Chapter 2
Discrete Fourier Transform

2.1 Definitions

The DFT and IDFT can be defined as follows.

2.1.1 DFT

$$X^{\mathrm{F}}(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \ldots, N-1, \quad N \text{ DFT coefficients} \tag{2.1a}$$

$$W_N = \exp\left(\frac{-j2\pi}{N}\right)$$

$$W_N^{kn} = \exp\left[\left(\frac{-j2\pi}{N}\right) kn\right]$$

where $x(n)$, $n = 0, 1, \ldots, N-1$ is a uniformly sampled sequence, T is sampling interval. $W_N = \exp(-j\,2\pi/N)$ is the N-th root of unity, and $X^{\mathrm{F}}(k)$, $k = 0, 1, \ldots, N-1$ is the k-th DFT coefficient. $j = \sqrt{-1}$.

2.1.2 IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X^{\mathrm{F}}(k) W_N^{-kn}, \quad n = 0, 1, \ldots, N-1, \quad N \text{ data samples} \tag{2.1b}$$

$$\left(W_N^{kn}\right)^* = W_N^{-kn} = \exp[(j2\pi/N)kn] \qquad e^{\pm j\theta} = \cos\theta \pm j\sin\theta$$

K.R. Rao et al., *Fast Fourier Transform: Algorithms and Applications*,
Signals and Communication Technology,
DOI 10.1007/978-1-4020-6629-0_2,

Superscript $*$ indicates complex conjugate operation. The DFT pair can be symbolically represented as:

$$x(n) \Leftrightarrow X^{\mathrm{F}}(k) \tag{2.2}$$

The normalization factor $1/N$ in (2.1b) can be equally distributed between DFT and IDFT (this is called unitary DFT) or it can be moved to the forward DFT i.e.,

2.1.3 Unitary DFT (Normalized)

$$\text{Forward} \quad X^{\mathrm{F}}(k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} x(n) W_N^{kn}, \quad k = 0, 1, \ldots, N-1 \tag{2.3a}$$

$$\text{Inverse} \quad x(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} X^{\mathrm{F}}(k) W_N^{-kn}, \quad n = 0, 1, \ldots, N-1 \tag{2.3b}$$

Alternatively,

$$\text{Forward} \quad X^{\mathrm{F}}(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) \exp\left(\frac{-j2\pi kn}{N}\right), \quad k = 0, 1, \ldots, N-1 \tag{2.4a}$$

$$\text{Inverse} \quad x(n) = \sum_{k=0}^{N-1} X^{\mathrm{F}}(k) \exp\left(\frac{j2\pi kn}{N}\right), \quad n = 0, 1, \ldots, N-1 \tag{2.4b}$$

While the DFT/IDFT as defined in (2.1), (2.3) and (2.4) is equally valid, for the sake of consistency, we will henceforth adopt (2.1). Hence

$$\text{DFT} \quad X^{\mathrm{F}}(k) = \sum_{n=0}^{N-1} x(n) \left[\cos\frac{2\pi kn}{N} - j\sin\frac{2\pi kn}{N}\right], \quad k = 0, 1, \ldots, N-1 \tag{2.5a}$$

$$\text{IDFT} \quad x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X^{\mathrm{F}}(k) \left[\cos\frac{2\pi kn}{N} + j\sin\frac{2\pi kn}{N}\right], \quad n = 0, 1, \ldots, N-1 \tag{2.5b}$$

Both $x(n)$ and $X^{\mathrm{F}}(k)$ are N-point sequences or length N sequences.

$$\underset{(N \times 1)}{\underline{x}(n)} = [x(0), x(1), \ldots, x(N-1)]^T \qquad N\text{-point data vector, and}$$

$$\underset{(N \times 1)}{\underline{X}^{\mathrm{F}}(k)} = \left[X^{\mathrm{F}}(0), X^{\mathrm{F}}(1), \ldots, X^{\mathrm{F}}(N-1)\right]^T \qquad N\text{-point DFT vector.}$$

$$W_N = \exp\left(\frac{-j2\pi}{N}\right) \qquad N\text{-th root of unity}$$

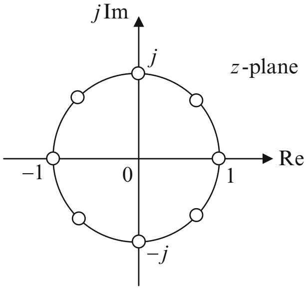


Fig. 2.1 The eight roots of unity distributed uniformly along the unit circle with the center at the origin in the z-plane

$$W_N^k = W_N^{k \bmod N}$$

$k \bmod N = k$ modulo $N =$ Remainder of $\left(\frac{k}{N}\right)$. For example 23 mod 5 = 3, $\frac{23}{5} = 4 + \frac{3}{5}$. Superscript T implies transpose.

$\sum_{k=0}^{N-1} W_N^k = 0$. All the N roots are distributed uniformly on the unit circle with the center at the origin. The sum of all the N roots of unity is zero. For example $\sum\limits_{k=0}^{7} W_8^k = 0$ (Fig. 2.1). In general, $\sum\limits_{k=0}^{N-1} W_N^{pk} = N\delta(p)$ where p is an integer.

The relationship between the Z-transform and DFT can be established as follows.

2.2 The Z-Transform

$X(z)$, the Z-transform of $x(n)$ is defined as

$$X(z) = \sum_{n=0}^{N-1} x(n)\, z^{-n} \tag{2.6a}$$

$f_s = \dfrac{1}{T} =$ sampling rate, $\left(\dfrac{\#\text{ of samples}}{\text{sec}}\right)$, or $T = \dfrac{1}{f_s}$, sampling interval in seconds.

$$X(e^{j\omega T}) = \sum_{n=0}^{N-1} x(n) \exp\left[\frac{-j2\pi f n}{f_s}\right] \tag{2.6b}$$

is $X(z)$ evaluated on the unit circle with center at the origin in the z-plane.

By choosing equally distributed points on this unit circle (2.6b) can be expressed as

$$X^{\mathrm{F}}(k) = \sum_{n=0}^{N-1} x(n) \exp\left[\frac{-j2\pi kn}{N}\right], \quad k = 0, 1, \ldots, N-1 \tag{2.7}$$

where $k = Nf/f_s$.

The Z-transform of $x(n)$ evaluated on the unit circle with center at the origin in the z-plane at equally distributed points, is the DFT of $x(n)$ (Fig. 2.3).

$X^{\mathrm{F}}(k)$, $k = 0, 1, \ldots, N-1$ represents the DFT of $\{x(n)\}$ at frequency $f = kf_{\mathrm{s}}/N$. It should be noted that because of the frequency folding, the highest frequency representation of $x(n)$ is at $X^{\mathrm{F}}\left(\frac{N}{2}\right)$ i.e., at $f = f_{\mathrm{s}}/2$. For example, let $N = 100, T = 1\ \mu\mathrm{s}$. Then $f_{\mathrm{s}} = 1$ MHz and the resolution in the frequency domain f_0 is 10^4 Hz. The highest frequency content is 0.5 MHz. These are illustrated below: (Figs. 2.2 through 2.4)

The resolution in the frequency domain is

$$f_0 = \frac{1}{NT} = \frac{1}{T_{\mathrm{R}}} = \frac{f_{\mathrm{s}}}{N} \tag{2.8}$$

$T_{\mathrm{R}} = NT$ is the record length. For a given N, one can observe the inverse relationship between the resolution in time T and resolution in frequency f_0. Note that $x(n)$ can be a uniformly sampled sequence in space also in which case T is in meters and $f_{\mathrm{s}} = \frac{1}{T} = \#$ of samples/meter.

The periodicity of the DFT can be observed from Fig. 2.3. As we trace $X^{\mathrm{F}}(k)$ at equally distributed points around the unit circle with center at the origin in the z-plane, *the DFT repeats itself* every time we go over the unit circle. Hence

$$X^{\mathrm{F}}(k) = X^{\mathrm{F}}(k + lN) \tag{2.9a}$$

where l is an integer. *The DFT assumes that* $x(n)$ *is also periodic* with period N i.e.,

$$x(n) = x(n + lN) \tag{2.9b}$$

For a practical example of Fig. 2.4, if a signal $x_1(n)$ is given as Fig. 2.5a, then its magnitude spectrum is Fig. 2.5b.

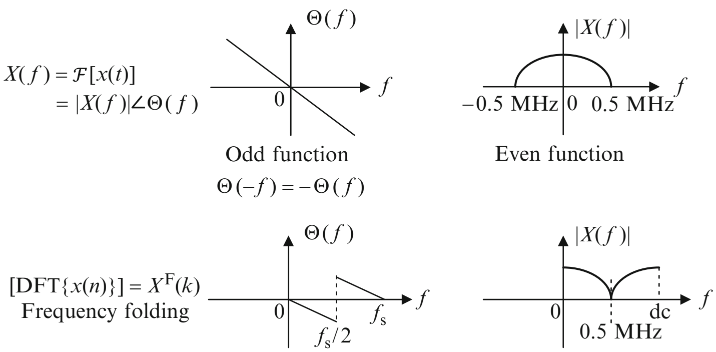


Fig. 2.2 Effect of frequency folding at $f_{\mathrm{s}}/2$

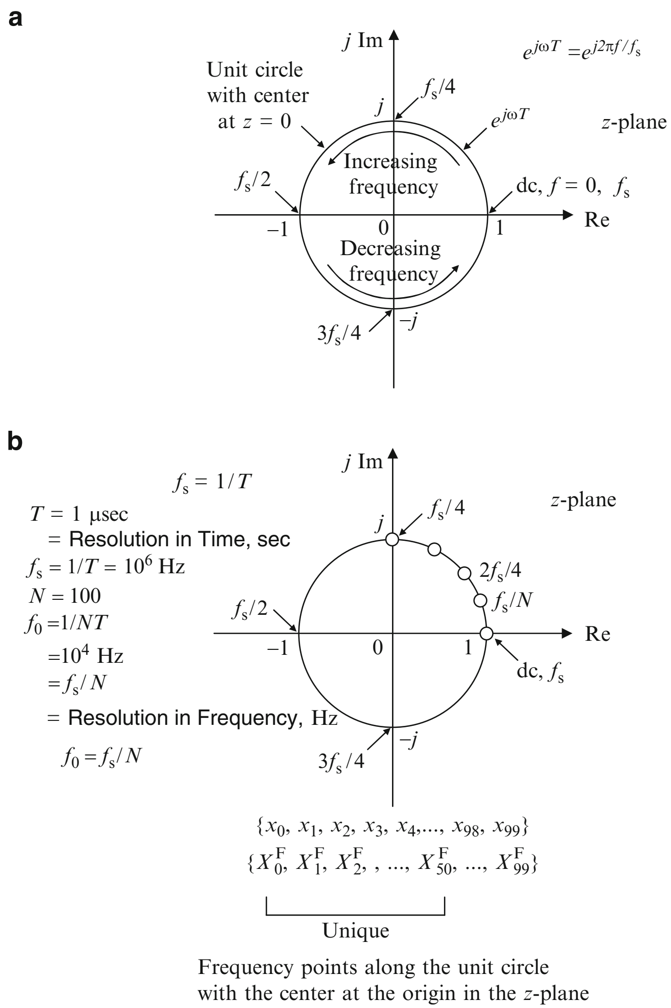


Fig. 2.3 **a** DFT of $x(n)$ is the Z-transform at N equally distributed points on the unit circle with the center at the origin in the z-plane. **b** specific example for $T = 1\ \mu s$ and $N = 100$

$$x_1(n) = \sin\left(\frac{2\pi}{T_1}n\right) = \sin\left(\frac{2\pi}{20}n\right), \quad n = 0, 1, \ldots, N-1, \quad N = 100$$

$$T_1 = 20 \times T = 20\,\mu\text{s} \qquad \text{Record length,} \quad T_R = N \times T = 100\,\mu\text{s}$$

$$f_s = 1/T = 1 \text{ MHz}, \ f_0 = \frac{f_s}{N} = \frac{1}{100} \text{ MHz}, \ f_1 = 5f_0 = \frac{5}{100} \text{ MHz}$$

