2

Optimality Conditions

Chapter 2

2.0 Introduction

In this chapter we will focus on necessary and sufficient optimality conditions for constrained problems.

As an introduction let us remind ourselves of the optimality conditions for *unconstrained* and *equality constrained* problems, which are commonly dealt with in basic Mathematics lectures.

We consider a real-valued function $f\colon D \longrightarrow \mathbb{R}$ with domain $D \subset \mathbb{R}^n$ and define, as usual, for a point $x_0 \in D$:

1) f has a *local minimum* in x_0

$$:\Longleftrightarrow\ \exists\, U \in \mathbb{U}_{x_0}\ \forall\, x \in U \cap D\ \ f(x) \ge f(x_0)$$

W. Forst and D. Hoffmann, *Optimization—Theory and Practice*,
Springer Undergraduate Texts in Mathematics and Technology,
DOI 10.1007/978-0-387-78977-4_2,

2) f has a *strict local minimum* in x_0

$$:\iff \exists\, U \in \mathbb{U}_{x_0}\ \forall\, x \in U \cap D \setminus \{x_0\}\ \ f(x) > f(x_0)$$

3) f has a *global minimum* in x_0

$$:\iff \forall\, x \in D\ \ f(x) \geq f(x_0)$$

4) f has a *strict global minimum* in x_0

$$:\iff \forall\, x \in D \setminus \{x_0\}\ \ f(x) > f(x_0)$$

Here, $\mathbb{U}_{x_0}$ denotes the neighborhood system of x_0.

We often say "x_0 is a *local minimizer* of f" or "x_0 is a *local minimum point of f*" instead of "f has a *local minimum* in x_0" and so on. The *minimizer* is a point $x_0 \in D$, the *minimum* is the corresponding value $f(x_0)$.

Necessary Condition

Suppose that the function f has a local minimum in $x_0 \in \overset{\circ}{D}$, that is, in an interior point of D. Then:

a) If f is differentiable in x_0, then $\nabla f(x_0) = 0$ holds.

b) If f is twice continuously differentiable in a neighborhood of x_0, then the Hessian $H_f(x_0) = \nabla^2 f(x_0) = \left(\frac{\partial^2 f}{\partial x_\nu \partial x_\mu}(x_0)\right)$ is positive semidefinite.

We will use the notation $f'(x_0)$ (to denote the derivative of f at x_0; as we know, this is a linear map from $\mathbb{R}^n$ to $\mathbb{R}$, read as a *row vector*) as well as the corresponding transposed vector $\nabla f(x_0)$ (gradient, *column vector*).

Points $x \in \overset{\circ}{D}$ with $\nabla f(x) = 0$ are called *stationary points*. At a stationary point there can be a local minimum, a local maximum or a *saddlepoint*. To determine that there is a local minimum at a stationary point, we use the following:

Sufficient Condition

Suppose that the function f is twice continuously differentiable in a neighborhood of $x_0 \in D$; also suppose that the necessary optimality condition $\nabla f(x_0) = 0$ holds and that the Hessian $\nabla^2 f(x_0)$ is positive definite. Then f has a strict local minimum in x_0.

The proof of this proposition is based on the Taylor theorem and we regard it as known from Calculus. Let us recall that a symmetric (n,n)-matrix A is *positive definite* if and only if all principal subdeterminants

$$\det \begin{pmatrix} a_{11} & \dots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \dots & a_{kk} \end{pmatrix} \qquad (k = 1, \dots, n)$$

are positive (cf. exercise 3).

Now let f be a real-valued function with domain $D \subset \mathbb{R}^n$ which we want to minimize subject to the *equality constraints*

$$h_j(x) = 0 \quad (j = 1, \ldots, p)$$

for $p < n$; here, let $h_1, \ldots, h_p$ also be defined on D. We are looking for local minimizers of f, that is, points $x_0 \in D$ which belong to the *feasible region*

$$\mathcal{F} := \big\{x \in D \mid h_j(x) = 0 \;\; (j = 1, \ldots, p)\big\}$$

and to which a neighborhood U exists with $f(x) \geq f(x_0)$ for all $x \in U \cap \mathcal{F}$.

Intuitively, it seems reasonable to solve the constraints for p of the n variables, and to eliminate these by inserting them into the objective function. For the *reduced objective function* we thereby get a nonrestricted problem for which under suitable assumptions the above necessary optimality condition holds.

After these preliminary remarks, we are now able to formulate the following *necessary optimality condition*: Lagrange **Multiplier Rule**

Let $D \subset \mathbb{R}^n$ be open and $f, h_1, \ldots, h_p$ continuously differentiable in D. Suppose that f has a local minimum in $x_0 \in \mathcal{F}$ subject to the constraints

$$h_j(x) = 0 \quad (j = 1, \ldots, p).$$

Let also the Jacobian $\left(\frac{\partial h_j}{\partial x_k}(x_0)\right)_{p,n}$ have rank p. Then there exist real numbers $\mu_1, \ldots, \mu_p$ — the so-called Lagrange multipliers *— with*

$$\nabla f(x_0) + \sum_{j=1}^{p} \mu_j \nabla h_j(x_0) = 0. \tag{1}$$

Corresponding to our preliminary remarks, a main tool in a *proof* would be the *Implicit Function Theorem*. We assume that interested readers are familiar with a proof from multidimensional analysis. In addition, the results will be generalized in theorem 2.2.5. Therefore we do not give a proof here, but instead illustrate the matter with the following simple problem, which was already introduced in chapter 1 (as example 5):

Example 1

With $f(x) := x_1 x_2^2$ and $h(x) := h_1(x) := x_1^2 + x_2^2 - 2$ for $x = (x_1, x_2)^T \in D := \mathbb{R}^2$ we consider the problem:

$$f(x) \longrightarrow \min \quad \text{subject to the constraint} \quad h(x) = 0\,.$$

We hence have $n = 2$ and $p = 1$.

Before we start, however, note that this problem can of course be solved very easily straight away: One inserts x_2^2 from the constraint $x_1^2 + x_2^2 - 2 = 0$ into $f(x)$ and thus gets a one-dimensional problem.

Points x meeting the constraint are different from 0 and thus also meet the rank condition. With $\mu := \mu_1$ the equation $\nabla f(x) + \mu \nabla h(x) = 0$ translates into

$$x_2^2 + \mu 2x_1 = 0 \quad \text{and} \quad 2x_1x_2 + \mu 2x_2 = 0\,.$$

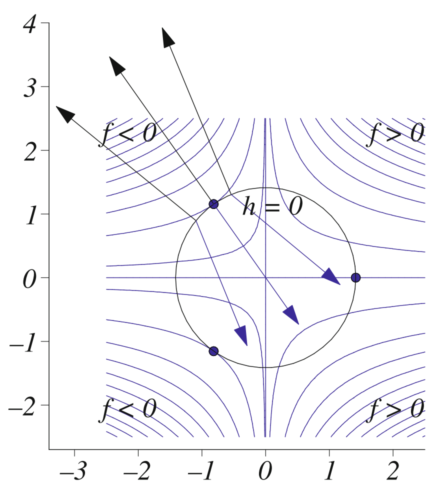
Multiplication of the first equation by x_2 and the second by x_1 gives

$$x_2^3 + 2\mu x_1 x_2 = 0 \quad \text{and} \quad 2x_1^2 x_2 + 2\mu x_1 x_2 = 0$$

and thus

$$x_2^3 = 2x_1^2 x_2\,.$$

For $x_2 = 0$ the constraint yields $x_1 = \pm\sqrt{2}$. Of these two evidently only $x_1 = \sqrt{2}$ remains as a potential minimizer. If $x_2 \neq 0$, we have $x_2^2 = 2x_1^2$ and hence with the constraint $3x_1^2 = 2$, thus $x_1 = \pm\sqrt{2/3}$ and then $x_2 = \pm 2/\sqrt{3}$. In this case the distribution of the zeros and signs of f gives that only $x = (-\sqrt{2/3}, \pm 2/\sqrt{3})^T$ remain as potential minimizers. Since f is continuous on the compact set $\{x \in \mathbb{R}^2 \mid h(x) = 0\}$, we know that there exists a global minimizer. Altogether, we get: f attains its global minimum at $(-\sqrt{2/3}, \pm 2/\sqrt{3})^T$, the point $(\sqrt{2}, 0)^T$ yields a local minimum. The following picture illustrates the gradient condition very well:



◁

The aim of our further investigations will be to generalize the Lagrange Multiplier Rule to *minimization problems with inequality constraints*:

$$(P) \qquad \begin{array}{l} f(x) \longrightarrow \min \quad \textit{subject to the constraints} \\ g_i(x) \le 0 \ \text{ for } \ i \in \mathcal{I} := \{1, \ldots, m\} \\ h_j(x) = 0 \ \text{ for } \ j \in \mathcal{E} := \{1, \ldots, p\} \end{array} .$$

With $m, p \in \mathbb{N}_0$ (hence, $\mathcal{E} = \emptyset$ or $\mathcal{I} = \emptyset$ are allowed), the functions $f, g_1, \ldots, g_m, h_1, \ldots, h_p$ are supposed to be continuously differentiable on an open subset D in $\mathbb{R}^n$ and $p \le n$. The set

$$\mathcal{F} := \big\{x \in D \mid g_i(x) \le 0 \ \text{ for } \ i \in \mathcal{I}, \ h_j(x) = 0 \ \text{ for } \ j \in \mathcal{E}\big\}$$

— in analogy to the above — is called the *feasible region* or *set of feasible points of* (P).

In most cases we state the problem in the slightly shortened form

$$(P) \quad \begin{cases} f(x) \longrightarrow \min \\ g_i(x) \le 0 \ \text{ for } \ i \in \mathcal{I} \\ h_j(x) = 0 \ \text{ for } \ j \in \mathcal{E} \end{cases} .$$

The *optimal value* $v(P)$ to problem (P) is defined as

$$v(P) := \inf \{f(x) : x \in \mathcal{F}\} .$$

We allow $v(P)$ to attain the extended values $+\infty$ and $-\infty$. We follow the standard convention that the infimum of the empty set is ∞. If there are feasible points x_k with $f(x_k) \longrightarrow -\infty \quad (k \longrightarrow \infty)$, then $v(P) = -\infty$ and we say problem (P) — or the function f on $\mathcal{F}$ — is unbounded from below.

We say x_0 is a *minimal point* or a *minimizer* if x_0 is feasible and $f(x_0) = v(P)$.

In order to formulate optimality conditions for (P), we will need some simple tools from *Convex Analysis*. These will be provided in the following section.

2.1 Convex Sets, Inequalities

In the following consider the space $\mathbb{R}^n$ for $n \in \mathbb{N}$ with the euclidean norm and let C be a nonempty subset of $\mathbb{R}^n$. The standard *inner product* or *scalar product* on $\mathbb{R}^n$ is given by $\langle x, y \rangle := x^T y = \sum_{\nu=1}^n x_\nu y_\nu$ for $x, y \in \mathbb{R}^n$. The *euclidean norm* of a vector $x \in \mathbb{R}^n$ is defined by $\|x\| := \|x\|_2 := \sqrt{\langle x, x \rangle}$.

Definition

a) C is called *convex* $:\iff \forall\, x_1, x_2 \in C \ \forall\, \lambda \in (0,1) \ \ (1-\lambda)x_1 + \lambda x_2 \in C$

b) C is called a *cone* (with *apex* 0) $:\iff \forall\, x \in C \ \forall\, \lambda > 0 \ \ \lambda x \in C$

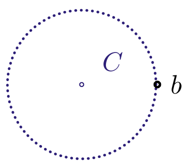
Remark

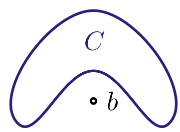
C is a *convex cone* if and only if:
$\forall\, x_1, x_2 \in C\ \forall\, \lambda_1, \lambda_2 > 0\ \ \lambda_1 x_1 + \lambda_2 x_2 \in C$

Proposition 2.1.1 (Separating Hyperplane Theorem)

Let C be closed and convex, and $b \in \mathbb{R}^n \setminus C$. Then there exist $p \in \mathbb{R}^n \setminus \{0\}$ and $\alpha \in \mathbb{R}$ such that $\langle p, x\rangle \geq \alpha > \langle p, b\rangle$ for all $x \in C$, that is, the hyperplane defined by $H := \{x \in \mathbb{R}^n \mid \langle p, x\rangle = \alpha\}$ strictly separates C and b. If furthermore C is a cone, we can choose $\alpha = 0$.

The following two little pictures show that none of the two assumptions that C is *convex* and *closed* can be dropped. The set C on the left is convex but not closed; on the right it is closed but not convex.





Proof: Since C is closed,

$$\delta := \delta(b, C) = \inf\left\{\|x - b\| : x \in C\right\}$$

is positive, and there exists a sequence (x_k) in C such that $\|x_k - b\| \longrightarrow \delta$. WLOG let $x_k \to q$ for a $q \in \mathbb{R}^n$ (otherwise use a suitable subsequence). Then q is in C with $\|p\| = \delta > 0$ for $p := q - b$.

For $x \in C$ and $0 < \tau < 1$ it holds that

$$\begin{aligned} \|p\|^2 = \delta^2 &\leq \|(1-\tau)q + \tau x - b\|^2 = \|q - b + \tau(x - q)\|^2 \\ &= \|p\|^2 + 2\tau\langle x - q, p\rangle + \tau^2\|x - q\|^2. \end{aligned}$$

From this we obtain

$$0 \leq 2\langle x - q, p\rangle + \tau\|x - q\|^2$$

and after passage to the limit $\tau \to 0$

$$0 \leq \langle x - q, p\rangle .$$

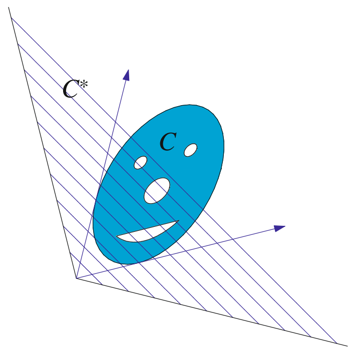
With $\alpha := \delta^2 + \langle b, p\rangle$ the first assertion $\langle p, x\rangle \geq \alpha > \langle p, b\rangle$ follows. If C is a *cone*, then for all $\lambda > 0$ and $x \in C$ the vectors $\frac{1}{\lambda}x$ and λx are also in C.

Therefore $\langle p, x\rangle = \lambda \left\langle p, \frac{1}{\lambda}x\right\rangle \ge \lambda\alpha$ holds and consequently $\langle p, x\rangle \ge 0$. $\lambda \langle p, x\rangle = \langle p, \lambda x\rangle \ge \alpha$ shows $0 \ge \alpha$, hence, $\langle p, b\rangle < \alpha \le 0$. $\square$

Definition

$$C^* := \Big\{y \in \mathbb{R}^n \mid \forall\, x \in C \;\; \langle y, x\rangle \ge 0\Big\}$$

is called the *dual cone* of C.



Remark C^* *is a closed, convex cone.*

We omit a *proof.* The statement is an immediate consequence of the definition of the dual cone.

As an *important application* let us now consider the following situation: Let $A = (a_1, \ldots, a_n) \in \mathbb{R}^{m\times n}$ be an (m, n)-matrix with columns $a_1, \ldots, a_n \in \mathbb{R}^m$.

Definition

$$\operatorname{cone}(A) := \operatorname{cone}(a_1, \ldots, a_n) := A\mathbb{R}^n_+ = \{Aw \mid w \in \mathbb{R}^n_+\}$$

is called the (positive) *conic hull* of $a_1, \ldots, a_n$.

Lemma 2.1.2

1) $\operatorname{cone}(A)$ *is a* closed, *convex cone.*

2) $\big(\operatorname{cone}(A)\big)^* = \big\{y \in \mathbb{R}^m \mid A^T y \ge 0\big\}$

Proof:

1) It is obvious that $C_n := \operatorname{cone}(a_1, \ldots, a_n)$ is a convex cone. We will prove that it is *closed* by means of induction over n:

For $n = 1$ the cone $C_1 = \{\xi_1 a_1 \mid \xi_1 \ge 0\}$ is — in the nontrivial case — a closed half line. For the induction step from n to $n+1$ we assume that every conic hull generated by not more than n vectors is closed.

Firstly, consider the case that

$$-a_j \in \operatorname{cone}(a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_{n+1}) \quad \text{for all } j = 1, \ldots, n+1 \,.$$

It follows that $C_{n+1} = \operatorname{span}\{a_1, \ldots, a_{n+1}\}$ and therefore obviously that C_{n+1} is closed:

The inclusion from left to right is trivial, and the other one follows, with $\xi_1, \ldots, \xi_{n+1} \in \mathbb{R}$ from

$$\sum_{j=1}^{n+1} \xi_j \, a_j \; = \; \sum_{j=1}^{n+1} |\xi_j| \operatorname{sign}(\xi_j) \, a_j \;.$$

Otherwise, assume WLOG $-a_{n+1} \notin \operatorname{cone}(a_1, \ldots, a_n) = C_n$; because of the induction hypothesis, C_n is closed and therefore $\delta := \delta(-a_{n+1}, C_n)$ is positive. Every $x \in C_{n+1}$ can be written in the form $x = \sum_{j=1}^{n+1} \xi_j \, a_j$ with $\xi_1, \ldots, \xi_{n+1} \in \mathbb{R}_+$. Then

$$\xi_{n+1} \leq \frac{\|x\|}{\delta}$$

holds because in the nontrivial case $\xi_{n+1} > 0$ this follows directly from

$$\|x\| \; = \; \xi_{n+1} \left\| \, - a_{n+1} - \underbrace{\sum_{j=1}^{n} \frac{\xi_j}{\xi_{n+1}} \, a_j}_{\in C_n} \right\| \geq \xi_{n+1} \, \delta \,.$$

Let $(x^{(k)})$ be a sequence in C_{n+1} and $x \in \mathbb{R}^m$ with $x^{(k)} \longrightarrow x$ for $k \to \infty$. We want to show $x \in C_{n+1}$: For $k \in \mathbb{N}$ there exist $\xi_1^{(k)}, \ldots, \xi_{n+1}^{(k)} \in \mathbb{R}_+$ such that

$$x^{(k)} \; = \; \sum_{j=1}^{n+1} \xi_j^{(k)} \, a_j \,.$$

As $(x^{(k)})$ is a convergent sequence, there exists an $M > 0$ such that $\|x^{(k)}\| \leq M$ for all $k \in \mathbb{N}$, and we get

$$0 \; \leq \; \xi_{n+1}^{(k)} \; \leq \; \frac{M}{\delta} \,.$$

WLOG let the sequence $\bigl(\xi_{n+1}^{(k)}\bigr)$ be convergent (otherwise, consider a suitable subsequence), and set $\xi_{n+1} := \lim \xi_{n+1}^{(k)}$. So we have

$$C_n \ni x^{(k)} - \xi_{n+1}^{(k)} \, a_{n+1} \longrightarrow x - \xi_{n+1} \, a_{n+1} \,.$$

By induction, C_n is closed, thus $x - \xi_{n+1} a_{n+1}$ is an element of C_n and consequently x is in C_{n+1}.

