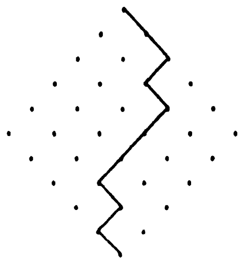
2 Combinations and Permutations

January 5. In his first lecture, Pólya discussed in general terms what combinatorics is about: The study of counting various combinations or configurations. He started with a problem based on the mystical sign known, appropriately, as an "abracadabra".

```
          A
        B   B
      R   R   R
    A   A   A   A
  C   C   C   C   C
A   A   A   A   A   A
  D   D   D   D   D
    A   A   A   A
      B   B   B
        R   R
          A
```

The question is, how many different ways are there to spell out "abracadabra", always going from one letter to an adjacent letter? Due to the way some letters (especially C and D) are found only in certain rows, it turns out the only ways to spell "abracadabra" start with the topmost 'A' and zig-zag down to the bottommost 'A'. If we think of the letters as points, then any spelling of "abracadabra" specifies a sequence of points forming a crooked line from the top to the bottom. One such line is shown below.



You can also think of this problem in terms of a network of streets in a city where all blocks are the same size. Then the problem becomes one of computing how many ways there are of getting from the northern corner to the southern corner in the minimum number (10) of blocks. (That 10 is the minimum can be seen from the fact that each block, in addition to taking us either east or west, takes us

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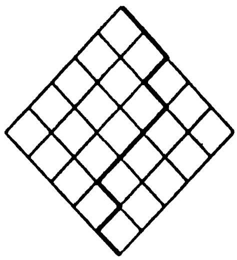
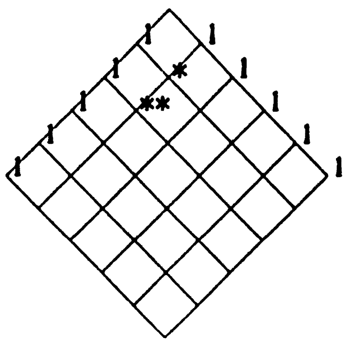
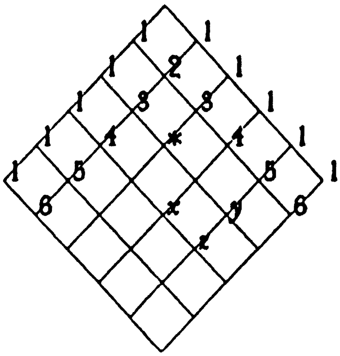
southward one-tenth the t tal southward distance between the two corners.)



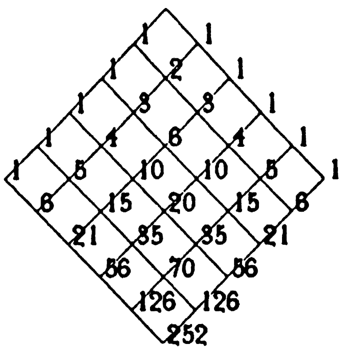
It was decided empirically (*i.e.*, by taking a vote) that there were more than 100 paths, but there was disagreement over whether there were more than 1000, so Pólya proceeded to approach the problem by more formal methods. He began by emphasising an important maxim which you should always consider when working on *any* problem: "*If you cannot solve the proposed problem, solve first a suitable related problem.*" In this instance, the related problem is that of computing how many different paths there are from the northern corner to various other corners, still restricting ourselves to travelling only southeast and southwest. For starters, there is only one path to each of the corners on the northeast edge, namely the path consisting of travelling always southeast and never southwest. Similarly, there is only one path to each of the corners on the northwest edge. We note these values by writing them next to the corners involved.



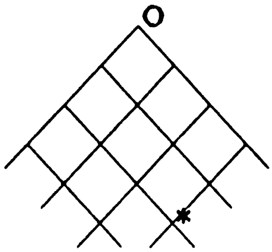
Now what about the corner marked with a *? You could get there by going one block southeast followed by one block southwest, or by going first southwest and then southeast. Similarly, to get to the corner marked **, you could go southeast, then southwest twice, or you could go southwest, then southeast, then southwest, or you could go southwest twice and then southeast. Moving down the diagonal in the manner and, by symmetry, the corresponding diagonal on the eastern side, we can fill in some more values.



