2

Isometries of $\mathbb{AR}^n$

In this chapter, we look at the properties of the affine Euclidean space $\mathbb{AR}^n$ as a metric space with the distance

$$d(a,b) = |\overrightarrow{ab}|.$$

An *isometry* of $\mathbb{AR}^n$ is a map s from $\mathbb{AR}^n$ onto $\mathbb{AR}^n$ that preserves distance,

$$d(sa,sb) = d(a,b) \text{ for all } a,b \in \mathbb{AR}^n.$$

We denote the group of all isometries of $\mathbb{AR}^n$ by $\operatorname{Isom} \mathbb{AR}^n$.

We warn the reader that in this chapter, we use some elementary notions from group theory.

2.1 Fixed Points of Groups of Isometries

The following simple result will be used later in the case of finite groups of isometries.

Theorem 2.1. *Let $W < \operatorname{Isom} \mathbb{AR}^n$ be a group of isometries of $\mathbb{AR}^n$. Assume that for some point $e \in \mathbb{AR}^n$, the orbit*

$$W \cdot e = \{\, we \mid w \in W \,\}$$

is finite. Then W fixes a point in $\mathbb{AR}^n$.

Proof. We shall use a very elementary property of triangles stated in Figure 2.1; its proof is left to the reader.

Set $E = W \cdot e$. For any point $x \in \mathbb{AR}^n$ set

$$m(x) = \max_{f \in E} d(x,f).$$

Take the point a where $m(x)$ reaches its minimum. We shall discuss the existence of the minimum a bit later (it is intuitively evident anyway). Meanwhile, we claim that the point a is unique, which would allow us to complete the proof of the theorem.

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In the triangle abc the segment cd is shorter than at least one of the sides ac and bc.

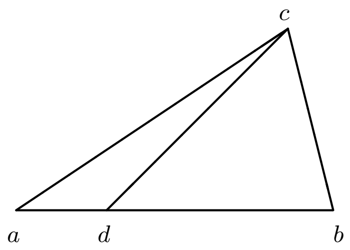

Fig. 2.1. For the proof of Theorem 2.1.

Proof of the claim. Indeed, if $b \neq a$ is another minimal point, take an inner point d of the segment $[a, b]$ and after that a point c such that $d(d, c) = m(d)$. We see from Figure 2.1 that for one of the points a and b, say a,

$$m(d) = d(d, c) < d(a, c) \leqslant m(a),$$

which contradicts to the minimal choice of a.

So we can return to the proof of the theorem. Since the group W permutes the points in E and preserves the distances in $\mathbb{AR}^n$, it preserves the function $m(x)$, i.e., $m(wx) = m(x)$ for all $w \in W$ and $x \in \mathbb{AR}^n$, and thus W should fix a (unique) point where the function $m(x)$ attains its minimum. The theorem is proven. However, to make the proof really watertight, we need to return to the issue of the existence of the minimum.

The existence of the minimum is intuitively clear; an accurate proof consists of the following two observations. Firstly, the function $m(x)$, being the supremum of a finite number of continuous functions $d(x, f)$, is itself continuous. Secondly, we can search for the minimum not over the entire space $\mathbb{AR}^n$, but only over the set

$$\{\, x \mid d(x, f) \leqslant m(a) \text{ for all } f \in E \,\},$$

for some $a \in \mathbb{AR}^n$. This set is closed and bounded, hence compact. But a continuous function on a compact set attains its extreme values. □

Notice that the proof that we have just given is a modification of a fixed-point theorem for a group acting on a space with a hyperbolic metric. J. Tits in one of his talks has attributed the proof to J.-P. Serre. An alternative (and more traditional) proof can be found in Exercise 2.2 on page 15.

2.2 Structure of Isom $\mathbb{AR}^n$

2.2.1 Translations

For every vector $\alpha \in \mathbb{R}^n$ one can define the map

$$\begin{aligned} t_\alpha : \mathbb{AR}^n &\longrightarrow \mathbb{AR}^n, \\ a &\mapsto a + \alpha. \end{aligned}$$

The map t_α is an isometry of $\mathbb{AR}^n$; it is called *translation through the vector* α. Translations of $\mathbb{AR}^n$ form a commutative group that is obviously isomorphic to the additive group of the vector space $\mathbb{R}^n$; we shall denote it by the same symbol $\mathbb{R}^n$ as the vector space.