2) The definitions of cone(A) and of the dual cone give immediately:

$$\begin{aligned}\big(\operatorname{cone}(A)\big)^* &= \big\{y \in \mathbb{R}^m \mid \forall\, v \in \operatorname{cone}(A) \;\; \langle v, y\rangle \ge 0\big\}\\ &= \big\{y \in \mathbb{R}^m \mid \forall\, w \in \mathbb{R}^n_+ \;\; \langle Aw, y\rangle \ge 0\big\}\\ &= \big\{y \in \mathbb{R}^m \mid \forall\, w \in \mathbb{R}^n_+ \;\; \big\langle w, A^T y\big\rangle \ge 0\big\}\\ &\stackrel{\checkmark}{=} \big\{y \in \mathbb{R}^m \mid A^T y \ge 0\big\}\end{aligned}$$ □

A crucial tool for the following considerations is the

Theorem of the Alternative (Farkas (1902))

For $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ the following are strong alternatives:

1) $\exists\, x \in \mathbb{R}^n_+ \;\; Ax = b$

2) $\exists\, y \in \mathbb{R}^m \;\; A^T y \ge 0 \;\wedge\; b^T y < 0$

Proof: *1)* $\Longrightarrow \neg$ *2)*: For $x \in \mathbb{R}^n_+$ with $Ax = b$ and $y \in \mathbb{R}^m$ with $A^T y \ge 0$ we have $b^T y = x^T A^T y \ge 0$.

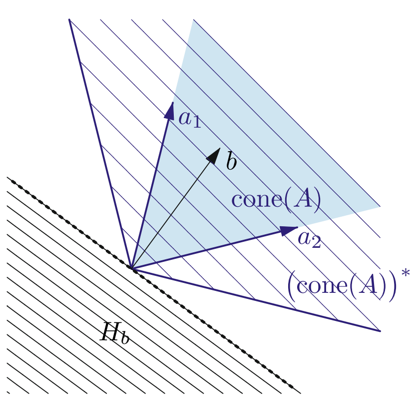
$\neg$ *1)* $\Longleftarrow$ *2)*: $C := \operatorname{cone}(A)$ is a closed convex cone which does not contain the vector b: Following the addendum in the Separating Hyperplane Theorem there exists a $y \in \mathbb{R}^m$ with $\langle y, x\rangle \ge 0 > \langle y, b\rangle$ for all $x \in C$, in particular $a_\nu^T y = \langle y, a_\nu\rangle \ge 0$, that is, $A^T y \ge 0$. □

If we *illustrate* the assertion, the theorem can be memorized easily: *1)* means nothing but $b \in \operatorname{cone}(A)$. With the open 'half space'

$$H_b := \big\{y \in \mathbb{R}^m \mid \langle y, b\rangle < 0\big\}$$

the condition *2)* states that $\big(\operatorname{cone}(A)\big)^*$ and H_b have a common point.

In the two-dimensional case, for example, we can illustrate the theorem with the following picture, which shows case *1)*:



If you rotate the vector b out of cone(A), you get case *2)*.

Chapter 2

2.2 Local First-Order Optimality Conditions

We want to take up the minimization problem (P) from page 39 again and use the notation introduced there. For $x_0 \in \mathcal{F}$, the index set

$$\mathcal{A}(x_0) := \left\{i \in \mathcal{I} \mid g_i(x_0) = 0\right\}$$

describes the *inequality restrictions which are active at* x_0.

The active constraints have a special significance: They restrict feasible corrections around a feasible point. If a constraint is *inactive* ($g_i(x_0) < 0$) at the feasible point x_0, it is possible to move from x_0 a bit in any direction without violating this constraint.

Definition

Let $d \in \mathbb{R}^n$ and $x_0 \in \mathcal{F}$. Then d is called the *feasible direction of* $\mathcal{F}$ *at* x_0 $:\Longleftrightarrow\ \exists\, \delta > 0\ \forall\, \tau \in [0,\, \delta]\ \ x_0 + \tau\, d \in \mathcal{F}$.

A 'small' movement from x_0 along such a direction gives feasible points.

The set of all feasible directions of $\mathcal{F}$ at x_0 is a *cone*, denoted by

$$\mathcal{C}_{fd}(x_0)\,.$$

Let d be a feasible direction of $\mathcal{F}$ at x_0. If we choose a δ according to the definition, then we have

$$\underbrace{g_i(x_0 + \tau\, d)}_{\le 0} \;=\; \underbrace{g_i(x_0)}_{=0} + \tau\, g_i'(x_0)\, d \,+\, \mathrm{o}(\tau)$$

for $i \in \mathcal{A}(x_0)$ and $0 < \tau \le \delta$. Dividing by τ and passing to the limit as $\tau \to 0$ gives $g_i'(x_0)\, d \le 0$. In the same way we get $h_j'(x_0)\, d = 0$ for all $j \in \mathcal{E}$.

Definition

For any $x_0 \in \mathcal{F}$

$$\mathcal{C}_\ell(P, x_0) := \left\{d \in \mathbb{R}^n \mid \forall\, i \in \mathcal{A}(x_0)\ g_i'(x_0)\, d \le 0,\ \forall\, j \in \mathcal{E}\ h_j'(x_0)\, d = 0\right\}$$

is called the *linearizing cone* of (P) at x_0. Hence, $\mathcal{C}_\ell(x_0) := \mathcal{C}_\ell(P, x_0)$ *contains at least all feasible directions of* $\mathcal{F}$ *at* x_0:

$$\mathcal{C}_{fd}(x_0) \subset \mathcal{C}_\ell(x_0)$$

The linearizing cone is not only dependent on the *set* of feasible points $\mathcal{F}$ but also on the *representation* of $\mathcal{F}$ (compare Example 4). We therefore write more precisely $\mathcal{C}_\ell(P, x_0)$.

Definition

For any $x_0 \in D$

$$\mathcal{C}_{dd}(x_0) := \left\{ d \in \mathbb{R}^n \mid f'(x_0)\, d < 0 \right\}$$

is called the *cone of descent directions of f at x_0*.

Note that 0 is not in $\mathcal{C}_{dd}(x_0)$; also, for all $d \in \mathcal{C}_{dd}(x_0)$

$$f(x_0 + \tau d) = f(x_0) + \tau \underbrace{f'(x_0)\, d}_{<0} + o(\tau)$$

holds and therefore, $f(x_0 + \tau d) < f(x_0)$ for sufficiently small $\tau > 0$.

Thus, $d \in \mathcal{C}_{dd}(x_0)$ guarantees that the objective function f can be reduced along this direction. Hence, for a local minimizer x_0 of (P) it necessarily holds that $\mathcal{C}_{dd}(x_0) \cap \mathcal{C}_{fd}(x_0) = \emptyset$.

We will illustrate the above definitions with the following

Example 1

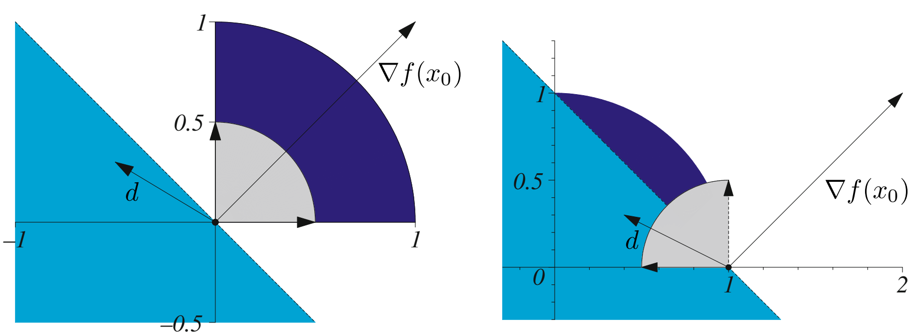
Let

$$\mathcal{F} := \left\{ x = (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1^2 + x_2^2 - 1 \le 0,\ -x_1 \le 0,\ -x_2 \le 0 \right\},$$

and f be defined by $f(x) := x_1 + x_2$. Hence, $\mathcal{F}$ is the part of the unit disk which lies in the first quadrant. The objective function f evidently attains a (strict, global) minimum at $(0,0)^T$.

In both of the following pictures $\mathcal{F}$ is colored in dark blue.

$x_0 := (0,0)^T$ $\qquad\qquad$ $x_0 := (1,0)^T$



a) Let $x_0 := (0,0)^T$. $g_1(x) := x_1^2 + x_2^2 - 1$, $g_2(x) := -x_1$ and $g_3(x) := -x_2$ give $\mathcal{A}(x_0) = \{2,3\}$. A vector $d := (d_1, d_2)^T \in \mathbb{R}^2$ is a *feasible direction* of $\mathcal{F}$ at x_0 if and only if $d_1 \geq 0$ and $d_2 \geq 0$ hold. Hence, the set $\mathcal{C}_{fd}(x_0)$ of feasible directions is a convex cone, namely, the first quadrant, and it is represented in the left picture by the gray angular domain. $g_2'(x_0) = (-1, 0)$ and $g_3'(x_0) = (0, -1)$ produce

$$\mathcal{C}_\ell(x_0) = \left\{d \in \mathbb{R}^2 \mid -d_1 \leq 0,\, -d_2 \leq 0\right\}.$$

Hence, in this example, the *linearizing cone* and the *cone of feasible directions* are the same. Moreover, the *cone of descent directions* $\mathcal{C}_{dd}(x_0)$ — colored in light blue in the picture — is, because of $f'(x_0)d = (1,1)d = d_1 + d_2$, an open half space and disjoint to $\mathcal{C}_\ell(x_0)$.

b) If $x_0 := (1, 0)^T$, we have $\mathcal{A}(x_0) = \{1, 3\}$ and $d := (d_1, d_2)^T \in \mathbb{R}^2$ is a *feasible direction* of $\mathcal{F}$ at x_0 if and only if $d = (0, 0)^T$ or $d_1 < 0$ and $d_2 \geq 0$ hold. The set of feasible directions is again a convex cone. In the right picture it is depicted by the shifted gray angular domain. Because of $g_1'(x_0) = (2, 0)$ and $g_3'(x_0) = (0, -1)$, we get

$$\mathcal{C}_\ell(x_0) = \left\{d \in \mathbb{R}^2 \mid d_1 \leq 0,\, d_2 \geq 0\right\}.$$

As we can see, in this case the *linearizing cone* includes the cone of feasible directions properly as a subset. In the picture the *cone of descent directions* has also been moved to x_0. We can see that it contains feasible directions of $\mathcal{F}$ at x_0. Consequently, f does *not* have a local minimum in x_0. ◁

Proposition 2.2.1

For $x_0 \in \mathcal{F}$ it holds that $\mathcal{C}_\ell(x_0) \cap \mathcal{C}_{dd}(x_0) = \emptyset$ if and only if there exist $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p$ such that

$$\nabla f(x_0) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x_0) + \sum_{j=1}^{p} \mu_j \nabla h_j(x_0) = 0 \tag{2}$$

and

$$\lambda_i\, g_i(x_0) = 0 \text{ for all } i \in \mathcal{I}. \tag{3}$$

Together, these conditions — $x_0 \in \mathcal{F}$, $\lambda \geq 0$, (2) and (3) — are called Karush–Kuhn–Tucker *conditions*, or *KKT conditions*. (3) is called the *complementary slackness condition* or *complementarity condition*. This condition of course means $\lambda_i = 0$ or (in the nonexclusive sense) $g_i(x_0) = 0$ for all

$i \in \mathcal{I}$. A corresponding pair (λ, μ) or the scalars $\lambda_1, \dots, \lambda_m, \mu_1, \dots, \mu_p$ are called LAGRANGE *multipliers.* The function L defined by

$$L(x, \lambda, \mu) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{j=1}^{p} \mu_j h_j(x) = f(x) + \lambda^T g(x) + \mu^T h(x)$$

for $x \in D$, $\lambda \in \mathbb{R}_+^m$ and $\mu \in \mathbb{R}^p$ is called the LAGRANGE *function* or *Lagrangian* of (P). Here we have combined the m functions g_i to a vector-valued function g and respectively the p functions h_j to a vector-valued function h.

Points $x_0 \in \mathcal{F}$ fulfilling (2) and (3) with a suitable $\lambda \in \mathbb{R}_+^m$ and $\mu \in \mathbb{R}^p$ play an important role. They are called KARUSH–KUHN–TUCKER *points*, or *KKT points.*

Owing to the complementarity condition (3), the multipliers λ_i corresponding to *inactive restrictions* at x_0 must be zero. So we can omit the terms for $i \in \mathcal{I} \setminus \mathcal{A}(x_0)$ from (2) and rewrite this condition as

$$\nabla f(x_0) + \sum_{i \in \mathcal{A}(x_0)} \lambda_i \nabla g_i(x_0) + \sum_{j=1}^{p} \mu_j \nabla h_j(x_0) = 0 . \qquad (2')$$

Proof: By definition of $\mathcal{C}_\ell(x_0)$ and $\mathcal{C}_{dd}(x_0)$ it holds that:

$$d \in \mathcal{C}_\ell(x_0) \cap \mathcal{C}_{dd}(x_0) \iff \begin{cases} f'(x_0)\,d < 0 \\ \forall\, i \in \mathcal{A}(x_0) \;\; g_i'(x_0)\,d \le 0 \\ \forall\, j \in \mathcal{E} \;\; h_j'(x_0)\,d = 0 \end{cases}$$

$$\iff \begin{cases} f'(x_0)\,d < 0 \\ \forall\, i \in \mathcal{A}(x_0) \;\; -g_i'(x_0)\,d \ge 0 \\ \forall\, j \in \mathcal{E} \;\; -h_j'(x_0)\,d \ge 0 \\ \forall\, j \in \mathcal{E} \;\; h_j'(x_0)\,d \ge 0 \end{cases}$$

With that the Theorem of the Alternative from section 2.1 directly provides the following equivalence:

$\mathcal{C}_\ell(x_0) \cap \mathcal{C}_{dd}(x_0) = \emptyset$ if and only if there exist $\lambda_i \ge 0$ for $i \in \mathcal{A}(x_0)$ and $\mu_j' \ge 0$, $\mu_j'' \ge 0$ for $j \in \mathcal{E}$ such that

$$\nabla f(x_0) = \sum_{i \in \mathcal{A}(x_0)} \lambda_i (-\nabla g_i(x_0)) + \sum_{j=1}^{p} \mu_j' (-\nabla h_j(x_0)) + \sum_{j=1}^{p} \mu_j'' \nabla h_j(x_0).$$

If we now set $\lambda_i := 0$ for $i \in \mathcal{I} \setminus \mathcal{A}(x_0)$ and $\mu_j := \mu_j' - \mu_j''$ for $j \in \mathcal{E}$, the above is equivalent to: There exist $\lambda_i \ge 0$ for $i \in \mathcal{I}$ and $\mu_j \in \mathbb{R}$ for $j \in \mathcal{E}$ with

$$\nabla f(x_0) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x_0) + \sum_{j=1}^{p} \mu_j \nabla h_j(x_0) = 0$$

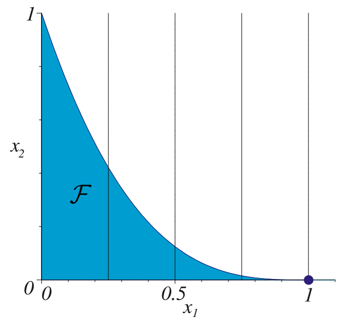
and

$$\lambda_i \, g_i(x_0) \;=\; 0 \;\; \text{for all} \;\; i \in \mathcal{I}\,. \qquad \square$$

So now the *question* arises whether not just $\mathcal{C}_{fd}(x_0) \cap \mathcal{C}_{dd}(x_0) = \emptyset$, but even $\mathcal{C}_\ell(x_0) \cap \mathcal{C}_{dd}(x_0) = \emptyset$ is true for any local minimizer $x_0 \in \mathcal{F}$. The following simple example gives a negative answer to this question:

Example 2 (Kuhn–Tucker (1951))

For $n = 2$ and $x = (x_1, x_2)^T \in \mathbb{R}^2 =: D$ let
$f(x) := -x_1\,,\; g_1(x) := x_2 + (x_1 - 1)^3\,,\; g_2(x) := -x_1$ and $g_3(x) := -x_2\,.$
For $x_0 := (1, 0)^T$, $m = 3$ and $p = 0$ we have:
$\nabla f(x_0) = (-1, 0)^T$, $\nabla g_1(x_0) = (0, 1)^T$, $\nabla g_2(x_0) = (-1, 0)^T$ and $\nabla g_3(x_0) = (0, -1)^T$.
Since $\mathcal{A}(x_0) = \{1, 3\}$, we get $\mathcal{C}_\ell(x_0) = \big\{(d_1, d_2)^T \in \mathbb{R}^2 \mid d_2 = 0\big\}$, as well as $\mathcal{C}_{dd}(x_0) = \big\{(d_1, d_2)^T \in \mathbb{R}^2 \mid d_1 > 0\big\}$; evidently, $\mathcal{C}_\ell(x_0) \cap \mathcal{C}_{dd}(x_0)$ is nonempty. However, the function f has a minimum at x_0 subject to the given constraints.



