations (2.12) and (2.13) into the factor market clearing equations and express the result in terms of two linear equations in the endogenous variables Y_1 and Y_2

$$\begin{aligned} B_{L1}Y_1 + B_{L2}Y_2 &= L \\ B_{K1}Y_1 + B_{K2}Y_2 &= K \end{aligned}$$

where B_{Lj} and B_{Kj} are input-output coefficients for sector j and, as noted above, defined as

$$\begin{aligned} B_{Lj} &\equiv C^j_w\left(W(p_1,p_2), R(p_1,p_2)\right),\ j=1,2 \\ B_{Kj} &\equiv C^j_r\left(W(p_1,p_2), R(p_1,p_2)\right),\ j=1,2 \end{aligned}$$

Solving this system yields the supply functions

$$Y_1 = \frac{B_{K2}L - B_{L2}K}{D_B} \tag{2.23}$$

$$Y_2 = \frac{-B_{K1}L + B_{L1}K}{D_B} \tag{2.24}$$

where

$$D_B \equiv B_{L1}B_{L2}\frac{r}{w}(K_2/L_2 - K_1/L_1)$$

The sign of $\partial Y_j/\partial L$ and $\partial Y_j/\partial K$ depends on the sign of D_B, which in turn depends on the relative factor intensity term, $K_2/L_2 - K_1/L_1$. If sector 2 is relatively capital intensive, then D_B is positive and $\partial Y_1/\partial L > 0$ and $\partial Y_2/\partial L < 0$, while $\partial Y_1/\partial K < 0$ and $\partial Y_2/\partial K > 0$. This establishes part (a) i.e., one sector will expand and the other will contract.

To establish that the industry which uses that factor relatively intensively will expand more proportionately than the proportionate increase in the endowment, appeal to the linear homogeneity properties of the supply functions (2.14). Given the supply functions are homogeneous of degree one in L and K, by Euler's theorem, endowment elasticities sum to unity

$$\varepsilon_L^{y_j} + \varepsilon_K^{y_j} = 1, \;\; j = 1, 2 \tag{2.25}$$

where $\varepsilon_i^{y_j}$ is the sector j output elasticity with respect to endowment $i = L, K$. For example, the elasticity of sector j output with respect to labor is equal to $\varepsilon_L^{y_j} \equiv Y_L^j\left(p_1, p_2, L, K\right)\left(L/Y_j\right)$. As with Stopler-Samuelson, for each equation in (2.25), one term must be negative, and the other positive and greater than one. Hence, the sector employing the factor intensively will expand more than proportionately to the increase in this factor's endowment, while the other industry will contract. This simple argument is left to the reader as an exercise.

2.3 Generalizing the basic model

Altering the dimensions of the basic model affect whether the zero profit conditions are sufficient to solve for factor prices as functions of traded good prices alone, and moreover, whether the solution is unique. Since the dynamic models discussed in later chapters are of various dimensions, this section generalizes aspects of the equilibrium conditions of the HOS model just presented. Let M_t denote the number of traded goods produced.

2.3.1 The case where $M_t = N$

Suppose first, that $M_t = N$ goods are produced. The zero profit conditions are

$$C^j(\mathbf{w}) = p_j, \; j = 1, \cdots, M_t \tag{2.26}$$

and factor market clearing requires

$$\sum_{j=1}^{M_t} C^j_{w_i}(\mathbf{w}) Y_j = V_i, \; i = 1, \cdots, N \tag{2.27}$$

In principle, the system (2.26) can be used to determine the factor rental rates w_i for each factor, $i \in I$. As with the 2×2 case, the equilibrium rental rates are independent of the factor endowments. Similar to the 2×2 case, this solution can be used to determine the equilibrium output levels for each sector by substituting the solution to (2.26) into (2.27) to determine the sectoral supplies as functions of prices and endowments.[5]

2.3.2 The case where $M_t < N$

Assume all M_t goods are produced and all N factors are employed. Then the entire system (2.26) and (2.27) of M_t+N equations is required to solve for the endogenous variables (w_i, Y_j) $i = 1, \cdots, N$ and $j = 1, \cdots, M_t$. Denote the result by the following rental rate functions

$$w_i = W^i(\mathbf{p}, \mathbf{V})$$

and supply functions

$$Y_j = Y^j(\mathbf{p}, \mathbf{V})$$

[5] A change in the endowment of factors will not affect factor prices provided the number of open sectors M_t remain unchanged. That is, provided the economy remains within its so called cone of diversification.