$$\frac{f_s}{2} = 50f_0 = \frac{1}{2}\text{MHz}$$

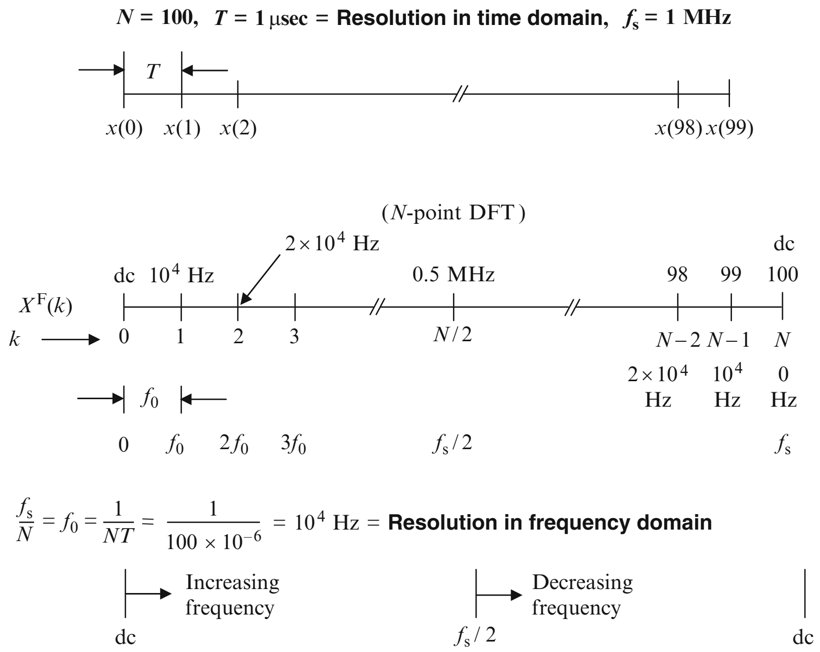


Fig. 2.4 Highest frequency representation of $x(n)$ is at $X^F(N/2)$ i.e., at $f = f_s/2$

The DFT/IDFT expressed in summation form in (2.1) can be expressed in vector-matrix form as

$$[\underline{X}^F(k)] = [F][\underline{x}(n)] \tag{2.10a}$$

where $[F]$ is the $(N \times N)$ DFT matrix described in (2.11).

$$\begin{array}{ccccc}
 & & \text{Columns} & & \text{Rows} \\
 & & n \to 0\ 1\ 2\ \cdots\ n\ \cdots\ N-1 & & k \\
 & & & & \downarrow \\
\begin{bmatrix} X^F(0) \\ X^F(1) \\ \vdots \\ X^F(k) \\ \vdots \\ X^F(N-1) \end{bmatrix} & = & \begin{bmatrix} \\ \\ W_N^{nk} \\ (n,k = 0,1,\ldots,N-1) \\ \\ \\ \end{bmatrix} & \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(n) \\ \vdots \\ x(N-1) \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ \vdots \\ k \\ \vdots \\ (N-1) \end{array} \\
(N \times 1) & & (N \times N) & (N \times 1) &
\end{array}$$

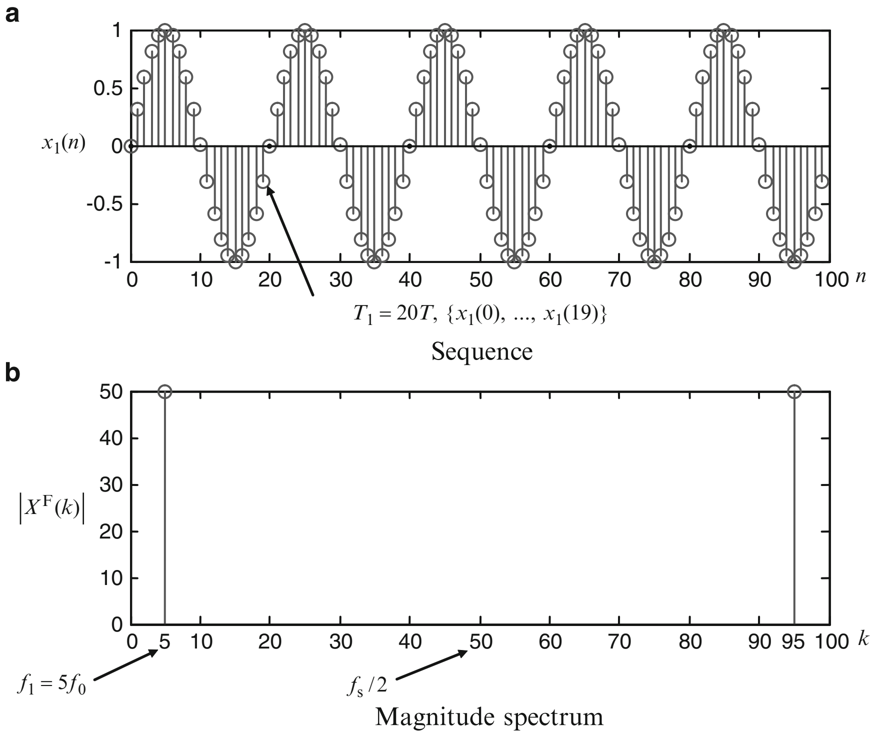


Fig. 2.5 Highest frequency representation of $x_1(n)$ is at $X^F(N/2)$ i.e., at $f = f_s/2$ and not at $f = f_s$ because of frequency folding (Fig. 2.2)

$$\text{IDFT} \quad [\underline{x}(n)] = \frac{1}{N}[F]^*[\underline{X}^F(k)] \tag{2.10b}$$

$$\begin{array}{ccccc}
 & & \begin{array}{c}\text{Columns}\\ k \to 0 \;\; 1 \;\; 2 \;\; \cdots \;\; k \;\; \cdots \;\; N-1\end{array} & & \begin{array}{c}\text{Rows}\\ n\\ \downarrow\end{array}\\
\begin{bmatrix} x(0)\\ x(1)\\ \vdots\\ x(n)\\ \vdots\\ x(N-1)\end{bmatrix} & = \dfrac{1}{N} & \begin{bmatrix} \\ W_N^{-nk}\\ (n,k = 0, 1, \ldots, N-1)\\ \\ \end{bmatrix} & \begin{bmatrix} X^F(0)\\ X^F(1)\\ \vdots\\ X^F(k)\\ \vdots\\ X^F(N-1)\end{bmatrix} & \begin{array}{c} 0\\ 1\\ \vdots\\ n\\ \vdots\\ (N-1)\end{array}\\
(N \times 1) & & (N \times N) & (N \times 1) &
\end{array}$$

The $(N \times N)$ DFT matrix $[F]$ is:

$$
\begin{array}{cc}
 & \begin{array}{ccccccccc} \text{Columns} & 0 & 1 & 2 & \cdots & n & \cdots & (N-1) & \text{Rows} \\ \rightarrow & & & & & & & & \downarrow \end{array} \\
\underset{(N \times N)}{[F]} = & \begin{bmatrix} W_N^0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ W_N^0 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ W_N^0 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^k & W_N^{2k} & \cdots & W_N^{k(N-1)} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ W_N^0 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{array}{c} 0 \\ 1 \\ 2 \\ \vdots \\ k \\ \vdots \\ (N-1) \end{array} \\
 & \text{DFT Matrix}
\end{array}
\tag{2.11}
$$

Note that

$$\left(\frac{1}{\sqrt{N}}[F]^*\right)\left(\frac{1}{\sqrt{N}}[F]\right) = [I_N] = (N \times N) \text{ unit matrix.}$$

Each row of $[F]$ is a basis vector (BV). The elements of $[F]$ in row l and column k are $W_N^{lk}, l, k = 0,\ 1, \ldots, N-1$. As $W_N = \exp(-j2\pi/N)$ is the Nth root of unity, of the N^2 elements in $[F]$, only N elements are unique, i.e., $W_N^l = W_N^{l \bmod N}$, where $l \bmod N$ implies remainder of l divided by N. For example, 5 mod 8 is 5, 10 mod 5 is 0, and 11 mod 8 is 3. Mod is the abbreviation for modulo. The following observations can be made about the DFT matrix:

1. $[F]$ is symmetric, $[F] = [F]^T$.
2. $[F]$ is unitary, $[F][F]^* = N[I_N]$, where $[I_N]$ is an unit matrix of size $(N \times N)$.

$$[F]^{-1} = \tfrac{1}{N}[F]^*, \quad [F][F]^{-1} = [I_N] = \text{unit matrix} = \begin{bmatrix} 1 & & & O \\ & 1 & & \\ & & \ddots & \\ O & & & 1 \end{bmatrix} \tag{2.12}$$

For example the (8×8) DFT matrix can be simplified as (here $W = W_8 = \exp(-j2\pi/8)$)

$$\begin{array}{c} \text{Columns} \rightarrow \\ \\ \\ \\ \\ \\ \\ \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & W & W^2 & W^3 & -1 & -W & -W^2 & -W^3 \\
1 & W^2 & -1 & -W^2 & 1 & W^2 & -1 & -W^2 \\
1 & W^3 & -W^2 & W & -1 & -W^3 & W^2 & -W \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -W & W^2 & -W^3 & -1 & W & -W^2 & W^3 \\
1 & -W^2 & -1 & W^2 & 1 & -W^2 & -1 & W^2 \\
1 & -W^3 & -W^2 & -W & -1 & W^3 & W^2 & W
\end{bmatrix}
\end{array}
\begin{array}{c} \text{Rows} \downarrow \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \tag{2.13}$$

Observe that $W_N^{N/2} = -1$ and $W_N^{N/4} = -j$. Also $\sum_{k=0}^{N-1} W_N^k = 0 =$ sum of all the N distinct roots of unity. These roots are uniformly distributed on the unit circle with center at the origin in the z-plane (Fig. 2.3).

2.3 Properties of the DFT

From the definition of the DFT, several properties can be developed.

1. *Linearity*: Given $x_1(n) \Leftrightarrow X_1^{\mathrm{F}}(k)$ and $x_2(n) \Leftrightarrow X_2^{\mathrm{F}}(k)$ then

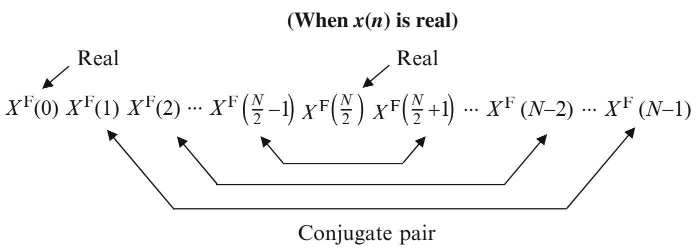
$$[a_1 x_1(n) + a_2 x_2(n)] \Leftrightarrow \left[a_1 X_1^{\mathrm{F}}(k) + a_2 X_2^{\mathrm{F}}(k)\right] \tag{2.14}$$

   where a_1 and a_2 are constants.

2. Complex conjugate theorem: For an N-point DFT, when $x(n)$ *is a real sequence*,

$$X^{\mathrm{F}}\left(\tfrac{N}{2} + k\right) = X^{\mathrm{F}^*}\left(\tfrac{N}{2} - k\right), k = 0, 1, \ldots, \tfrac{N}{2} \tag{2.15}$$

This implies that both $X^{\mathrm{F}}(0)$ and $X^{\mathrm{F}}(N/2)$ are real. By expressing $X^{\mathrm{F}}(k)$ in polar form as $X^{\mathrm{F}}(k) = |X^{\mathrm{F}}(k)| \exp[j\Theta(k)]$, it is evident that $|X^{\mathrm{F}}(k)|$ versus k is an even function and $\Theta(k)$ versus k is an odd function around $N/2$ in the frequency domain (Fig. 2.4). $|X^{\mathrm{F}}(k)|$ and $\Theta(k)$ are called the magnitude spectrum and phase spectrum respectively. Of the N DFT coefficients only $(N/2) + 1$ coefficients are independent. $|X^{\mathrm{F}}(k)|^2, k = 0, 1, \ldots, N-1$ is the power spectrum. This is an even function around $N/2$ in the frequency domain (Fig. 2.6).



