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Worlds Out of Nothing

A Course in the History of Geometry in the 19th Century

Springer

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Preface

In 1789 geometry was at a low ebb. Euler had written on it – he wrote on everything – but his successor, Joseph Louis Lagrange was much more of an algebraist and an analyst. The great flowering of French mathematics that may be said to have started with Lagrange's arrival in Paris in 1787 at the age of 51 and continued with Laplace, Legendre, and then the generation of Cauchy, Fourier and Poisson, accomplished much and innovated widely, but less in geometry than in other areas. Only Gaspard Monge stood out, both as an original geometer and as an inspiring teacher. In 1914, at the other end of what historians call the long 19th century, geometry was not only a major branch of mathematics with several new branches springing from it, it could claim to have been one of the most provocative and challenging in its implications for the nature of mathematics. There were whole new geometries, some contesting the centrality of Euclidean geometry and therefore the framework for all of mechanics, others arguably yet more fundamental, and pervading more and more of the domain of pure mathematics. This transformation of geometry is the theme of this book.

This book discusses the ideas, the people, and the way they (the people and the ideas) fit into larger pictures. It starts with the rediscovery of projective geometry in post-revolutionary France by Poncelet, a student of Monge. It then takes in the discovery of non-Euclidean geometry by Bolyai in Austria–Hungary and Lobachevskii in Russia in the 1820s. The reception of Poncelet's work was poor, that of the work of Bolyai and Lobachevskii dismal, and we consider why this was. With these clear examples in place showing the importance of the social in the reception of mathematical ideas, the book then considers how matters turned around. For projective geometry, the algebraic methods of Möbius and Plücker are prominent because they not only led to the resolution of a major paradox facing Poncelet's work they also opened up the little-studied domain of plane curves of degree 3 or more. For non-Euclidean geometry, the crucial ideas are those of Gauss, of Riemann, however obscurely they were presented, and of Beltrami. By the 1870s, the new geometries were secure. Projective geometry was fast becoming the central topic in geometry, non-Euclidean geometry had been given a presentation acceptable to mathematicians at least, and geometry was a topic of growing importance in mathematics. The book moves towards its conclusion with a consideration of Klein's view of geometry, the subsequent growth of axiomatic projective geometry, first in Italy and soon afterwards in the hands of David Hilbert, and with the work of Henri Poincaré on non-Euclidean geometry, including his conventionalism. The book ends by briefly examining the formal aspect of mathematics, the relation to physics in the work of Einstein, and some thoughts about the truth, if any, of mathematics.

The book offers a number of unfamiliar ways in which mathematics can be thought about. Its relations to philosophy and to physics are noted and at times explicitly discussed, and indeed quite deliberately the book discusses some of those aspects of 19th-century geometry of interest to philosophers of mathematics today. Because the book is about mathematics but is not a mathematics course, original motivations matter, individuals' critical judgements matter and forming an overview matters. There is an important place in a modern mathematics degree for imperfect understanding (as opposed to education exclusively delivered in bite-sized chunks designed to promote technical mastery) because imperfect understanding is all we can bring to most issues. No one understands enough about all the major problems of the day, but all of us must grapple with them on pain of being irresponsible. We can, sometimes, form what we must admit is an imperfect grasp of what the experts say, and base our actions on that. In a much more modest way, a book that invites you to think about how imperfect judgements can most reasonably be formed has a certain justification. It is equally true that writing about mathematics is a valuable skill. Almost every graduate will go on to spend a great deal of time writing: writing reports, bids for money, research papers, policy statements, teaching documents.

We lack a recent history of projective geometry and of many, if not all, of its major protagonists. For too long now, readers have had to fall back on Coolidge's *A history of geometrical methods*, first published in 1940 [43], supplemented by a very small number of more specialised studies. Coolidge's book has its uses – it is mathematically accurate – but it slights the social aspects, rewrites the mathematics, and can degenerate to a list of results. This book therefore offers the first recent treatment of the work of Poncelet, although the reader should consult Bos et al. [22] for a splendidly thorough account of Poncelet's porism. The argument presented here that Plücker's resolution of the duality paradox was decisive for the future of projective geometry seems to be new, and indeed Plücker's contribution has been rather marginalised. The argument that research in projective geometry was partly driven by a desire to make it fundamental is new. I note it here in such matters as the production of a projective definition of conic sections. It should be followed through to a proper study of projective transformations and the axiomatisations of the subject; I hope to complete such an analysis in a forthcoming paper.