As with Equation (2.15), express the GDP function as

$$G(\mathbf{p}, \mathbf{V}) = \sum_{j=1}^{M_t} p_j Y^j(\mathbf{p}, \mathbf{V}) \tag{2.28}$$

where (2.28) satisfies properties G1 – G5. Assuming (2.28) is differentiable,

$$\begin{aligned} w_i &= W^i(\mathbf{p}, \mathbf{V}) = G_{v_i}(\mathbf{p}, \mathbf{V}), \ i = 1, \cdots, N & (2.29) \\ Y_j &= Y^j(\mathbf{p}, \mathbf{V}) = G_{p_j}(\mathbf{p}, \mathbf{V}), \ j = 1, \cdots, M_t & (2.30) \end{aligned}$$

2.3.3 Comparative statics

These results apply for the case where $M_t \leq N$. Since (2.28) is homogeneous of degree one in prices and homogeneous of degree one in factor endowments, it follows that the factor rental rate price elasticities, defined as, $\varepsilon_{p_j}^{w_i} = W_{p_j}^i(\mathbf{p}, \mathbf{V})(p_j/w_i)$, sum to unity

$$\sum_{j=1}^{M_t} \varepsilon_{p_j}^{w_i} = 1, \ i = 1, \cdots, N \tag{2.31}$$

This result is analogous to (2.22). Likewise, the output endowment elasticities

$$\sum_{j=1}^{M_t} \varepsilon_{V_i}^{y_j} = 1, \ i = 1, \cdots, N \tag{2.32}$$

which is analogous to (2.25), also sum to unity. The supply elasticity of output j with regard to factor endowment i is defined as $\varepsilon_{V_i}^{y_j} = Y_{V_i}^j(\mathbf{p}, \mathbf{V})(V_i/Y_j)$.

While the factor rental rate elasticities in (2.31) sum to unity, it is not necessary for any term to be greater than one or for any term to be negative as was the case in HOS model. Thus, when the price of a good increases, no factor price need increase by a greater percentage than the percentage change in the output price, nor does a factor price need fall.

However, if there exists a product whose price p_j increase causes the ith factor rental rate to increase in greater proportion than the change in output price, i.e., if an elasticity $\varepsilon_{p_j}^{w_i}$ is greater than unity, then there must exist at least one other output $j' \neq j$ such that an increase in its price will cause w_i to fall, $\varepsilon_{p_{j'}}^{w_i} < 0$.

A similar result applies to the Rybczynski theorem. It is not necessary for any term in (2.32) to be greater than one, or for any term to be negative. If an endowment increases, the output of all goods could increase less than proportionately to the increase in the endowment. If one output does increase in greater proportion than the increase in endowment, then some other output must fall.

Finally, one can show the equilibrium rate of return to factor i is non-increasing in its own endowment, i.e.,

$$\frac{\partial W^i(\cdot)}{\partial V_i} = \frac{\partial^2 G(\cdot)}{\partial V_i^2} \leq 0,$$

while the equilibrium supply of output j is non-decreasing in its own price, i.e.,

$$\frac{\partial Y^j(\cdot)}{\partial p_j} = \frac{\partial^2 G(\cdot)}{\partial p_j^2} \geq 0$$

Since the rental rate functions are homogeneous of degree zero in endowments and the supply functions of degree zero in prices, their respective endowment and price elasticities sum to zero.

2.4 The special case of a home (non-traded) good

A dynamic three-sector model in which two goods are traded and one is only traded domestically is developed in later chapters. The presence of a fixed factor and a home-good are useful for studying the effects of a sector specific resource, such as land, on a country's transition to long-run equilibrium. Such models are also useful in studying the effects of (i) government deficit spending, (ii) foreign aid, and (iii) remittances from workers living abroad on relative prices and on the corresponding allocation of resources from traded to home-good production. This section considers the static version of such a model, the basic form and comparative static properties of which are used in later chapters.

2.4.1 The environment

The economy is small, open and competitive. It produces three final goods, agriculture, manufacturing, and the home-good, indexed respectively by $j = a, m, s$. The agricultural and manufacturing goods are traded internationally at given world prices p_a and p_m, while the price of the home-good p_s is determined domestically. The economy is endowed with labor L, capital K, and land H. Capital and labor are economy-wide factors, while land is employed only in agriculture. Here, $\mathbf{V} = (K, L, H) \in \mathbb{R}^3_{++}$. As such, land is a resource specific to agriculture in the sense that its services can be rented in and out among firms in agriculture, but land is not used by firms in the other two sectors. In this case, $M = N = 3$, $M_t = 2$, and hence $M_t < N$. As before, households exchange the services of labor, capital, and land for wages w, capital rents r, and land rents π, where w, r, and π are each per-unit returns. All resulting income is used by households to purchase agricultural, manufacturing, and the home-good, denoted Q_a, Q_m, and Q_s respectively.

