2
Poncelet (and Pole and Polar)

Figure 2.1 Poncelet

2.1 Poncelet reminisces

Listen to his words:

> Following the example of a celebrated contemporary novelist, whose statue stands at the entrance to the room where our academicians hold their private meetings to glorify, without doubt, a political and religious system from the day before yesterday and which is still fashionable today, I could have entitled this work, which is purely mathematical, *Memoirs from beyond the tomb*. It is, in fact, the fruit of the meditations of a young lieutenant of the engineers, left for dead on the fatal battlefield of Krasnoy, not far from Smolensk, and for a long time strewn with the bodies of the French army. There, in that terrible retreat from Moscow, seven thousand Frenchmen, exhausted by hunger, cold and fatigue, under the orders of the unfortunate Marshal Ney, came, deprived of all artillery, on the 18th of November 1812, the an-

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> niversary of the Russian Saint Michael, to fight a furious, bloody and final combat with twenty-five thousand soldiers, fresh and equipped with forty cannons of Field-Marshal Prince Miliradowitch, who himself would soon become the victim of a military conspiracy hatched in the bosom of the modern capital of the Muscovite Tsars. But the adoption of such an ostentatious title, however justifiable it might seem, would seem with good reason to be a ridiculous plagiarism, an overweening imitation with perhaps a permitted licence, of the avowed leader of the romantic novel in our France, at a time of moral perturbation as much political as literary. A similar title, besides, would suggest of this modest book neither the serious and reserved habits of the author, still less the character, the aptitudes, the tastes which presume a sincere love of the truths of geometry, whose profound culture calls for a spirit disengaged from all foreign passion and, one might say, of any earthly interest.
>
> Now, such is precisely, and in some way inevitably, the moral and mathematical position of the author of this work in the distant prisons of Russia. Much later, when he appeared to neglect the study of this geometry in favour of teaching the mechanical and industrial sciences, he had in reality no other purpose but to make it useful to the working class and the youth of our schools; he wished to inspire them with a love of the eternal truths of science, a hatred of the intrigue and the sophisticated subtleties of a greedy charlatanism, which signals an epoch where, among the conquests of the modern spirit, one deplores with sorrow the aberrations, the passion for money which dishonours our character, our customs, and even our national literature. Finally, if in the honourable steps of Vauban and Belidor, of Bézout, of Borda and Coulomb, of Daniel Bernoulli, of Euler and so many other illustrious benefactors of humanity, he has attempted to make useful to the class of artists or engineers, in writing for the general public in such a way as to avoid the reproaches too often and rightly addressed to the members of the profession.

The novelist referred to is François-Auguste-René, Vicompte de Chateaubriand (1768–1848), one of France's first Romantic authors. Chateaubriand had initially refused to side with the Royalists in the French Revolution, but eventually did so (after the flight of Louis XVI in June 1791) and was wounded in action. He left for England in May 1793, where he wrote his *Essais sur les révolutions* (*Essays on revolutions*, 1797) [34]. In 1800 he returned to Paris, and in 1802 to the traditional Christianity he had once disclaimed. *Encyclopaedia Britannica* [63] comments that "His apologetic treatise extolling Christianity, *Le génie du christianisme* (*The genius of Christianity*, 1802) [35], won favour both with the Royalists and with Napoleon Bonaparte, who was just then concluding a concordat with the papacy and restoring Roman Catholicism as the state religion in France." Napoleon made him first secretary to the embassy to Rome on the strength of it, but Chateaubriand resigned in 1804 in protest at Napoleon's execution of a supposed conspirator, and threw himself into the literary life with many love affairs. The Bourbon monarchy that reigned from 1814 to 1830 favoured him with many appointments, but after then he lived a private life. His *Mémoires d'outre-tombe* (*Memoirs from beyond the tomb*, 1849–50) [36], was written for posthumous publication. It is an account of his emotional life, mixed up with contemporary French history, his unstinting appreciation of women and his sensitivity to nature; it succeeds (or succeeded) in evoking vividly the spirit of the Romantic epoch, and became his best-remembered work.