I have written in various places on various aspects of the history of non-Euclidean geometry and I did not want to burden this book with too much repetition. I continue to hold a restrained view of what Gauss knew; however, Erhard Scholz urges a different view forcefully [219]. I have tried to draw attention to some of the better recent investigations into Riemann, but much had to be set aside. I was happy to be prodded into rethinking what I knew about Beltrami's papers, which make an unexpected appearance in Klein's study of projective geometry.

For the later period, Toepell's invaluable study of Hilbert's route into projective geometry [236] made that work easier, and Marchisotto's work in progress on Pieri will do much to illuminate the Italian contributions (also helpfully discussed recently by Avellone et al. [5] and Bottazzini [23]). But we are particularly fortunate to have *David Hilbert's lectures on the foundations of geometry, 1891–1902* edited by M. Hallett and U. Majer finally in print [107]. This book not only reproduces the early lecture courses Hilbert gave that led up to his famous *Grundlagen der Geometrie* of 1899 [119], and which are already very informative about how Hilbert worked and reworked his ideas, it has an excellent, instructive commentary. Speculations at the end of the present book about Einstein's ideas, and about the philosophical implications of the new geometries, owe much to the work of other people, but I believe they are fresh in the present context.

However, I must stress that this book grew out of a course in which I lectured on some, even most, of the content of the 30 principal chapters to be found here, including the revision material, and invited students to read the rest. It is my hope that the book can be used in the same way, and various selections of the material are possible. It is not an attempt to write a complete history of geometry in the 19th century, for the simple reason that too much is missing. Quite simply, a manageable course that reflects the coherence of the developments and captures enough of their significance to be worthwhile must leave things out. I could have said more about Monge's mathematical work, for which see Taton's book [235] and the relevant chapter in Lützen's book on Liouville [158]. Research needs to be done on Chasles' presentation of projective geometry, and the way his work eclipsed that of Poncelet. Similarly, Steiner's contribution is slighted here and does not seem to have been studied by historians of mathematics recently. Von Staudt was almost wholly omitted from the course – amends are made here in the Appendix.

Not entirely for reasons of space, I have also omitted Grassmann entirely.[1] He came to regard his first book, of 1844, as "an utter failure" [95, p. 19] and radically revised it in 1862. This was both an extension of the former and a reworking of it in more mathematical, less philosophical dress, because, as Grassmann put it in his 1862 book, of "the difficulty ... the study of that work caused the reader on account of what they believed to be its more philosophical than mathematical form." [96, p. xiii] As he noted in the foreword to the second edition of his first book in 1878, its reception only began in 1867 with a favourable notice by Hankel and more importantly by Clebsch.

Almost every social and contextual issue raised could have been discussed further. When much fuller histories of this material are written, such matters as the how, when and where of what was done in geometry in the 19th century will have to be examined, audiences analysed, many gaps filled in.

On a positive note, the 19th century is a remarkable century in the history of mathematics. It is the century in which our ideas of mathematics changed most radically, and in which many of the ideas that form the undergraduate syllabus were created (rigorous analysis, group theory, linear algebra, and much of the geometry that this book is particularly about). I hope readers will find it interesting to see some of the ideas which may have occupied them at university emerge in their historical context.

Mathematics is a great cultural adventure. It's been going for over 5,000 years as a written activity[2], it shows no signs of stopping. This book offers a chance to examine some of the things that make mathematics important, and to obtain some kind of an overview of that activity. If we ask why anyone should care about all this old material, one answer takes us to another question: what is attractive about mathematics? It is notoriously hard to tell non-mathematicians what the appeal of the subject is. Another answer is that, up to a point, we live in a society that sells us certain ideas about mathematics in particular and education in general as intrinsically interesting, and marginalises others. So we can notice that society offers us certain loaded choices, and ask how, and why. In this way, we start to understand mathematics as a social enterprise, and begin to think about what mathematics is, and why.

I have taken the opportunity to make a number of changes to the book. Athanase Papadopoulos is preparing the first English translation of Lobachevskii's *Pangéométrie* of 1855 (see [156]) and he was kind enough to show me his work in progress. This has led me to reconsider what I have written about

[1] Grassmann's major works were published in 1844 [93] and 1862 [94], English translations are in [95] and in [96], respectively.