2.4.2 Behavior of households and firms

As with Section 2.2, households hold identical, homothetic preferences satisfying Assumption 1. Hence, the "community" indirect utility function is given by

$$\mathcal{V} = \mathrm{v}\left(p_a, p_m, p_s\right)\left(wL + rK + \pi H\right),$$

and the corresponding Marshallian demand functions are:

$$Q_j = q^j\left(p_a, p_m, p_s\right)\left(wL + rK + \pi H\right),\ j = a, m, s$$

Firms within each sector are atomistic, identical, and hold technologies satisfying Assumption 2. Firms producing the manufactured and home-goods employ technology $f^j : \mathbb{R}^2_{++} \rightarrow \mathbb{R}_+$ defined as $y_j = f^j\left(\ell_j, k_j\right),\ j = m, s$. The corresponding sector level total cost functions are given by

$$TC_j = C^j\left(w, r\right) Y_j,\ j = m, s$$

Rather than specifying the corresponding cost function for firms producing the agricultural good, we use the sectoral value-added function (2.6). This approach can be shown to reduce the dimensionality of the problem and simplifies the comparative statics of the model. Represent the agricultural technology by the production function $f^a : \mathbb{R}^3_{++} \to \mathbb{R}_+$, defined as $y_a = f^a(\ell_a, k_a, h)$, where ℓ_a, k_a, and h are the respective levels of labor, capital, and land employed by an agricultural firm. Given the land endowment H is fixed for the sector, and given f^a is linearly homogeneous in all inputs, the sectoral aggregate technology, denoted $\mathcal{F}^a(\cdot)$, exhibits decreasing returns to scale in L_a and K_a.

Define the agricultural value-added function as

$$\pi^a(p_a, w, r)\, H \equiv \max_{L_a, K_a} \{p_a \mathcal{F}^a(L_a, K_a, H) - wL_a - rK_a\}$$

where H is specific to the sector, and hence, not treated as a choice variable at the sector level. By Hotelling's lemma, agriculture's partial equilibrium supply function is given by

$$y^a(p_a, w, r)\, H = \pi^a_{p_a}(p_a, w, r)\, H \tag{2.33}$$

As noted in the first section, a perfectly competitive land market among producers implies that in equilibrium, the shadow price of an additional unit of land, $\pi^a(p_a, w, r)$, is equal to the land rental rate that clears the market for land among individual producers. Thus, firms in this sector earn zero profits since, in equilibrium, the value of output is exhausted by payments to factors

$$p_a Y_a = wL_a + rK_a + \pi^a(p_a, w, r)\, H$$

2.4.3 The characterization of equilibrium

Restricting analysis to the case where all sectors are open, i.e. each $Y_j > 0$, equilibrium is defined by the positive values

$$(w, r, p_s, Y_m, Y_s) \in \mathbb{R}^5_{++}$$

satisfying the following conditions: two zero profit conditions in output markets,

$$C^j(w,r) - p_j = 0,\ \ j = m, s$$

labor and capital market clearing

$$\begin{aligned}\sum_{j=m,s} C_w^j(w,r)\,Y_j - \pi_w^a(p_a,w,r)\,H &= L\\ \sum_{j=m,s} C_r^j(w,r)\,Y_j - \pi_r^a(p_a,w,r)\,H &= K\end{aligned}$$

and clearing of the domestic market for the home-good

$$q^s(p_a,p_m,p_s)(wL + rK + \pi H) = Y_s \tag{2.34}$$

where $\pi = \pi^a(p_a,w,r)$.

The model's endogenous variables can be obtained as follows. Similar to the HOS model, the two zero profit equations can be used to express the rate of return to capital and labor as a function of the traded good price p_m, and the home-good price p_s. Express the result as

$$w = W(p_m,p_s) \tag{2.35}$$

$$r = R(p_m,p_s) \tag{2.36}$$

We determine the value of p_s shortly. Substitute (2.35) and (2.36) into the factor market clearing equations to obtain Y_m and Y_s represented by

$$Y_j = Y^j(p_a,p_m,p_s,L,K,H),\ \ j = m, s \tag{2.37}$$

The supply function for agriculture can be expressed in output price alone by substituting the rental rate equations (2.35) and (2.36) into the partial equilibrium supply function (2.33)