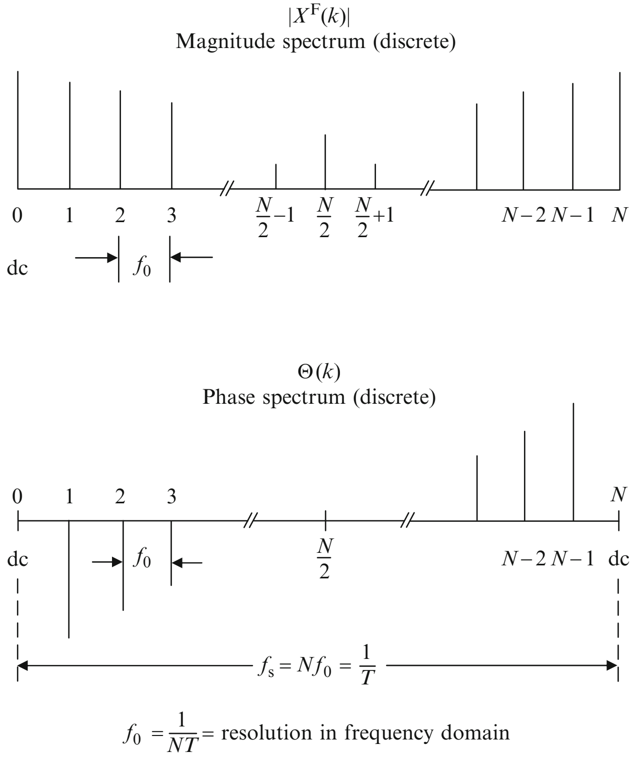


Fig. 2.6 Magnitude and phase spectra when $x(n)$ is a real sequence

$$X^{\mathrm{F}}\left(\frac{N}{2}+k\right) = \sum_{n=0}^{N-1} x(n) W_N^{\left(\frac{N}{2}+k\right)n}$$

$$X^{\mathrm{F}^*}\left(\frac{N}{2}-k\right) = \left[\sum_{n=0}^{N-1} x(n) W_N^{\left(\frac{N}{2}-k\right)n}\right]^* = \sum_{n=0}^{N-1} x(n) W_N^{\left(\frac{N}{2}+k\right)n} = X^{\mathrm{F}}\left(\frac{N}{2}+k\right)$$

since $W_N^{N/2} = \exp\left(\frac{-j2\pi}{N}\frac{N}{2}\right) = e^{-j\pi} = -1, \quad \left(W_N^{-kn}\right)^* = W_N^{kn}.$

$$X^{\mathrm{F}}(0) = \sum_{n=0}^{N-1} x(n), \text{ dc coefficient} \quad \boxed{\frac{1}{N}\sum_{n=0}^{N-1} x(n) : \text{ Mean of } x(n)}$$

$$X^{\mathrm{F}}\left(\frac{N}{2}\right) = \sum_{n=0}^{N-1} x(n) W_N^{n\frac{N}{2}} = \sum_{n=0}^{N-1} x(n)(-1)^n$$

Figure 2.7 illustrates the even and odd function properties of $X^{\mathrm{F}}(k)$ at $k = N/2$, when $x(n)$ is real.

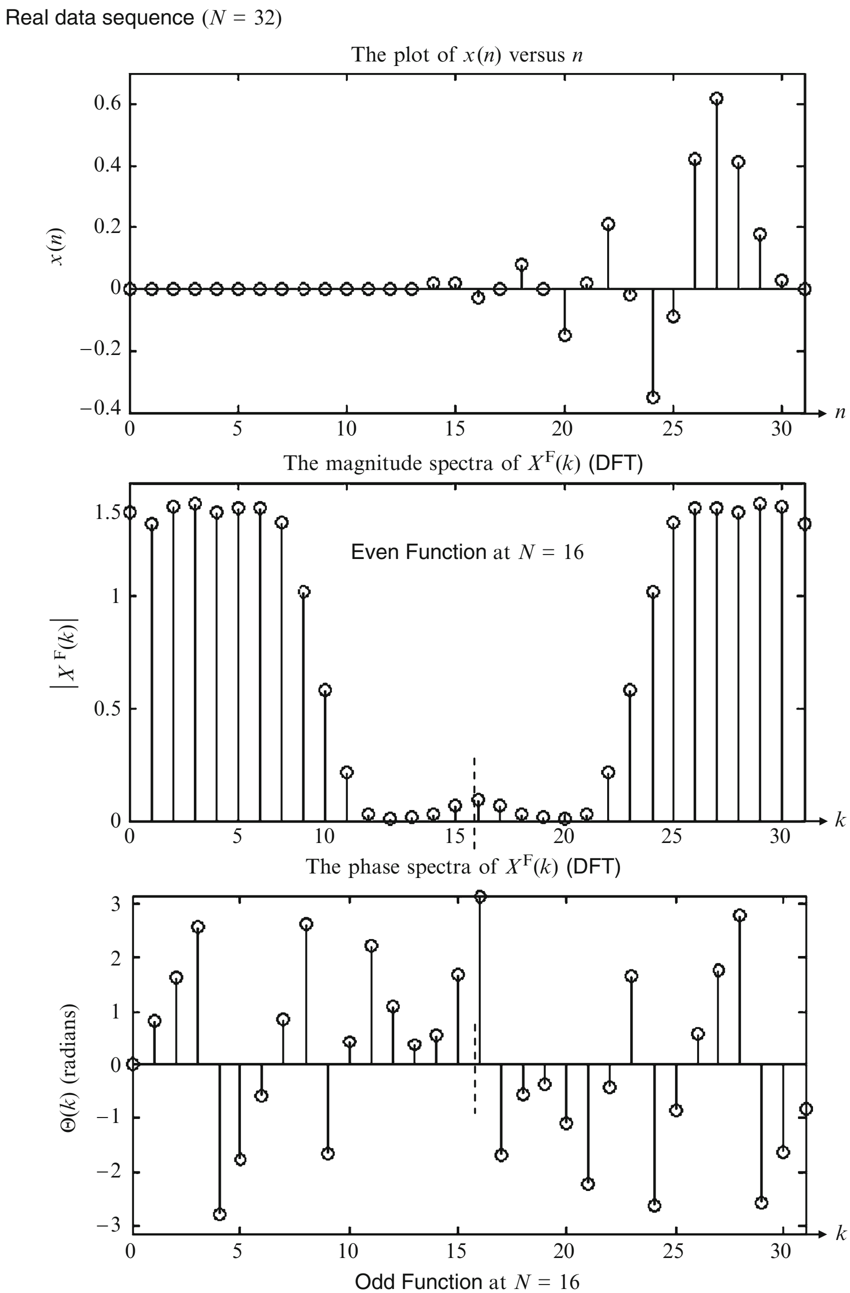


Fig. 2.7 Even and odd function properties of $X^{\mathrm{F}}(k)$ at $k = N/2$, when $x(n)$ is real

3. Parseval's theorem

This property is valid for all unitary transforms.

$$\sum_{n=0}^{N-1} x(n)x^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{F}}(k)X^{\mathrm{F}^*}(k) \tag{2.16a}$$

Energy of the sequence $\{x(n)\}$ is preserved in the DFT domain.

Proof:

$$\begin{aligned}\sum_{n=0}^{N-1} |x(n)|^2 &= \sum_{n=0}^{N-1} x(n)x^*(n) = \frac{1}{N}\sum_{n=0}^{N-1} x^*(n)\sum_{k=0}^{N-1} X^{\mathrm{F}}(k)\ W_N^{-nk} \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{F}}(k)\sum_{n=0}^{N-1} x^*(n)W_N^{-nk} = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{F}}(k)\left(\sum_{n=0}^{N-1} x(n)W_N^{nk}\right)^* \\ &= \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{F}}(k)X^{\mathrm{F}^*}(k) = \frac{1}{N}\sum_{k=0}^{N-1} \left|X^{\mathrm{F}}(k)\right|^2\end{aligned} \tag{2.16b}$$

4. Circular shift

Given: $x(n) \Leftrightarrow X^{\mathrm{F}}(k)$, then

$$x(n+h) \Leftrightarrow X^{\mathrm{F}}(k)\ W_N^{-hk} \tag{2.17}$$

$x(n+h)$ is shifted circularly to the left by h sampling intervals in the time domain i.e., $\{x_{n+h}\}$ is $(x_h, x_{h+1},\ x_{h+2},\ \ldots, x_{N-1},\ x_0,\ x_1,\ \ldots, x_{h-1})$.

Proof:

$$\begin{aligned}\mathrm{DFT}\,[x(n+h)] &= \sum_{n=0}^{N-1} x(n+h)\ W_N^{nk},\ \text{let } m = n+h \\ &= \sum_{m=h}^{N+h-1} x(m)\ W_N^{(m-h)k} \\ &= \left(\sum_{m=h}^{N+h-1} x(m)W_N^{mk}\right)W_N^{-hk} \\ &= X^{\mathrm{F}}(k)W_N^{-hk}\end{aligned} \tag{2.18}$$

since $\sum_{m=h}^{N+h-1} x(m)\,W_N^{mk} = X^{\mathrm{F}}(k),\ k = 0, 1, \ldots, N-1.$

As $\left|W_N^{-hk}\right| = 1$, the DFT of a circularly shifted sequence and the DFT of the original sequence are related only by phase. Magnitude spectrum and hence power spectrum are invariant to the circular shift of a sequence.

5. DFT of $\left[x(n)\exp\left(\frac{j2\pi hn}{N}\right)\right]$

$$\text{DFT}\left[x(n)\exp\left(\frac{j2\pi hn}{N}\right)\right] = \sum_{n=0}^{N-1}\left(x(n)W_N^{-hn}\right)W_N^{kn} = \sum_{n=0}^{N-1} x(n)\, W_N^{(k-h)n} = X^{\text{F}}(k-h) \tag{2.19}$$

Since $X^{\text{F}}(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}$, $k = 0, 1, \ldots, N-1$, $X^{\text{F}}(k-h)$, is $X^{\text{F}}(k)$ circularly shifted by h sampling intervals in the frequency domain. For the special case when $h = N/2$

$$\begin{aligned} &\text{DFT}\left[x(n)\exp\left(\tfrac{j2\pi}{N}\tfrac{N}{2}n\right)\right] = \text{DFT}\left[(-1)^n x(n)\right], \quad e^{\pm j\pi n} = (-1)^n \\ &\quad = \text{DFT}\left\{x(0), -x(1), x(2), -x(3), x(4), \ldots, (-1)^{N-1}\, x(N-1)\right\} = X^{\text{F}}\left(k - \tfrac{N}{2}\right) \end{aligned} \tag{2.20}$$

Here $\{x(0), x(1), x(2), x(3), \ldots, x(N-1)\}$ is the original N-point sequence.

In the discrete frequency spectrum, the dc component is now shifted to its midpoint (Fig. 2.8). To the left and right sides of the midpoint, frequency increases (consider these as positive and negative frequencies).

6. DFT of a permuted sequence [B1]

$$\{x(pn)\} \Leftrightarrow \{X^{\text{F}}(qk)\} \qquad 0 \le p, q \le N-1 \tag{2.21}$$

Let the sequence $x(n)$ be permuted, with n replaced by pn modulo N, where $0 \le p \le N-1$ and p is an integer relatively prime to N. Then the DFT of $x(pn)$ is given by

$$A^{\text{F}}(k) = \sum_{n=0}^{N-1} x(pn)W^{nk} \tag{2.22}$$

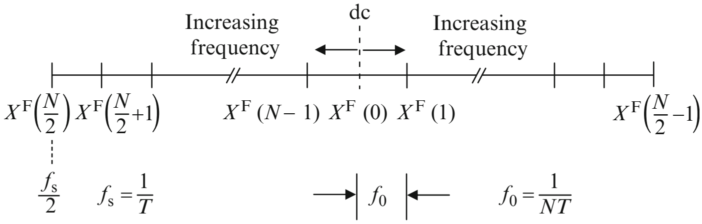


Fig. 2.8 DFT of $[(-1)^n x(n)]$. The dc coefficient is at the center with increasing frequencies to its left and right

When a and b have no common factors other than 1, they are said to be relatively prime and denoted as $(a, b) = 1$. Since $(p, N) = 1$, we can find an integer q such that $0 \le q \le N - 1$ and $qp \equiv (1 \text{ modulo } N)$.[1] Equation (2.22) is not changed if n is replaced by qn modulo N. We then have

$$\begin{aligned} A^{\mathrm{F}}(k) &= \sum_{n=0}^{N-1} x(pqn) W^{nqk} \\ &= \sum_{n=0}^{N-1} x(aNn + n) W^{nqk} \quad \text{since } qp \equiv (1 \text{ modulo } N), \text{ or } qp = aN + 1 \\ &= \sum_{n=0}^{N-1} x(n) W^{n(qk)} = X^{\mathrm{F}}(qk) \quad \text{as } x(n) \text{ is periodic from (2.9b)} \end{aligned} \tag{2.23}$$

where a is an integer.
Example 2.1 For $N = 8$, p is $\{3, 5, 7\}$. $q = 3$ for $p = 3$. $q = 5$ for $p = 5$. $q = 7$ for $p = 7$. For $p = 3$, $pn = \{0, 3, 6, 1, 4, 7, 2, 5\}$. Since $p = q = 3$, $pn = qk$ when $n = k$. For $p = 5$, $pn = \{0, 5, 2, 7, 4, 1, 6, 3\}$. For $p = 7$, $pn = \{0, 7, 6, 5, 4, 3, 2, 1\}$.