[2] My thanks to Eleanor Robson for telling me that the earliest mathematics exercises for trainee scribes are from late fourth-millennium Uruk (a city in southern Iraq).

Lobachevskii, to make a short mention of his path to his great discovery and his publications of 1829, and to make some mention of his *Pangéométrie*. I hope readers will read that work in full, when it becomes available in its new, English, translation with its richly informative commentary. I also thank him for telling me, what I should have known, that the University of Kasan was not a backwater but in fact the second-oldest university in Russia, founded after Moscow and before St Petersburg, and was well staffed from its inception. This, of course, helps explain why Martin Bartels was there and was one of Lobachevskii's teachers.

Marvin Greenberg, whose *Euclidean and non-Euclidean geometries* I warmly recommend for its many insights from the standpoint of modern geometry, has spurred me in numerous ways to reconsider what I wrote, and I thank him for making me look again at Saccheri's arguments, at Janos Bolyai's squaring of the circle, and at several other matters. Readers may also profitably consult his paper Greenberg, M.J. 2010 Old and new results in the foundations of elementary plane Euclidean and non-Euclidean geometries, *American Mathematical Monthly*, March 198–219.

I would also like to draw attention here to Robin Hartshorne's *Geometry: Euclid and beyond* for its particularly lucid explanations of many mathematical topics that lie just beyond this book, including, for example, the subject of non-Archimedean geometries, which relate in fascinating ways to the topic of parallelism.

Conversations with a number of historians of mathematics have sharpened my opinions on a number of topics. It turns out to be harder than I had thought to say when the subject of projective geometry acquired a local habitation and a name. From the time of Poncelet onwards people spoke of projective properties of figures and of their projections, but as these ideas came to constitute a definite subject it was often called "synthetic geometry". I am not aware of the term "projective geometry" in the writings of Chasles, Hesse, Steiner, or von Staudt. Klein spoke of projective geometry in his Erlangen programme as late as 1872; the first to call a book by that name may be Cremona in 1873.

A number of people wrote in to tell me of numerous smaller, mostly typographical, errors in the book. I thank them all and apologise to everyone. I particularly want to thank Dirk Schlimm and Marvin Greenberg, but also, among the students on the MSc course at the Open University in various years, Peter Lush and Dave Sixsmith.

Very amusingly, it has turned out that the picture said to be of Legendre (Figure 7.3 in the first printing) is not of the mathematician Adrien-Marie Legendre at all, but of an entirely unrelated namesake. The whole story, with information about how it was discovered, is told in Peter Duren's article "Changing Faces: The mistaken portrait of Legendre" in the December 2009 issue of the

Notices of the American Mathematical Society, 56, 1440-1444. A newly discovered cartoon of Legendre, shown on the front cover of the journal, is now thought to be the only likeness we have of the mathematician.

How to use this book

This book is the fruit of a secondment to the University of Warwick for one term a year from 2001 to 2004, where I taught it as an option to third- and fourth-year students. To underline the fact that this book can be used as a history course, in which opinions play a major role, and not a dispassionate mathematics course, I have retained the informal, chatty style of the lectures. In it I have tried to present everything the student needs to read except for some primary sources that are available over the Web, and also to give detailed advice (to students and their instructors) on how to tackle the assignments.

A book on the history of geometry studies a body of ideas and practices historically. What is discussed is the production and reception of ideas, and how this was affected by the social context. The ideas are those of mathematics, the practices those of mathematicians. The mathematics involved is a substantial mathematical diet, even at the University of Warwick, where students may study non-Euclidean geometry. However, I make the claim that a course based on this book can be done, and done well, by anyone willing to think about these ideas and that all the necessary mathematics is presented here. This is a defensible claim because the ideas here are to be understood as a historian would treat them, and not as a mathematician would. That is to say, they are to be understood so that arguments about their production and reception can be understood. Insofar as these arguments require that an informed opinion be held about some mathematical argument, that piece of mathematics belongs in a history course, but the whole mathematical package (precise definitions, proper proofs, abundant practice with techniques) does not.