$\triangleleft$

Lemma 2.2.2

For $x_0 \in \mathcal{F}$ it holds that: $\mathcal{C}_\ell(x_0) \cap \mathcal{C}_{dd}(x_0) = \emptyset \iff \nabla f(x_0) \in \mathcal{C}_\ell(x_0)^*$

Proof:

$$\begin{aligned} \mathcal{C}_\ell(x_0) \cap \mathcal{C}_{dd}(x_0) = \emptyset &\iff \forall\, d \in \mathcal{C}_\ell(x_0) \;\; \langle \nabla f(x_0), d \rangle \;=\; f'(x_0)\, d \geq 0 \\ &\iff \nabla f(x_0) \in \mathcal{C}_\ell(x_0)^* \end{aligned} \qquad \square$$

The cone $\mathcal{C}_{fd}(x_0)$ of all feasible directions is too small to ensure general optimality conditions. Difficulties may occur due to the fact that *the boundary of $\mathcal{F}$ is curved.* Therefore, we have to consider a set which is less intuitive but bigger and with more suitable properties. To attain this goal, it is useful to state the *concept of being tangent to a set* more precisely:

Definition

A sequence (x_k) *converges in direction* d to x_0

$$:\iff \;\; x_k \;=\; x_0 + \alpha_k (d + r_k) \text{ with } \alpha_k \downarrow 0 \text{ and } r_k \to 0\,.$$

We will use the following *notation:* $x_k \xrightarrow{d} x_0$

$x_k \xrightarrow{d} x_0$ simply means: There exists a sequence of positive numbers (α_k) such that $\alpha_k \downarrow 0$ and

$$\frac{1}{\alpha_k}(x_k - x_0) \longrightarrow d \text{ for } k \longrightarrow \infty\,.$$

Definition

Let M be a nonempty subset of $\mathbb{R}^n$ and $x_0 \in M$. Then

$$\mathcal{C}_t\,(M, x_0) \;:=\; \left\{d \in \mathbb{R}^n \mid \exists\,(x_k) \in M^{\mathbb{N}} \;\; x_k \xrightarrow{d} x_0\right\}$$

is called the *tangent cone* of M at x_0. The vectors of $\mathcal{C}_t\,(M, x_0)$ are called *tangents* or *tangent directions* of M at x_0.

Of main interest is the special case

$$\mathcal{C}_t(x_0) := \mathcal{C}_t\,(\mathcal{F}, x_0)\,.$$

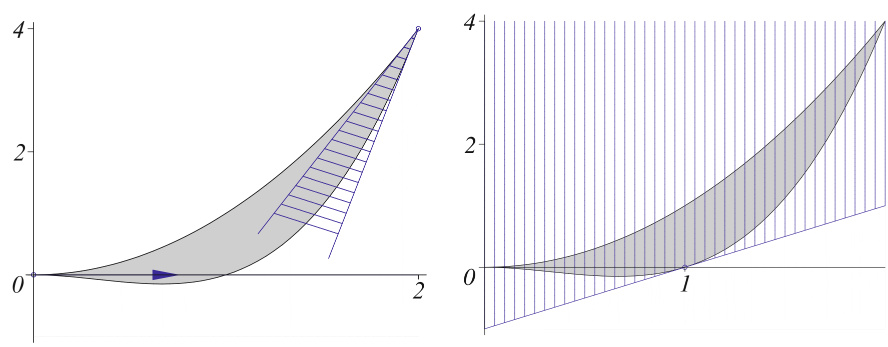
Example 3

a) The following two figures illustrate the cone of tangents for

$$\mathcal{F} \;:=\; \left\{x = (x_1, x_2)^T \in \mathbb{R}^2 \;\mid\; x_1 \geq 0,\, x_1^2 \geq x_2 \geq x_1^2\,(x_1 - 1)\right\}$$

and the points $x_0 \in \left\{(0,0)^T, (2,4)^T, (1,0)^T\right\}$. For convenience the origin is translated to x_0. The reader is invited to verify this:

$x_0 = (0,0)^T$ *and* $x_0 = (2,4)^T$ $\qquad$ $x_0 = (1,0)^T$



b) $\mathcal{F} := \left\{x \in \mathbb{R}^n \mid \|x\|_2 = 1\right\}$: $\quad \mathcal{C}_t(x_0) = \left\{d \in \mathbb{R}^n \mid \langle d, x_0\rangle = 0\right\}$

c) $\mathcal{F} := \left\{x \in \mathbb{R}^n \mid \|x\|_2 \leq 1\right\}$: Then $\mathcal{C}_t(x_0) = \mathbb{R}^n$ if $\|x_0\|_2 < 1$ holds, and $\mathcal{C}_t(x_0) = \left\{d \in \mathbb{R}^n \mid \langle d, x_0\rangle \leq 0\right\}$ if $\|x_0\|_2 = 1$.

These assertions have to be proven in exercise 10. $\triangleleft$

Chapter 2

Lemma 2.2.3

1) $\mathcal{C}_t(x_0)$ *is a closed cone,* $0 \in \mathcal{C}_t(x_0)$ *.*

2) $\overline{\mathcal{C}_{fd}(x_0)} \subset \mathcal{C}_t(x_0) \subset \mathcal{C}_\ell(x_0)$

Proof: The proof of *1)* is to be done in exercise 9.

2) First inclusion: As the tangent cone $\mathcal{C}_t(x_0)$ is closed, it is sufficient to show the inclusion $\mathcal{C}_{fd}(x_0) \subset \mathcal{C}_t(x_0)$. For $d \in \mathcal{C}_{fd}(x_0)$ and 'large' integers k it holds that $x_0 + \frac{1}{k}d \in \mathcal{F}$. With $\alpha_k := \frac{1}{k}$ and $r_k := 0$ this shows $d \in \mathcal{C}_t(x_0)$.

Second inclusion: Let $d \in \mathcal{C}_t(x_0)$ and $(x_k) \in \mathcal{F}^{\mathbb{N}}$ be a sequence with $x_k = x_0 + \alpha_k(d + r_k)$, $\alpha_k \downarrow 0$ and $r_k \to 0$. For $i \in \mathcal{A}(x_0)$

$$\underbrace{g_i(x_k)}_{\leq 0} = \underbrace{g_i(x_0)}_{=0} + \alpha_k g_i'(x_0)(d + r_k) + o(\alpha_k)$$

produces the inequality $g_i'(x_0)d \leq 0$. In the same way we get $h_j'(x_0)d = 0$ for $j \in \mathcal{E}$. □

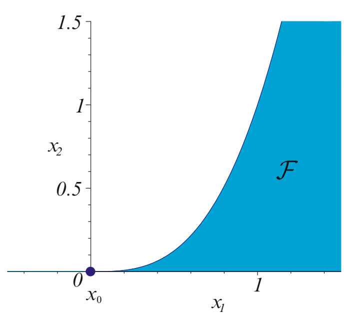
Now the *question* arises whether $\mathcal{C}_t(x_0) = \mathcal{C}_\ell(x_0)$ always holds. The following example gives a negative answer:

Example 4

a) Consider $\mathcal{F} := \left\{x \in \mathbb{R}^2 \mid -x_1^3 + x_2 \leq 0\,,\ -x_2 \leq 0\right\}$ and $x_0 := (0,0)^T$. In this case $\mathcal{A}(x_0) = \{1, 2\}$. This gives
$\mathcal{C}_\ell(x_0) = \left\{d \in \mathbb{R}^2 \mid d_2 = 0\right\}$ and $\mathcal{C}_t(x_0) = \left\{d \in \mathbb{R}^2 \mid d_1 \geq 0\,,\ d_2 = 0\right\}$.
The last statement has to be shown in exercise 10.

b) Now let $\mathcal{F} := \left\{x \in \mathbb{R}^2 \mid -x_1^3 + x_2 \leq 0\,,\ -x_1 \leq 0\,,\ -x_2 \leq 0\right\}$ and $x_0 := (0,0)^T$. Then $\mathcal{A}(x_0) = \{1, 2, 3\}$ and therefore
$\mathcal{C}_\ell(x_0) = \left\{d \in \mathbb{R}^2 \mid d_1 \geq 0\,,\ d_2 = 0\right\} = \mathcal{C}_t(x_0)$.

Hence, *the linearizing cone is dependent on the representation of the set of feasible points* $\mathcal{F}$ which is the same in both cases!



◁

Lemma 2.2.4

For a local minimizer x_0 *of* (P) *it holds that* $\nabla f(x_0) \in \mathcal{C}_t(x_0)^*$, *hence* $\mathcal{C}_{dd}(x_0) \cap \mathcal{C}_t(x_0) = \emptyset$.

Geometrically this condition states that for a local minimizer x_0 of (P) the angle between the gradient and any tangent direction, especially any feasible direction, does not exceed $90°$.

Proof: Let $d \in \mathcal{C}_t(x_0)$. Then there exists a sequence $(x_k) \in \mathcal{F}^{\mathbb{N}}$ such that $x_k = x_0 + \alpha_k(d + r_k)$, $\alpha_k \downarrow 0$ and $r_k \longrightarrow 0$.

$$0 \le f(x_k) - f(x_0) = \alpha_k f'(x_0)(d + r_k) + \mathrm{o}(\alpha_k)$$

gives the result $f'(x_0)d \ge 0$. □

The principal result in this section is the following:

Theorem 2.2.5 (Karush–Kuhn–Tucker)

Suppose that x_0 *is a local minimizer of* (P), *and the constraint qualification*[1] $\mathcal{C}_\ell(x_0)^* = \mathcal{C}_t(x_0)^*$ *is fulfilled. Then there exist vectors* $\lambda \in \mathbb{R}^m_+$ *and* $\mu \in \mathbb{R}^p$ *such that*

$$\nabla f(x_0) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x_0) + \sum_{j=1}^{p} \mu_j \nabla h_j(x_0) = 0 \quad \textit{and}$$
$$\lambda_i \, g_i(x_0) = 0 \ \ \textit{for} \ i = 1, \ldots, m\,.$$

Proof: If x_0 is a local minimizer of (P), it follows from lemma 2.2.4 with the help of the presupposed constraint qualification that

$$\nabla f(x_0) \in \mathcal{C}_t(x_0)^* = \mathcal{C}_\ell(x_0)^*;$$

lemma 2.2.2 yields $\mathcal{C}_\ell(x_0) \cap \mathcal{C}_{dd}(x_0) = \emptyset$ and the latter together with proposition 2.2.1 gives the result. □

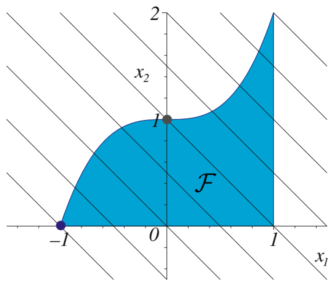
In the presence of the presupposed constraint qualification $\mathcal{C}_t(x_0)^* = \mathcal{C}_\ell(x_0)^*$ the condition $\nabla f(x_0) \in \mathcal{C}_t(x_0)^*$ of lemma 2.2.4 transforms to $\nabla f(x_0) \in \mathcal{C}_\ell(x_0)^*$. This claim can be confirmed with the aid of a simple linear optimization problem:

Example 5 (Kleinmichel (1975))

For $x = (x_1, x_2)^T \in \mathbb{R}^2$ we consider the problem

$$f(x) := x_1 + x_2 \longrightarrow \min$$
$$-x_1^3 + x_2 \le 1$$
$$x_1 \le 1\,,\ -x_2 \le 0$$

[1] Guignard (1969)



and ask whether the feasible points $x_0 := (-1, 0)^T$ and $\widetilde{x_0} := (0, 1)^T$ are local minimizers. (The examination of the picture shows immediately that this is not the case for $\widetilde{x_0}$, and that the objective function f attains a (strict, global) minimum at x_0. But we try to forget this for a while.) We have $\mathcal{A}(x_0) = \{1, 3\}$. In order to show that $\nabla f(x_0) \in \mathcal{C}_\ell(x_0)^*$, hence, $f'(x_0)d \geq 0$ for all $d \in \mathcal{C}_\ell(x_0)$, we compute $\min\limits_{d \in \mathcal{C}_\ell(x_0)} f'(x_0)d$. So we have the following linear problem:

$$\begin{aligned} d_1 + d_2 &\longrightarrow \min \\ -3d_1 + d_2 &\leq 0 \\ -d_2 &\leq 0 \end{aligned}$$

Evidently it has the minimal value 0; lemma 2.2.2 gives that $\mathcal{C}_\ell(x_0) \cap \mathcal{C}_{dd}(x_0)$ is empty. Following proposition 2.2.1 there exist $\lambda_1, \lambda_3 \geq 0$ for x_0 satisfying

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_1 \begin{pmatrix} -3 \\ 1 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

The above yields $\lambda_1 = \frac{1}{3}$, $\lambda_3 = \frac{4}{3}$.

For $\widetilde{x_0}$ we have $\mathcal{A}(\widetilde{x_0}) = \{1\}$. In the same way as the above this leads to the subproblem

$$\begin{aligned} d_1 + d_2 &\longrightarrow \min \\ d_2 &\leq 0 \end{aligned}$$

whose objective function is unbounded; therefore $\mathcal{C}_\ell(\widetilde{x_0}) \cap \mathcal{C}_{dd}(\widetilde{x_0}) \neq \emptyset$.

So $\widetilde{x_0}$ is not a local minimizer, but the point x_0 remains as a candidate. $\triangleleft$

Convex Functions

Convexity plays a central role in optimization. We already had some simple results from Convex Analysis in section 2.1. Convex optimization problems — the functions f and g_i are supposed to be convex and the funcions h_j affinely linear — are by far easier to solve than general nonlinear problems. These assumptions ensure that the problems are well-behaved. They have two significant properties: *A local minimizer is always a global one. The KKT conditions are sufficient for optimality.* A special feature of *strictly* convex functions is that they have at most one minimal point. But convex functions also play an important role in problems that are not convex. Therefore a simple and short treatment of convex functions is given here:

Definition

Let $D \subset \mathbb{R}^n$ be nonempty and convex. A real-valued function f defined on at least D is called *convex* on D if and only if

$$f\big((1-\tau)x + \tau y\big) \le (1-\tau) f(x) + \tau f(y)$$

holds for all $x, y \in D$ and $\tau \in (0,1)$. f is called *strictly convex* on D if and only if

$$f\big((1-\tau)x + \tau y\big) < (1-\tau) f(x) + \tau f(y)$$

for all $x, y \in D$ with $x \neq y$ and $\tau \in (0,1)$. The addition "on D" will be omitted, if D is the domain of definition. We say f is *concave* (on D) iff $-f$ is convex, and *strictly concave* (on D) iff $-f$ is strictly convex.

For a concave function the line segment joining two points on the graph is never above the graph.

Let $D \subset \mathbb{R}^n$ be nonempty and convex and $f\colon D \longrightarrow \mathbb{R}$ a *convex* function.

Properties

1) If f attains a local minimum at a point $x^* \in D$, then $f(x^*)$ is the global minimum.

2) f is continuous in $\overset{\circ}{D}$.

3) The function φ defined by $\varphi(\tau) := \frac{f(x+\tau h) - f(x)}{\tau}$ for $x \in \overset{\circ}{D}$, $h \in \mathbb{R}^n$ and sufficiently small, positive τ is isotone, that is, order-preserving.

4) For D open and a differentiable f it holds that $f(y) - f(x) \ge f'(x)(y-x)$ for all $x, y \in D$.

With the function f defined by $f(x) := 0$ for $x \in [0,1)$ and $f(1) := 1$ we can see that assertion *2)* cannot be extended to the whole of D.

Proof:

1) If there existed an $\overline{x} \in D$ such that $f(\overline{x}) < f(x^*)$, then we would have

$$f((1-\tau)x^* + \tau \overline{x}) \le (1-\tau) f(x^*) + \tau f(\overline{x}) < f(x^*)$$

for $0 < \tau \le 1$ and consequently a contradiction to the fact that f attains a local minimum at x^*.

2) For $x_0 \in \overset{\circ}{D}$ consider the function ψ defined by $\psi(h) := f(x_0 + h) - f(x_0)$ for $h \in \mathbb{R}^n$ with a sufficiently small norm $\|h\|_\infty$: It is clear that the function ψ is convex. Let $\varrho > 0$ such that for

$$K := \{h \in \mathbb{R}^n \mid \|h\|_\infty \le \varrho\}$$

it holds that $x_0+K\subset \overset{\circ}{D}$. Evidently, there exist $m\in\mathbb{N}$ and $a_1,\dots,a_m\in\mathbb{R}^n$ with $K = \operatorname{conv}(a_1,\dots,a_m)$ (convex hull). Every $h\in K$ may be represented as $h=\sum_{\mu=1}^m \gamma_\mu a_\mu$ with $\gamma_\mu\ge 0$ satisfying $\sum_{\mu=1}^m\gamma_\mu = 1$. With

$$\alpha := \begin{cases} \max\{|\psi(a_\mu)| \mid \mu=1,\dots,m\}, & \text{if positive}\\ 1 & , \text{ otherwise}\end{cases}$$

we have $\psi(h)\le\sum_{\mu=1}^m\gamma_\mu\psi(a_\mu)\le\alpha$. Now let $\varepsilon\in(0,\alpha]$. Then firstly for all $h\in\mathbb{R}^n$ with $\|h\|_\infty\le\varepsilon\varrho/\alpha$ we have

$$\psi(h) = \psi\Big(\big(1-\tfrac{\varepsilon}{\alpha}\big)0+\tfrac{\varepsilon}{\alpha}\big(\tfrac{\alpha}{\varepsilon}h\big)\Big) \le \frac{\varepsilon}{\alpha}\psi\big(\tfrac{\alpha}{\varepsilon}h\big) \le \varepsilon$$

and therefore with

$$0 = \psi(0) = \psi\Big(\frac12 h-\frac12 h\Big) \le \frac12\psi(h)+\frac12\psi(-h)$$

$\psi(h)\ge-\psi(-h)\ge-\varepsilon$, hence, all together $|\psi(h)|\le\varepsilon$.

3) Since f is convex, we have

$$\begin{aligned} f(x+\tau_0 h) &= f\big(\big(1-\tfrac{\tau_0}{\tau_1}\big)x+\tfrac{\tau_0}{\tau_1}(x+\tau_1 h)\big)\\ &\le \big(1-\tfrac{\tau_0}{\tau_1}\big)f(x)+\tfrac{\tau_0}{\tau_1}f(x+\tau_1 h)\end{aligned}$$

for $0<\tau_0<\tau_1$. Transformation leads to

$$\frac{f(x+\tau_0 h)-f(x)}{\tau_0} \le \frac{f(x+\tau_1 h)-f(x)}{\tau_1}.$$

4) This follows directly from *3)* (with $h=y-x$):

$$f'(x)h = \lim_{\tau\to 0+}\frac{f(x+\tau h)-f(x)}{\tau} \le \frac{f(x+h)-f(x)}{1} \qquad \square$$

Constraint Qualifications

The condition $\mathcal{C}_\ell(x_0)^* = \mathcal{C}_t(x_0)^*$ is very abstract, extremely general, but not easily verifiable. Therefore, for practical problems, we will try to find regularity assumptions called *constraint qualifications* (CQ) which are more specific, easily verifiable, but also somewhat restrictive.