Example 2.2 Let $N = 8$ and $p = 3$. Then $q = 3$. Let $x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$ and let

$$A^{\mathrm{F}}(K) = X^{\mathrm{F}}(qk) = \{X^{\mathrm{F}}(0), X^{\mathrm{F}}(3), X^{\mathrm{F}}(6), X^{\mathrm{F}}(1), X^{\mathrm{F}}(4), X^{\mathrm{F}}(7), X^{\mathrm{F}}(2), X^{\mathrm{F}}(5)\}$$

Then

$$\begin{aligned} a(n) &= N\text{-point IDFT of } [A^{\mathrm{F}}(\mathrm{k})] \\ &= \{0, 3, 6, 1, 4, 7, 2, 5\} = x(pn) \end{aligned}$$

2.4 Convolution Theorem

Circular convolution of two periodic sequences in time/spatial domain is equivalent to multiplication in the DFT domain. Let $x(n)$ and $y(n)$ be two real periodic sequences with period N. Their circular convolution is given by

$$\begin{aligned} z_{\mathrm{con}}(m) &= \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, y(m-n) \qquad m = 0, 1, \ldots, N-1 \\ &= x(n) * y(n) \end{aligned} \tag{2.24a}$$

[1] See problem 2.21(a).

In the DFT domain this is equivalent to

$$Z^{\mathrm{F}}_{\mathrm{con}}(k) = \frac{1}{N} X^{\mathrm{F}}(k) Y^{\mathrm{F}}(k) \tag{2.24b}$$

where $x(n) \Leftrightarrow X^{\mathrm{F}}(k)$, $y(n) \Leftrightarrow Y^{\mathrm{F}}(k)$, and $z_{\mathrm{con}}(m) \Leftrightarrow Z^{\mathrm{F}}_{\mathrm{con}}(k)$.

Proof:
DFT of $z_{\mathrm{con}}(m)$ is

$$\begin{aligned}
&\sum_{m=0}^{N-1} \left[\frac{1}{N} \sum_{n=0}^{N-1} x(n) y(m-n) \right] W_N^{mk} \\
&= \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, W_N^{nk} \sum_{m=0}^{N-1} y(m-n)\, W_N^{(m-n)k} \\
&= \frac{1}{N} X^{\mathrm{F}}(k) Y^{\mathrm{F}}(k) \\
\mathrm{IDFT}&\left[\frac{1}{N} X^{\mathrm{F}}(k) X^{\mathrm{F}}(k) \right] = z_{\mathrm{con}}(m)
\end{aligned}$$

To obtain a noncircular or aperiodic convolution using the DFT, the two sequences $x(n)$ and $y(n)$ have to be extended by adding zeros. Even though this results in a circular convolution, for one period it is the same as the noncircular convolution (Fig. 2.9). This technique can be illustrated as follows:

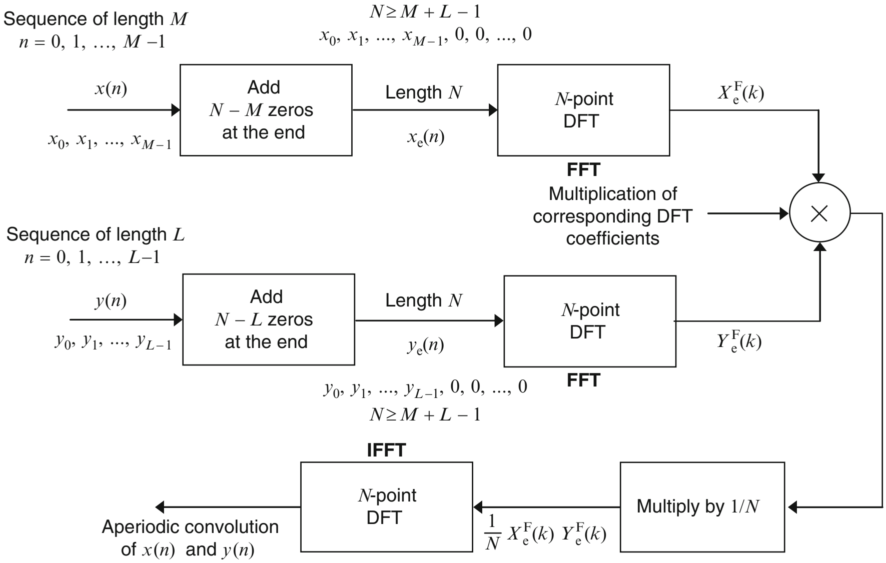


Fig. 2.9 Aperiodic convolution using the DFT/IDFT. $x_e(n)$ and $y_e(n)$ are extended sequences

Given: $\{x(n)\} = \{x_0, x_1, \ldots, x_{M-1}\}$, an M-point sequence and $\{y(n)\} = \{y_0, y_1, \ldots, y_{L-1}\}$, an L-point sequence, obtain their circular convolution.

1. Extend $x(n)$ by adding $N - M$ zeros at the end i.e., $\{x_e(n)\} = \{x_0, x_1, \ldots, x_{M-1}, 0, 0, \ldots, 0\}$, where $N \geq M + L - 1$.
2. Extend $y(n)$ by adding $N - L$ zeros at the end i.e., $\{y_e(n)\} = \{y_0, y_1, \ldots, y_{L-1}, 0, 0, \ldots, 0\}$.
3. Apply N-point DFT to $\{x_e(n)\}$ to get $\{X_e^F(k)\}$, $k = 0, 1, \ldots, N-1$.
4. Carry out step 3 on $\{y_e(n)\}$ to get $\{Y_e^F(k)\}$, $k = 0, 1, \ldots, N-1$.
5. Multiply $\frac{1}{N}X_e^F(k)$ and $Y_e^F(k)$ to get $\frac{1}{N}X_e^F(k)Y_e^F(k)$, $k = 0, 1, \ldots, N-1$.
6. Apply N-point IDFT to $\{\frac{1}{N}X_e^F(k)\,Y_e^F(k)\}$ to get $z_{\mathrm{con}}(m)$, $m = 0, 1, \ldots, N-1$, which is the aperiodic convolution of $\{x(n)\}$ and $\{y(n)\}$.

Needless to say all the DFTs/IDFTs are implemented via the fast algorithms (see Chapter 3). Note that $\{X_e^F(k)\}$ and $\{Y_e^F(k)\}$ are N-point DFTs of the extended sequences $\{x_e(n)\}$ and $\{y_e(n)\}$ respectively.

Example 2.3 Periodic and Nonperiodic (Fig. 2.10)

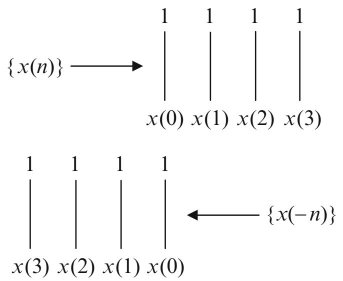
Discrete convolution of two sequences $\{x(n)\}$ and $\{y(n)\}$.

$$\begin{aligned}
&z_{\mathrm{con}}(m) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)y(m-n) \\
&x(n) \qquad n = 0, 1, \ldots, M-1 \\
&y(n) \qquad n = 0, 1, \ldots, L-1 \qquad (\text{where } N = L + M - 1) \\
&z_{\mathrm{con}}(m) \qquad m = 0, 1, \ldots, N-1
\end{aligned}$$

This is illustrated with a sample example as follows:

Example 2.4

Let $x(n)$ be $\{1, 1, 1, 1\}$ $n = 0, 1, 2, 3$ i.e., $L = 4$. Convolve $x(n)$ with itself.



$$z_{\mathrm{con}}(m) = \frac{1}{N}\sum_{n=0}^{N-1} x(n)\,x(m-n)$$

$$\begin{aligned}
z_{\mathrm{con}}(0) &= \frac{1}{8}\sum_{n=0}^{7} x(n)\,x(-n) \\
&= 1/8
\end{aligned}$$

1. Keep $\{x(n)\}$ as it is.
2. Reflect $\{y(n)\}$ around 0 to get $\{y(-n)\}$.

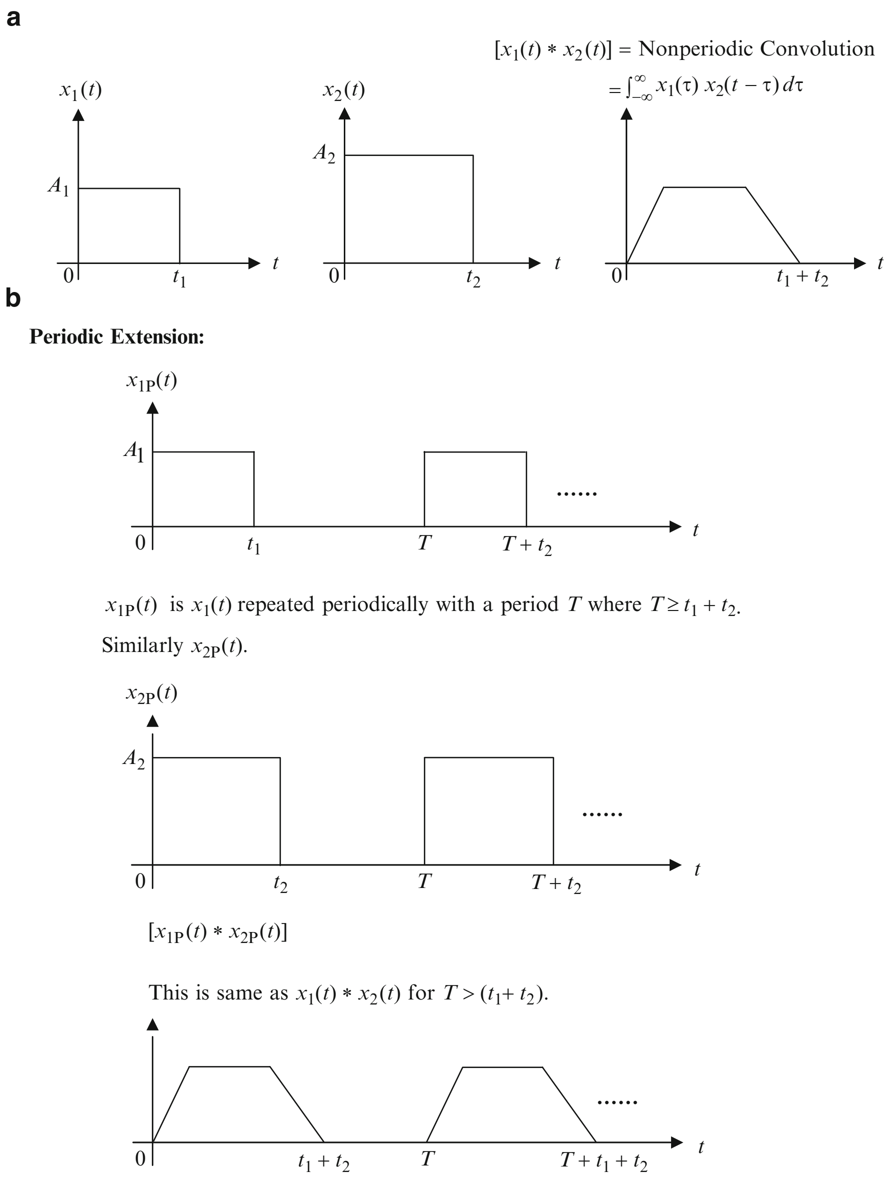
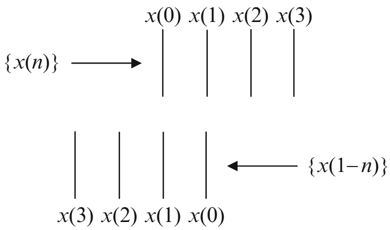
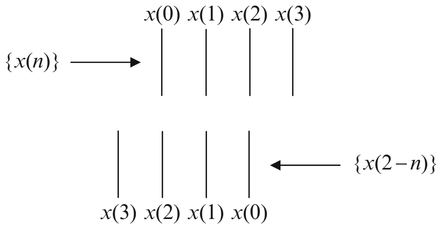