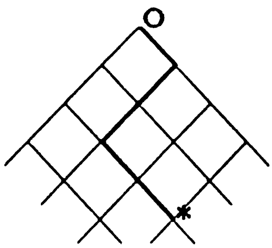
Had we tried to go much further like this, it would probably have gotten rather tiresome, so instead we came up with a general observation regarding an arbitrary corner, such as the one marked z above. If we know that there are x ways to get to the corner just northwest of z, and y ways to get to the corner northeast of z, then there are $x+y$ ways to get to z, since to get there we must first get to either x or y, after which there's only one way to continue on to z. For instance, there are 3+3=6 paths to the corner marked $*$. This general rule provides us with an easy method to finish computing the number of paths to the southern corner. The first homework assignment was to complete this computation. It comes as no surprise that everyone got it right. For the record, here it is.



The numbers we've been computing are known as binomial coefficients, for reasons we'll get to eventually. The arrangement of the numbers, when cut off by any horizontal line so as to form a triangular pattern, is known as Pascal's triangle. (Pascal referred to it as "the arithmetical triangle".) The numbers are uniquely defined by the boundary condition (the 1's along the edges) together with the recursion formula (each number not on the edge is the sum of the two above it). In addition to this recursion formula, which defines each number in terms of earlier ones, there is another way to look at the situation. Here's a small chunk of the street network we've been working with:



Suppose we want to know the number of different paths (of minimum length) from the origin O to the starred corner. Each such path must consist of 5 blocks, of which exactly 3 go to the right (as seen from above). If we specify which 3 of the 5 blocks will go to the right, we uniquely specify the path. For instance, if we choose the 1st, 4th, and 5th blocks, we get this path:



Conversely, each path from O to * specifies a unique set of 3 blocks that go to the right. So the number of paths is the same as the number of ways of choosing 3 blocks out of the total 5. Euler's notation for this sort of thing is $\binom{5}{3}$ or, in general, $\binom{n}{r}$, denoting the number of ways of choosing a subset of size r from a set of size n.

This is usually read "n-choose-r". (Another name often heard to describe this value, but which recently has fallen out of favor, is that used by Jacob Bernoulli: the combinations of n elements taken r at a time.) Computing this value is the first problem of combinatorics.

Next we come to some basic rules for working with multiple sets. The rules are fairly simple (as basic rules are wont to be), but are nevertheless very important (again as basic rules are wont to be). First off, suppose that out of a set of possibilities, A, it is possible to choose any one of m different elements. From another set, B, it is possible to choose any one of n elements. We wish to select an element from either A or B; we don't care which. *Assuming A and B have no elements in common*, there are $m+n$ possible choices.

Next, suppose the elements of A are $a_1, a_2, \ldots, a_m$, and the elements of B are $b_1, b_2, \ldots, b_n$. We wish to select two elements, one from each set, in a specific order (say, first one from A and then one from B). This operation is known as the Cartesian product of the two sets, due to its relationship with the rectangular (Cartesian) coordinate system. For instance, if A has three elements and B has two, there are six possible pairs: (a_1,b_1), (a_1,b_2), (a_2,b_1), (a_2,b_2), (a_3,b_1), and (a_3,b_2). In general, there are $m \cdot n$ possibilities.

Finally, take a more general case of the Cartesian product. Suppose that, having chosen a_1, we then have a choice among a set of elements $b_{11}, b_{12}, \ldots, b_{1n}$. If we start by choosing a_2, we then have a choice from a *different* set: $b_{21}, b_{22}, \ldots, b_{2n}$, and so on. In general, the possibilities for b differ depending upon our choice for a, *but there are always n of them.* As long as the number of possibilities for b is constant, the total number of pairs (a_i,b_j) is still $m \cdot n$. We'll see an application of this in a moment.