For the moment we will consider the case that we *only* have *inequality constraints*. Hence, $\boxed{\mathcal{E}=\emptyset}$ and $\mathcal{I}=\{1,\dots,m\}$ with an $m\in\mathbb{N}_0$. Linear constraints pose fewer problems than nonlinear constraints. Therefore, we will assume the partition

$$\mathcal{I} = \mathcal{I}_1 \uplus \mathcal{I}_2.$$

If and only if $i \in \mathcal{I}_2$ let $g_i(x) = a_i^T x - b_i$ with suitable vectors a_i and b_i, that is, g_i is 'linear', more precisely affinely linear. Corresponding to this partition, we will also split up the set of active constraints $\mathcal{A}(x_0)$ for $x_0 \in \mathcal{F}$ into

$$\mathcal{A}_j(x_0) := \mathcal{I}_j \cap \mathcal{A}(x_0) \text{ for } j = 1, 2\,.$$

We will now focus on the following *Constraint Qualifications*:

(GCQ) Guignard *Constraint Qualification*: $\mathcal{C}_\ell(x_0)^* = \mathcal{C}_t(x_0)^*$

(ACQ) Abadie *Constraint Qualification*: $\mathcal{C}_\ell(x_0) = \mathcal{C}_t(x_0)$

(MFCQ) Mangasarian–Fromovitz *Constraint Qualification*:

$$\exists\, d \in \mathbb{R}^n \begin{cases} g_i'(x_0)\,d < 0 & \text{for } i \in \mathcal{A}_1(x_0) \\ g_i'(x_0)\,d \le 0 & \text{for } i \in \mathcal{A}_2(x_0) \end{cases}$$

(SCQ) Slater *Constraint Qualification*:
The functions g_i are *convex* for all $i \in \mathcal{I}$ and
$\exists\, \tilde{x} \in \mathcal{F}$ $g_i(\tilde{x}) < 0$ for $i \in \mathcal{I}_1$.

The conditions $g_i'(x_0)\,d < 0$ and $g_i'(x_0)\,d \le 0$ each define half spaces. (MFCQ) means nothing else but that the intersection of all of these half spaces is nonempty.

We will prove (SCQ) $\Longrightarrow$ (MFCQ) $\Longrightarrow$ (ACQ) .

The constraint qualification (GCQ) introduced in theorem 2.2.5 is a trivial consequence of (ACQ).

Proof: (SCQ) $\Longrightarrow$ (MFCQ): From the properties of convex and affinely linear functions and the definition of $\mathcal{A}(x_0)$ we get:

$$\begin{aligned} g_i'(x_0)(\tilde{x} - x_0) &\le g_i(\tilde{x}) - g_i(x_0) = g_i(\tilde{x}) < 0 \quad \text{for} \quad i \in \mathcal{A}_1(x_0) \\ g_i'(x_0)(\tilde{x} - x_0) &= g_i(\tilde{x}) - g_i(x_0) = g_i(\tilde{x}) \le 0 \quad \text{for} \quad i \in \mathcal{A}_2(x_0)\,. \end{aligned}$$

(MFCQ) $\Longrightarrow$ (ACQ): Lemma 2.2.3 gives that $\mathcal{C}_t(x_0) \subset \mathcal{C}_\ell(x_0)$ and $0 \in \mathcal{C}_t(x_0)$ always hold. Therefore it remains to prove that $\mathcal{C}_\ell(x_0) \setminus \{0\} \subset \mathcal{C}_t(x_0)$. So let $d_0 \in \mathcal{C}_\ell(x_0) \setminus \{0\}$. Take d as stated in (MFCQ). Then for a sufficiently small $\lambda > 0$ we have $d_0 + \lambda d \neq 0$. Since d_0 is in $\mathcal{C}_\ell(x_0)$, it follows that

$$\begin{aligned} g_i'(x_0)(d_0 + \lambda d) < 0 \quad &\text{for} \quad i \in \mathcal{A}_1(x_0) \quad \text{and} \\ g_i'(x_0)(d_0 + \lambda d) \le 0 \quad &\text{for} \quad i \in \mathcal{A}_2(x_0)\,. \end{aligned}$$

For the moment take a fixed λ. Setting $u := \frac{d_0 + \lambda d}{\|d_0 + \lambda d\|_2}$ produces

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$$g_i(x_0+tu) = \underbrace{g_i(x_0)}_{=0} + t\,\underbrace{g_i'(x_0)\,u}_{<0} + o(t) \quad \text{for} \quad i \in \mathcal{A}_1(x_0) \quad \text{and}$$
$$g_i(x_0+tu) = \underbrace{g_i(x_0)}_{=0} + t\,\underbrace{g_i'(x_0)\,u}_{\le 0} \qquad \text{for} \quad i \in \mathcal{A}_2(x_0)\,.$$

Thus, we have $g_i(x_0+tu) \le 0$ for $i \in \mathcal{A}(x_0)$ and $t > 0$ sufficiently small. For the indices $i \in \mathcal{I} \setminus \mathcal{A}(x_0)$ this is obviously true. Hence, there exists a $t_0 > 0$ such that $x_0 + tu \in \mathcal{F}$ for $0 \le t \le t_0$. For the sequence (x_k) defined by $x_k := x_0 + \frac{t_0}{k} u$ it holds that $x_k \xrightarrow{u} x_0$. Therefore, $u \in \mathcal{C}_t(x_0)$ and consequently $d_0 + \lambda d \in \mathcal{C}_t(x_0)$. Passing to the limit as $\lambda \longrightarrow 0$ yields $d_0 \in \overline{\mathcal{C}_t(x_0)}$. Lemma 2.2.3 or respectively exercise 9 gives that $\mathcal{C}_t(x_0)$ is closed. Hence, $d_0 \in \mathcal{C}_t(x_0)$. □

Now we will consider the *general case*, where there may also occur *equality* constraints. In this context one often finds the following *linear independence constraint qualification* in the literature:

(LICQ) The vectors $\big(\nabla g_i(x_0) \mid i \in \mathcal{A}(x_0)\big)$ and $\big(\nabla h_j(x_0) \mid j \in \mathcal{E}\big)$ are linearly independent.

(LICQ) greatly reduces the number of active inequality constraints. Instead of (LICQ) we will now consider the following weaker constraint qualification which is a variant of (MFCQ), and is often cited as the Arrow–Hurwitz–Uzawa constraint qualification:

(AHUCQ) There exists a $d \in \mathbb{R}^n$ such that $\begin{cases} g_i'(x_0)\,d < 0 & \text{for } \; i \in \mathcal{A}(x_0), \\ h_j'(x_0)\,d = 0 & \text{for } \; j \in \mathcal{E}\,, \end{cases}$
and the vectors $\big(\nabla h_j(x_0) \mid j \in \mathcal{E}\big)$ are linearly independent.

We will show: (LICQ) $\Longrightarrow$ (AHUCQ) $\Longrightarrow$ (ACQ)

Proof: (LICQ) $\Longrightarrow$ (AHUCQ): (AHUCQ) follows, for example, directly from the solvability of the system of linear equations

$$\begin{aligned} g_i'(x_0)\,d &= -1 \quad \text{for} \;\; i \in \mathcal{A}(x_0), \\ h_j'(x_0)\,d &= \;\;0 \quad \text{for} \;\; j \in \mathcal{E}. \end{aligned}$$

(AHUCQ) $\Longrightarrow$ (ACQ): Lemma 2.2.3 gives that again we only have to show $d_0 \in \mathcal{C}_t(x_0)$ for all $d_0 \in \mathcal{C}_\ell(x_0) \setminus \{0\}$. Take d as stated in (AHUCQ). Then we have $d_0 + \lambda d =: w \neq 0$ for a sufficiently small $\lambda > 0$ and thus

$$\begin{aligned} g_i'(x_0)\,w &< 0 \quad \text{for} \;\; i \in \mathcal{A}(x_0) \quad \text{and} \\ h_j'(x_0)\,w &= 0 \quad \text{for} \;\; j \in \mathcal{E}. \end{aligned}$$

Denote

$$A := \big(\nabla h_1(x_0), \dots, \nabla h_p(x_0)\big) \in \mathbb{R}^{n\times p} \ .$$

For that $A^T A$ is regular because $\operatorname{rank}(A) = p$. Now consider the following system of linear equations dependent on $u \in \mathbb{R}^p$ and $t \in \mathbb{R}$:

$$\varphi_j(u,t) := h_j(x_0 + Au + tw) = 0 \quad (j = 1, \dots, p)$$

For the corresponding vector-valued function φ we have $\varphi(0,0) = 0$, and because of

$$\tfrac{\partial \varphi_j}{\partial u_i}(u,t) = h_j'(x_0 + Au + tw)\nabla h_i(x_0)\,,$$

we are able to solve $\varphi(u,t) = 0$ locally for u, that is, there exist a nullneighborhood $U_0 \subset \mathbb{R}$ and a continuously differentiable function $u\colon U_0 \longrightarrow \mathbb{R}^p$ satisfying

$$\begin{aligned} &u(0) = 0\ , \\ &h_j(\underbrace{x_0 + Au(t) + tw}_{=:\, x(t)}) = 0 \ \text{ for } \ t \in U_0 \quad (j = 1, \dots, p)\,. \end{aligned}$$

Differentiation with respect to t at $t = 0$ leads to

$$h_j'(x_0)\big(Au'(0) + w\big) = 0 \quad (j = 1, \dots, p)$$

and consequently — considering that $h_j'(x_0)w = 0$ and $A^T A$ is regular — to $u'(0) = 0$. Then for $i \in \mathcal{A}(x_0)$ it holds that

$$g_i(x(t)) = g_i(x_0) + t\, g_i'(x_0)x'(0) + \mathrm{o}(t) = t\, g_i'(x_0)\big(Au'(0) + w\big) + \mathrm{o}(t)\,.$$

With $u'(0) = 0$ we obtain

$$g_i(x(t)) = t\Big(g_i'(x_0)w + \frac{\mathrm{o}(t)}{t}\Big)$$

and the latter is negative for $t > 0$ sufficiently small.

Hence, there exists a $t_1 > 0$ with $x(t) \in \mathcal{F}$ for $0 \le t \le t_1$. From

$$x\left(\frac{t_1}{k}\right) = x_0 + \frac{t_1}{k}\Big(w + \underbrace{A\,\frac{u(t_1/k)}{t_1/k}}_{\longrightarrow 0 \ (k\to\infty)}\Big)$$

for $k \in \mathbb{N}$ we get $x\left(\frac{t_1}{k}\right) \xrightarrow{w} x_0$; this yields $w = d_0 + \lambda d \in \mathcal{C}_t(x_0)$ and also by passing to the limit as $\lambda \to 0$

$$d_0 \in \overline{\mathcal{C}_t(x_0)} = \mathcal{C}_t(x_0)\ . \qquad \square$$

Convex Optimization Problems

Firstly suppose that $C \subset \mathbb{R}^n$ is nonempty and the functions $f, g_i \colon C \longrightarrow \mathbb{R}$ are *arbitrary* for $i \in \mathcal{I}$. We consider the *general* optimization problem

$$(P) \quad \begin{cases} f(x) \longrightarrow \min \\ g_i(x) \le 0 \ \text{ for } \ i \in \mathcal{I} := \{1, \ldots, m\} \end{cases} .$$

In the following section the Lagrangian L to (P) defined by

$$L(x, \lambda) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) = f(x) + \langle \lambda, g(x) \rangle \quad \text{for} \quad x \in C \ \text{ and } \ \lambda \in \mathbb{R}^m_+$$

will play an important role. As usual we have combined the m functions g_i to a vector-valued function g.

Definition

A pair $(x^*, \lambda^*) \in C \times \mathbb{R}^m_+$ is called a *saddlepoint* of L if and only if

$$L(x^*, \lambda) \le L(x^*, \lambda^*) \le L(x, \lambda^*)$$

holds for all $x \in C$ and $\lambda \in \mathbb{R}^m_+$, that is, x^* minimizes $L(\,\cdot\,, \lambda^*)$ and λ^* maximizes $L(x^*, \,\cdot\,)$.

Lemma 2.2.6

If (x^*, λ^*) is a saddlepoint of L, then it holds that:

- *x^* is a global minimizer of (P).*
- *$L(x^*, \lambda^*) = f(x^*)$*
- *$\lambda_i^* g_i(x^*) = 0$ for all $i \in \mathcal{I}$.*

Proof: Let $x \in C$ and $\lambda \in \mathbb{R}^m_+$. From

$$0 \ge L(x^*, \lambda) - L(x^*, \lambda^*) = \langle \lambda - \lambda^*, g(x^*) \rangle \tag{4}$$

we obtain for $\lambda := 0$

$$\langle \lambda^*, g(x^*) \rangle \ge 0 . \tag{5}$$

With $\lambda := \lambda^* + e_i$ we get — also from (4) —

$$g_i(x^*) \le 0 \ \text{ for all } \ i \in \mathcal{I}, \ \text{ that is, } \ g(x^*) \le 0 . \tag{6}$$

Because of (6), it holds that $\langle \lambda^*, g(x^*) \rangle \le 0$. Together with (5) this produces

$$\langle \lambda^*, g(x^*) \rangle = 0 \ \text{ and hence, } \ \lambda_i^* g_i(x^*) = 0 \ \text{ for all } \ i \in \mathcal{I} .$$

For $x \in \mathcal{F}$ it follows that

$$f(x^*) = L(x^*, \lambda^*) \le L(x, \lambda^*) = f(x) + \langle \lambda^*, \underbrace{g(x)}_{\le 0} \rangle \le f(x) .$$

Therefore x^* is a global minimizer of (P). □

We assume now that C is open and convex and the functions $f, g_i : C \longrightarrow \mathbb{R}$ are continuously differentiable and *convex* for $i \in \mathcal{I}$. In this case we write more precisely (CP) instead of (P).

Theorem 2.2.7

If the Slater *constraint qualification holds and x^* is a minimizer of (CP), then there exists a vector $\lambda^* \in \mathbb{R}^m_+$ such that (x^*, λ^*) is a saddlepoint of L.*

Proof: Taking into account our observations from page 55, theorem 2.2.5 gives that there exists a $\lambda^* \in \mathbb{R}^m_+$ such that

$$0 = L_x(x^*, \lambda^*) \text{ and } \langle \lambda^*, g(x^*) \rangle = 0 .$$

With that we get for $x \in C$[1]

$$L(x, \lambda^*) - L(x^*, \lambda^*) \ge L_x(x^*, \lambda^*)(x - x^*) = 0$$

and

$$L(x^*, \lambda^*) - L(x^*, \lambda) = -\langle \underbrace{\lambda}_{\ge 0}, \underbrace{g(x^*)}_{\le 0} \rangle \ge 0 .$$

Hence, (x^*, λ^*) is a saddlepoint of L. □

The following example shows that the Slater constraint qualification is essential in this theorem:

Example 6

With $n = 1$ and $m = 1$ we regard the *convex problem*

$$(P) \quad \begin{cases} f(x) := -x \longrightarrow \min \\ g(x) := x^2 \le 0 \end{cases} .$$

The only feasible point is $x^* = 0$ with value $f(0) = 0$. So 0 minimizes $f(x)$ subject to $g(x) \le 0$.

$L(x, \lambda) := -x + \lambda x^2$ for $\lambda \ge 0, x \in \mathbb{R}$. There is no $\lambda^* \in [0, \infty)$ such that (x^*, λ^*) is a saddlepoint of L. ◁

The following important observation shows that neither constraint qualifications nor second-order optimality conditions, which we will deal with in the

[1] By the convexity of f and g_i the function $L(\,\cdot\,, \lambda^*)$ is convex.

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next section, are needed for a *sufficient* condition for general *convex optimization problems*:

Suppose that $f, g_i, h_j\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ are continuously differentiable functions with f and g_i *convex* and h_j (affinely) *linear* ($i \in \mathcal{I}, j \in \mathcal{E}$), and consider the following *convex optimization problem*[2]

$$(CP) \quad \begin{cases} f(x) \longrightarrow \min \\ g_i(x) \le 0 \quad \text{for} \quad i \in \mathcal{I} \\ h_j(x) = 0 \quad \text{for} \quad j \in \mathcal{E} \end{cases} .$$

We will show that *for this special kind of problem every KKT point already gives a (global) minimum*:

Theorem 2.2.8

Suppose $x_0 \in \mathcal{F}$ and there exist vectors $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p$ such that

$$\nabla f(x_0) + \sum_{i=1}^m \lambda_i \nabla g_i(x_0) + \sum_{j=1}^p \mu_j \nabla h_j(x_0) = 0 \quad \textit{and}$$
$$\lambda_i\, g_i(x_0) = 0 \;\; \textit{for}\;\; i = 1, \ldots, m,$$

then (CP) attains its global minimum at x_0.

The *Proof* of this theorem is surprisingly simple:

Taking into account *4)* on page 53, we get for $x \in \mathcal{F}$:

$$\begin{aligned} f(x) - f(x_0) &\underset{f \text{ convex}}{\ge} f'(x_0)(x - x_0) \\ &= -\sum_{i=1}^m \lambda_i\, g_i'(x_0)(x - x_0) - \sum_{j=1}^p \mu_j \underbrace{h_j'(x_0)(x - x_0)}_{= h_j(x) - h_j(x_0) = 0} \\ &\underset{g_i \text{ convex}}{\ge} -\sum_{i=1}^m \lambda_i (g_i(x) - g_i(x_0)) = -\sum_{i=1}^m \lambda_i\, g_i(x) \ge 0 \end{aligned} \qquad \square$$

The following example shows that *even if we have convex problems the KKT conditions are not necessary for minimal points:*

Example 7

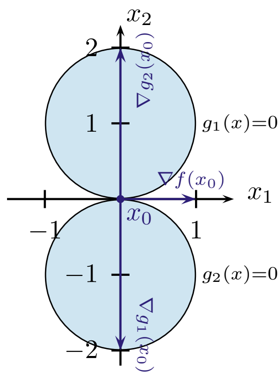
With $n = 2$, $m = 2$ and $x = (x_1, x_2)^T \in D := \mathbb{R}^2$ we consider:

---

[2] Since the functions h_j are assumed to be (affinely) linear, exercise 6 gives that this problem can be written in the form from page 58 by substituting the two inequalities $h_j(x) \le 0$ and $-h_j(x) \le 0$ for every equation $h_j(x) = 0$.

$$(P) \quad \begin{cases} f(x) := x_1 \longrightarrow \min \\ g_1(x) := x_1^2+(x_2-1)^2-1 \le 0 \\ g_2(x) := x_1^2+(x_2+1)^2-1 \le 0 \end{cases}$$

Obviously, only the point $x_0 := (0,0)^T$ is feasible. Hence, x_0 is the (global) minimal point. Since $\nabla f(x_0) = (1,0)^T$, $\nabla g_1(x_0) = (0,-2)^T$ and $\nabla g_2(x_0) = (0,2)^T$, the gradient condition of the KKT conditions is *not* met. f is *linear*, the functions g_ν are *convex*. Evidently, however, the SLATER condition is not fulfilled.



