Preface

This expository text contains an elementary treatment of finite groups generated by reflections. There are many good books on this subject; in particular, a book by Humphreys [Hum] provides an excellent introduction to the theory, which is very much alive and under active development; see, for example, the recent survey by Dolgachev [Dol].

The main reason why we decided to write another text is not mathematical but pedagogical: we wished to emphasize the intuitive elementary geometric aspects of the theory of reflection groups. In the theory of reflection groups, the underlying ideas of many proofs can be presented by simple drawings much better than by a dry verbal exposition. Probably for the reason of their extreme simplicity these elementary arguments are mentioned in most books only briefly and tangentially.

The second reason for the existence of this book is a remarkable feature of the theory of reflection groups: its principal objects can be defined right on the spot in the most intuitive way. We give first an informal description:

> Imagine a few semitransparent mirrors in ordinary three-dimensional space. Mirrors (more precisely, their images) multiply by reflecting in each other, as in a kaleidoscope or a hall of mirrors. Of special interest are systems of mirrors that generate only finitely many reflected images. Such finite systems of mirrors happen to be one of the cornerstones of modern mathematics and lie at the core of many mathematical theories.

As usual, the theory is actually concerned with the more general case of n-dimensional Euclidean space, with mirrors being $(n-1)$-dimensional hyperplanes rather than two-dimensional planes. To that end, we give a formal definition:

> A system of hyperplanes (mirrors) Σ in Euclidean space $\mathbb{R}^n$ is called *closed* if for any two mirrors M_1 and M_2 in Σ, the mirror image of M_2 in M_1 also belongs to Σ.

Thus, the principal objects of the theory are *finite closed systems of mirrors*. In more general terms, the theory can be described as the geometry of multiple mirror images.

Instead of a closed mirror system, one can operate with the group of transformations generated by all its reflections—and this is a more traditional approach to the theory. We prefer to emphasize the role of mirrors, since we hope that this allows us to fully engage the reader's geometric intuition. This approach is well known and exploited in Chapter 5, §3 of Bourbaki's classical text [Bou]; see also Vinberg's exposition [Vin]. We have combined it with Tits's theory of chamber complexes [Tits] and thus made the exposition of the theory almost entirely geometrical.

Finally, we cannot escape the fact that the theory of finite reflection groups leads to their full classification. In the resulting list, two of the four infinite series of reflections groups, the symmetric groups $\mathrm{Sym}_{n+1} = A_n$ and hyperoctahedral groups BC_n, are groups of symmetries of two of the most common regular polytopes, the regular n-dimensional simplex and the n-dimensional cube. The third series, D_n, is a slight modification of BC_n, while the fourth one, $G_2(n)$, is the group of symmetries of the regular planar n-gon. Therefore the theory of reflection groups mostly deals with very concrete objects; why should we avoid an equally concrete down-to-earth approach to its development?

This is why we tried to include (frequently in the form of exercises) as many elementary facts about concrete groups as possible. We feel that this is well justified even if judged from the "global" viewpoint of mathematics as a whole. Indeed, finite reflection groups form one of the cornerstones of modern algebra and geometry. Even the simplest observations about particular groups have fundamental implications, for example for the structure of Lie groups and representation theory. A mathematician working in one such area normally maintains a whole menagerie of facts about reflection groups. We believe that students will find them interesting and amusing.

We hope that our approach allows the novice mathematician easy access to the theory of reflection groups. This aspect of the book makes it close to Grove and Benson [GB]. We realize, however, that, since classical geometry has almost completely disappeared from schools and university curricula, we need to smuggle it back in and provide the student reader with a modicum of Euclidean geometry and the theory of convex polyhedra. We do not wish to appeal to the reader's geometric intuition without trying first to help him or her develop it.

In particular, we decided to saturate the book with visual material. Our sketches and diagrams are intentionally left very unsophisticated; the book was tested in a lecture course at the University of Manchester, and most pictures, in their even less sophisticated versions, were first drawn on the blackboard. There was no point in drawing pictures that could not be reproduced by students and reused in their homework. Pictures are not for decoration, they are indispensable (though perhaps greasy and soiled) tools of the trade.