A permutation is an ordering of a set of objects. For instance, given the set of three numbers $\{1,2,3\}$, we could order them in any of 6 different ways: $\{1,2,3\}$, $\{1,3,2\}$, $\{2,1,3\}$, $\{2,3,1\}$, $\{3,1,2\}$, or $\{3,2,1\}$. The number of different permutations of n elements is denoted by P_n. Hence $P_3 = 6$. We also see fairly easily that $P_1 = 1$ and $P_2 = 2$. At this point Pólya stated another important maxim: "*The beginning of most discoveries is to recognise a pattern.*" There is a pattern to the three numbers we've got so far; to make it more apparent, we can rewrite them as follows:

$$P_1 = 1 = 1$$
$$P_2 = 2 = 1 \cdot 2$$
$$P_3 = 6 = 1 \cdot 2 \cdot 3.$$

We conjecture that $P_n = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$. This product is called <u>n factorial</u> and is usually written "$n!$". Now we need to prove our conjecture. Well, *suppose* it's true that $P_n = n!$. Then what would P_{n+1} be? It is the number of ways of ordering $n+1$ objects. The $n+1^{st}$ object could be in any one of $n+1$ positions. Whichever of these positions we choose, the remaining n objects can be ordered in any of P_n ways. Using the generalisation of the Cartesian product rule, we conclude that the total number of ways we can order $n+1$ objects is $(n+1) \cdot P_n$. Therefore, if $P_n = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$, then $P_{n+1} = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \cdot (n+1) = (n+1)!$. But we know that $P_3 = 3!$, so taking $n=3$ we conclude that $P_4 = 4!$. Knowing this, we can take $n=4$ and conclude that $P_5 = 5!$, and so on. For any finite n, we can prove that $P_n = n!$ by starting at P_3 and chugging away for a while. This method of proof, which Pólya describes as "a diabolic way of proving things", is called <u>mathematical induction</u>. It is extremely useful since it saves you from having to figure out the formula you're proving. If you can make a "lucky guess" as to what the answer is, you may be able to prove it by induction.

January 10. Pólya began the lecture by reviewing the material from the previous lecture. In doing so he brought out some points that hadn't been explicitly stated before. In particular, there's the formal definition of the binomial coefficients:

Boundary condition: $\binom{n}{0} = \binom{n}{n} = 1$

Recursion: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.
[n and r integers, $0<r<n+1$]

Similarly, P_n can be defined by boundary conditions and recursion:

Boundary condition: $P_1 = 1! = 1$

Recursion: $P_n = n! = nP_{n-1}$.

If we apply this recursion formula with $n=1$, we find that $P_1 = 1 \cdot P_0$. Hence we define $P_0 = 0! = 1$.

From here, we move on to look at something Pólya called a "variation", a word you may immediately forget. It is defined as follows. Given a set of n objects, we wish to choose r of them in some order. That is, choosing the first object and then the second would be considered different from choosing the second and then the first. How many such variations are there? One approach is to start by choosing some object to be the first one selected. There are n choices. For each choice, there are $n-1$ choices for the second object. Thus, by the product rule, there are $n(n-1)$ choices for the first two objects together. For each such pair, there are $(n-2)$ objects remaining from which to choose the third object. So there are $n(n-1)(n-2)$ choices for the first three objects. Continuing in this manner, we find that there are $n(n-1)(n-2)\ldots(n-r+1)$ variations.

We can often learn something by solving a problem in two different ways, so here's a second approach. We first choose the subset of r objects from among the n. We know there are $\binom{n}{r}$ ways to do this. We then choose the ordering for the r objects. We know how many ways there are to do this, too; it's P_r. So there are $\binom{n}{r} P_r$ variations. But this answer must be the same as the one we got the other way. Therefore $\binom{n}{r}\cdot P_r = n(n-1)(n-2)\ldots(n-r+1)$. So we have learned something new:

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{1\cdot 2\cdot 3\cdot\ldots\cdot r}.$$

(Note that, in the second form, the sum of 'corresponding' terms in the numerator and denominator is always $n+1$; this can be a useful mnemonic for remembering what the last term in the numerator is.) For example, the number that we computed for the first homework assignment is $\binom{10}{5}$, which by this formula is $(10\cdot9\cdot8\cdot7\cdot6)/(1\cdot2\cdot3\cdot4\ 5) = (10\cdot9\cdot7\cdot6)/(1\cdot3\cdot5) = (2\cdot9\cdot7\cdot6)/3 = 2\cdot9\cdot7\cdot2 = 252$. It's always a good idea to test out a formula on some special cases where we already know the answer, so let's look at $\binom{n}{n}$ and $\binom{n}{0}$. We have

$$\binom{n}{n} = \frac{n(n-1)(n-2)\ldots 1}{1\cdot 2\cdot 3\cdot\ldots\cdot n},$$

which, since the numerator and denominator have all the same factors, albeit in different orders, indeed equals 1. $\binom{n}{0}$, however, poses a bit of a problem, since the numerator has no factors. By defining the product of zero factors to be equal to 1 (just as $0! = 1$) we find that $\binom{n}{0} = 1$ as expected.