Of course, one could also argue from proposition 2.2.1: The cones

$$\mathcal{C}_{dd}(x_0) = \left\{d \in \mathbb{R}^2 \mid f'(x_0)d < 0\right\} = \left\{d \in \mathbb{R}^2 \mid d_1 < 0\right\}$$

and

$$\mathcal{C}_\ell(x_0) = \left\{d \in \mathbb{R}^2 \mid \forall\, i \in \mathcal{A}(x_0)\;\, g_i'(x_0)d \le 0\right\} = \left\{d \in \mathbb{R}^2 \mid d_2 = 0\right\}$$

are clearly not disjoint. $\triangleleft$

2.3 Local Second-Order Optimality Conditions

To get a finer characterization, it is natural to examine the effects of second-order terms near a given point too. The following second-order results take the 'curvature' of the feasible region in a neighborhood of a 'candidate' for a minimizer into account. The necessary second-order condition $s^T H s \ge 0$ and the sufficient second-order condition $s^T H s > 0$ for the Hessian H of the Lagrangian with respect to x regard only certain subsets of vectors s.

Suppose that the functions f, g_i and h_j are twice continuously differentiable.

Theorem 2.3.1 (Necessary second-order condition)

Suppose $x_0 \in \mathcal{F}$ *and there exist* $\lambda \in \mathbb{R}_+^m$ *and* $\mu \in \mathbb{R}^p$ *such that*

$$\nabla f(x_0) + \sum_{i=1}^m \lambda_i \nabla g_i(x_0) + \sum_{j=1}^p \mu_j \nabla h_j(x_0) = 0 \quad \textit{and}$$
$$\lambda_i \, g_i(x_0) = 0 \quad \textit{for all} \quad i \in \mathcal{I}.$$

If (P) *has a local minimum at* x_0, *then*

$$s^T \Big(\nabla^2 f(x_0) + \sum_{i=1}^m \lambda_i \nabla^2 g_i(x_0) + \sum_{j=1}^p \mu_j \nabla^2 h_j(x_0)\Big) s \geq 0$$

holds for all $s \in \mathcal{C}_{t+}(x_0)$, *where*

$$\mathcal{F}_+ := \mathcal{F}_+(x_0) := \big\{x \in \mathcal{F} \mid g_i(x) = 0 \ \textit{ for all } \ i \in \mathcal{A}_+(x_0)\big\} \quad \textit{with}$$
$$\mathcal{A}_+(x_0) := \big\{i \in \mathcal{A}(x_0) \mid \lambda_i > 0\big\} \quad \textit{and}$$
$$\mathcal{C}_{t+}(x_0) := \mathcal{C}_t(\mathcal{F}_+, x_0) = \big\{d \in \mathbb{R}^n \mid \exists\, (x_k) \in \mathcal{F}_+^{\mathbb{N}} \ x_k \xrightarrow{d} x_0\big\}.$$

With the help of the Lagrangian L the second and fifth lines can be written more clearly

$$\nabla_x L(x_0, \lambda, \mu) = 0 ,$$

respectively

$$s^T \nabla_{x\,x}^2 L(x_0, \lambda, \mu)\, s \geq 0 .$$

Proof: It holds that

$$\lambda_i \, g_i(x) = 0 \ \text{ for all } \ x \in \mathcal{F}_+$$

because we have $\lambda_i = 0$ for $i \in \mathcal{I} \setminus \mathcal{A}_+(x_0)$ and $g_i(x) = 0$ for $i \in \mathcal{A}_+(x_0)$, respectively.

With the function φ defined by

$$\varphi(x) := f(x) + \sum_{i=1}^m \lambda_i \, g_i(x) + \sum_{j=1}^p \mu_j \, h_j(x) = L(x, \lambda, \mu)$$

for $x \in D$ this leads to the following relation:

$$\varphi(x) = f(x) \ \text{ for } \ x \in \mathcal{F}_+ .$$

x_0 gives a local minimum of f on $\mathcal{F}$, therefore one of φ on $\mathcal{F}_+$.

Now let $s \in \mathcal{C}_{t+}(x_0)$. Then by definition of the tangent cone there exists a sequence $(x^{(k)})$ in $\mathcal{F}_+$, such that $x^{(k)} = x_0 + \alpha_k(s + r_k)$, $\alpha_k \downarrow 0$ and $r_k \to 0$.

By assumption $\nabla\varphi(x_0) = 0$. With the TAYLOR theorem we get

$$\varphi(x_0) \le \varphi\big(x^{(k)}\big) = \varphi(x_0) + \alpha_k \underbrace{\varphi'(x_0)}_{=0}(s+r_k) + \frac{1}{2}\,\alpha_k^2\,(s+r_k)^T\nabla^2\varphi\big(x_0+\tau_k(x^{(k)}-x_0)\big)\,(s+r_k)$$

for all sufficiently large k and a suitable $\tau_k \in (0,1)$.

Dividing by $\alpha_k^2/2$ and passing to the limit as $k\to\infty$ gives the result

$$s^T\,\nabla^2\varphi(x_0)\,s \ge 0. \qquad \square$$

In the following example we will see that $x_0 := (0,0,0)^T$ is a stationary point. With the help of theorem 2.3.1 we want to show that the necessary condition for a minimum is *not* met.

Example 8

$$\begin{aligned} f(x) &:= x_3 - \frac{1}{2}x_1^2 \longrightarrow \min \\ g_1(x) &:= -x_1^2 - x_2 - x_3 \le 0 \\ g_2(x) &:= -x_1^2 + x_2 - x_3 \le 0 \\ g_3(x) &:= -x_3 \le 0 \end{aligned}$$

For the point $x_0 := (0,0,0)^T$ we have $f'(x_0) = (0,0,1)$, $\mathcal{A}(x_0) = \{1,2,3\}$ and $g_1'(x_0) = (0,-1,-1)$, $g_2'(x_0) = (0,1,-1)$, $g_3'(x_0) = (0,0,-1)$.
We start with the gradient condition:

$$\nabla_x L(x_0,\lambda) = \begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda_1\begin{pmatrix}0\\-1\\-1\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\-1\end{pmatrix} + \lambda_3\begin{pmatrix}0\\0\\-1\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\iff \begin{cases} -\lambda_1+\lambda_2 = 0 \\ -\lambda_1-\lambda_2-\lambda_3 = -1 \end{cases}$$

$$\iff \lambda_2 = \lambda_1\,,\quad \lambda_3 = 1-2\lambda_1$$

For $\lambda_1 := 1/2$ we obtain $\lambda = (1/2, 1/2, 0)^T \in \mathbb{R}^3_+$ and $\lambda_i\,g_i(x_0) = 0$ for $i \in \mathcal{I}$.
Hence, we get $\mathcal{A}_+(x_0) = \{1,2\}$,

$$\mathcal{F}_+ = \big\{x \in \mathbb{R}^3 \mid g_1(x) = g_2(x) = 0\,,\ g_3(x) \le 0\big\} = \{(0,0,0)^T\}$$

and therefore $\mathcal{C}_{t+}(x_0) = \{(0,0,0)^T\}$. *In this way* no decision can be made!

Setting $\lambda_1 := 0$ we obtain respectively $\lambda = e_3$, $\mathcal{A}_+(x_0) = \{3\}$, $\mathcal{F}_+ = \{x \in \mathcal{F} \mid x_3 = 0\}$, $\mathcal{C}_{t+}(x_0) = \big\{\alpha\,e_1 \mid \alpha \in \mathbb{R}\big\}$ and

$$H := \nabla^2 f(x_0) + \nabla^2 g_3(x_0) = \begin{pmatrix}-1&0&0\\0&0&0\\0&0&0\end{pmatrix}.$$

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H is negative definite on $\mathcal{C}_{t+}(x_0)$. Consequently there is *no* local minimum of (P) at $x_0 = 0$. $\lhd$

In order to expand the second-order necessary condition to a sufficient condition, we will now have to make stronger assumptions.

Before we do that, let us recall that there will remain a 'gap' between these two conditions. This fact is well-known (even for real-valued functions of *one* variable) and is usually demonstrated by the functions f_2, f_3 and f_4 defined by

$$f_k(x) := x^k \text{ for } x \in \mathbb{R}, k = 2, 3, 4,$$

at the point $x_0 = 0$.

The following **Remark** can be proven in the same way as *2)* in lemma 2.2.3:

$$\mathcal{C}_{t+}(x_0) \subset \mathcal{C}_{\ell+}(x_0) := \left\{ s \in \mathbb{R}^n \;\middle|\; \begin{array}{l} g_i'(x_0)\,s = 0 \text{ for } i \in \mathcal{A}_+(x_0) \\ g_i'(x_0)\,s \le 0 \text{ for } i \in \mathcal{A}(x_0) \setminus \mathcal{A}_+(x_0) \\ h_j'(x_0)\,s = 0 \text{ for } j \in \mathcal{E} \end{array} \right\}$$

Theorem 2.3.2 (Sufficient second-order condition)

Suppose $x_0 \in \mathcal{F}$ and there exist vectors $\lambda \in \mathbb{R}^m_+$ and $\mu \in \mathbb{R}^p$ such that

$$\nabla_x L(x_0, \lambda, \mu) = 0 \quad \textit{and} \quad \lambda^T g(x_0) = 0\,.$$

Furthermore, suppose that

$$s^T \nabla^2_{xx} L(x_0, \lambda, \mu)\, s > 0$$

for all $s \in \mathcal{C}_{\ell+}(x_0) \setminus \{0\}$. Then (P) attains a strict local minimum at x_0.

Proof (indirect): If f does *not* have a strict local minimum at x_0, then there exists a sequence $(x^{(k)})$ in $\mathcal{F} \setminus \{x_0\}$ with $x^{(k)} \longrightarrow x_0$ and $f(x^{(k)}) \le f(x_0)$. For $s_k := \frac{x^{(k)} - x_0}{\|x^{(k)} - x_0\|_2}$ it holds that $\|s_k\|_2 = 1$. Hence, there exists a convergent subsequence. WLOG suppose $s_k \longrightarrow s$ for an $s \in \mathbb{R}^n$. With $\alpha_k := \|x^{(k)} - x_0\|_2$ we have $x^{(k)} = x_0 + \alpha_k s_k$ and WLOG $\alpha_k \downarrow 0$. From

$$f(x_0) \ge f(x^{(k)}) = f(x_0) + \alpha_k f'(x_0)\, s_k + \mathrm{o}(\alpha_k)$$

it follows that

$$f'(x_0)\, s \le 0\,.$$

For $i \in \mathcal{A}(x_0)$ and $j \in \mathcal{E}$ we get in the same way:

$$\underbrace{g_i(x^{(k)})}_{\le 0} = \underbrace{g_i(x_0)}_{=0} + \alpha_k\, g_i'(x_0)\, s_k + \mathrm{o}(\alpha_k) \implies g_i'(x_0)\, s \le 0$$
$$\underbrace{h_j(x^{(k)})}_{=0} = \underbrace{h_j(x_0)}_{=0} + \alpha_k\, h_j'(x_0)\, s_k + \mathrm{o}(\alpha_k) \implies h_j'(x_0)\, s = 0$$

With the assumption $\nabla_x L(x_0, \lambda, \mu) = 0$ it follows that

$$\underbrace{f'(x_0)\, s}_{\le 0} + \underbrace{\sum_{i=1}^{m} \lambda_i\, g_i'(x_0)\, s}_{} + \sum_{j=1}^{p} \mu_j\, \underbrace{h_j'(x_0)\, s}_{=0} = 0$$
$$= \sum_{i \in \mathcal{A}_+(x_0)} \lambda_i\, \underbrace{g_i'(x_0)\, s}_{\le 0}$$

and from that $g_i'(x_0)\, s = 0$ for all $i \in \mathcal{A}_+(x_0)$.

Since $\|s\|_2 = 1$, we get $s \in \mathcal{C}_{\ell+}(x_0) \setminus \{0\}$. For the function φ defined by

$$\varphi(x) := f(x) + \sum_{i=1}^{m} \lambda_i\, g_i(x) + \sum_{j=1}^{p} \mu_j\, h_j(x) = L(x, \lambda, \mu)$$

it holds by assumption that $\nabla\varphi(x_0) = 0$.

$$\varphi(x^{(k)}) = \underbrace{f(x^{(k)})}_{\le f(x_0)} + \sum_{i=1}^{m} \lambda_i\, \underbrace{g_i(x^{(k)})}_{\le 0} + \sum_{j=1}^{p} \mu_j\, \underbrace{h_j(x^{(k)})}_{=0} \le f(x_0) = \varphi(x_0)$$

The Taylor theorem yields

$$\varphi(x^{(k)}) = \varphi(x_0) + \alpha_k\, \underbrace{\varphi'(x_0)}_{=0}\, s_k + \frac{1}{2}\, \alpha_k^2\, s_k^T\, \nabla^2\varphi\big(x_0 + \tau_k(x^{(k)} - x_0)\big)\, s_k$$

with a suitable $\tau_k \in (0, 1)$. From this we deduce, as usual, $s^T \nabla^2\varphi(x_0)\, s \le 0$. With $s \in \mathcal{C}_{\ell+}(x_0) \setminus \{0\}$ we get a contradiction to our assumption. □

The following example gives a simple illustration of the necessary and sufficient second-order conditions of theorems 2.3.1 and 2.3.2:

Example 9 (Fiacco and McCormick (1968))

$f(x) := (x_1 - 1)^2 + x_2^2 \longrightarrow \min$

$g_1(x) := x_1 - \varrho x_2^2 \le 0$

We are looking for a $\varrho > 0$ such that $x_0 := (0, 0)^T$ is a local minimizer of the problem: With $\nabla f(x_0) = (-2, 0)^T, \nabla g_1(x_0) = (1, 0)^T$ the condition $\nabla_x L(x_0, \lambda, \mu) = 0$ firstly yields $\lambda_1 = 2$.

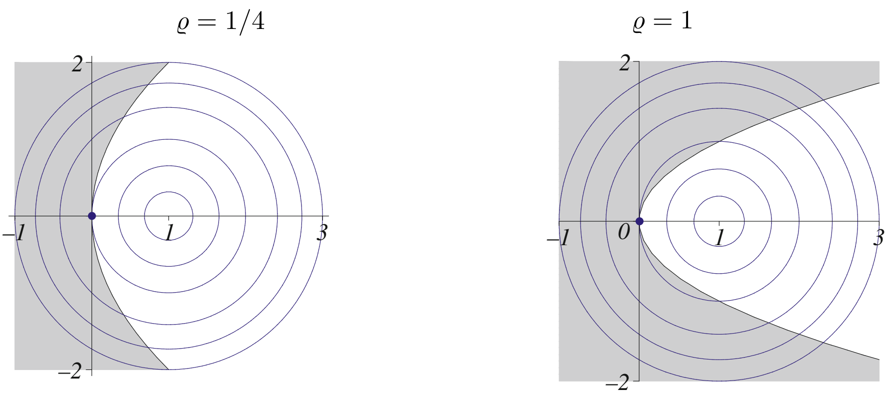
In this case (MFCQ) is fulfilled with $\mathcal{A}_1(x_0) = \mathcal{A}(x_0) = \{1\} = \mathcal{A}_+(x_0)$. We have

$$\mathcal{C}_{\ell+}(x_0) = \{d \in \mathbb{R}^2 \mid d_1 = 0\} = \mathcal{C}_{t+}(x_0)\,.$$

The matrix

$$\nabla^2 f(x_0) + 2\nabla^2 g_1(x_0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 \\ 0 & -2\varrho \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & 1-2\varrho \end{pmatrix}$$

is negative definite on $\mathcal{C}_{t+}(x_0)$ for $\varrho > 1/2$. Thus the second-order necessary condition of theorem 2.3.1 is violated and so there is no local minimum at x_0. For $\varrho < 1/2$ the Hessian is positive definite on $\mathcal{C}_{\ell+}(x_0)$. Hence, the sufficient conditions of theorem 2.3.2 are fulfilled and thus there is a strict local minimum at x_0. When $\varrho = 1/2$, this result is not determined by the second-order conditions; but we can confirm it in the following simple way: $f(x) = (x_1-1)^2 + x_2^2 = x_1^2 + 1 + (x_2^2 - 2x_1)$. Because of $x_2^2 - 2x_1 \ge 0$ this yields $f(x) \ge 1$ and $f(x) = 1$ only for $x_1 = 0$ and $x_2^2 - 2x_1 = 0$. Hence, there is a strict local minimum at x_0.



◁

2.4 Duality

Duality plays a crucial role in the theory of optimization and in the development of corresponding computational algorithms. It gives insight from a theoretical point of view but is also significant for computational purposes and economic interpretations, for example shadow prices. We shall concentrate on some of the more basic results and limit ourselves to a particular duality — Lagrange duality — which is the most popular and useful one for many purposes.

Given an arbitrary optimization problem, called *primal problem*, we consider a problem that is closely related to it, called the Lagrange *dual problem*. Several properties of this dual problem are demonstrated in this section. They help to provide strategies for solving the primal and the dual problem. The Lagrange dual problem of large classes of important nonconvex optimization problems can be formulated as an easier problem than the original one.

Lagrange Dual Problem

With $n \in \mathbb{N}$, $m, p \in \mathbb{N}_0$, $\emptyset \neq C \subset \mathbb{R}^n$, functions $f \colon C \longrightarrow \mathbb{R}$, $g = (g_1, \ldots, g_m)^T \colon C \longrightarrow \mathbb{R}^m$, $h = (h_1, \ldots, h_p)^T \colon C \longrightarrow \mathbb{R}^p$ and the feasible region

$$\mathcal{F} := \big\{x \in C \mid g(x) \le 0,\ h(x) = 0\big\}$$

we regard the *primal problem* in standard form:

$$(P) \quad \begin{cases} f(x) \longrightarrow \min \\ x \in \mathcal{F} \end{cases}$$

There is a certain flexibility in defining a given problem: Some of the constraints $g_i(x) \le 0$ or $h_j(x) = 0$ can be included in the definition of the set C.