2.2.2 Orthogonal Transformations

When we fix an orthonormal coordinate system in $\mathbb{AR}^n$ with the origin o, a point $a \in \mathbb{AR}^n$ can be identified with its *position vector* $\alpha = \overrightarrow{oa}$. This allows us to identify $\mathbb{AR}^n$ and $\mathbb{R}^n$. Every orthogonal linear transformation w of the Euclidean vector space $\mathbb{R}^n$ can be treated as a transformation of the affine space $\mathbb{AR}^n$. Moreover, this transformation is an isometry, because by the definition of an orthogonal transformation w,

$$w\alpha \cdot w\alpha = \alpha \cdot \alpha;$$

hence $|w\alpha| = |\alpha|$ for all $\alpha \in \mathbb{R}^n$. Therefore we have, for $\alpha = \overrightarrow{oa}$ and $\beta = \overrightarrow{ob}$,

$$d(wa, wb) = |w\beta - w\alpha| = |w(\beta - \alpha)| = |\beta - \alpha| = d(a, b).$$

The group of all orthogonal linear transformations of $\mathbb{R}^n$ is called the *orthogonal group* and denoted by $\mathbb{O}_n$.

Theorem 2.2. *The group of all isometries of* $\mathbb{AR}^n$ *which fix the point o coincides with the orthogonal group* $\mathbb{O}_n$.

Proof. Let s be an isometry of $\mathbb{AR}^n$ that fixes the origin o. We have to prove that when we treat s as a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, the following conditions are satisfied: for all $\alpha, \beta \in \mathbb{R}^n$,

- $s(k\alpha) = k \cdot s\alpha$ for any constant $k \in \mathbb{R}$;
- $s(\alpha + \beta) = s\alpha + s\beta$;
- $s\alpha \cdot s\beta = \alpha \cdot \beta$.

If a and b are two points in $\mathbb{AR}^n$, then by Exercise 2.3, the segment $[a, b]$ can be characterized as the set of all points x such that

$$d(a, b) = d(a, x) + d(x, b).$$

So the terminal point a' of the vector $c\alpha$ for $k > 1$ is the only point satisfying the conditions

$$d(o, a') = k \cdot d(0, a) \quad \text{and} \quad d(o, a) + d(a, a') = d(o, a').$$

If now $sa = b$, then since the isometry s preserves distances and fixes the origin o, the point $b' = sa'$ is the only point in $\mathbb{AR}^n$ satisfying

$$d(o,b') = k \cdot d(0,b) \quad \text{and} \quad d(o,b) + d(b,b') = d(o,b').$$

Hence $s \cdot k\alpha = \overrightarrow{ob'} = k\beta = k \cdot s\alpha$ for $k > 0$. The cases $k \leqslant 0$ and $0 < k \leqslant 1$ require only minor adjustments in the above proof and are left to the reader as an exercise. Thus s preserves multiplication by scalars.

The additivity of s, i.e., the property

$$s(\alpha + \beta) = s\alpha + s\beta,$$

follows, in an analogous way, from the observation that the vector $\delta = \alpha + \beta$ can be constructed in two steps: starting with the terminal points a and b of the vectors α and β, we first find the midpoint of the segment $[a,b]$ as the unique point c such that

$$d(a,c) = d(c,b) = \frac{1}{2} d(a,b),$$

and then set $\delta = 2\overrightarrow{oc}$. A detailed justification of this construction is left to the reader as an exercise.

Since s preserves distances, it preserves lengths of the vectors. But from $|s\alpha| = |\alpha|$ it follows that

$$s\alpha \cdot s\alpha = \alpha \cdot \alpha$$

for all $\alpha \in \mathbb{R}^n$. Now we apply the additivity of s and observe that

$$\begin{aligned}
(\alpha + \beta) \cdot (\alpha + \beta) &= s(\alpha + \beta) \cdot s(\alpha + \beta) \\
&= (s\alpha + s\beta) \cdot (s\alpha + s\beta) \\
&= s\alpha \cdot s\alpha + 2s\alpha \cdot s\beta + s\beta \cdot s\beta \\
&= \alpha \cdot \alpha + 2s\alpha \cdot s\beta + \beta \cdot \beta.
\end{aligned}$$

On the other hand,

$$(\alpha + \beta) \cdot (\alpha + \beta) = \alpha \cdot \alpha + 2\alpha \cdot \beta + \beta \cdot \beta.$$

Comparing these two equations, we see that

$$2s\alpha \cdot s\beta = 2\alpha \cdot \beta$$

and

$$s\alpha \cdot s\beta = \alpha \cdot \beta.$$

$\square$

Theorem 2.3. *Every isometry of a real affine Euclidean space $\mathbb{AR}^n$ is a composition of a translation and an orthogonal transformation. The group* $\operatorname{Isom} \mathbb{AR}^n$ *of all isometries of $\mathbb{AR}^n$ is a* semidirect product *of the group $\mathbb{R}^n$ of all translations and the orthogonal group $\mathbb{O}_n$,*

$$\operatorname{Isom} \mathbb{AR}^n = \mathbb{R}^n \rtimes \mathbb{O}_n.$$

This means that we have the following decomposition of $\operatorname{Isom} \mathbb{AR}^n$*:*

$$\operatorname{Isom} \mathbb{AR}^n = \mathbb{R}^n \cdot \mathbb{O}_n, \quad \mathbb{R}^n \lhd \operatorname{Isom} \mathbb{AR}^n, \quad \textit{and} \quad \mathbb{R}^n \cap \mathbb{O}_n = 1.$$