$$Y_a = Y^a(p_a,p_m,p_s)\,H = y^a(p_a,W(p_m,p_s),R(p_m,p_s))\,H \tag{2.38}$$

GDP can be expressed as a function of factor payments using (2.35), (2.36), and (2.37) as follows,

$$G\left(p_a, p_m, p_s, L, K, H\right) = W\left(p_m, p_s\right) L + R\left(p_m, p_s\right) K + \pi^a\left(p_a, W\left(p_m, p_s\right), R\left(p_m, p_s\right)\right) H \tag{2.39}$$

or equivalently by

$$G\left(p_a, p_m, p_s, L, K, H\right) = \sum_{j=m,s} p_j Y^j\left(p_m, p_s, L, K\right) + p_a Y^a \left(p_a, p_m, p_s\right) H \tag{2.40}$$

The GDP function can also be derived from the maximization problem

$$G\left(p_a, p_m, p_s, L, K, H\right) \equiv \max_{L_j, K_j}\left\{\sum_{j=m,s} p_j \mathcal{F}^j(L_j, K_j) + p_a \mathcal{F}^a\left(L_a, K_a; H\right)\right\} \tag{2.41}$$

subject to the resource constraints

$$L \geq \sum_{j=a,m,s} L_j, \; K \geq \sum_{j=a,m,s} K_j$$

where $G(\cdot)$ satisfies properties G1 – G5.

The remaining endogenous variable is p_s. In the home-good market clearing equation (2.34), substitute (2.39) for factor payments, and (2.37) for home-good supply Y_s and solve for p_s. We focus on the role of the home-good market in the next section.

2.4.4 Selected comparative statics

The major departure from the HOS model is the presence of the home-good market. Changes in world prices and changes in endowments have direct effects on factor rental rates and output supply that are similar to those of the HOS model. However, since these variables affect the market for home-goods, they also have indirect effects on supply and factor rental rates that are transmitted through changes in the home-good price.

The price of the home-good

As noted in discussing consumer and firm behavior, both the home-good demand function, expressed as

$$Q^s\left(p_a,p_m,p_s,L,K,H\right) \equiv q^s\left(p_m,p_a,p_s\right) G\left(p_a,p_m,p_s,L,K,H\right) \tag{2.42}$$

and the supply function,

$$Y_s = Y^s\left(p_a,p_m,p_s,L,K,H\right) \equiv G_{p_s}\left(p_a,p_m,p_s,L,K,H\right) \tag{2.43}$$

are homogeneous of degree zero in prices (p_a,p_m,p_s), and homogeneous of degree one in endowments (L,K,H).

Equate home-good demand to home-good supply, and express the resulting equation in elasticity form

$$\tilde{p}_s = \sum_{j=a,m} \varepsilon_j^P \tilde{p}_j + \sum_{i=L,K,H} \varepsilon_i^P \tilde{v}_i \tag{2.44}$$

Here, $\tilde{v}_L = dL/L$, $\tilde{v}_K = dK/K$, $\tilde{v}_H = dH/H$, and

$$\begin{aligned} \varepsilon_j^P &= P_{p_j}\left(\cdot\right)\frac{p_j}{P^s\left(\cdot\right)},\ j=a,m \\ \varepsilon_i^P &= P_{v_i}\left(\cdot\right)\frac{v_i}{P^s\left(\cdot\right)},\ i=L,K,H \end{aligned}$$

respectively, define the elasticities of the two traded good prices and the elasticities of the three factor endowments.

In the Appendix we show the home-good price is: (i) homogeneous of degree one in traded good prices, implying

$$\sum_{j=m,a} \varepsilon_j^P = 1 \tag{2.45}$$

and (ii) homogeous of degree zero in endowments, implying

$$\sum_{i=L,K,H} \varepsilon_i^P = 0 \tag{2.46}$$

By (2.45), either ε_a^P and ε_m^P are both positive and sum to one, or one of the elasticities is negative and the other greater than one. What is the implication of one of these elasticities being negative? Let $j = m$ be the imported good. Then all else constant, an increase in p_m is referred to as a negative change in the country's terms of trade. An increase in this price can decrease real income, and "pull" more resources into production of the import competing good Y_m. In this case, both home-good supply, and demand fall, i.e., $\partial Y^s(\cdot)/\partial p_m < 0$ and $\partial Q_s/\partial p_m < 0$. If demand falls more than supply, then excess demand for the home-good declines, implying

$$\partial \tilde{p}_s/\partial \tilde{p}_m < 0 \tag{2.47}$$

in which case ε_m^P is negative. Conversely, an improvement in the country's terms of trade can cause the home-good price to increase in greater proportion than the price of the export good.