The long quotation is from Jean-Victor Poncelet in 1862, in the preface of a book (*Applications d'analyse et de géométrie* [205]), which was an annotated set of the notes he made while a prisoner in Russia in 1813 and 1814. He fought in the terrible battle at Krasnoy but was taken prisoner, and survived the years that followed only by luck. In the winter of 1812 it became so cold that even the mercury in the thermometers froze (which occurs at a temperature of $-39°$C). He managed to get to the hospital at Saratov, where he remained a prisoner until the defeat of Napoleon and the Treaty of Paris was signed on 30 May 1814; his journey home took two and a half months, and he arrived in Metz on 7 September. In prison there was nothing to restore him to health but the April sun, and there, to distract his spirits, he resumed his study of mathematics, even though there was not even the distant echo of the profound analytical works of Euler, Bernoulli, Huygens, Newton and d'Alembert, not to mention the more recent and no less admirable work of Lagrange, Legendre, Laplace, Monge and their disciples.[1]

He recalled that he had graduated from the École Polytechnique in November 1810, and left the Applied Engineering School in Metz in March 1812 to work on the fortifications of the Dutch island of Walcheren. At the École Polytechnique he had acquired a taste for the work of Monge, Carnot and Brianchon, but, completely cut off as he was in Saratov, he knew nothing of their recent work published before his return to France in 1814. This is why he occupied himself summarising all he knew of the mathematical sciences in notebooks that he then distributed to his fellow prisoners who wanted to finish an education disrupted by the incessant military campaigns.[2]

[1] For a biography of Poncelet, see Tribout [239], who points out that much original documentation about Poncelet was destroyed in the First World War.

[2] Carnot's study of the properties of pairs of intersecting curves had been published in 1806 as the third of his books aimed at revising and extending the science of geometry. Lazare Carnot (1753–1823) was another mathematician and scientist ac-

Poncelet was struck by the observation that the elementary parts of the differential and integral calculus and of algebra had left the most vivid impression on his mind, and he could rediscover the basic results about areas and volumes, even though he had forgotten them. It seemed that those ideas would not be forgotten at any stage in life. But the complicated and laborious methods, whatever their interest or scientific merit, the abstract and spiny proofs which have been introduced into mathematics, and which would never have been recommended by Lagrange, Laplace, or Monge in their admirable lectures in the early École Polytechnique and École Normale – they vanished entirely. As for mechanics, Poncelet confessed that apart from purely geometric theorems on the composition of forces he remembered nothing. Galileo's laws left absolutely no trace in his mind. It was in vain that he tried to write the differential equations of motion with respect to the coordinate axes, which is why, when in charge of creating a course in mechanics at the Applied Engineering School in Metz he became an innovator by conviction and a reformer by necessity.

Poncelet's purpose in publishing these notes in 1862, the 50th anniversary of his capture, was not just to cheat death by writing a book that would survive. Poncelet, like so many old men, was fighting his old battles one more time. These notebooks were the proof that he had had the priority for several discoveries over others who had not been deprived of their liberty in the service of France. He had argued his case before, when these theorems had been published; with this book he would argue them from beyond the grave.

Poncelet had published several theorems in the years 1817 to 1832, and one remarkable book. Not all of these results were contested by Gergonne, one of his rivals, and most were published in Gergonne's *Annales de Mathématiques Pures et Appliquées*, the only journal at the time entirely devoted to mathematics. (The journals of the learned societies at the time covered all of science, loosely divided into topics in some journals but not others.) The first of these results goes to this day by the name of Poncelet's porism; what it is will be made clear in due course. Another, published in 1818 (no. 2 in Poncelet's list[3]), established that the number of tangents common to two curves of degrees m and k is in general, and at most, $mk\,(m-1)\,(k-1)$.

The third was a novel solution to a problem first raised by the ancient Greek geometer Apollonius: find a circle tangent to three given circles. Yet another was a long article, published in 1820, covering some of the same material later

tively involved in politics. His defence of Paris, in 1794, when he was in charge of the revolutionary army, had earned him the popular title of "Organiser of the Victory". For a thorough account of his life and work, see Jean and Nicole Dhombres, *Lazare Carnot* [52].