Proof. The proof is an almost immediate corollary of the previous result. Indeed, if $w \in \operatorname{Isom}\mathbb{AR}^n$ is an arbitrary isometry, take the translation $t = t_\alpha$ through the position vector $\alpha = \overrightarrow{o, wo}$ of the point wo. Then $to = wo$ and $o = t^{-1}wo$. Thus the map $s = t^{-1}w$ is an isometry of $\mathbb{AR}^n$ that fixes the origin o and, by Theorem 2.2, belongs to $\mathbb{O}_n$. Hence $w = ts$ and

$$\operatorname{Isom}\mathbb{AR}^n = \mathbb{R}^n\mathbb{O}_n.$$

Obviously $\mathbb{R}^n \cap \mathbb{O}_n = 1$, and we need to check only that $\mathbb{R}^n \lhd \operatorname{Isom}\mathbb{AR}^n$. But this follows from the observation that for any orthogonal transformation s,

$$st_\alpha s^{-1} = t_{s\alpha}$$

(Exercise 2.5), and consequently we have, for any isometry $w = ts$ with $t \in \mathbb{R}^n$ and $s \in \mathbb{O}_n$,

$$wt_\alpha w^{-1} = ts \cdot t_\alpha \cdot s^{-1}t^{-1} = t \cdot t_{s\alpha} \cdot t^{-1} = t_{s\alpha} \in \mathbb{R}^n.$$

Given an isometry $w = ts$ of $\mathbb{AR}^n$ with $t \in \mathbb{R}^n$ and $s \in \mathbb{O}_n$, we say that w *preserves orientation* if $\det s = +1$, and *changes orientation* if $\det s = -1$.

Exercises

2.1. Prove the property of triangles in $\mathbb{AR}^2$ stated in Figure 2.1.

2.2.* Barycenter. There is a more traditional approach to Theorem 2.1. If

$$F = \{ f_1, \ldots, f_k \}$$

is a finite set of points in $\mathbb{AR}^n$, its *barycenter* b is a point defined by the condition

$$\sum_{j=1}^{k} \overrightarrow{bf_j} = 0.$$

1. Prove that a finite set F has a unique barycenter.
2. Further, prove that the barycenter b is the point where the function

$$M(x) = \sum_{j=1}^{k} d(x, f_j)^2$$

takes its minimal value. In particular, if the set F is invariant under the action of a group W of isometries, then W fixes the barycenter b.

2.3. If a and b are two points in $\mathbb{AR}^n$, then the segment $[a, b]$ can be characterized as the set of all points x such that

$$d(a, b) = d(a, x) + d(x, b).$$

2.4. Draw a diagram illustrating the construction of $\alpha + \beta$ in the proof of Theorem 2.2, and fill in the details of the proof.

2.5. Prove that if t_α is a translation through the vector α and s is an orthogonal transformation then

$$st_\alpha s^{-1} = t_{s\alpha}.$$

2.6. Prove the following generalization of Theorem 2.1: if a group $W < \operatorname{Isom} \mathbb{AR}^n$ has a bounded orbit on $\mathbb{AR}^n$ then W fixes a point.

ELATIONS. A map $f : \mathbb{AR}^b \longrightarrow \mathbb{AR}^n$ is called an *elation* if there is a constant k such that for all $a, b \in \mathbb{AR}^n$,

$$d(f(a), f(b)) = kd(a, b).$$

An isometry is a special case $k = 1$ of an elation. The constant k is called the *coefficient* of the elation f.

2.7. An elation of $\mathbb{AR}^n$ with the coefficient k is the composition of a translation, an orthogonal transformation, and a map of the form

$$\begin{aligned} \mathbb{R}^n &\longrightarrow \mathbb{R}^n, \\ \alpha &\mapsto k\alpha. \end{aligned}$$

2.8.* Prove that an elation of $\mathbb{AR}^n$ preserves angles: if it sends points a, b, c to the points a', b', c', respectively, then $\angle abc = \angle a'b'c'$.

2.9.* Prove that elations can be characterized as maps from $\mathbb{AR}^n$ onto $\mathbb{AR}^n$ that send straight lines to straight lines and preserveve perpendicularity.

2.10. The group of all elations of $\mathbb{AR}^n$ is isomorphic to $\mathbb{R}^n \rtimes (\mathbb{O}_n \times \mathbb{R}^{>0})$, where $\mathbb{R}^{>0}$ is the group of positive real numbers with respect to multiplication.

ISOMETRIES OF $\mathbb{AR}^3$.

2.11.* EULER'S THEOREM is a classical observation: if an isometry of $\mathbb{AR}^3$ has a fixed point and preserves orientation then it is a rotation about an axis.

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