Fig. 2.10 Nonperiodic convolution **a** of $x_1(t)$ and $x_2(t)$ is same as periodic convolution. **b** of $x_{1P}(t)$ and $x_{2P}(t)$ over one period

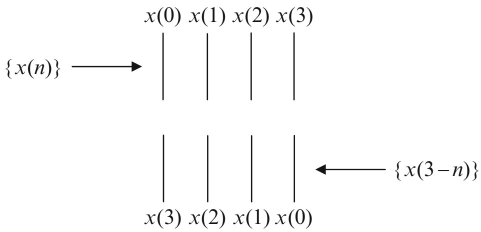
3. Multiply $x(n)$ and $y(-n)$ and add.
4. Shift $\{y(-n)\}$ to the right by one sampling interval. Multiply $x(n)$ and $y(1-n)$ and add.
5. Shift $\{y(-n)\}$ by two sampling intervals. Multiply and add.
6. Shift $\{y(-n)\}$ by three sampling intervals. Multiply and add, and so on.



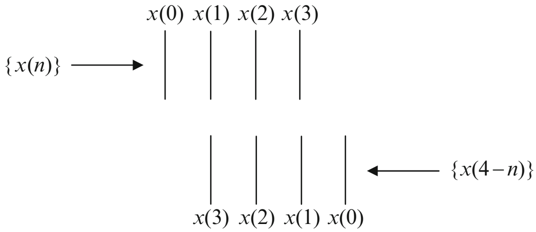
$$z_{\mathrm{con}}(1) = \frac{1}{8}\sum_{n=0}^{7} x(n)\,x(1-n)$$
$$= 2/8$$



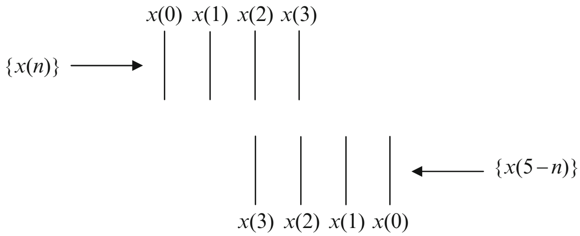
$$z_{\mathrm{con}}(3) = \frac{1}{8}\sum_{n=0}^{7} x(n)\,x(2-n)$$
$$= 3/8$$



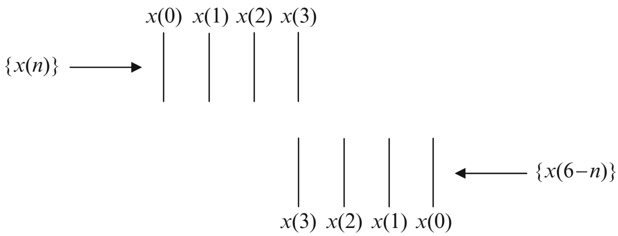
$$z_{\mathrm{con}}(3) = \frac{1}{8}\sum_{n=0}^{7} x(n)\,x(3-n)$$
$$= 4/8$$



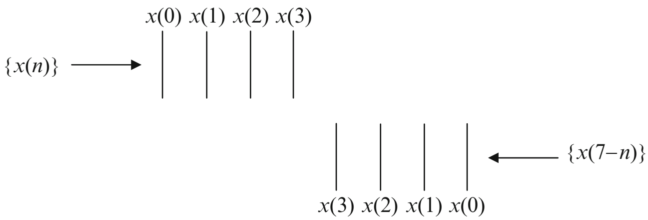
$$z_{\text{con}}(4) = \frac{1}{8}\sum_{n=0}^{7} x(n)\,x(4-n)$$
$$= 3/8$$



$$z_{\text{con}}(5) = \frac{1}{8}\sum_{n=0}^{7} x(n)\,x(5-n)$$
$$= 2/8$$

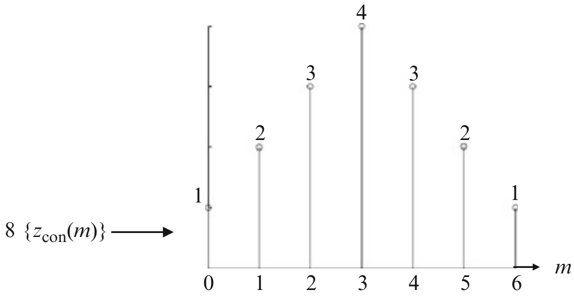


$$z_{\text{con}}(6) = \frac{1}{8}\sum_{n=0}^{7} x(n)\,x(6-n)$$
$$= 1/8$$



$$z_{\text{con}}(7) = \frac{1}{8}\sum_{n=0}^{7} x(n)\,x(7-n) = 0$$

$$z_{\text{con}}(m) = 0, \qquad m \geq 7 \text{ and } m \leq (-1)$$



Convolution of a uniform sequence with itself yields a triangular sequence. This is an aperiodic or noncircular convolution. To obtain this through the DFT/IDFT, extend both $\{x(n)\}, n = 0, 1, \ldots, L-1$ and $\{y(n)\}, n = 0, 1, \ldots, M-1$ by adding zeros at the end such that $N \geq (L + M - 1)$ (Fig. 2.9).

2.4.1 Multiplication Theorem

Multiplication of two periodic sequences in time/spatial domain is equivalent to circular convolution in the DFT domain (see Problem 2.16).

2.5 Correlation Theorem

Similar to the convolution theorem, an analogous theorem exists for the correlation (Fig. 2.11). Circular correlation of two real periodic sequences $x(n)$ and $y(n)$ is given by

$$\begin{aligned} z_{\text{cor}}(m) &= \frac{1}{N}\sum_{n=0}^{N-1} x^*(n)y(m+n) \qquad \text{when } x(n) \text{ is complex} \\ &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)y(m+n) \qquad m = 0, 1, \ldots, N-1 \end{aligned} \tag{2.25a}$$

as $x(n)$ is usually real. This is the same process as in discrete convolution except $\{y(n)\}$ is not reflected. All other properties hold. In the DFT domain this is equivalent to

$$Z^{\text{F}}_{\text{cor}}(k) = \frac{1}{N} X^{\text{F}^*}(k) Y^{\text{F}}(k). \quad \text{Note } Z^{\text{F}}_{\text{cor}}(k) \neq \frac{1}{N} X^{\text{F}}(k) Y^{\text{F}^*}(k) \tag{2.25b}$$

$x(n)$	1	2	3	0	0
$y(m-n)$					
$m=0$	1	0	0	3	2
$m=1$	2	1			3
$m=2$	3	2	1		
$m=3$		3	2	1	
$m=4$			3	2	1
$z_{\text{con}}(m)$	1	4	10	12	9

1	1	3	0	0	
					$y(m+n)$
1	2	3	0	0	$m=0$
2	3			1	$m=1$
3			1	2	$m=2$
		1	2	3	$-m=3=3\pm N=-2$
	1	2	3		$m=4=4\pm N=-1$
12	5	3	3	7	$z_{\text{cor}}(m)$
3	7	12	5	3	$\underline{x}=[1\ 1\ 3]$ = xcorr $(\underline{y}, \underline{x})$

wrapped around

$$z_{\text{con}}(m)=\frac{1}{N}\sum_{n=0}^{N-1}x(n)y(m-n) \qquad z_{\text{cor}}(m)=\frac{1}{N}\sum_{n=0}^{N-1}x(n)y(m+n)$$

$$m=0,\ 1,\ \ldots,\ N-1$$

$$z_{\text{con}}^{\text{F}}(k)=\frac{1}{N}X^{\text{F}}(k)Y^{\text{F}}(k) \qquad z_{\text{cor}}^{\text{F}}(k)=\frac{1}{N}X^{\text{F}^*}(k)Y^{\text{F}}(k)$$

Aperiodic convolution by using circular convolution from multiplication in DFT

Aperiodic correlation by using circular correlation from multiplication in DFT

Fig. 2.11 Relationship between convolution and correlation theorems

Proof.

$$\text{DFT of } z_{\text{cor}}(m) \text{ is } \sum_{m=0}^{N-1}\left[\frac{1}{N}\sum_{n=0}^{N-1}x(n)y(m+n)\right]W_N^{mk}$$

$$=\frac{1}{N}\sum_{n=0}^{N-1}x(n)W_N^{-nk}\sum_{m=0}^{N-1}y(m+n)\,W_N^{(m+n)k},\ \text{let } m+n=l$$

$$=\frac{1}{N}\underbrace{\sum_{n=0}^{N-1}x(n)\,W_N^{-nk}}\ \underbrace{\sum_{l=n}^{N-1+n}y(l)\,W_N^{lk}}$$

$$=\frac{1}{N}X^{\text{F}^*}(k)Y^{\text{F}}k).\ \text{Here } z_{\text{cor}}(m)\Leftrightarrow Z_{\text{cor}}^{\text{F}}(k)$$

As in the case of the convolution, to obtain a noncircular (aperiodic) correlation through the DFT/IDFT, both $\{x(n)\}$ and $\{y(n)\}$ must be extended by adding zeros at the end such that $N \geq M+L-1$, where M and L are lengths of the sequences $\{x(n)\}$ and $\{y(n)\}$ respectively. Even though this results in a circular correlation, for one period it is the same as the noncircular correlation.

By taking complex conjugation of $X^{\text{F}}(k)$ in Fig. 2.9, the same block diagram can be used to obtain an aperiodic correlation.

$$\boxed{\text{IDFT}\left[\frac{1}{N}X^{\text{F}*}(k)Y^{\text{F}}(k)\right]=z_{\text{cor}}(m)} \tag{2.26}$$

This technique can be illustrated as follows.

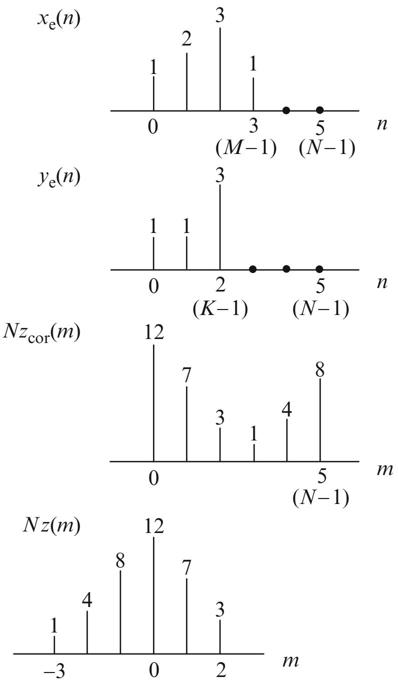
Given: $\{x(n)\} = \{x_0, x_1, \ldots, x_{M-1}\}$, an M-point sequence and $\{y(n)\} = \{y_0, y_1, \ldots, y_{L-1}\}$, an L-point sequence, obtain their circular correlation.