[3] This list is Poncelet's report on his own work, the *Notice analytique sur les travaux de M. Poncelet*, Paris 1834 [204].

treated in the book of 1822 on conic sections and quadric surfaces. We shall see that this book is his most remarkable and lasting claim to originality. In 1829 he wrote an important paper extending the theory of pole and polar to curves of degree greater than 2.

With all this in mind, let's go back over the long passage above that opened this chapter. Unexpected, isn't it, in a mathematical book? Personally, I love these long French sentences. You can easily imagine the old man grabbing you by the arm, telling you of the lofty mission of his life that began so near to death so many years ago, on the field that marked the final defeat of the country's greatest military leader. Don't you feel, just as certainly, the literary power of this? Who is this strange, passionate, eccentric old man, straight out of fiction?

Listen to the text:

> Memoirs from beyond the tomb ... the meditations of a young man left for dead ... seven thousand exhausted men, ill equipped, facing twenty-five thousand well-equipped soldiers. ... And yet, you know, the Russian Field-Marshal dies, and I, young Poncelet survived. ... And I (if you can believe it) am "the author of a modest book, and a person of serious and reserved habits".

Did Poncelet himself really believe it? Or is that his deluded self-image? Or is it a permissible view of himself, and yet false? People are different people at different times. What did it mean to escape death by the merest chance, when so many you know died, and then to endure two years in limbo, from the age of 24 to 26, not knowing for most of that time what your future would be? Denied, in any case, the pleasures of the prime of one's life. To find survival in mathematics, and perhaps to dedicate oneself to it on one's return.

> A love of the eternal truths of science, a hatred of the intrigue and the sophisticated subtleties of a greedy charlatanism.

Some things matter; some things don't. The eternal truths of science ... this modest book. False modesty? An honest recognition that, amid all the desolation of the Russian prison, there came no great mark of redemption? Or in the end did Poncelet make eternal mathematics?

Some things matter; some things don't: to have seen mercury freeze, and to survive even that. What does that count for, when one is old and thinks over one's life?

Poncelet's antithesis was between the eternal truths of geometry and the grubby world of intrigue and charlatanism. Real enough, perhaps. But we have to live in it. Did Poncelet pass beyond any earthly interest? His mathematics, shortly to be described, was not to everyone's taste. The mathematician may like it, the scientist may not. But Poncelet eschewed the world of recondite learning in favour of popular instruction. Even when he moved over to the study of machines, this was, he now said, “to make [geometry] useful to the working class and the youth of our schools; and to inspire them with a love of the eternal truths of science” [202]. They go together, it seems, utility and eternal truth.

And who will be helped? The working class, and the young in general, of France. They will be rescued, Poncelet evidently hoped, from “the passion for money which dishonours our character, our customs, and even our national literature”. “Ours” here means that of France. The whole peroration at the start is about the French defeat. It is Frenchmen who die, bravely we must presume, and glory with it. French glory. It is France that stands in a moral swamp in 1862. It is French dignity that Poncelet has spent his life, in his way, trying to restore. There are some typically French notes struck, here and later. The pantheon of great names: Lagrange, Legendre, Laplace and Monge, the great days of the École Polytechnique and the École Normale; Metz, where the Applied Engineering School was.

Finally, the mathematics. The honest recognition that only the elementary bits survived, and all that subtle stuff went clean out of his head. Poncelet, at least, was not one of those mathematicians who only wake up when the rest of us find it too difficult. He was one of those mathematicians who remade things, made them new, made them according to his rules. The two years as a prisoner in Russia made him an original, someone who did things his way. I think that fits with the rest of the person he made himself become between 1814 and 1867, when he died. But he also had a dislike of clever, tricky mathematics. He won’t teach that kind of thing, if he can avoid it. He wants to be understood by the common man, if you like. Is this part of his dislike of “sophisticated subtleties”?