Substituting the two inequalities $h_j(x) \le 0$ and $-h_j(x) \le 0$ for every equation $h_j(x) = 0$ we can assume WLOG $p = 0$. Then we have

$$\mathcal{F} = \big\{x \in C \mid g(x) \le 0\big\}.$$

The *Lagrangian function* L is defined as a weighted sum of the objective function and the constraint functions, defined by

$$L(x, \lambda) := f(x) + \lambda^T g(x) = f(x) + \langle \lambda, g(x)\rangle = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x)$$

for $x \in C$ and $\lambda = (\lambda_1, \ldots, \lambda_m)^T \in \mathbb{R}^m_+$.

The vector λ is called the *dual variable* or *multiplier* associated with the problem. For $i = 1, \ldots, m$ we refer to λ_i as the *dual variable* or *multiplier* associated with the inequality constraint $g_i(x) \le 0$.

The Lagrange *dual function,* or *dual function,* φ is defined by

$$\varphi(\lambda) := \inf_{x \in C} L(x, \lambda)$$

on the *effective domain* of φ

$$\mathcal{F}_D := \Big\{\lambda \in \mathbb{R}^m_+ \mid \inf_{x \in C} L(x, \lambda) > -\infty\Big\}.$$

The Lagrange *dual problem*, or *dual problem*, then is defined by

$$(D) \quad \begin{cases} \varphi(\lambda) \longrightarrow \max \\ \lambda \in \mathcal{F}_D \end{cases}.$$

In the general case, the dual problem may not have a solution, even if the primal problem has one; conversely, the primal problem may not have a solution, even if the dual problem has one:

Example 10

For both examples let $C := \mathbb{R}, m := 1$ and $p := 0$:

a)
$$(P) \quad \begin{cases} f(x) := x + 2010 \longrightarrow \min \\ g(x) := \frac{1}{2}x^2 \le 0 \end{cases}$$

1. $x^* := 0$ is the only feasible point. Thus
   $\inf\{f(x) \mid x \in \mathcal{F}\} = f(0) = 2010$.
2. $L(x,\lambda) := f(x) + \lambda g(x) = x + 2010 + \frac{\lambda}{2}x^2 \quad (\lambda \ge 0,\, x \in \mathbb{R})$
   $\mathcal{F}_D = \mathbb{R}_{++}$ (for $\lambda > 0$: parabola opening upwards; for $\lambda = 0$: unbounded from below): $\varphi(\lambda) = 2010 - \frac{1}{2\lambda}$

b)
$$(P) \quad \begin{cases} f(x) := \exp(-x) \longrightarrow \min \\ g(x) := -x \le 0 \end{cases}$$

1. We have $\inf\{f(x) \mid x \in \mathcal{F}\} = \inf\{\exp(-x) \mid x \ge 0\} = 0$, but there exists no $x \in \mathcal{F} = \mathbb{R}_+$ with $f(x) = 0$.
2. $L(x,\lambda) := f(x) + \lambda g(x) = \exp(-x) - \lambda x \quad (\lambda \ge 0)$ shows $\mathcal{F}_D = \{0\}$ with $\varphi(0) = 0$. So we have $\sup\{\varphi(\lambda) \mid \lambda \in \mathcal{F}_D\} = 0 = \varphi(0)$. $\triangleleft$

The dual objective function φ — as the pointwise infimum of a family of affinely linear functions — is always a *concave function*, even if the initial problem is not convex. Hence the dual problem can always be written $(\varphi \mapsto -\varphi)$ as a *convex minimum problem:*

Remark *The set $\mathcal{F}_D$ is convex, and φ is a concave function on $\mathcal{F}_D$.*

Proof: Let $x \in C$, $\alpha \in [0,1]$ and $\lambda, \mu \in \mathcal{F}_D$:

$$\begin{aligned} L(x, \alpha\lambda + (1-\alpha)\mu) &= f(x) + \langle \alpha\lambda + (1-\alpha)\mu, g(x)\rangle \\ &= \alpha\big(f(x) + \langle\lambda, g(x)\rangle\big) + (1-\alpha)\big(f(x) + \langle\mu, g(x)\rangle\big) \\ &= \alpha L(x,\lambda) + (1-\alpha)L(x,\mu) \\ &\ge \alpha\varphi(\lambda) + (1-\alpha)\varphi(\mu) \end{aligned}$$

This inequality has two implications: $\alpha\lambda + (1-\alpha)\mu \in \mathcal{F}_D$, and further, $\varphi(\alpha\lambda + (1-\alpha)\mu) \ge \alpha\varphi(\lambda) + (1-\alpha)\varphi(\mu)$. $\square$

As we shall see below, the dual function yields lower bounds on the optimal value

$$p^* := v(P) := \inf(P) := \inf\{f(x) : x \in \mathcal{F}\}$$

of the primal problem (P). The optimal value of the dual problem (D) is defined by

$$d^* := v(D) := \sup(D) := \sup\{\varphi(\lambda) : \lambda \in \mathcal{F}_D\}.$$

We allow $v(P)$ and $v(D)$ to attain the extended values $+\infty$ and $-\infty$ and follow the standard convention that the infimum of the empty set is ∞ and the supremum of the empty set is $-\infty$. If there are feasible points x_k with $f(x_k) \to -\infty$ $(k \to \infty)$, then $v(P) = -\infty$ and we say problem (P) — or the function f on $\mathcal{F}$ — is unbounded from below. If there are feasible points λ_k with $\varphi(\lambda_k) \to \infty$ $(k \to \infty)$, then $v(D) = \infty$ and we say problem (D) — or the function φ on $\mathcal{F}_D$ — is unbounded from above. The problems (P) and (D) always have optimal values — possibly ∞ or $-\infty$. The question is whether or not they have *optimizers,* that is, there exist feasible points achieving these values. If there exists a feasible point achieving $\inf(P)$, we sometimes write $\min(P)$ instead of $\inf(P)$, accordingly $\max(D)$ instead of $\sup(D)$ if there is a feasible point achieving $\sup(D)$. In example 10, *a)* we had $\min(P) = \sup(D)$, in example 10, *b)* we got $\inf(P) = \max(D)$.

What is *the relationship between d^* and p^**? The following theorem gives a first answer:

Weak Duality Theorem

If x is feasible to the primal problem (P) and λ is feasible to the dual problem (D), then we have $\varphi(\lambda) \le f(x)$. *In particular*

$$d^* \le p^*.$$

Proof: Let $x \in \mathcal{F}$ and $\lambda \in \mathcal{F}_D$:

$$\varphi(\lambda) \le L(x, \lambda) = f(x) + \underbrace{\lambda^T}_{\ge 0} \underbrace{g(x)}_{\le 0} \le f(x)$$

This implies immediately $d^* \le p^*$. □

Although very easy to show, the weak duality result has useful implications: For instance, it implies that the primal problem has no feasible points if the optimal value of (D) is ∞. Conversely, if the primal problem is unbounded from below, the dual problem has no feasible points. Any feasible point λ to the dual problem provides a lower bound $\varphi(\lambda)$ on the optimal value p^* of problem (P), and any feasible point x to the primal problem (P) provides an upper bound $f(x)$ on the optimal value d^* of problem (D). One aim is to generate *good* bounds. This can help to get termination criteria for algorithms: If one has a feasible point x to (P) and a feasible point λ to (D), whose values are close together, then these values must be close to the optima in both problems.

Corollary

If $f(x^*) = \varphi(\lambda^*)$ for some $x^* \in \mathcal{F}$ and $\lambda^* \in \mathcal{F}_D$, then x^* is a minimizer to the primal problem (P) and λ^* is a maximizer to the dual problem (D).

Proof:

$\varphi(\lambda^*) \leq \sup\{\varphi(\lambda) \mid \lambda \in \mathcal{F}_D\} \leq \inf\{f(x) \mid x \in \mathcal{F}\} \leq f(x^*) = \varphi(\lambda^*)$

Hence, equality holds everywhere, in particular
$f(x^*) = \inf\{f(x) \mid x \in \mathcal{F}\}$ and $\varphi(\lambda^*) = \sup\{\varphi(\lambda) \mid \lambda \in \mathcal{F}_D\}$. □

The difference $p^* - d^*$ is called the *duality gap*. If this duality gap is zero, that is, $p^* = d^*$, then we say that *strong duality* holds. We will see later on: If the functions f and g are convex (on the convex set C) and a certain constraint qualification holds, then one has strong duality. In nonconvex cases, however, a duality gap

$$p^* - d^* > 0$$

has to be expected. The following examples illustrate the necessity of making more demands on f, g and C to get a close relation between the problems (P) and (D):

Example 11 With $n := 1$, $m := 1$:

a) $d^* = -\infty,\ p^* = \infty$ $\quad C := \mathbb{R}_+ ,\ f(x) := -x,\ g(x) := \pi \quad (x \in C)$:

$L(x, \lambda) = -x + \lambda\pi \quad (x \in C, \lambda \in \mathbb{R}_+)$
$\mathcal{F} = \emptyset,\ p^* = \infty;\ \inf\limits_{x \in C} L(x, \lambda) = -\infty,\ \mathcal{F}_D = \emptyset,\ d^* = -\infty$

b) $d^* = 0,\ p^* = \infty$ $\quad C := \mathbb{R}_{++},\ f(x) := x,\ g(x) := x \quad (x \in C)$:

$L(x, \lambda) = x + \lambda x = (1 + \lambda)x$
$\mathcal{F} = \emptyset,\ p^* = \infty;\ \mathcal{F}_D = \mathbb{R}_+,\ \varphi(\lambda) = 0,\ d^* = 0$

c) $-\infty = d^* < p^* = 0$

$C := \mathbb{R},\ f(x) := x^3,\ g(x) := -x \quad (x \in \mathbb{R})$:
$\mathcal{F} = \mathbb{R}_+ ,\ p^* = \min(P) = 0$
$L(x, \lambda) = x^3 - \lambda x \quad \big(x \in \mathbb{R},\ \lambda \geq 0\big)$
$\mathcal{F}_D = \emptyset ,\ d^* = -\infty$

d) $d^* = \max(D) < \min(P) = p^*$

$C := [0, 1] ,\ f(x) := -x^2,\ g(x) := 2x - 1 \quad (x \in C)$:
$\mathcal{F} = [0, 1/2],\ p^* = \min(P) = f(1/2) = -1/4$
$L(x, \lambda) = -x^2 + \lambda(2x - 1) \quad \big(x \in [0, 1],\ \lambda \geq 0\big)$
For $\lambda \in \mathcal{F}_D = \mathbb{R}_+$ we get

$$\varphi(\lambda) = \min\big(L(0, \lambda), L(1, \lambda)\big) = \min\big(-\lambda, \lambda - 1\big) = \begin{cases} -\lambda \ , & \lambda \geq 1/2 \\ \lambda - 1 \, , & \lambda < 1/2 \end{cases}$$

and hence, $d^* = \max(D) = \varphi(1/2) = -1/2$. $\triangleleft$

With $m, n \in \mathbb{N}$, a real (m,n)-matrix A, vectors $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ we consider a *linear problem* in standard form, that is,

$$(P) \quad \begin{cases} c^T x \longrightarrow \min \\ Ax = b, \ x \geq 0 \end{cases} .$$

The LAGRANGE dual problem of this linear problem is given by

$$(D) \quad \begin{cases} b^T \mu \longrightarrow \max \\ A^T \mu \leq c \end{cases} .$$

Proof: With $f(x) := c^T x$, $h(x) := b - Ax$ $\quad (x \in \mathbb{R}^n_+ =: C)$ we have

$$L(x, \mu) = \langle c, x \rangle + \langle \mu, b - Ax \rangle = \langle \mu, b \rangle + \big\langle x, c - A^T \mu \big\rangle \quad (\mu \in \mathbb{R}^m).$$

$$\inf_{x \in C} \big\{ \langle \mu, b \rangle + \big\langle x, c - A^T \mu \big\rangle \big\} \overset{\checkmark}{=} \begin{cases} b^T \mu \, , & \text{if} \quad A^T \mu \leq c \\ -\infty \, , & \text{else} \end{cases}$$ $\square$

It is easy to verify that the LAGRANGE dual problem of (D) — transformed into standard form — is again the primal problem (cf. exercise 18).

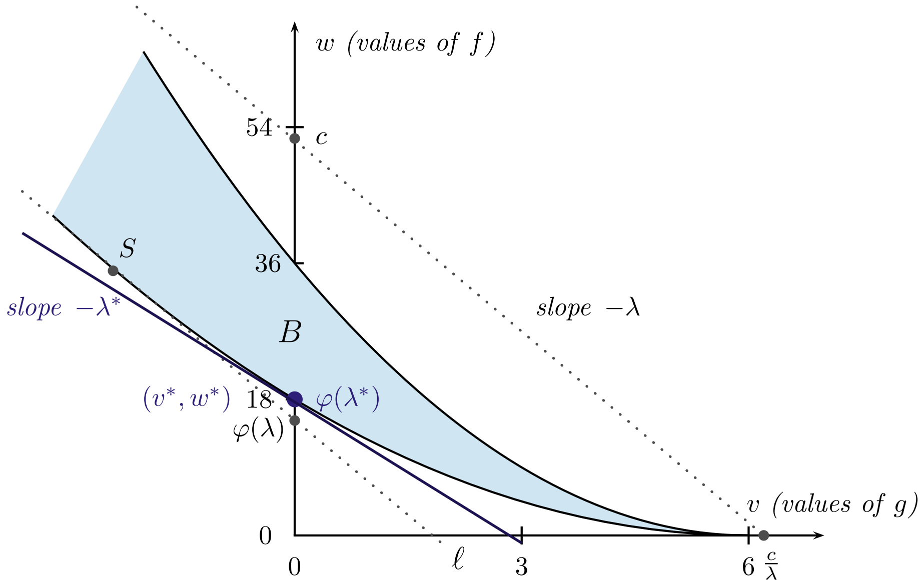
Geometric Interpretation

We give a geometric interpretation of the dual problem that helps to find and understand examples which illustrate the various possible relations that can occur between the primal and the dual problem. This visualization can give insight in theoretical results. For the sake of simplicity, we consider only the case $m = 1$, that is, only *one* inequality constraint:

We look at the image of C under the map (g, f), that is,

$$B := \big\{ (g(x), f(x)) \mid x \in C \big\}.$$

In the *primal problem* we have to find a pair $(v, w) \in B$ with minimal ordinate w in the (v, w)-plane, that is, the point (v, w) in B which minimizes w subject to $v \leq 0$. It is the point (v^*, w^*) — the image under (g, f) of the minimizer x^* to problem (P) — in the following figure, which illustrates a typical case for $n = 2$:



To get $\varphi(\lambda)$ for a fixed $\lambda \geq 0$, we have to minimize $L(x, \lambda) = f(x) + \lambda g(x)$ over $x \in C$, that is, $w + \lambda v$ over $(v, w) \in B$.

For any constant $c \in \mathbb{R}$, the equation $w + \lambda v = c$ describes a straight line with slope $-\lambda$ and intercept c on the w-axis. Hence we have to find the lowest line with slope $-\lambda$ which intersects the region B (move the line $w + \lambda v = c$ parallel to itself as far down as possible while it touches B). This leads to the line ℓ tangent to B at the point S in the figure. (The region B has to lie above the line and to touch it.) Then the intercept on the w-axis gives $\varphi(\lambda)$.

The geometric description of the *dual problem* (D) is now clear: Find the value λ^* which defines the slope of a tangent to B intersecting the ordinate at the highest possible point.

Example 12

Let $n := 2$, $m := 1$, $C := \mathbb{R}^2_+$ and $x = (x_1, x_2)^T \in C$:

$$(P) \quad \begin{cases} f(x) := x_1^2 + x_2^2 \longrightarrow \min \\ g(x) := 6 - x_1 - x_2 \leq 0 \end{cases}$$

$g(x) \leq 0$ implies $6 \leq x_1 + x_2$. The equality $6 = x_1 + x_2$ gives
$f(x) = x_1^2 + (6 - x_1)^2 = 2\left((x_1 - 3)^2 + 9\right)$.

The minimum is attained at $x^* = (3, 3)$ with $f(x^*) = 18$: $\quad \min(P) = 18$

$$\begin{aligned} L(x, \lambda) &= x_1^2 + x_2^2 + \lambda(-x_1 - x_2 + 6) \quad \big(\lambda \geq 0,\, x \in C\big) \\ &= (x_1 - \lambda/2)^2 + (x_2 - \lambda/2)^2 + 6\lambda - \lambda^2/2 \end{aligned}$$

So we get the minimum for $x_1 = x_2 = \lambda/2$ with value $6\lambda - \lambda^2/2$.

$\varphi(\lambda) = 6\lambda - \lambda^2/2$ describes a parabola, therefore we get the maximum at $\lambda = 6$ with value $\varphi(\lambda) = 18$: $\max(D) = 18$

To get the region $B := \{(g(x), f(x)) : x \in C\}$, we proceed as follows:
For $x \in C$ we have $v := g(x) \leq 6$. The equation $-x_1 - x_2 + 6 = v$ gives $x_2 = -x_1 + 6 - v$ and further

$$\begin{aligned} f(x) &= x_1^2 + x_2^2 = x_1^2 + (x_1 + (v-6))^2 \\ &= 2x_1^2 + 2(v-6)x_1 + (v-6)^2 \\ &= 2\big(x_1 + (v-6)/2\big)^2 + (v-6)^2/2 \geq (v-6)^2/2 \end{aligned}$$

with equality for $x_1 = -(v-6)/2$.

$$f(x) = 2x_1 \underbrace{(x_1 + v - 6)}_{\leq 0} + (v-6)^2 \leq (v-6)^2$$

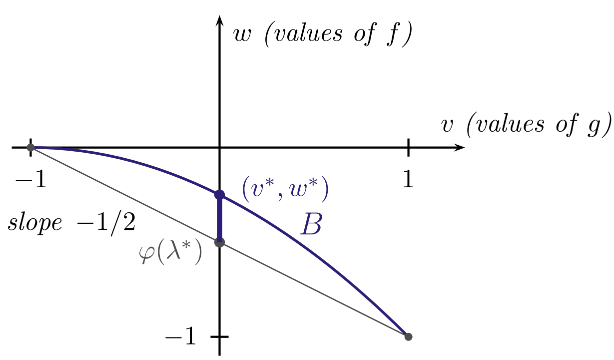
with equality for $x_1 = 0$. So we have

$$B = \big\{(v, w) \mid v \leq 6,\ (v-6)^2/2 \leq w \leq (v-6)^2\big\}. \qquad \triangleleft$$

The attentive reader will have noticed that this example corresponds to the foregoing figure.