Condition (2.46) is useful in understanding the change in home-good prices in the process of economic growth. For instance, later we show if an economy's initial capital stock is less than its long-run equilibrium level, the stock of capital grows at a rate that exceeds the rate of growth in the labor force. If the home-good sector is labor intensive relative to the other two sectors, then the elasticity ε_K^P is positive, while ε_L^P is negative. In this case, as capital accumulates, the home-good price grows over time. This growth in the price of home-good dominates the negative effect of growth in the labor force. Effectively, p_s must increase in order to compete for the labor resources that are otherwise made more productive in sectors that are relatively more capital intensive than the home-good sector.

Finally, since p_s influences equilibrium factor rental rates and the equilibrium supply of manufacturing and agricultural output, it follows that changes in p_s can have *indirect* effects on these variables in the sense that home-good price effects are transmitted to (2.35), (2.36), (2.37), and (2.38) via (2.44). We now turn to these issues.

Home-good price effects on factor rental rates and supply

Since the home-good price is homogeneous of degree one in traded good prices, the factor rental rate equations (2.35) and

(2.36) remain homogeneous of degree one in traded good prices. Thus, while the rental rate elasticities sum to unity as in (2.22), the effect of a change in the price of a traded good on w and r are now more complicated. We have

$$\tilde{w} = \varepsilon^{w}_{p_m}\tilde{p}_m + \varepsilon^{w}_{p_s}\tilde{p}_s \tag{2.48}$$

$$\tilde{r} = \varepsilon^{r}_{p_m}\tilde{p}_m + \varepsilon^{r}_{p_s}\tilde{p}_s \tag{2.49}$$

where the change in home-good price, $\tilde{p}_s$, is given by (2.44).

This linkage also applies to the supply functions (2.37) and (2.38). In elasticity terms

$$\tilde{Y}_j = \varepsilon^{Y_j}_{p_m}\tilde{p}_m + \varepsilon^{Y_j}_{p_a}\tilde{p}_a + \varepsilon^{Y_j}_{p_s}\tilde{p}_s + \varepsilon^{Y_j}_{L}\tilde{L} + \varepsilon^{Y_j}_{K}\tilde{K}, \;\; j = m, s \tag{2.50}$$

where the land endowment H is assumed constant. The elasticities $\left(\varepsilon^{Y_j}_{p_a},\ \varepsilon^{Y_j}_{p_m},\ \varepsilon^{Y_j}_{p_s}\right)$ are the supply response of sector j to changes in output prices p_m, p_a, and p_s. For the case of agriculture, expressing (2.33) in elasticity terms gives

$$\tilde{Y}_a = \varepsilon^{Y_a}_{p_a}\tilde{p}_a + \varepsilon^{Y_a}_{w}\tilde{w} + \varepsilon^{Y_a}_{r}\tilde{r},$$

and substituting (2.48) and (2.49) into the above expression yields

$$\tilde{Y}_a = \varepsilon^{Y_a}_{p_a}\tilde{p}_a + \varepsilon^{Y_a}_{w}\left(\varepsilon^{w}_{p_m}\tilde{p}_m + \varepsilon^{w}_{p_s}\tilde{p}_s\right) + \varepsilon^{Y_a}_{r}\left(\varepsilon^{r}_{p_m}\tilde{p}_m + \varepsilon^{r}_{p_s}\tilde{p}_s\right) \tag{2.51}$$

Here, the elasticities $\left(\varepsilon^{Y_a}_{w}, \varepsilon^{Y_a}_{r}\right)$ are the agricultural sector's supply elasticities with respect to factor rental rates. The supply functions (2.50) and (2.51) are homogeneous of degree zero in prices, and hence the respective elasticities for each equation sum to zero.

The indirect effects in the case of (2.50) occur through the adjustment of the home-good price as determined by (2.57). In the case of Y_a, the indirect effects are transmitted through the labor and capital markets, which in turn are influenced by adjustments in the home-good price.