2.2 Poncelet’s mathematics

What, then, was Poncelet’s original mathematics? I shan’t take us into his study of machines. Poncelet’s porism will be discussed briefly below; a porism – the term is Greek – is a striking thing: a problem that either cannot be solved, or has infinitely many solutions. The paper of 1818 (no. 2 in Poncelet’s list) is on the number of tangents common to two curves, and the study of curves other than conics was very little understood at that time. Since conic sections were Greek in origin, it’s clear that any exploration of this topic was like entering a new continent. Then we go back to a problem posed by the ancient Greek geometer Apollonius: find a circle tangent to three given circles. This won’t detain us. Suffice it to say that the classical solutions are long and difficult.

I have already said that the long article, published in 1820, and the book of 1822 on conic sections and quadric surfaces are his most remarkable. The paper of 1829 on the theory of pole and polars for curves of degree greater than 2 is, it turns out, another major paper opening up the study of curves other than conics.

Let us open the book of 1822 [202]. It too has an account of the circumstances in which he was led to his discovery of projective geometry, during the months following his capture by enemy troops during Napoleon's ill-fated invasion of Russia.

> This book is the result of researches which I undertook in the spring of 1813 in the prisons of Russia: deprived of every kind of book and help, and the proper facilities, above all distracted by the misfortunes of my country, I was unable to give it all the perfection desirable. However, I had at the time found the fundamental theorems in my work: that is to say the principles of central projection of figures in general and conic sections in particular, the principles of secants and tangents common to those curves, those of polygons which are circumscribed or inscribed to them.

And he gave this account of the aim of his work:

> The point of this book, voluminous as it may appear, is less to increase the number of properties [of figures] than to indicate the route by which they are found. In a word, I have sought above all to perfect the method of proof and discovery in elementary geometry.

In his *Treatise on the projective properties of figures* [202] (see the extract below), Poncelet contrasted what he called the geometry of particulars with analytic geometry, which he also called algebraic analysis. He found algebraic analysis to be well developed, but the geometry of particulars ("individual curves and surfaces") to be lacking in some respects. He regretted that the arguments used in synthetic geometry lacked generality. By contrast with algebra, which can handle negatives and even imaginary magnitudes, synthetic geometry "is more timid or more severe". For instance, if three points A, B, C lie on a line and C is between A and B then we write $AC + CB = AB$, but if it lies outside it we write $AC - CB = AB$ or even $CB - CA = AB$. So if in the course of a proof a perpendicular DC from D to AB falls inside or outside AB the whole argument must be carried on in two variant forms.

Poncelet gave examples of what he thought was an acceptable general rule for applying the same argument to different figures. If one figure could be obtained from another by changing it by insensible degrees, say by an arbitrary continuous movement, then he considered it obvious that some properties of the first figure would persist through these changes to the final figure, provided, of course, that one took note of the fact that some quantities (which could be specified in advance) change in size, vanish, or become negative. He called this proposal the principle or law of continuity. It is, to be frank, somewhat vague. It was never made rigorous by Poncelet, and it was strongly attacked as soon as it was published, as we shall see. He admitted it led to paradoxes; for example, where are the common points – which he called ideal points – to two seemingly non-intersecting circles? According to his principle of continuity, these points should be obtained by a continuous movement of a pair of intersecting circles. But, he said, the paradoxes do not go away if you use algebraic, rather than geometric, analysis. So the problem was to explain them directly and not to let them halt progress.

It is possible to feel here that Poncelet has worked something long and difficult out for himself, building on the insights obtained in Saratov. He is in possession of not just a theorem, but a theory – a whole way of thinking about projective geometry. Naturally he believes that it speaks clearly, directly and accessibly to everyone. It is not the spiny stuff that does not stay in the mind, but clear basic principles. Nothing sophisticated, only limpid truth. If we find it obscure, well, give it time seems to have been his view.

The terms Poncelet applied to geometry ("analytic", "synthetic", "algebraic") are not entirely easy to use; they shift their meanings a little from user to user and date to date. The principal distinction being made is between analytic or algebraic geometry, which can even be called coordinate geometry, on the one hand, and synthetic geometry on the other. Synthetic geometry then means geometry (loosely) in the style of Euclid's *Elements* in which what is discussed are curves, lines, angles and areas, and algebra is avoided.