Example 13 We look once more at example 11, *d)*:
$B := \{(g(x), f(x)) \mid x \in C\} = \big\{\big(2x-1, -x^2\big) \mid 0 \leq x \leq 1\big\}$
$v := g(x) = 2x - 1 \in [-1, 1]$ gives $x = (1+v)/2$, hence,
$w := f(x) = -(1+v)^2/4$.

Duality Gap



$\triangleleft$

Chapter 2

Chapter 2

Saddlepoints and Duality

For the following *characterization of strong duality* neither convexity nor differentiability is needed:

Theorem 2.4.1

Let x^* be a point in C and $\lambda^* \in \mathbb{R}^m_+$. Then the following statements are equivalent:

a) (x^*, λ^*) *is a saddlepoint of the* LAGRANGE *function* L.

b) x^* *is a minimizer to problem* (P) *and* λ^* *is a maximizer to problem* (D) *with*

$$f(x^*) = L(x^*, \lambda^*) = \varphi(\lambda^*)\,.$$

In other words: *A saddlepoint of the Lagrangian L exists if and only if the problems (P) and (D) have the same value and admit optimizers, that is,*

$$\min(P) = \max(D)\,.$$

Proof: First, we show that *a)* implies *b)*:

$$\begin{aligned} L(x^*, \lambda^*) &= \inf_{x \in C} L(x, \lambda^*) \le \sup_{\lambda \in \mathbb{R}^m_+} \inf_{x \in C} L(x, \lambda) \\ &\overset{\checkmark}{\le} \inf_{x \in C} \sup_{\lambda \in \mathbb{R}^m_+} L(x, \lambda) \le \sup_{\lambda \in \mathbb{R}^m_+} L(x^*, \lambda) = L(x^*, \lambda^*) \end{aligned}$$

Consequently, $\infty > \varphi(\lambda^*) = \inf_{x \in C} L(x, \lambda^*) = \sup_{\lambda \in \mathbb{R}^m_+} L(x^*, \lambda) = L(x^*, \lambda^*)$. By lemma 2.2.6 we know already: x^* is a minimizer of (P) with $f(x^*) = L(x^*, \lambda^*)$. *b)* now follows by the corollary to the weak duality theorem.

Conversely, suppose now that *b)* holds true:

$$\begin{aligned} \varphi(\lambda^*) &= \inf\{L(x, \lambda^*) \mid x \in C\} \le L(x^*, \lambda^*) \\ &= f(x^*) + \langle \lambda^*, g(x^*) \rangle \le f(x^*) \end{aligned} \tag{7}$$

We have $\varphi(\lambda^*) = f(x^*)$, by assumption. Therefore, equality holds everywhere in (7), especially, $\langle \lambda^*, g(x^*) \rangle = 0$. This leads to

$L(x^*, \lambda^*) = f(x^*) \le L(x, \lambda^*)$ for $x \in C$ and

$L(x^*, \lambda) = f(x^*) + \langle \lambda, g(x^*) \rangle \le f(x^*) = L(x^*, \lambda^*)$ for $\lambda \in \mathbb{R}^m_+$. $\square$

Perturbation and Sensitivity Analysis

In this subsection, we discuss how changes in parameters affect the solution of the primal problem. This is called *sensitivity analysis*. How sensitive are the minimizer and its value to 'small' perturbations in the data of the problem? If parameters change, sensitivity analysis often helps to avoid having to solve a problem again.

For $u \in \mathbb{R}^m$ we consider the 'perturbed' optimization problem

$$(P_u) \quad \begin{cases} f(x) \longrightarrow \min \\ x \in \mathcal{F}_u \end{cases}$$

with the feasible region

$$\mathcal{F}_u := \left\{x \in C \mid \; g(x) \le u\right\}.$$

The vector u is called the *'perturbation vector'*. Obviously we have $(P_0) = (P)$.

If a variable u_i is positive, this means that we 'relax' the i-th constraint $g_i(x) \le 0$ to $g_i(x) \le u_i$; if u_i is negative we tighten this constraint.

We define the *perturbation* or *sensitivity function*

$$p\colon \mathbb{R}^m \longrightarrow \mathbb{R} \cup \{-\infty, \infty\}$$

associated with the problem (P) by

$$p(u) := \inf\{f(x) \mid x \in \mathcal{F}_u\} = \inf\{f(x) \mid x \in C,\ g(x) \le u\} \quad \text{for } u \in \mathbb{R}^m$$

(with $\inf \emptyset := \infty$). Obviously we have $p(0) = p^*$.

The function p gives the minimal value of the problem (P_u) as a function of 'perturbations' of the right-hand side of the constraint $g(x) \le 0$.

Its *effective domain* is given by the set

$$\operatorname{dom}(p) := \{u \in \mathbb{R}^m \mid p(u) < \infty\} \overset{\checkmark}{=} \{u \in \mathbb{R}^m \mid \exists\, x \in C\ g(x) \le u\}.$$

Obviously the function p *is antitone*, that is, order-reversing: If the vector u increases, the feasible region $\mathcal{F}_u$ increases and so p decreases (in the weak sense).

Remark

If the original problem (P) *is convex, then the effective domain* $\operatorname{dom}(p)$ *is convex and the perturbation function* p *is convex on it.*

Since $-\infty$ is possible as a value for p on $\operatorname{dom}(p)$, *convexity* here means the convexity of the *epigraph*[3]

$$\operatorname{epi}(p) := \{(u, z) \in \mathbb{R}^m \times \mathbb{R} \mid u \in \operatorname{dom}(p),\ p(u) \le z\}$$

[3] The prefix 'epi' means 'above'. A *real-valued* function p is convex if and only if the set $\operatorname{epi}(p)$ is convex (cf. exercise 8).

Chapter 2

Proof: The convexity of $\mathrm{dom}(p)$ and p is given immediately by the convexity of the set C and the convexity of the function g:

Let $u, v \in \mathrm{dom}(p)$ and $\varrho \in (0,1)$. For $\alpha, \beta \in \mathbb{R}$ with $p(u) < \alpha$ and $p(v) < \beta$ there exist vectors $x, y \in C$ with $g(x) \le u$, $g(y) \le v$ and $f(x) < \alpha$, $f(y) < \beta$. The vector $\widetilde{x} := \varrho x + (1-\varrho) y$ belongs to C with

$$g(\widetilde{x}) \le \varrho g(x) + (1-\varrho) g(y) \le \varrho u + (1-\varrho) v =: \widetilde{u}$$

and

$$f(\widetilde{x}) \le \varrho f(x) + (1-\varrho) f(y) < \varrho\alpha + (1-\varrho)\beta .$$

This shows $p(\widetilde{u}) \le f(\widetilde{x}) < \varrho\alpha + (1-\varrho)\beta$, hence, $p(\widetilde{u}) \le \varrho p(u) + (1-\varrho) p(v)$. □

Remark

We assume that strong duality holds and that the dual optimal value is attained. Let λ^* be a maximizer to the dual problem (D). Then we have

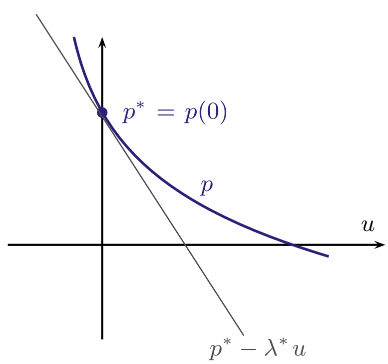
$$p(u) \ge p(0) - \langle \lambda^*, u \rangle \quad \textit{for all } u \in \mathbb{R}^m .$$

Proof: For a given $u \in \mathbb{R}^m$ and any feasible point x to the problem (P_u), that is, $x \in \mathcal{F}_u$, we have

$$p(0) = p^* = d^* = \varphi(\lambda^*) \le f(x) + \langle \lambda^*, g(x) \rangle \le f(x) + \langle \lambda^*, u \rangle .$$

From this follows $p(0) \le p(u) + \langle \lambda^*, u \rangle$. □

This inequality gives a lower bound on the optimal value of the perturbed problem (P_u). The hyperplane given by $z = p(0) - \langle \lambda^*, u \rangle$ 'supports' the epigraph of the function p at the point $(0, p(0))$. For a problem with only one inequality constraint the inequality shows that the affinely linear function $u \mapsto p^* - \lambda^* u$ $(u \in \mathbb{R})$ lies below the graph of p and is tangent to it at the point $(0, p^*)$.



We get the following rough sensitivity results:

If λ_i^* is 'small', relaxing the i-th constraint causes a small decrease of the optimal value $p(u)$. Conversely, if λ_i^* is 'large', tightening the i-th constraint causes a large increase of the optimal value $p(u)$.

Under the assumptions of the foregoing remark we have:

Remark

If the function p is differentiable[4] *at the point $u = 0$, then the maximizer λ^* of the dual problem (D) is related to the gradient of p at $u = 0$:*

$$\nabla p(0) = -\lambda^*$$

Here the LAGRANGE multipliers λ_i^* are exactly the *local sensitivities* of the function p with respect to perturbations of the constraints.

Proof: The differentiability at the point $u = 0$ gives:
$p(u) = p(0) + \langle \nabla p(0), u \rangle + r(u)\, \|u\|$ with $r(u) \to 0$ for $\mathbb{R}^m \ni u \to 0$.
Hence we obtain $-\langle \nabla p(0) + \lambda^*, u \rangle \le r(u)\, \|u\|$. We set $u := -t\,[\nabla p(0) + \lambda^*]$ for $t > 0$ and get $t\, \|\nabla p(0) + \lambda^*\|^2 \le t\, \|\nabla p(0) + \lambda^*\|\; r\big(-t\,[\nabla p(0) + \lambda^*]\big)$. This shows: $\|\nabla p(0) + \lambda^*\| \le r\big(-t\,[\nabla p(0) + \lambda^*]\big)$. Passage to the limit $t \to 0$ yields $\nabla p(0) + \lambda^* = 0$. $\square$

For the rest of this section we consider only the special case of a *convex optimization problem*, where the functions f and g are *convex* and continuously differentiable and the set C is convex.

Economic Interpretation of Duality

The equation

$$\nabla p(0) = -\lambda^*$$

or

$$-\frac{\partial p}{\partial u_i}(0) = \lambda_i^* \quad \text{for} \quad i = 1, \ldots, m$$

leads to the following interpretation of dual variables in economics:

The components λ_i^* of the LAGRANGE multiplier λ^* are often called *shadow prices* or *attribute costs*. They represent the *'marginal' rate of change* of the optimal value

$$p^* = v(P) = \inf(P)$$

[4] *Subgradients* generalize the concept of gradient and are helpful if the function p is *not* differentiable at the point $u = 0$. We do not pursue this aspect and its relation to the concept of *stability*.

of the primal problem (P) with respect to changes in the constraints. They describe the incremental change in the value p^* per unit increase in the right-hand side of the constraint.

If, for example, the variable $x \in \mathbb{R}^n$ determines how an enterprise 'operates', the objective function f describes the cost for some production process, and the constraint $g_i(x) \leq 0$ gives a bound on a special resource, for example labor, material or space, then $p^*(u)$ shows us how much the costs (and with it the profit) change when the resource changes. λ_i^* determines approximately how much fewer costs the enterprise would have, for a 'small' increase in availability of the i-th resource. Under these circumstances λ_i^* has the dimension of dollars (or euros) per unit of capacity of the i-th resource and can therefore be regarded as a value per unit resource. So we get the maximum price we should pay for an additional unit of u_i.

Strong Duality

Below we will see: If the Slater constraint qualification holds and the original problem is convex, then we have strong duality, that is, $p^* = d^*$. We see once more: The class of convex programs is a class of 'well-behaved' optimization problems. Convex optimization is relatively 'easy'.

We need a slightly different separation theorem (compared to proposition 2.1.1). We quote it without proof (for a proof see, for example: [Fra], p. 49f):

Separation Theorem

Given two disjoint nonempty convex sets $\mathcal{V}$ and $\mathcal{W}$ in $\mathbb{R}^k$, there exist a real α and a vector $p \in \mathbb{R}^k \setminus \{0\}$ with

$$\langle p, v \rangle \geq \alpha \text{ for all } v \in \mathcal{V} \quad \text{and} \quad \langle p, w \rangle \leq \alpha \text{ for all } w \in \mathcal{W}.$$

In other words: *The hyperplane $\{x \in \mathbb{R}^k \mid \langle p, x \rangle = \alpha\}$ separates $\mathcal{V}$ and $\mathcal{W}$.*

The **example**

$$\mathcal{V} := \left\{x = (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 \leq 0\right\} \quad \text{and}$$
$$\mathcal{W} := \left\{x = (x_1, x_2)^T \in \mathbb{R}^2 \mid x_1 > 0,\, x_1 x_2 \geq 1\right\}$$

(with separating 'line' $x_1 = 0$) shows that the sets cannot be 'strictly' separated.

Strong Duality Theorem

Suppose that the Slater *constraint qualification*

$$\exists\, \tilde{x} \in \mathcal{F} \;\; g_i(\tilde{x}) < 0 \;\; \text{for all } i \in \mathcal{I}_1$$

holds for the convex problem (P). Then we have strong duality, and the value of the dual problem (D) is attained if $p^* > -\infty$.

In order to simplify the *proof,* we verify the theorem under the slightly stronger condition

$$\exists\, \tilde{x} \in \mathcal{F} \;\; g_i(\tilde{x}) < 0 \;\; \text{for all } i \in \mathcal{I}\,.$$

For an extension of the proof to the (refined) SLATER *constraint qualification* see for example [Rock], p. 277.

Proof: There exists a feasible point, hence we have $p^* < \infty$. If $p^* = -\infty$, then we get $d^* = -\infty$ by the weak duality theorem. Hence, we can suppose that p^* *is finite.* The two sets

$$\mathcal{V} := \{(v,w) \in \mathbb{R}^m \times \mathbb{R} \mid \exists\, x \in C \;\; g(x) \le v \;\text{ and }\; f(x) \le w\}$$

$$\mathcal{W} := \{(0,w) \in \mathbb{R}^m \times \mathbb{R} \mid w < p^*\}$$

are nonempty and *convex.* By the definition of p^* they are *disjoint:* Let (v,w) be in $\mathcal{W} \cap \mathcal{V}$: $(v,w) \in \mathcal{W}$ shows $v = 0$ and $w < p^*$. For $(v,w) \in \mathcal{V}$ there exists an $x \in C$ with $g(x) \le v = 0$ and $f(x) \le w < p^*$, which is a contradiction to the definition of p^*.

The quoted separation theorem gives the existence of a pair

$(\lambda, \mu) \in \mathbb{R}^m \times \mathbb{R} \setminus \{(0,0)\}$ and an $\alpha \in \mathbb{R}$ such that:

$$\langle \lambda, v\rangle + \mu w \ge \alpha \;\text{ for all } (v,w) \in \mathcal{V} \quad \text{and} \tag{8}$$

$$\langle \lambda, v\rangle + \mu w \le \alpha \text{ for all } (v,w) \in \mathcal{W} \tag{9}$$

From (8) we get $\lambda \ge 0$ and $\mu \ge 0$. (9) means that $\mu w \le \alpha$ for all $w < p^*$, hence $\mu p^* \le \alpha$. (8) and the definition of $\mathcal{V}$ give for any $x \in C$:

$$\langle \lambda, g(x)\rangle + \mu f(x) \ge \alpha \ge \mu p^* \tag{10}$$

For $\boxed{\mu = 0}$ we get from (10) that $\langle \lambda, g(x)\rangle \ge 0$ for any $x \in C$, especially $\langle \lambda, g(\widetilde{x})\rangle \ge 0$ for a point $\widetilde{x} \in C$ with $g_i(\widetilde{x}) < 0$ for all $i \in \mathcal{I}$. This shows $\lambda = 0$ arriving at a *contradiction* to $(\lambda, \mu) \ne (0,0)$. So we have $\boxed{\mu > 0}$: We divide the inequality (10) by μ and obtain

$$L\big(x, \lambda/\mu\big) \ge p^* \;\text{ for any }\; x \in C\,.$$

From this follows $\varphi\big(\lambda/\mu\big) \ge p^*$. By the weak duality theorem we have $\varphi\big(\lambda/\mu\big) \le d^* \le p^*$. This shows strong duality and that the dual value is attained. $\square$

Strong duality can be obtained for some special nonconvex problems too: It holds for any optimization problem with quadratic objective function and one quadratic inequality constraint, provided SLATER's constraint qualification holds. See for example [Bo/Va], Appendix B.

Exercises

1. *Orthogonal Distance Line Fitting*

Consider the following *approximation problem* arising from quality control in manufacturing using coordinate measurement techniques [Ga/Hr]. Let

$$M := \{(x_1, y_1), (x_2, y_2), \dots, (x_m, y_m)\}$$

be a set of $m \in \mathbb{N}$ given points in $\mathbb{R}^2$. The task is to find a line L

$$L(c, n_1, n_2) := \big\{(x, y) \in \mathbb{R}^2 \mid c + n_1 x + n_2 y = 0\big\}$$

in Hessian normal form with $n_1^2 + n_2^2 = 1$ which *best approximates* the point set M such that the *sum of squares of the distances of the points from the straight line* becomes minimal. If we calculate $r_j := c + n_1 x_j + n_2 y_j$ for a point (x_j, y_j), then $|r_j|$ is its distance to L.

a) Formulate the above problem as a constrained optimization problem.

b) Show the existence of a solution and determine the optimal parameters c, n_1 and n_2 by means of the Lagrange multiplier rule. Explicate when and in which sense these parameters are uniquely defined.

c) Find a (minimal) example which consists of three points and has infinitely many optimizers.

d) Solve the optimization problem with *Matlab*® or *Maple*® and test your program with the following data (cf. [Ga/Hr]):

x_j	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
y_j	0.2	1.0	2.6	3.6	4.9	5.3	6.5	7.8	8.0	9.0

2. *a)* Solve the optimization problem

$$f(x_1, x_2) := 2x_1 + 3x_2 \longrightarrow \max$$
$$\sqrt{x_1} + \sqrt{x_2} = 5$$

using Lagrange multipliers (cf. [Br/Ti]).

b) Visualize the contour lines of f as well as the set of feasible points, and mark the solution. Explain the result!