1. Extend $x(n)$ by adding $N - M$ zeros at the end i.e., $\{x_e(n)\} = \{x_0, x_1, \ldots, x_{M-1}, 0, 0, \ldots, 0\}$, where $N \geq M + L - 1$.
2. Extend $y(n)$ by adding $N - L$ zeros at the end i.e., $\{y_e(n)\} = \{y_0, y_1, \ldots, y_{L-1}, 0, 0, \ldots, 0\}$.
3. Apply N-point DFT to $\{x_e(n)\}$ to get $\{X_e^F(k)\}$, $k = 0, 1, \ldots, N-1$.
4. Carry out step 3 on $\{y_e(n)\}$ to get $\{Y_e^F(k)\}$, $k = 0, 1, \ldots, N-1$.
5. Multiply $\frac{1}{N} X_e^{F^*}(k)$ and $Y_e^F(k)$ to get $\frac{1}{N} X_e^{F^*}(k)\} Y_e^F(k)$, $k = 0, 1, \ldots, N-1$.
6. Apply N-point IDFT to $\left\{\frac{1}{N} X_e^{F^*}(k)\, Y_e^F(k)\right\}$ to get $z_{cor}(m)$, $m = 0, 1, \ldots, N-1$ which is the aperiodic correlation of $\{x(n)\}$ and $\{y(n)\}$.
7. Since $z_{cor}(m)$ for $0 \leq m \leq N-1$ are the values of correlation at different lags, with positive and negative lags stored in a wrap-around order, we can get $z(m)$ as follows. If $M \geq L$ and $N = M + L - 1$

$$\begin{aligned} z(m) &= z_{cor}(m), & 0 \leq m \leq L-1 \\ z(m-N) &= z_{cor}(m), & M \leq m \leq N-1 \end{aligned} \tag{2.27}$$

Example 2.5

Given: $\{x(n)\} = \{1, 2, 3, 1\}$, an M-point sequence and $\{y(n)\} = \{1, 1, 3\}$, an L-point sequence, obtain their aperiodic correlation.



1. Extend $x(n)$ by adding $N-M$ zeros at the end i.e., $\{x_e(n)\} = \{1,2,3,1,0,0\}$, where $N = L + M - 1$.
2. Extend $y(n)$ by adding $N-L$ zeros at the end i.e., $\{y_e(n)\} = \{1,1,3,0,0,0\}$.
3. Apply N-point DFT to $\{x_e(n)\}$ to get $\{X_e^F(k)\}$, $k = 0, 1, \ldots, N-1$.
4. Carry out step (3) on $\{y_e(n)\}$ to get $\{Y_e^F(k)\}$, $k = 0, 1, \ldots, N-1$.
5. Multiply $X_e^{F^*}(k)$ and $Y_e^F(k)$ to get $X_e^{F^*}(k)Y_e^F(k)$, $k = 0, 1, \ldots, N-1$.
6. Apply N-point IDFT to $\left\{X_e^{F^*}(k)Y_e^F(k)\right\}$ to get $N\{z_{\mathrm{cor}}(m)\} = \{12, 7, 3, 1, 4, 8\}$ which is the aperiodic correlation of $\{x(n)\}$ and $\{y(n)\}$.
7. Since $z_{\mathrm{cor}}(m)$ for $0 \le m \le 6$ are the values of correlation in a wrap-around order, we can get $N\ \{z(m)\} = \{1, 4, 8, 12, 7, 3\}$ for $-2 \le m \le 3$ according to (2.27).

Example 2.6: Implement Example 2.5 by MATLAB.

```
x = [1 2 3 1] ;
y = [1 1 3] ;
x_e = [1 2 3 1 0 0] ;                      % (1)
y_e = [1 1 3 0 0 0] ;                      % (2)
Z = conj (fft (x_e) ) .* fft (y_e) ;       % (5)
z = ifft (Z)                               % (6) [12 7 3 1 4 8]
```

If we wrap around the above data, we get the same with the result of the following aperiodic correlation.

```
xcorr(y, x)                                % [1 4 8 12 7 3]
```

2.6 Overlap-Add and Overlap-Save Methods

Circular convolution of two sequences in time/spatial domain is equivalent to multiplication in the DFT domain. To obtain an aperiodic convolution using the DFT, the two sequences have to be extended by adding zeros as described in Section 2.3.

2.6.1 The Overlap-Add Method

When an input sequence of infinite duration $x(n)$ is convolved with the finite-length impulse response of a filter $y(n)$, the sequence to be filtered is segmented into sections $x_r(n)$ and the filtered sections $z_r(m)$ are fitted together in an appropriate way. Let the rth section of an input sequence be $x_r(n)$, an impulse response be $y(n) = \left\{\frac{1}{4}, \frac{1}{2}, \frac{1}{4}\right\}$, and the rth section of the filtered sequence be $z_r(m)$. The procedure to get the filtered sequence $z(m)$ is illustrated with an example in Fig. 2.12.

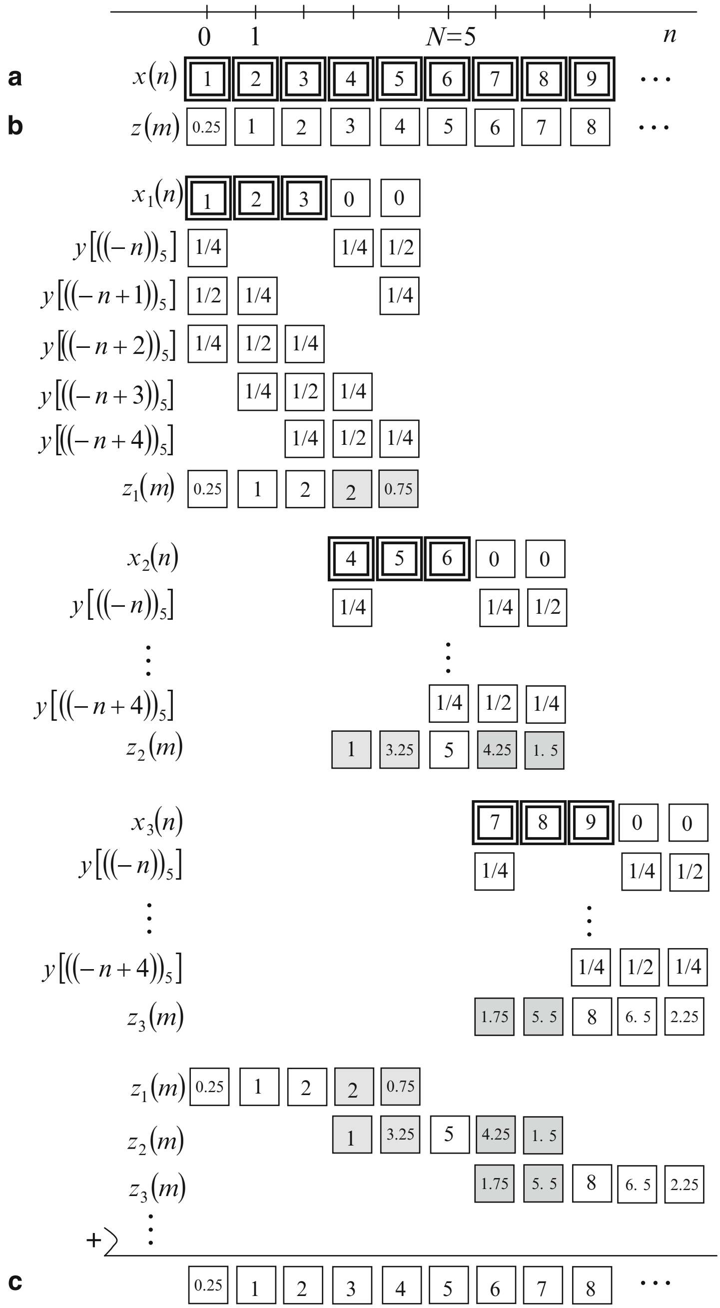


Fig. 2.12 Aperiodic convolution using the overlap-add method. **a** The sequence $x(n)$ to be convolved with the sequence $y(n)$. **b** The aperiodic convolution of $x(n)$ and $y(n)$. **c** Aperiodic convolution using the DFT/IDFT. Figure 2.13 shows how to use the DFT/IDFT for this example. $N = L + M - 1 = 5, L = M = 3$

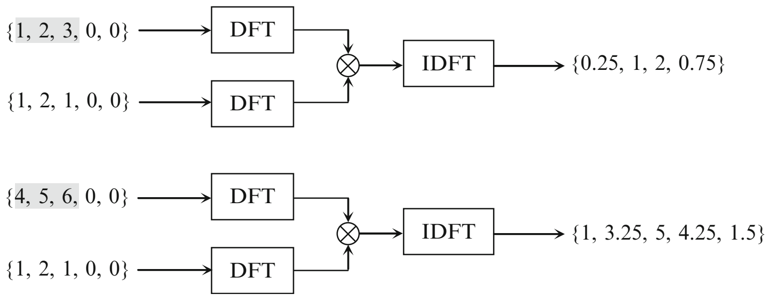


Fig. 2.13 Obtaining aperiodic convolution from circular convolution using the overlap-add method. $N = L + M - 1$, $L = M = 3$. Note that the DFTs and IDFTs are implemented via fast algorithms (see Chapter 3)

The filtering of each segment can then be implemented using the DFT/IDFT as shown in Fig. 2.13.

Discrete convolution of two sequences $\{x_r(n)\}$ and $\{y(n)\}$ can be computed as follows:

$$\begin{aligned} z_r(m) &= \sum_{n=0}^{N-1} x_r(n) y\left[((m-n))_N\right] \\ x_r(n) &\quad n = 0, 1, \ldots, L-1 \\ y\left[((n))_N\right] &\quad n = 0, 1, \ldots, M-1 \quad (\text{where}\, N = L + M - 1) \\ z_r(m) &\quad m = 0,\ 1, \ldots, N-1 \end{aligned} \tag{2.28}$$

In (2.28) the second sequence $y\left[((m-n))_N\right]$ is circularly time reversed and circularly shifted with respect to the first sequence $x_r(n)$. The sequence $y\left[((n))_N\right]$ is shifted modulo N. The sequences $x_r(n)$ have L nonzero points and $(M-1)$ zeros to have a length of $(L+M-1)$ points. Since the beginning of each input section is separated from the next by L points and each filtered section has length $(L+M-1)$, the filtered sections will overlap by $(M-1)$ points, and the overlap samples must be added (Fig. 2.12). So this procedure is referred to as the *overlap-add method*. This method can be done in MATLAB by using the command $z =$ FFTFILT (y, x).

An alternative fast convolution procedure, called the *overlap-save method*, corresponds to carrying out an L-point circular convolution of an M-point impulse response $y(n)$ with an L-point segment $x_r(n)$ and identifying the part of circular convolution that corresponds to an aperiodic convolution. After the first $(M-1)$ points of each output segment are discarded, the consecutive output segments are combined to form the output (Fig. 2.14). Each consecutive input section consists of $(L-M+1)$ new points and $(M-1)$ points so that the input sections overlap (see [G2]).