2.3 Poncelet, *Traité des propriétés projectives des figures*, 1822 [202, pp. xix–xxvii]

> In ordinary geometry, which one often calls synthetic, the principles are quite otherwise, the development is more timid or more severe. The figure is described, one never loses sight of it, one always reasons with quantities and forms that are real and existing, and one never draws consequences which cannot be depicted in the imagination or before one's eyes by sensible objects. One stops when those objects cease to have a positive, absolute existence, a physical existence. Rigour is even pushed to the point of not admitting the consequences of an argument, established for a certain general disposition of the objects of a figure,

for another equally general disposition of those objects which has every possible analogy with the first. In a word, in this restrained geometry one is forced to reproduce the entire series of primitive arguments from the moment where a line and a point have passed from the right to the left of one another, etc.

Now here precisely is in fact the weakness; here is what so strongly puts it below the new geometry, especially analytic geometry. If it was possible to apply implicit reasoning having abstracted from the figure, if only it was possible to apply the consequences of that kind of reasoning, this state of things would not exist, and ordinary geometry, without needing to employ the calculus and the signs of algebra, would rise to become in all respects the rival of analytic geometry, even if, as we have said already, it was not possible to conserve the explicit form of the reasoning.

Let us consider an arbitrary figure in a general position and indeterminate in some way, taken from all those that one can consider without breaking the laws, the conditions, the relationships which exist between the diverse parts of the system. Let us suppose, having been given this, that one finds one or more relations or properties, be they metric or descriptive, belong to the figure by drawing on ordinary explicit reasoning, that is to say by the development of an argument that in certain cases is the only one regards as rigorous. Is it not evident that if, keeping the same given things, one can vary the primitive figure by insensible degrees by imposing on certain parts of the figure a continuous but otherwise arbitrary movement, is it not evident that the properties and relations found for the first system, remain applicable to successive states of the system, provided always that one has regard for certain particular modifications that may intervene, as when certain quantities vanish or change their sense or sign, etc., modifications which it will always be easy to recognise a priori and by infallible rules? . . .

Now this principle, regarded as an axiom by the wisest mathematicians, one can call the principle or law of continuity for mathematical relationships involving abstract and depicted magnitudes.

In the last analysis the principle of continuity has been admitted in its full extent and without any restriction by different geometers, who have employed it either overtly or tacitly, because without it they would be plunged into all the metaphysical considerations of imaginaries which have always been driven back from the narrow sanctuary of rational geometry. Its explicit use in this science is almost always limited to real states of a system which is transformed by insensible degrees. And even there it gives rise to the infinitely little and the infinitely great which geometers still seek, in our day, to banish from the domain of the exact sciences.

[...]

However, it will still not be difficult to establish this principle in an entirely direct and rigorous manner, with the aid of a calculus just like algebra the certainty of which is not the least to be doubted in our time, thanks to two centuries of efforts and success!

In any case will it be necessary, and will one not immediately admit the principle of continuity in its full extent into rational geometry, as one does at once in algebraic calculus and then in the application of calculus to geometry, if it is not a means of proof but rather as a means of discovery or invention? Is it not at least as necessary to point out the resources employed at various times by men of genius for discovering the truth, as the feeble efforts they have then been obliged to use to prove them according to intellectual taste, either timid or less capable of bringing them home?

Finally, what harm can result, above all if one is restrained in one's conclusions, if one never uses half-truths, if one never admits analogy or induction, which are often deceptive, and which it is not necessary to confound with the principle of continuity? In fact, analogy and induction conclude from the particular to the general, from a series of isolated facts not necessarily related, in a word discontinuous, to a general and constant fact. The law of continuity, on the contrary, starts from a general state and some sort of indeterminacy of the system (that is to say that the conditions which govern it are never replaced by still more general ones) and they remain in a series of similar states going from one to the other by insensible gradations. One insists, besides, that the objects to which it is applied are by their nature continuous or submit to laws which can be regarded as such. Certain objects can even change their position by a series of variations undergone in the system, others can move away to infinity or approach to insensible distances, etc.; the general relations survive all the modifications without ceasing to apply to the system.