That said, the mathematics in a course on the history of mathematics should not be mysterious, and the assessment was devised to minimise this risk. It is given here much in the form it was given to the Warwick university students, and at about the same time (see Chapters 12, 21 and 31, where my ideas about assessment in general and these assessment questions in particular are set out). The opening question each year asked for a short explanation of the reception of the work of either Poncelet or Bolyai and Lobachevskii. This tests the student's ability to digest some six chapters of material and isolate the key features in about 500 words. The word length in this assignment is deliberately short so that the assignment is hard and a process of analysis and selection must take place. The second assignment (1,500 words) asked that they talk their way through either Salmon's classification of quartic curves, or Cremona's projective study of conics and duality, or Lobachevskii's 1840 presentation of non-Euclidean geometry (students are encouraged to use the Poincaré disc model). Real mathematical understanding is required to do this well. The Salmon material, which closely follows Plücker's original presentation of these ideas, was new to the students, quite tricky and not free of error. The Cremona passage is in a kind of mathematics they cannot have seen anywhere else, and the Lobachevskii question asks them to explain in what way the disc model clarifies what Lobachevskii left obscure. (The students knew by then that this lack of clarity was not a major reason for Lobachevskii's ideas doing so badly in his lifetime.) It is only with some idea of the validity of a disc model that the Lobachevskii passage becomes readable at this level, and even then it is not easy. The final assignment was to summarise the key developments in geometry in the long 19th century and account for them. It is evident that the mathematical ideas have not come from nowhere, and have not succeeded simply on their own merits; the purpose of this assignment was to give the students a chance to demonstrate some overall mastery of a slew of ideas and relate them to contemporary opinions about their worth.

Inevitably, answers to such questions are more like opinions (let us hope, well-supported opinions) than they are facts. There *are* facts in history, a great many of them, but how they are organised, what weight to attach to them, is a matter for each historian to decide. Being a historian is a matter of producing arguments: establishing that these facts are evidence relevant to a certain claim; that the conclusion they support is this and not that; that the confidence one can have in the conclusion is strong or weak. So the answers to historical questions are a mixture of facts and arguments, and one aspect of reading history is sorting out facts from judgements in any argument. Different historians, different people, inevitably and rightly differ among themselves in the judgements they reach.

I have tried to suggest a range of source material, but inevitably language is a problem. The original sources for this material are overwhelmingly German and French, and I was able on occasion to find French students willing to go further with their study of Chasles. There is probably more worth in including Italian, and even Russian or Latin, than English sources, and even allowing for the existence of translations (and providing some more), there is not enough that English students can read. To that extent, even the splendid, and ever growing, Digital Mathematics Library [53] is not enough. On the other hand, the seductive Web ever beckons, with items from the wonderful to the downright odd. Safest to say that used with thought, and in conjunction with other material, it is a fine resource.

The fact that this book can be used as a course explains some other possibly unexpected features. I did not ask the students to read Lobachevskii's booklet of 1840 in detail until quite late, because it is surely too hard without something like a disc model in hand. I could have provided more mathematical examples at every stage, but I have tried to steer a line between providing enough information for it to be possible for a student to make an informed judgement about, say, what Plücker did, and teaching how to dualise a wide range of curves. That would require wrestling with resolvents; but what a historian of mathematics has to do is reach an informed opinion and I judged that eloquent examples could profitably substitute for theorems. Students were invited to familiarise themselves with these examples all the same. For the benefit of users of this book, some mathematics is presented very swiftly in various places so that other approaches can be taken to this material.

The best way to find out about the lives of mathematicians is to consult the *Dictionary of scientific biography* [90] and then to go to the sources mentioned there. Entries vary from the short to almost book length; that for Laplace is a multi-authored work of 130 pages. The Web is another source of information, of uneven merit.

I should like to thank the staff at both The Open University and the University of Warwick who made it possible for me to teach this course at Warwick. Above all, I should like to thank the students at Warwick. Each year, well over a hundred enrolled, completed the course and reported positively on their experiences. Most were taking degrees in mathematics, some in mathematics and statistics, some in physics, some in philosophy. Individually and collectively they demonstrated a hugely encouraging desire to grapple with difficult material. They did so successfully and often impressively, and there are many points in this book that I owe to their persistence and indeed to their insights. I am also happy to thank Nicholas Jackson for his help with everything to do with producing the index.

The manuscript then passed to Jennifer Harding, whose careful and astute editing did much to sharpen this book, and whom I am also happy to thank. I also thank Aaron Wilson for his fine work on the illustrations. At the same time Springer sent it to no less than six referees, and while they all reported positively on the book, which was gratifying, several took the time to identify failings and to suggest improvements, and for those comments I am particularly grateful. My good friend Erhard Scholz also read the book with his customary care and his erudition has saved me from errors I would blush to admit to; I offer him heartfelt thanks.

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