This is our conscious choice; indeed, what matters—and it is part of our teaching philosophy—is that the pictures are *reproducible*. Even if the reader has very modest drawing skills, he or she should be able to draw similar pictures as a way of facilitating his or her mathematical work. Moreover, we even included in our book a short chapter "The Forgotten Art of Blackboard Drawing." Without attempting to reinvent descriptive geometry, we give there some advice on making usable mathematical drawings;

the crucial piece of advice is that you have to treat your sketch as a mathematical object.

This book contains a number of exercises of different levels of difficulty; a star * marks more difficult exercises. Some of the exercises may look irrelevant to the subject of the book and are included for the sole purpose of developing the student's geometric intuition.

Outline of the Book and Dependencies between Chapters

Part I contains mostly standard material from linear algebra, with a brief discussion of polyhedra and polyhedral cones, topics usually addressed in courses on linear programming (Chapters 3 and 4). A more experienced reader can skip this material and later return to it for reference.

The book as such starts in Part II. In Chapter 5, we introduce reflections and their mirrors, and in Chapter 6, the main concept of the book: closed systems of mirrors. Chapter 7 contains some very elementary group theory, namely discussion of the structure of dihedral groups. Geometrically, they are finite systems of mirrors in the Euclidean plane. Chapter 8 introduces dual objects, namely root systems. Following our principles to put as much flesh on the theoretical bones as possible, Chapter 9 discusses, in great but elementary detail, the root and mirror systems A_{n-1}, BC_n, D_n.

Mirrors cut a space into *chambers*; the set of all such chambers, together with the action of the reflection group on it, is known as a *Coxeter complex*. Part III of the book studies Coxeter complexes. Chapter 14 deviates somewhat from the classical treatment of reflection groups and can be skipped on first reading.

Part IV deals with the classification of reflection groups; Chapter 17 contains lists of root systems and their detailed properties.

The novice reader may wish first to build some geometric intuition; in that case, we advise him or her to read first Chapters 6 and 7 and then move to Part V. The latter contains an independent and elementary treatment of 3-dimensional reflection groups. To read it, one needs only some basic linear algebra and group theory, and the more technical material from Part I can be temporarily skipped.

Prerequisites and Use as a Course Text

This book was carefully designed to be accessible to graduate and senior undergraduate students; we tried to calibrate its level to be usable as a course text in third and fourth year of master of mathematics degree courses in England. Hence the formal prerequisites for reading the book are very modest.

We assume the reader's solid knowledge of linear algebra, especially the theory of orthogonal transformations in real Euclidean spaces.

We use a modicum of topological concepts: open and closed subsets in the Euclidean space $\mathbb{R}^n$.

We also assume that the reader is familiar with the following basic notions of group theory:

> the order of a finite group; normal subgroups and factor groups; homomorphisms and isomorphisms; generators and relations; standard notation for permutations and rules for their multiplication; cyclic groups; action of a group on a set; the orbit-stabilizer theorem.

You can find this material in any introductory text on the subject. We highly recommend a book by Armstrong [Arm] for a first reading.

If this book is used as a text for a lecture course, then Grove and Benson [GB] and Humphreys [Hum] are obvious sources for auxiliary reading and extension of topics touched on in this book. In a few cases, our book borrows mathematical ideas from the other two, although our pedagogical approach is, as a rule, different.

Acknowledgments

The early versions of the text were carefully read by Robert Sandling, Richard Booth, Neil White, and Alexander Elashvili, who suggested many corrections and improvements.

The first author expresses his special thanks to his PhD students Christine Altseimer, Ayşe Berkman, and Richard Booth. Although working on completely different projects in combinatorics and group theory, each of them spent many hours toiling at the diagram of the root system of type B_3 (Figure 9.6 on page 71), which was drawn (and stayed there for months) at the blackboard in his office.

Part of the material from this book was used in the monograph [BGW]; we thank Israel Gelfand and Neil White whose comments helped to improve the present book, too.

In July 2007, the book was tested in an informal intensive course for Turkish students in Nesin Matematik Köyü in the village of Şirince in Turkey, and in April 2009 – in lectures to undegraduate students of MidEastern Technical University in Ankara arranged under provisions of the Erasmus Teaching Mobility Programme.

Our special thanks go to David Kramer for his very careful editing of the text.

Manchester,
22 June 2009

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http://www.springer.com/978-0-387-79065-7

Mirrors and Reflections
The Geometry of Finite Reflection Groups
Borovik, A.V.; Borovik, A.
2010, XII, 172 p. 74 illus., Softcover
ISBN: 978-0-387-79065-7