Another way we can get this explicit form for the binomial coefficients is by using mathematical induction. We assume it's true for small n (we can check this by hand) and then show that, if it's true for n, it's true for $n+1$. The first problem on the second homework assignment was to carry out this proof. Here it is: We assume that, for some value of n,

$$\binom{n}{r} = \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot r}$$

for all values of r. Substituting $r-1$ for r, we find

$$\binom{n}{r-1} = \frac{n(n-1)(n-2)\dots(n-r+2)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (r-1)}.$$

By the definition of the binomial coefficients, we know that

$$\begin{aligned}
\binom{n+1}{r} &= \binom{n}{r-1} + \binom{n}{r} \\
&= \frac{n(n-1)(n-2)\dots(n-r+2)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (r-1)} + \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot r} \\
&= \frac{n(n-1)(n-2)\dots(n-r+2) \cdot r}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (r-1) \cdot r} + \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot r} \\
&= \frac{n(n-1)(n-2)\dots(n-r+2) \cdot (r + n-r+1)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot (r-1) \cdot r} \\
&= \frac{(n+1)n(n-1)(n-2)\dots(n-r+2)}{1 \cdot 2 \cdot 3 \cdot \dots \cdot r},
\end{aligned}$$

which is the formula we're trying to prove (with $n+1$ substituted for n). Hence, if the formula is true for n, it's true for $n+1$. This, combined with the fact that it's true for $n=1$, means it is true for all finite n. (Actually, there's a minor flaw in this proof. To wit, the recursion formula cannot be used to compute $\binom{n}{n}$ or $\binom{n}{0}$, since it would involve coefficients outside the range $0 \le r \le n$. However, we've already shown separately that these two special cases satisfy the formula, so we're all right.)

A more compact way to write the formula for the binomial coefficient can be derived by multiplying both the numerator and denominator by the factors $(n-r)$, $(n-r-1)$, and so on down to 1.

$$\binom{n}{r} = \frac{n(n-1)(n-2)\ldots(n-r+1)\cdot(n-r)(n-r-1)\ldots 2\cdot 1}{(1\cdot 2\cdot 3\cdot\ldots\cdot r)(1\cdot 2\cdot\ldots\cdot(n-r-1)\cdot(n-r))}$$
$$= \frac{n!}{r!(n-r)!}$$

Notice that, based on this formula, it is immediately apparent that $\binom{n}{r} = \binom{n}{n-r}$. This was to be expected, since by the method of its construction Pascal's triangle is clearly symmetric.

Next, we consider n houses. They are built identically, because it's easier that way. But then, to make them look different, they are painted different colors: r of them are painted red, s of them yellow, and the remaining t of them green. In how many ways can we assign the colors to the houses? We first choose which houses will be painted red; there are $\binom{n}{r}$ ways to make this choice. Whatever choice we make, there are $n-r$ houses left, of which we choose s to be painted yellow; there are $\binom{n-r}{s}$ ways to do this. At this point we have no choices left to make, since all the rest must be green (that is, $r+s+t=n$). So what do we have? By the product rule, there are $\binom{n}{r}\binom{n-r}{s}$ ways to paint the houses. Using the formula we worked out a moment ago, we find

$$\binom{n}{r}\binom{n-r}{s} = \frac{n!}{r!(n-r)!}\cdot\frac{(n-r)!}{s!(n-r-s)!}.$$

But $n-r-s=t$, and the $(n-r)!$ factors cancel, leaving us with

$$\frac{n!}{r!s!t!},$$

which is, fortunately, symmetric with respect to r, s, and t. (The alternative to its being symmetric would be for it to be wrong, since the original problem was symmetric.) This sort of formula is called a multinomial coefficient.

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