3. Let $n \in \mathbb{N}$ and $A = (a_{\nu,\mu})$ be a real symmetric (n, n)-matrix with the submatrices A_k

$$A_k := \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1k} \\ a_{21} & a_{22} & \dots & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k1} & a_{k2} & \dots & a_{kk} \end{pmatrix} \quad \text{for} \quad k \in \{1, \dots, n\}.$$

Then the following statements are equivalent:

a) A is positive definite.

b) $\exists\, \delta > 0\ \forall\, x \in \mathbb{R}^n\ x^T Ax \geq \delta \|x\|^2$

c) $\forall\, k \in \{1, ..., n\}\ \det A_k > 0$

4. Consider a function $f\colon\ \mathbb{R}^n \longrightarrow \mathbb{R}$.

a) If f is differentiable, then the following holds:
f convex $\iff \forall x, y \in \mathbb{R}^n\ f(y) - f(x) \geq f'(x)(y - x)$

b) If f is twice continuously differentiable, then:
f convex $\iff \forall x \in \mathbb{R}^n\ \nabla^2 f(x)$ positive semidefinite

c) What do the corresponding characterizations of strictly convex functions look like?

5. In the "colloquial speech" of mathematicians one can sometimes hear the following statement: *"Strictly convex functions always have exactly one minimizer."*

However, is it really right to use this term so carelessly? Consider two typical representatives $f_i\colon\ \mathbb{R}^2 \longrightarrow \mathbb{R}$, $i \in \{1, 2\}$:

$$f_1(x, y) = x^2 + y^2$$
$$f_2(x, y) = x^2 - y^2$$

Visualize these functions and plot their contour lines. Which function is convex? Show this analytically as well. Is the above statement correct?

Let $D_j \subset \mathbb{R}^2$ for $j \in \{1, 2, 3, 4, 5\}$ be a region in $\mathbb{R}^2$ with

$$\begin{aligned} D_1 &:= \{(x, y) \in \mathbb{R}^2\ :\ x_1^2 + x_2^2\ \leq\ 0.04\} \\ D_2 &:= \{(x, y) \in \mathbb{R}^2\ :\ (x_1 - 0.55)^2 + (x_2 - 0.7)^2\ \leq\ 0.04\} \\ D_3 &:= \{(x, y) \in \mathbb{R}^2\ :\ (x_1 - 0.55)^2 + x_2^2\ \leq\ 0.04\}. \end{aligned}$$

The outer boundary of the regions D_4 and D_5 is defined by

$$\begin{aligned} x &= 0.5(0.5 + 0.2\cos(6\vartheta))\cos\vartheta + x_c \\ y &= 0.5(0.5 + 0.2\cos(6\vartheta))\sin\vartheta + y_c \end{aligned}\ ,\quad \vartheta \in [0, 2\pi)\,,$$

where $(x_c, y_c) = (0, 0)$ for D_4 and $(x_c, y_c) = (0, -0.7)$ for D_5.

If we now restrict the above functions f_i to D_j ($i \in \{1, 2\}$, $j \in \{1, 2, 3, 4, 5\}$), does the statement about the uniqueness of the minimizers still hold? Find all the minimal points, where possible! Where do they lie? Which role does the convexity of the region and the function play?

6. Show that a function $f\colon\ \mathbb{R}^n \longrightarrow \mathbb{R}$ is affinely linear if and only if it is convex as well as concave.

7. Let X be a real vector space. For $m \in \mathbb{N}$ and $x_1, \dots, x_m \in X$ let

$$\operatorname{conv}(x_1, \dots, x_m) := \left\{ \sum_{i=1}^{m} \lambda_i \, x_i \mid \lambda_1, \dots, \lambda_m > 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Verify that the following assertions hold for a nonempty subset $A \subset X$:

a) A convex $\iff \forall \, m \in \mathbb{N} \; \forall \, a_1, \dots, a_m \in A \quad \operatorname{conv}(a_1, \dots, a_m) \subset A$

b) Let A be convex and $f\colon A \longrightarrow \mathbb{R}$ a convex function. For $x_1, x_2, \dots, x_m \in A$ and $x \in \operatorname{conv}(x_1, \dots, x_m)$ in a representation as given above, it then holds that

$$f\left(\sum_{i=1}^{m} \lambda_i x_i \right) \le \sum_{i=1}^{m} \lambda_i f(x_i) \, .$$

c) The intersection of an arbitrary number of convex sets is convex. Consequently there exists the smallest convex superset $\operatorname{conv}(A)$ of A, called the *convex hull* of A.

d) It holds that $\operatorname{conv}(A) = \bigcup\limits_{\substack{m \in \mathbb{N} \\ a_1, \dots, a_m \in A}} \operatorname{conv}(a_1, \dots, a_m)$.

e) CARATHÉODORY's lemma:

For $X = \mathbb{R}^n$ it holds that $\operatorname{conv}(A) = \bigcup\limits_{\substack{m \le n+1 \\ a_1, \dots, a_m \in A}} \operatorname{conv}(a_1, \dots, a_m)$.

f) In which way does this lemma have to be modified for $X = \mathbb{C}^n$?

g) For $X \in \{\mathbb{R}^n, \mathbb{C}^n\}$ and A compact the convex hull $\operatorname{conv}(A)$ is also compact.

8. For a nonempty subset $D \subset \mathbb{R}^n$ and a function $f\colon D \longrightarrow \mathbb{R}$ let

$$\operatorname{epi}(f) := \{(x, y) \in D \times \mathbb{R} : f(x) \le y\}$$

be the *epigraph* of f. Show that for a convex set D we have

$$f \text{ convex} \iff \operatorname{epi}(f) \text{ convex.}$$

9. Prove part *1)* of lemma 2.2.3 and additionally show the following assertions for $\mathcal{F}$ *convex* and $x_0 \in \mathcal{F}$:

a) $\mathcal{C}_{fd}(x_0) = \{\mu(x - x_0) \mid \mu > 0, x \in \mathcal{F}\}$

b) $\mathcal{C}_t(x_0) = \overline{\mathcal{C}_{fd}(x_0)}$

c) $\mathcal{C}_t(x_0)$ is convex.

10. Prove for the *tangent cones* of the following sets

$$\begin{aligned}
\mathcal{F}_1 &:= \{x \in \mathbb{R}^n \mid \|x\|_2 = 1\}, \\
\mathcal{F}_2 &:= \{x \in \mathbb{R}^n \mid \|x\|_2 \le 1\}, \\
\mathcal{F}_3 &:= \{x \in \mathbb{R}^2 \mid -x_1^3 + x_2 \le 0, -x_2 \le 0\}:
\end{aligned}$$

a) For $x_0 \in \mathcal{F}_1$ it holds that $\mathcal{C}_t(x_0) = \{d \in \mathbb{R}^n \mid \langle d, x_0 \rangle = 0\}$.

b) For $x_0 \in \mathcal{F}_2$ we have $\mathcal{C}_t(x_0) = \begin{cases} \mathbb{R}^n\,, & \|x_0\|_2 < 1\,, \\ \{d \in \mathbb{R}^n \mid \langle d, x_0 \rangle \le 0\}\,, & \|x_0\|_2 = 1\,. \end{cases}$

c) For $x_0 := (0,0)^T \in \mathcal{F}_3$ it holds that $\mathcal{C}_t(x_0) = \{d \in \mathbb{R}^2 \mid d_1 \ge 0, d_2 = 0\}$.

11. With $f(x) := x_1^2 + x_2^2$ for $x \in \mathbb{R}^2$ consider

$$(P) \quad \begin{cases} f(x) \longrightarrow \min \\ -x_2 \le 0 \\ x_1^3 - x_2 \le 0 \\ x_1^3(x_2 - x_1^3) \le 0 \end{cases}$$

and determine the linearizing cone, the tangent cone and the respective dual cones at the (strict global) minimal point $x_0 := (0,0)^T$.

12. Let x_0 be a feasible point of the optimization problem (P). According to page 56 *it holds that* (LICQ) $\Longrightarrow$ (AHUCQ) $\Longrightarrow$ (ACQ).

Show by means of the following examples (with $n = m = 2$ and $p = 0$) that these two implications do not hold in the other direction:

a) $f(x) := x_1^2+(x_2+1)^2$, $g_1(x) := -x_1^3-x_2$, $g_2(x) := -x_2$, $x_0 := (0,0)^T$

b) $f(x) := x_1^2+(x_2+1)^2$, $g_1(x) := x_2-x_1^2$, $g_2(x) := -x_2$, $x_0 := (0,0)^T$

13. Let the following optimization problem be given:

$$\begin{aligned} & f(x) \longrightarrow \min\,, \quad x \in \mathbb{R}^2 \\ & g_1(x_1,x_2) := 3(x_1-1)^3 - 2x_2 + 2 \le 0 \\ & g_2(x_1,x_2) := (x_1-1)^3 + 2x_2 - 2 \le 0 \\ & g_3(x_1,x_2) := -x_1 \le 0 \\ & g_4(x_1,x_2) := -x_2 \le 0 \end{aligned}$$

a) Plot the feasible region.

b) Solve the optimization problem for the following objective functions:

(i) $f(x_1,x_2) := (x_1-1)^2 + (x_2 - \frac{3}{2})^2$

(ii) $f(x_1,x_2) := (x_1-1)^2 + (x_2-4)^2$
Regard the objective function on the 'upper boundary' of $\mathcal{F}$.

(iii) $f(x_1,x_2) := (x_1 - \frac{5}{4})^2 + (x_2 - \frac{5}{4})^2$
Do the KKT conditions hold at the optimal point?

Hint: In addition illustrate these problems graphically.

14. *Optimal Location of a Rescue Helicopter* (see example 4 of chapter 1) *a)* Formulate the minimax problem

$$d_{\max}(x,y) := \max_{1\le j\le m} \sqrt{(x-x_j)^2+(y-y_j)^2}$$

as a quadratic optimization problem

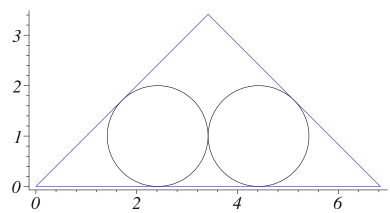
$$\begin{cases} f(x,y,\varrho) \longrightarrow \min \\ g_j(x,y,\varrho) \le 0 \qquad (j=1,\ldots,m) \end{cases}$$

(with f quadratic, g_j linear). You can find some *hints* on page 13.

b) Visualize the function $d_{\max}$ by plotting its contour lines for the points $(0,0)$, $(5,-1)$, $(4,6)$, $(1,3)$.

c) Give the corresponding Lagrangian. Solve the problem by means of the KARUSH–KUHN–TUCKER conditions.

15. Determine a triangle with minimal area containing two disjoint disks with radius 1. WLOG let $(0,0)$, $(x_1,0)$ and (x_2,x_3) with $x_1,x_3 \ge 0$ be the vertices of the triangle; (x_4,x_5) and (x_6,x_7) denote the centers of the disks.



a) Formulate this problem as a minimization problem in terms of seven variables and nine constraints (see [Pow 1]).

b) $x^* = \left(4+2\sqrt{2}, 2+\sqrt{2}, 2+\sqrt{2}, 1+\sqrt{2}, 1, 3+\sqrt{2}, 1\right)^T$ is a solution of this problem; calculate the corresponding LAGRANGE multipliers λ^*, such that the KARUSH–KUHN–TUCKER conditions are fulfilled.

c) Check the sufficient second-order optimality conditions for (x^*,λ^*).

16. Find the point $x \in \mathbb{R}^2$ that lies closest to the point $p := (2,3)$ under the constraints $g_1(x) := x_1 + x_2 \le 0$ and $g_2(x) := x_1^2 - 4 \le 0$.

a) Illustrate the problem graphically.

b) Verify that the problem is convex and fulfills (SCQ).

c) Determine the KKT points by differentiating between three cases: none is active, exactly the first one is active, exactly the second one is active.

d) Now conclude with theorem 2.2.8.

The problem can of course be solved elementarily. We, however, want to practice the theory with simple examples.

17. In a small power network the power r runs through two different channels. Let x_i be the power running through channel i for $i = 1, 2$. The total loss is given by the function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ with

$$f(x_1, x_2) := x_1 + \frac{1}{2}\left(x_1^2 + x_2^2\right).$$

Determine the current flow such that the total loss stays minimal. The constraints are given by $x_1 + x_2 = r\,,\ \ x_1 \ge 0\,,\, x_2 \ge 0\,.$

18. Verify in the *linear case* that the LAGRANGE dual problem of (D) (cf. p. 71) — transformed into standard form — is again the primal problem.

19. Consider the optimization problem (cf. [Erik]):

$$\begin{cases} f(x) \,:=\, \sum\limits_{i=1}^{n} x_i \log(\frac{x_i}{p_i}) \longrightarrow \min\,, \quad x \in \mathbb{R}^n \\ A^T x \,=\, b\,,\, x \ge 0 \end{cases}$$

where $A \in \mathbb{R}^{n\times m}, b \in \mathbb{R}^m$ and $p_1, p_2, \ldots, p_n \in \mathbb{R}_{++}$ are given. Let further $0 \ln 0$ be defined as 0. Prove:

a) The dual problem is given by

$$\varphi(\lambda) := b^T\lambda - \sum_{i=1}^{n} p_i \exp(e_i^T A\lambda - 1) \longrightarrow \max\,, \quad \lambda \in \mathbb{R}^m\,.$$

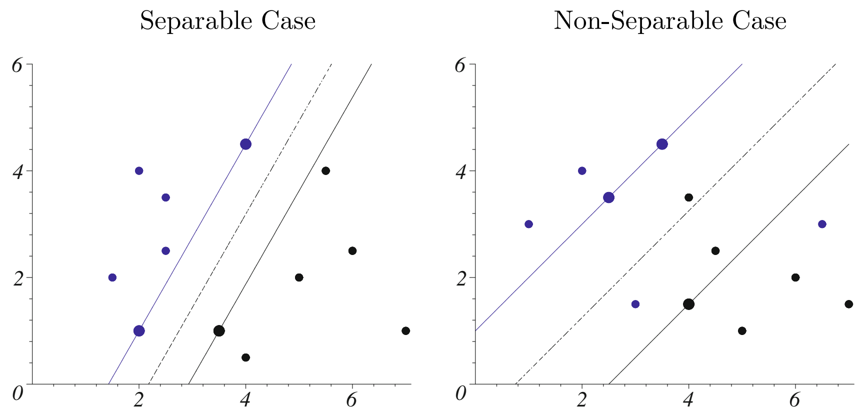
b) $\nabla\varphi(\lambda) \,=\, b - A^T x$ with $x_i = p_i \exp(e_i^T A\lambda - 1)$.

c) $\nabla^2\varphi(\lambda) \,=\, -A^T X A$, where $X = \mathrm{Diag}(x)$ with x from *b)*.

20. *Support Vector Machines* (cf. [Cr/Sh])

Support vector machines have been extensively used in *machine learning* and *data mining applications* such as classification and regression, *text categorization* as well as *medical applications*, for example *breast cancer diagnosis*. Let two classes of patterns be given, i. e., samples of observable characteristics which are represented by points x_i in $\mathbb{R}^n$. The patterns are given in the form (x_i, y_i), $i = 1, \ldots, m$, with $y_i \in \{1, -1\}$. $y_i = 1$ means that x_i belongs to class 1; otherwise x_i belongs to class 2. In the simplest case we are looking for a *separating hyperplane* described by $\langle w, x\rangle + \beta = 0$ with $\langle w, x_i\rangle + \beta \ge 1$ if $y_i = 1$ and $\langle w, x_i\rangle + \beta \le -1$ if $y_i = -1$. These conditions can be written as $y_i\big(\langle w, x_i\rangle + \beta\big) \ge 1$ $(i = 1, \ldots, m)$. We aim to maximize the 'margin' (distance) $2/\sqrt{\langle w, w\rangle}$ between the two hyperplanes $\langle w, x\rangle + \beta = 1$ and $\langle w, x\rangle + \beta = -1$. This gives a linearly constrained convex quadratic minimization problem

$$\begin{cases} \frac{1}{2}\langle w, w\rangle \longrightarrow \min \\ y_i\big(\langle w, x_i\rangle + \beta\big) \ge 1 \quad (i = 1, \ldots, m)\,. \end{cases} \tag{11}$$



In the case that the two classes are *not* linearly separable (by a hyperplane), we introduce nonnegative penalties ξ_i for the 'misclassification' of x_i and minimize both $\langle w, w\rangle$ and $\sum_{i=1}^m \xi_i$. We solve this optimization problem in the following way with *soft margins*

$$(P) \quad \begin{cases} \frac{1}{2}\langle w, w\rangle + C\sum_{i=1}^m \xi_i \to \min \\ y_i\big(\langle w, x_i\rangle + \beta\big) \ge 1 - \xi_i,\ \xi_i \ge 0 \quad (i = 1, \dots, m). \end{cases} \tag{12}$$

Here, C is a weight parameter of the penalty term.

a) Introducing the dual variables $\lambda \in \mathbb{R}_+^m$, derive the LAGRANGE dual problem to (P):

$$(D) \quad \begin{cases} -\frac{1}{2}\sum_{i,j=1}^m y_i y_j \langle x_i, x_j\rangle \lambda_i \lambda_j + \sum_{i=1}^m \lambda_i \longrightarrow \max \\ \sum_{i=1}^m y_i \lambda_i = 0, \quad 0 \le \lambda_i \le C \quad (i = 1, \dots, m) \end{cases} \tag{13}$$

Compute the coefficients $w \in \mathbb{R}^n$ and $\beta \in \mathbb{R}$ of the separating hyperplane by means of the dual solution λ and show

$w = \sum_{j=1}^m y_j \lambda_j x_j, \quad \beta = y_j - \langle w, x_j\rangle \quad$ if $0 < \lambda_j < C$.

Vectors x_j with $\lambda_j > 0$ are called *support vectors*.

b) Calculate a support vector 'machine' for *breast cancer diagnosis* using the file wisconsin-breast-cancer.data from the *Breast Cancer Wisconsin Data Set* (cf. http://archive.ics.uci.edu/ml/). The file wisconsin-breast-cancer.names gives information on the data set: It contains 699 instances consisting of 11 attributes. The first attribute gives the sample code number. Attributes 2 through 10 describe the medical status and give a 9-dimensional vector x_i. The last attribute is the class attribute ("2" for *benign*, "4" for *malignant*). Sixteen samples have a missing attribute, denoted by "?". Remove these samples from the data set. Now split the data into two portions: The first 120 instances are used as training data. Take software of your choice to solve the quadratic problem (P), using the penalty parameter $C = 1000$. The remaining instances are used to evaluate the 'performance' of the *classifier* or *decision function* given by $f(x) := \operatorname{sgn}\{\langle w, x\rangle + \beta\}$.

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