Traded good price effects on rental rates and supply

Let manufacturing be capital intensive relative to agriculture and the home-good, and assume manufacturing is an import competing sector. In this case, an increase in the price of manufactured goods, $\tilde{p}_m > 0$, amounts to a negative change in the country's terms of trade. It follows from Stopler-Samuelson that $\varepsilon^r_{p_m} > 0$ and $\varepsilon^w_{p_m} < 0$. Since the factor rental equations are homogeneous of degree zero, it follows that $\varepsilon^r_{p_s} < 0$ and $\varepsilon^w_{p_s} > 0$. The direct effect of $\tilde{p}_m > 0$ on rental rates is given by the elasticities $\varepsilon^w_{p_m}$, and $\varepsilon^r_{p_m}$ while the indirect effects are given by the product terms $\varepsilon^w_{p_s}\tilde{p}_s$ and $\varepsilon^r_{p_s}\tilde{p}_s$. However, $\tilde{p}_s$ is determined by the $\varepsilon^P_m\tilde{p}_m$ term in (2.44). If $\varepsilon^P_m > 0$, then in this case

$$\tilde{w} = \varepsilon^w_{p_m}\tilde{p}_m + \varepsilon^w_{p_s}\varepsilon^P_m\tilde{p}_m < 0 \tag{2.52}$$

and

$$\tilde{r} = \varepsilon^r_{p_m}\tilde{p}_m + \varepsilon^r_{p_s}\varepsilon^P_m\tilde{p}_m > 0 \tag{2.53}$$

In general, depending upon the sign of ε^P_m, the indirect effect can augment or lessen the direct effect of a change $\tilde{p}_m$ on rental rates. For instance, in the case considered here, suppose ε^P_m is negative. Then, wages fall and capital rental rates rise by a greater amount than predicted by the Stopler-Samuelson theorem.

A change in the price of the agricultural good causes a change in rental prices according to

$$\begin{aligned} \tilde{w} &= \varepsilon^w_{p_s}\left(\varepsilon^P_a\tilde{p}_a\right) \\ \tilde{r} &= \varepsilon^r_{p_s}\left(\varepsilon^P_a\tilde{p}_a\right) \end{aligned}$$

As in the case of (2.31), since the functions for w and r remain homogeneous of degree one in the prices of traded goods, when the price of a good increases, there is no need for any factor price increase to be proportionately greater than the output price increase, and no need for a factor price to fall. However, if an output price increase causes a factor rental rate to increase by a greater proportion than the change in the traded good price, then the rents to at least one factor must fall.

The effects of a change in the manufacturing price on supply (2.50) are transmitted through the terms $\varepsilon_{p_m}^{Y_j}\tilde{p}_m + \varepsilon_{p_s}^{Y_j}\varepsilon_m^P\tilde{p}_m$. For $\tilde{p}_m > 0$ and $\varepsilon_m^P > 0$, manufacturing experiences a positive direct effect, $\varepsilon_{p_m}^{Y_m} > 0$, and a positive indirect effect, $\varepsilon_{p_s}^{Y_m}\varepsilon_m^P\tilde{p}_m$. The home-good experiences a negative direct effect, $\varepsilon_{p_m}^{Y_s}\tilde{p}_m < 0$, and a negative indirect effect, $\varepsilon_{p_s}^{Y_s}\varepsilon_m^P\tilde{p}_m < 0$. The effects on agriculture are transmitted through factor markets. However, in the case considered here, the decreasing wage has a positive effect on agricultural output while the increasing capital rental rate has a negative effect on output. The net effect depends upon the share of labor relative to capital in total cost: if agriculture is labor intensive, output can increase.

Endowment effects on rental rates and supply

The differential of (2.48), (2.49), and (2.50) with respect to endowments, can be shown to yield the following expressions:

$$
\begin{aligned}
\tilde{w} &= \varepsilon_{p_s}^{w}\left(\varepsilon_L^P\tilde{L} + \varepsilon_K^P\tilde{K} + \varepsilon_H^P\tilde{H}\right) \\
\tilde{r} &= \varepsilon_{p_s}^{r}\left(\varepsilon_L^P\tilde{L} + \varepsilon_K^P\tilde{K} + \varepsilon_H^P\tilde{H}\right) \\
\tilde{Y}_j &= \varepsilon_{p_s}^{Y_j}\left(\varepsilon_L^P\tilde{L} + \varepsilon_K^P\tilde{K} + \varepsilon_H^P\tilde{H}\right) + \varepsilon_L^{Y_j}\tilde{L} + \varepsilon_K^{Y_j}\tilde{K},\ j = m, s \\
\tilde{Y}_a &= \varepsilon_w^{Y_a}\tilde{w} + \varepsilon_r^{Y_a}\tilde{r}
\end{aligned}
$$

each of which show the indirect effects of changes in endowments on factor rental rates and supply. Here, we utilize the endowment components of the home-good price equation (2.44).