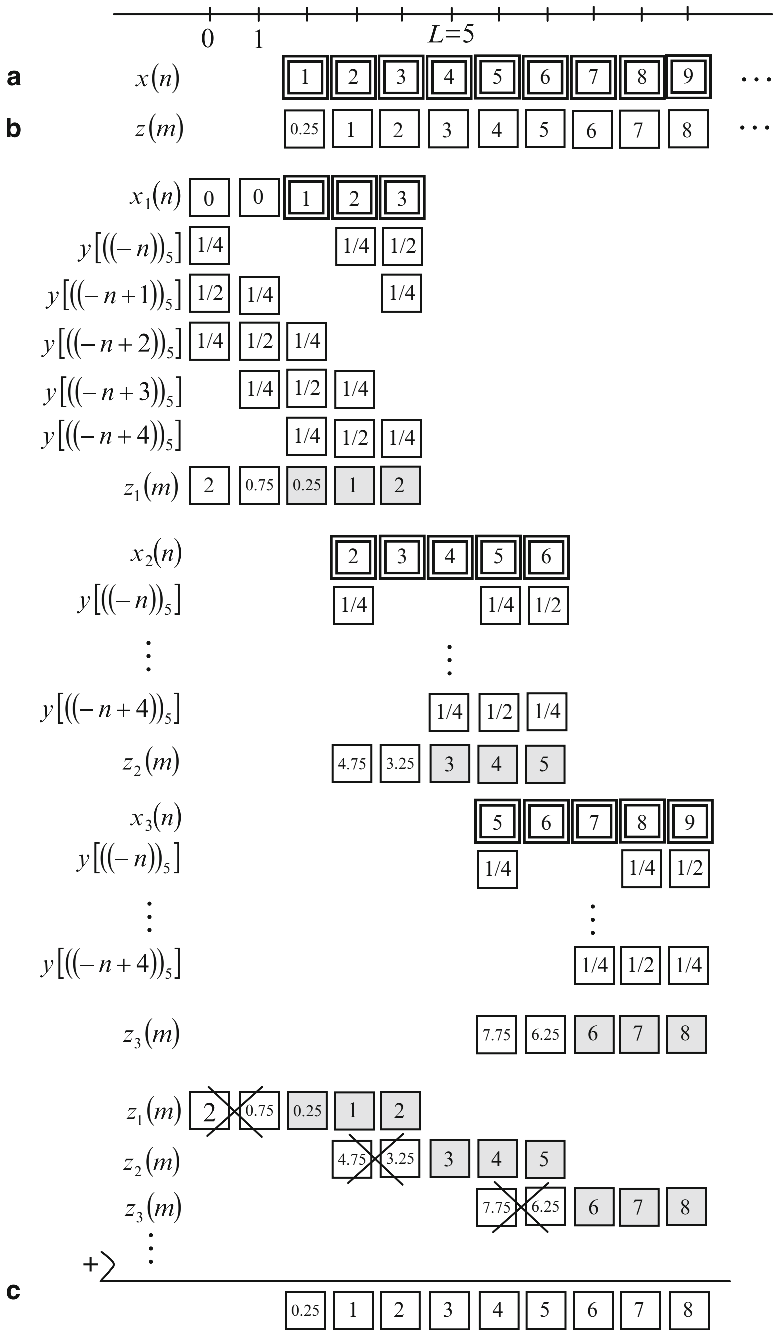


Fig. 2.14 Aperiodic convolution using the overlap-save method. **a** The sequence $x(n)$ to be convolved with the sequence $y(n)$. **b** The aperiodic convolution of $x(n)$ and $y(n)$. **c** Aperiodic convolution using the DFT/IDFT. $L = 5$, $M = 3$

2.7 Zero Padding in the Data Domain

To obtain the nonperiodic convolution/correlation of two (one of length L and another of length M) sequences using the DFT/IDFT approach, we have seen, the two sequences must be extended by adding zeros at the end such that their lengths are $N \geq L + M - 1$ (Fig. 2.9). It is appropriate to query the effects of zero padding in the frequency (DFT) domain. Let $\{x(n)\} = \{x_0, x_1, \ldots, x_{M-1}\}$, an M-point sequence be extended by adding $M - N$ zeros at the end of $\{x(n)\}$. The extended sequence is $\{x_e(n)\} = \{x_0, x_1, \ldots, x_{M-1}, 0, \ldots, 0\}$, where

$$x_e(n) = 0, \quad M \leq n \leq N - 1$$

The DFT of $\{x_e(n)\}$ is

$$X_e^F(k) = \sum_{n=0}^{N-1} x_e(n) W_N^{nk}, \quad k = 0, 1, \ldots, N - 1 \tag{2.29a}$$

$$= \sum_{n=0}^{M-1} x(n) W_N^{nk} \tag{2.29b}$$

Note that the subscript e in $X_e^F(k)$ stands for the DFT of the extended sequence $\{x_e(n)\}$. The DFT of $\{x(n)\}$ is

$$X^F(k) = \sum_{m=0}^{M-1} x(m) W_M^{mk}, \quad k = 0, 1, \ldots, M - 1 \tag{2.30}$$

Inspection of (2.29) and (2.30) shows that adding zeros to $\{x(n)\}$ at the end results in interpolation in the frequency domain.

Indeed when $N = PM$ where P is an integer $X_e^F(k)$ is an interpolated version $X^F(k)$ by the factor P. Also from (2.29b) and (2.30), $X_e^F(kP) = X^F(k)$ i.e.,

$$\sum_{n=0}^{M-1} x(n) \exp\left(\frac{-j2\pi nkP}{PM}\right) = \sum_{n=0}^{M-1} x(n) \exp\left(\frac{-j2\pi nk}{M}\right)$$

The process of adding zeros at the end of a data sequence $\{x(n)\}$ is called zero padding and is useful in a detailed representation in the frequency domain. No additional insight in the data domain is gained as IDFT of (2.29b) and (2.30) yields $\{x(n)\}$ and $\{x_e(n)\}$ respectively. Zero padding in the frequency domain (care must be taken in adding zeros to $X^F(k)$ because of the conjugate symmetry property Eq. [2.15]) is however not useful.

Example 2.7 If $N = 8$ and $M = 4$, then $P = 2$. Let

$$\{x(n)\} = \{1, 2, 3, 4\}$$

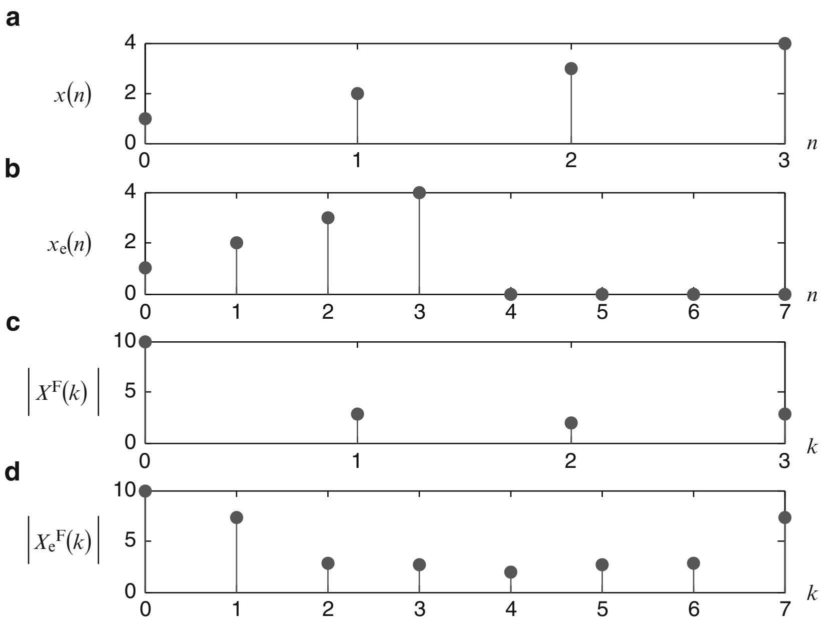


Fig. 2.15 Magnitude spectra of **a** a sequence and **b** the extended sequence are **c** and **d** respectively

then

$$\{X^F(k)\} = \{10, -2 + j2, -2, -2 - j2\}$$

$$\{x_e(n)\} = \{1, 2, 3, 4, 0, 0, 0, 0\}$$

$$\{X_e^F(k)\} = \{10, -0.414 - j7.243, -2 + j2, 2.414 - j1.243, -2, \\ 2.414 + j1.243, -2 - j2, -0.414 + j7.243\}$$

See Fig. 2.15. For a detailed representation of frequency response, zeros are added to input data (Fig. 2.16).

2.8 Computation of DFTs of Two Real Sequences Using One Complex FFT

Given two real sequences $x(n)$ and $y(n)$, $0 \leq n \leq N-1$, their DFTs can be computed using one complex FFT. Details are as follows:

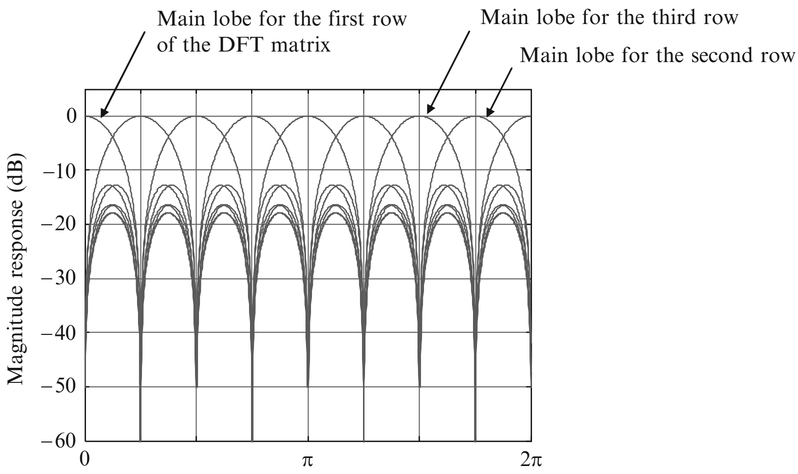


Fig. 2.16 Zeros are added at the end of each eight-point basis vector (BV) of the DFT for frequency response of size 1,024

Form a complex sequence

$$p(n) = x(n) + jy(n), \tag{2.31}$$

N-point DFT of $p(n)$ is

$$\begin{aligned} P^{F}(k) &= \sum_{n=0}^{N-1} p(n) W_N^{nk}, \quad k = 0, 1, \ldots, N-1 \\ &= \sum_{n=0}^{N-1} x(n) W_N^{nk} + j \sum_{n=0}^{N-1} y(n) W_N^{nk} \\ &= X^{F}(k) + jY^{F}(k) \end{aligned} \tag{2.32}$$

$$\begin{aligned} \text{Then} \quad P^{F^*}(N-k) &= X^{F^*}(N-k) - jY^{F^*}(N-k) \\ &= X^{F}(k) - jY^{F}(k) \end{aligned} \tag{2.33}$$

as $x(n)$ and $y(n)$ are real sequences. Hence

$$X^{F}(k) = \frac{1}{2}\left[P^{F}(k) + P^{F^*}(N-k)\right] \tag{2.34a}$$

$$Y^{F}(k) = \frac{1}{2}\left[P^{F^*}(N-k) - P^{F}(k)\right] \tag{2.34b}$$

Here we used the property that when $x(n)$ is real

$$X^{F^*}(N-k) = X^{F}(k) \tag{2.35}$$

Proof:

$$X^{F^*}(N-k) = \sum_{n=0}^{N-1} \left[x(n)W_N^{(N-k)n}\right]^* = \sum_{n=0}^{N-1} \left[x(n)W_N^{-kn}\right]^*$$
$$= \sum_{n=0}^{N-1} x(n)W_N^{kn} = X^F(k), \quad \text{since} \quad W_N^{Nn} = 1$$

2.9 A Circulant Matrix Is Diagonalized by the DFT Matrix

2.9.1 Toeplitz Matrix

Toeplitz matrix is a square matrix. On any NW to SE diagonal column the elements are the same.
Example:

$$\begin{array}{ccc} \text{NW} & & \text{NE} \\ & \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_1 & a_2 & a_3 \\ a_6 & a_5 & a_1 & a_2 \\ a_7 & a_6 & a_5 & a_1 \end{bmatrix} & \\ \text{SW} & & \text{SE} \end{array}$$

(4×4) Toeplitz matrix

e.g., c = [1 5 6 7]; first column, $N = 4$
r = [1 2 3 4]; first row
Toeplitz (c, r)

2.9.2 Circulant Matrix

A circulant matrix (CM) is such that each row is a circular shift of the previous row. A CM can be classified as left CM or right CM depending on the circular shift is to the right or left respectively. In our development we will consider only the right circular shift.

$$\begin{array}{ccc} & \text{NW} \qquad\qquad\qquad\qquad \text{NE} & \\ [H] = & \begin{bmatrix} h_0 & h_1 & h_2 & \cdots & h_{N-1} \\ h_{N-1} & h_0 & h_1 & \cdots & h_{N-2} \\ h_{N-2} & h_{N-1} & h_0 & \cdots & h_{N-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_1 & h_2 & h_3 & \cdots & h_0 \end{bmatrix} & \\ & \text{SW} \qquad\qquad\qquad\qquad \text{SE} & \end{array} \tag{2.36}$$

$[H]_{m,n} = [h_{(m-n) \bmod N}]$ element of $[H]$ in row m and column $n, 0 \le m, n \le N-1$.

e.g., c = [0 2 1]; first column, $N = 3$
r = [0 1 2]; first row
Toeplitz (c, r)