The only difficulty consists, as we have seen, in understanding fully what one wants to convey with the word general or indeterminate or particular state of a system. Now in each case the distinction is easy. For example, a line which meets another in a plane, is in a general state by comparison with the case where it becomes perpendicular or parallel to that line. Similarly a line (straight or curved) which meets another in a plane, is in a general or indeterminate state with regard to that other and the same thing takes place even when it ceases to meet it, provided that the two states do not suppose any particular relation of size or position between these lines. The contrary will evidently hold when they become tangents, or asymptotic, or parallel etc.; they will then be in a particular state with regard to the primitive state.

2.3.1 Commentary

Poncelet's account of these mysterious points of intersection was obscure. It was not equivalent to what a geometer relying on algebraic methods would say – namely, that such circles meet in complex points – we note only that in this respect Poncelet's presentation of his ideas, however "geometric", was not likely to replace the algebraic formulations of the previous hundred years. Nor indeed did it. One interpretation of what it means is given below.

So far in this account, Poncelet could appear as something of a crank, an oddball who did well in the French academic system of the early 19th century. It is time to begin to explain why he merits the attention of historians of mathematics, and the attention of mathematicians in his day and subsequently.

Let us start with the idea of the polar line of a point with respect to a conic. This is a topic worth following in more detail, and Poncelet made considerable use of it. It is not original to him, indeed the idea goes back to Apollonius, and was taken up in an original way by Brianchon before Poncelet, as he happily admitted. If a line ℓ meets a conic at Q_1 and Q_2 then the tangents to the conic at Q_1 and Q_2 meet at point P, say, called the pole of ℓ, which lies, of course, outside the conic. He deduced that to each line ℓ one can associate a point P, and the converse is also true; given a point P outside the conic, one can draw through P two tangents to the conic. Label the points where they touch the conic Q_1 and Q_2; then to P one can associate the line Q_1Q_2, the polar of P.

What happens if you start with a line that does not meet the conic? Well, the line is made up of points. Let P be one of them, and construct the polar line Q_1Q_2. Let P' be another, and construct its polar line $Q_1'Q_2'$. Let P'' be a third, and construct the polar line $Q_1''Q_2''$. Then rather wonderfully, it turns out that these three polar lines meet in a point, let us call it Q (from which it follows that the polar line of each point on the line ℓ passes through the point Q). So we have a natural candidate for the pole of the line: the point Q.

This is the occasion to state my policy on old proofs in this book on the history of mathematics. In principle, every fragment we have of the past is part of the evidence. In practice, for the 19th century, there's too much evidence, and a selection has to be made. Sometimes I shall select a proof as evidence.

I shall then make it clear what I take it to be evidence of. Sometimes, and this business of poles and polars is a case in point, I just want you to know what the words mean and that the statements are true. Generally then I shall give a modern proof, which may be quite different from the ones used at the time. Such proofs are presented in order to make the story easier to understand. In this case, the proof displays a profound insight into the geometry, which is part of our historical case for the importance of Poncelet.

2.4 Pole, polar and duality

Result 1

With respect to the unit circle, $x^2 + y^2 = 1$, the point $P = (u, v)$, the pole, has the polar line $\ell : ux + vy = 1$.

Suppose that the point lies outside the circle, and the two tangents, call them t and t', from it to the circle meet the circle at the points $T = (a, b)$ and $T' = (a', b')$ respectively. The equations of these tangents are $ax + by = 1$ and $a'x + b'y = 1$. Now, (u, v) lies on t and t' if and only if $au + bv = 1$ and $a'u + b'v = 1$, and (u, v) is the unique point satisfying those two equations. However, $(a, b), (a', b')$ lie on the line $px + qy = 1$ if and only if (p, q) satisfies $ap + bq = 1$ and $a'p + b'q = 1$, so, by uniqueness, $(p, q) = (u, v)$ and the line through T and T' has the equation $ux + vy = 1$. This line is called the polar line (or polar) of the point (u, v), and that point is called the pole of the line.