Consider the case where manufacturing is the most capital intensive sector while the home-good sector is the most labor intensive. Then $\varepsilon_K^{Y_m}, \varepsilon_L^{Y_s} > 0$, and $\varepsilon_L^{Y_m}, \varepsilon_K^{Y_s} < 0$. For this case, as stated above, the elasticity ε_K^P is positive, and ε_L^P is negative. Now, for purpose of the growth models presented in future chapters, consider the additional condition that growth in the capital stock exceeds the growth in labor, $\tilde{K} > \tilde{L}$. In this environment, the net effect of labor and capital accumulation on growth in the price of the home-good is positive, $\tilde{p}_s > 0$.

Under these circumstances, w increases and r falls. This result implies an increase in the productivity of labor as the capital to labor ratio increases over time, while the productivity of capital falls. Manufacturing output is affected negatively by the indirect effect of an increase in the home-good price, as determined by $\varepsilon_{p_s}^{Y_m}\left(\varepsilon_L^P\tilde{L}+\varepsilon_K^P\tilde{K}\right)$. Output is affected by Rybczynski effects, one of which is negative, $\varepsilon_L^{Y_m}\tilde{L}$, and other positive $\varepsilon_K^{Y_m}\tilde{K}$. Since manufacturing is capital intensive, it is possible for the capital effects to dominate.

The home-good sector output is affected in almost the opposite way. The home-good price effect $\varepsilon_{p_s}^{Y_s}\left(\varepsilon_L^P\tilde{L}+\varepsilon_K^P\tilde{K}\right)$ is positive, while the net factor accumulation effect, as determined by $\varepsilon_L^{Y_j}\tilde{L}+\varepsilon_K^{Y_j}\tilde{K}$ can be negative. However, the net price effect can dominate the factor accumulation effect so that growth in disposable income leads to increased consumption of the home-good, albeit at a higher price of the home-good. In this way, the home-good is competing for resources allocated to the production of traded goods so that the price ratio of traded to home-goods falls.

The effect on agricultural output once again depends on not only the magnitude of changes in w and r, but also on the sector's relative factor intensity. Changes in agricultural output, and the employment of labor and capital need not be monotonic as the labor and capital variables evolve over time.

Although most of the comparative static results in this section are ambiguous, all of the effects discussed above can be measured when a structural model is fit to data. Knowledge of these effects is crucial to explaining the evolution of a modeled economy.

2.5 Appendix: determinants of home-good price

We proceed as in Chipman (2007) to confirm (2.45). Totally differentiate expressions (2.42) and (2.43)

$$\begin{aligned}
dQ^s &= Q^s_{p_a}dp_a + Q^s_{p_m}dp_m + Q^s_{p_s}dp_s \\
dY^s &= Y^s_{p_a}dp_a + Y^s_{p_m}dp_m + Y^s_{p_s}dp_s
\end{aligned}$$

and convert to elasticities

$$\begin{aligned}\frac{dQ^s}{Q^s} &= Q^s_{p_a}\frac{p_a}{Q^s}\frac{dp_a}{p_a} + Q^s_{p_m}\frac{p_m}{Q^s}\frac{dp_m}{p_m} + Q^s_{p_s}\frac{p_s}{Q^s}\frac{dp_s}{p_s} \\ &= \sum_{j=a,m} \varepsilon_j^{Q_s}\tilde{p}_j + \varepsilon_{p_s}^{Q_s}\tilde{p}_s = 0 \end{aligned} \tag{2.54}$$

$$\begin{aligned}\frac{dY^s}{Y^s} &= Y^s_{p_a}\frac{Y^s}{p_a}\frac{dp_a}{p_a} + Y^s_{p_m}\frac{Y^s}{p_m}\frac{dp_m}{p_m} + Y^s_{p_s}\frac{Y^s}{p_s}\frac{dp_s}{p_s} \\ &= \sum_{j=a,m} \varepsilon_j^{Y_s}\tilde{p}_j + \varepsilon_{p_s}^{Y_s}\tilde{p}_s = 0 \end{aligned} \tag{2.55}$$

Define

$$\sum_{j=a,m} \varepsilon_j^{Q_s} = \sum_{j=a,m} Q^s_{p_j}\frac{p_j}{Q^s}; \quad \sum_{j=a,m} \varepsilon_j^{Y_s} = \sum_{j=a,m} Y^s_{p_j}\frac{Y^s}{p_j}$$

Given $Q^s(\cdot)$ and $Y^s(\cdot)$ are both homogeneous of degree zero in prices, it follows that

$$\varepsilon_{p_s}^{Q_s} = -\sum_{j=a,m} \varepsilon_{p_j}^{Q_s} \text{ and } \varepsilon_{p_s}^{Y_s} = -\sum_{j=a,m} \varepsilon_j^{Y_s}$$

and, hence we can rewrite (2.54) and (2.55) to obtain

$$\begin{aligned} &\sum_{j=a,m} \varepsilon_j^{Q_s}\tilde{p}_j - \sum_{j=a,m} \varepsilon_j^{Q_s}\tilde{p}_s \\ &= \sum_{j=a,m} \varepsilon_j^{Y_s}\tilde{p}_j - \sum_{j=a,m} \varepsilon_j^{Y_s}\tilde{p}_s = 0 \end{aligned} \tag{2.56}$$

Collecting terms in (2.56) and solving for $\tilde{p}_s$ gives

$$\sum_{j=a,m} \left(\varepsilon_j^{Y_s} - \varepsilon_j^{Q_s}\right)\tilde{p}_s = \sum_{j=a,m}^{y} \left(\varepsilon_j^{Y_s} - \varepsilon_j^{Q_s}\right)\tilde{p}_j$$

$$\Rightarrow \tilde{p}_s = \frac{\sum_{j=a,m}^{y} \left(\varepsilon_j^{Y_s} - \varepsilon_j^{Q_s}\right)\tilde{p}_j}{\sum_{j=a,m} \left(\varepsilon_j^{Y_s} - \varepsilon_j^{Q_s}\right)} \tag{2.57}$$

For a uniform rate of increase in traded good prices, $\tilde{p}_j = \tilde{p}$, $j = a, m$, (2.57) becomes

$$\tilde{p}_s = \frac{\sum_{j=a,m}^{y} \left(\varepsilon_j^{Y_s} - \varepsilon_j^{Q_s}\right)}{\sum_{j=a,m} \left(\varepsilon_j^{Y_s} - \varepsilon_j^{Q_s}\right)} \tilde{p}$$

The price of the home-good thus increases by the same proportion as the increase in world prices, establishing the result that relative prices remain unchanged, i.e.,

$$\tilde{p}_s = \tilde{p}$$

which establishes the claim that the home-good price is homogeneous of degree one in prices, expression (2.45).

We next consider the effect of endowments on home-good price. Totally differentiating demand (2.42) and supply (2.43) with respect to endowments, and expressing the result in elasticity terms yields

$$\varepsilon_{p_s}^{Q_s} \sum_{z=L,K,H} \varepsilon_z^{P} \tilde{z} + \sum_{z=L,K,H} \varepsilon_z^{Q_s} \tilde{z} = \varepsilon_s^{Y_s} \sum_{z=L,K,H} \varepsilon_z^{P} \tilde{z} + \sum_{z=L,K,H} \varepsilon_z^{Y_s} \tilde{z} \tag{2.58}$$

where the elasticities are: the direct price elasticity of home-good demand

$$\varepsilon_{p_s}^{Q_s} = Q_{p_s}^{s}(\cdot) \frac{p_s}{Q^s(\cdot)}$$

the endowment elasticities of the home-good price

$$\varepsilon_z^{P} = P_z^{s}(\cdot) \frac{z}{P^s(\cdot)}, \quad z = L, K, H$$

the endowment elasticities of home-good demand, (2.42)

$$\varepsilon_z^{Q_s} = Q_z^{s}(\cdot) \frac{z}{Q^s(\cdot)}, \quad z = L, K, H$$

and the endowment elasticities of home-good supply, (2.43)

$$\varepsilon_z^{Y_s} = Y_z^{s}(\cdot) \frac{z}{Y^s(\cdot)}, \quad z = L, K, H$$

Given a proportionate change in each endowment, i.e., $\tilde{L} = \tilde{K} = \tilde{H} = z^o$, and rearranging the terms in expression (2.58) gives

$$\left(\varepsilon_{p_s}^{Q_s} - \varepsilon_s^{Y_s}\right) \sum_{z=L,K,H} \varepsilon_z^P z^o = \sum_{z=L,K,H} \left(\varepsilon_z^{Y_s} - \varepsilon_z^{Q_s}\right) z^o \tag{2.59}$$

Solving for $\sum\limits_{z=L,K,H} \varepsilon_z^P$ yields

$$\sum_{z=L,K,H} \varepsilon_z^P = \frac{\sum\limits_{z=L,K,H} \left(\varepsilon_z^{Y_s} - \varepsilon_z^{Q_s}\right)}{\varepsilon_{p_s}^{Q_s} - \varepsilon_s^{Y_s}} \tag{2.60}$$

Since both demand and supply are homogeneous of degree one in endowments, the numerator of (2.60) is zero, thus establishing (2.46).

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