Now consider the line $xu + yv = 1$. A typical point on it has coordinates $\left(t, \frac{1-tu}{v}\right)$.

The polar line of this point is $xt + y\left(\frac{1-tu}{v}\right) = 1$.

Consider the points for which $t = 0$ and $t = 1$: $\left(0, \frac{1}{v}\right)$ and $\left(1, \frac{1-u}{v}\right)$ respectively. The corresponding polar lines are $y\left(\frac{1}{v}\right) = 1$ and $x + y\left(\frac{1-u}{v}\right) = 1$ respectively, which clearly meet at (u, v) – solve for y, to obtain $y = v$, and substitute that in the equation for x.

This implies that the polar lines of points on the line $xu + yv = 1$ all pass through the point (u, v), as you can check directly.

Result 2

The polar lines of points on the line $xu + yv = 1$ all pass through the point (u, v).

So (by Result 1) we can start with a point P outside the circle and obtain its polar line. We can then take points on that line (most of which, after all, lie outside the circle) and take their polar lines: they all meet at P.

This is what is called a duality: from a point obtain a line, from a line obtain a point. Do it twice and the original point is returned.

We extend this to points inside the circle entirely formally, so to each point we have a line, and to each line we have a point, and duality applies. We can even start with the line, pass to its pole, and obtain the polar line of that point: it will be the line we began with.

Some consequences of, and observations about, this result: for example, if the point (u, v) lies on the circle, its polar line is the tangent to the circle at that point. You can see by drawing a diagram that as the point P gets closer to the circle, the corresponding points Q and Q', also get closer to P, and the line joining them gets closer to being a tangent.

On the other hand, not every line has an equation of the form $xu + yv = 1$. Those that don't are those of the form $xu+yv = 0$, the lines through the origin. And indeed, what is the polar line of the origin? The point $(0, 0)$ would seem to have the line $0x + 0y = 1$, which is $0 = 1$, as its polar, but that's nonsense. So there's a problem with such lines.

Can you prove both these results without algebra? Yes. Much of it goes back to Apollonius, and everything that was needed was in place in the 17th century. Poncelet knew that much of it was in a fine book by La Hire, his *Sectiones conicae* of 1685 [144], and Brianchon had published some ideas about poles and polars before his journey to Russia. He knew of Desargues, however, only from secondary accounts, because at that time no copy of Desargues' major work had apparently survived, and he praised him handsomely.[4]

Does this just work for circles? Indeed not. It certainly works for any conic section. You can see that from the algebra. It would become more complicated, but the main result, the passage from pole to polar and back, all of duality, would survive. If you don't believe that, you have three options:

1. slog through the algebra for a general conic;
2. find algebraically a transformation that maps the circle to the conic, and use it to find the formulae for pole and polar;
3. work out a way of seeing it. That is for the next chapter.

One final remark, why the names pole and polar? They are faintly reminiscent of geography – and so they should be. Their origins are in spherical trigonometry,

[4] Desargues' main work on geometry, his *Brouillon Projet* was published in an edition of 50 copies in 1639 which were as good as lost by Poncelet's time. Desargues' theorem was described in a work by Abraham Bosse in 1648, Pascal and La Hire also mentioned Desargues by name in their work on what we would call projective geometry, and in these tenuous ways his reputation was kept alive. But it was only in the mid-19th century that Michel Chasles found a copy of the *Brouillon Projet* made by La Hire in 1679, and this was published by Poudra in his edition of Desargues' works in 1864 [51]. The only surviving original copy of the *Brouillon Projet* was found by P. Moisy and communicated to R. Taton in 1951; it forms the basis of the edition of 1951 [234]. For an English translation, see [75].

as Chemla has described [37]. To each point on a sphere there is a natural great circle that comes with it: join the point to its diametrically opposite point, there is exactly one great circle perpendicular to that diameter. In the case of the north pole, that great circle is the equator. Conversely, to every great circle there are two points naturally associated with it, and they are antipodal – at opposite ends of a diameter of the sphere. That is where the word "pole" comes from in "pole and polar". "Polar" is a natural word to choose, given that "equator" is too strong.

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