2.9.3 A Circulant Matrix Is Diagonalized by the DFT Matrix

Let $[H]$ be an $(N \times N)$ circulant matrix defined in (2.36) [B6, J13]. Define $\boldsymbol{\phi}_k$ as the basis vectors of the DFT matrix $[W_N^{nk}]$, $n,\ k = 0, 1, \ldots, N-1$

$$\underset{(N\times 1)}{\boldsymbol{\phi}_k} = \left(1, W_N^{-k}, W_N^{-2k}, \cdots, W_N^{-(N-1)k}\right)^T \qquad W_N = \exp\left(\tfrac{-j2\pi}{N}\right) \tag{2.37}$$

$$k\text{th basis vector}, \quad k = 0, 1, \ldots, N-1$$

Note that the basis vectors $\boldsymbol{\phi}_k$ are columns of $\left[W_N^{nk}\right]^*$ (see Problem 2.24(b)). Define

$$[[H]\boldsymbol{\phi}_k]_m = \sum_{n=0}^{N-1} h_{m-n} W_N^{-kn}, \quad m, k = 0, 1, \ldots, N-1 \qquad \boxed{\underset{(N\times N)}{\left[W_N^{kn}\right]}, \quad (N \times N) \text{ DFT matrix} \quad n, k = 0, 1, \ldots, N-1} \tag{2.38}$$

This is the mth row of $[H]$ post multiplied by $\boldsymbol{\phi}_k$. Thus it is a scalar.
Let $m - n = l$, then

$$\begin{aligned}
[[H]\boldsymbol{\phi}_k]_m &= \sum_{l=m}^{m-N+1} h_l W_N^{-k(m-1)} \\
&= W_N^{-km}\left(\sum_{l=m}^{m-N+1} h_l W_N^{kl}\right)
\end{aligned} \tag{2.39}$$

$$\begin{aligned}
[[H]\boldsymbol{\phi}_k]_m &= W_N^{-km}\left(\sum_{l=-N+m+1}^{-1} h_l W_N^{kl} + \sum_{l=0}^{m} h_l W_N^{kl}\right) \\
&= W_N^{-km}\left(\sum_{l=-N+m+1}^{-1} h_l W_N^{kl} + \sum_{l=0}^{N-1} h_l W_N^{kl} - \sum_{l=m+1}^{N-1} h_l W_N^{kl}\right) \\
&= W_N^{-km}(I + II - III) \\
&= W_N^{-km} \sum_{l=0}^{N-1} h_l W_N^{kl}, \quad \text{since} \quad I = III
\end{aligned} \tag{2.40}$$

Proof. In I let $p = l + N,\ l = p - N$

$$I = \sum_{p=m+1}^{N-1} h_{p-N} W_N^{k(p-N)} = \sum_{p=m+1}^{N-1} h_p W_N^{kp} \tag{2.41}$$

since $h_{p-N} = h_p$ and $W_N^{-kN} = 1$.

$$\begin{aligned}
[[H]\boldsymbol{\phi}_k]_m &= \sum_{n=0}^{N-1} h_{m-n} W_N^{-kn}, \qquad m, k = 0, 1, \ldots, N-1. \\
&= W_N^{-km} \sum_{l=0}^{N-1} h_l W_N^{kn}, \\
&= \boldsymbol{\phi}_k(m)\, \lambda_k
\end{aligned} \tag{2.42}$$

where $\lambda_k = \sum_{l=0}^{N-1} h_l W_N^{kl}, k = 0, 1, \ldots, N-1$, is the kth eigenvalue of $[H]$. This is the N-point DFT of the first row of $[H]$.

$$\underset{(N\times N)}{[H]}\ \underset{(N\times 1)}{\boldsymbol{\phi}_k} = \underset{(1\times 1)}{\lambda_k}\ \underset{(N\times 1)}{\boldsymbol{\phi}_k}, \qquad k = 0, 1, \ldots, N-1 \tag{2.43}$$

The N column vectors are combined to from

$$\begin{gathered}
[H](\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \ldots) = (\lambda_0 \boldsymbol{\phi}_0, \lambda_1 \boldsymbol{\phi}_1, \ldots) = (\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \ldots)\ \mathrm{diag}\,(\lambda_0,\ \lambda_1, \ldots) \\
[H][\Phi] = [\Phi]\ \mathrm{diag}\,(\lambda_0,\ \lambda_1, \ldots,\ \lambda_{N-1})
\end{gathered} \tag{2.44}$$

Pre multiply both sides of (2.44) by $[\Phi]^{-1}$.

$$\begin{aligned}
[\Phi]^{-1}[H][\Phi] &= \frac{1}{N}[\Phi]^*[H][\Phi] \\
&= \frac{1}{N}\left[W_N^{nk}\right][H]\left[W_N^{nk}\right]^* \\
&= \mathrm{diag}\,(\lambda_0,\ \lambda_1, \ldots, \lambda_{N-1})
\end{aligned} \tag{2.45}$$

where

$$\underset{(N\times N)}{[\Phi]} = (\underset{(N\times 1)}{\boldsymbol{\phi}_0}, \underset{(N\times 1)}{\boldsymbol{\phi}_1}, \ldots, \underset{(N\times 1)}{\boldsymbol{\phi}_k}, \ldots, \underset{(N\times 1)}{\boldsymbol{\phi}_{N-1}}) = \left[W_N^{nk}\right]^*$$

$$\boldsymbol{\phi}_k = \left(1, W_N^{-k}, W_N^{-2k}, \ldots, W_N^{-(N-1)k}\right)^T, \qquad k = 0, 1, \ldots, N-1$$

$[\Phi]^{-1} = \frac{1}{N}[\Phi]^*$, $[\Phi]$ is a unitary matrix

$$[H] = [\Phi]\,\mathrm{diag}(\lambda_0, \lambda_1, \ldots, \lambda_{N-1})[\Phi]^{-1} \tag{2.46}$$

$$\underset{(N\times N)}{[\Phi]^*} = (\underset{(N\times 1)}{\boldsymbol{\phi}_0^*}, \underset{(N\times 1)}{\boldsymbol{\phi}_1^*}, \ldots, \underset{(N\times 1)}{\boldsymbol{\phi}_{N-1}^*}) = [W_N^{nk}] \qquad (N \times N)\ \text{DFT matrix}$$

Equation (2.45) shows that the basis vectors of the DFT are eigenvectors of a circular matrix. Equation (2.45) is similar to the 2-D DFT of $[H]$ (see Eg. [5.6a]) except a complex conjugate operation is applied to the second DFT matrix in (2.45).

2.10 Summary

This chapter has defined the discrete Fourier transform (DFT) and several of its properties. Fast algorithms for efficient computation of the DFT, called fast Fourier transform (FFT), are addressed in the next chapter. These algorithms have been instrumental in ever increasing applications in diverse disciplines. They cover the gamut from radix-2/3/4 DIT, DIF, DIT/DIF to mixed radix, split-radix and prime factor algorithms.

2.11 Problems

Given $x(n) \Leftrightarrow X^{\mathrm{F}}(k)$ and $y(n) \Leftrightarrow Y^{\mathrm{F}}(k)$. Here both are N point DFTs.

2.1 Show that the DFT is a unitary transform i.e.,

$$[W]^{-1} = \frac{1}{N}[W]^{*}$$

where $[W]$ is the $(N \times N)$ DFT matrix.

2.2 Let $x(n)$ be real. $N = 2^n$. Then show that $X^{\mathrm{F}}\left(\frac{N}{2} - k\right) = X^{\mathrm{F}^*}\left(\frac{N}{2} + k\right)$, $k = 0, 1, \ldots, N/2$. What is the implication of this?

2.3 Show that $\sum_{n=0}^{N-1} x(n)y^{*}(n) = \frac{1}{N}\sum_{k=0}^{N-1} X^{\mathrm{F}}(k)Y^{\mathrm{F}^*}(k)$.

2.4 Show that $\sum_{n=0}^{N-1} x^2(n) = \sum_{k=0}^{N-1} \left|X^{\mathrm{F}}(k)\right|^2$. There may be a constant. Energy is preserved under a unitary transformation.

2.5 Show that $(-1)^n x(n) \Leftrightarrow X^{\mathrm{F}}\left(k - \frac{N}{2}\right)$.

2.6 Let $N = 4$. Explain fftshift(fft(fftshift(x))) in terms of the DFT coefficients $X^{\mathrm{F}}(k)$ or fft(x).

2.7 Show that $\sum_{n=0}^{N-1} W_N^{(r-k)n} = \begin{cases} N, & r = k \\ 0, & r \neq k \end{cases}$.

2.8 *Modulation/Frequency Shifting* Show that $\left[x(n)\exp\left(\frac{j2\pi k_0 n}{N}\right)\right] \Leftrightarrow X^{\mathrm{F}}(k - k_0)$. $X^{\mathrm{F}}(k - k_0)$ is $X^{\mathrm{F}}(k)$ shifted circularly to the right by k_0 along k (frequency domain).

2.9 *Circular Shift* Show that $x(n - n_0) \Leftrightarrow X^{\mathrm{F}}(k)\exp\left(\frac{-j2\pi k n_0}{N}\right)$. What is the implication of this?

2.10 *Circular Shift* Show that $\delta(n + n_0) \Leftrightarrow \exp\left(\frac{j2\pi k n_0}{N}\right)$ with $N = 4$ and $n_0 = 2$.

Here Kronecker delta function $\delta(n) = 1$ for $n = 0$ and $\delta(n) = 0$ for $n \neq 0$.

2.11 *Time Scaling* Show that $x(an) \Leftrightarrow \frac{1}{a} X^{\mathrm{F}}\left(\frac{k}{a}\right)$, '$a$' is a constant. What is the implication of this?

2.12 Show that $X^{\mathrm{F}}(k) = X^{\mathrm{F}^*}(-k)$, when $x(n)$ is real.

2.13 Show that $x^*(n) \Leftrightarrow X^{\mathrm{F}^*}(-k)$.

2.14 *Time Reversal* In two different ways show that $x(-n) \Leftrightarrow X^{\mathrm{F}}(-k)$.

(a) Use the DFT permutation property of (2.21) and $(N, N-1) = 1$ for any integer N.

(b) Use the definition of the DFT pair in (2.1).

Use the definition of the DFT pair in (2.1) for Problems 2.15–2.17.

2.15 The discrete convolution of two sequences $x_1(n)$ and $x_2(n)$ is defined as

$$y(n) = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m)\, x_2(n-m), \quad n = 0, 1, \ldots, N-1$$

Both $x_1(n)$ and $x_2(n)$ are N-point sequences. Show that $Y^{\mathrm{F}}(k) = \frac{1}{N} X_1^{\mathrm{F}}(k) X_2^{\mathrm{F}}(k)$, $(k = 0, 1, \ldots, N-1)$. $Y^{\mathrm{F}}(k), X_1^{\mathrm{F}}(k)$ and $X_2^{\mathrm{F}}(k)$ are the N-point DFTs of $y(n)$, $x_1(n)$ and $x_2(n)$, respectively.

2.16 The circular convolution of $X_1^{\mathrm{F}}(k)$ and $X_2^{\mathrm{F}}(k)$ is defined as

$$Y^{\mathrm{F}}(k) = \frac{1}{N}\sum_{m=0}^{N-1} X_1^{\mathrm{F}}(m) X_2^{\mathrm{F}}(k-m), \quad k = 0, 1, \ldots, N-1$$

Show that $y(n) = \frac{1}{N} x_1(n) x_2(n)$, $n = 0, 1, \ldots, N-1$. $Y^{\mathrm{F}}(k)$, $X_1^{\mathrm{F}}(k)$ and $X_2^{\mathrm{F}}(k)$ are the N-point DFTs of $y(n)$, $x_1(n)$ and $x_2(n)$, respectively.

2.17 The discrete correlation of $x_1(n)$ and $x_2(n)$ is defined as

$$z(n) = \frac{1}{N}\sum_{m=0}^{N-1} x_1(m)\, x_2(n+m), \quad n = 0, 1, \ldots, N-1$$

Show that $Z^{\mathrm{F}}(k) = \frac{1}{N} X_1^{\mathrm{F}^*}(k) X_2^{\mathrm{F}}(k)$, $k = 0, 1, \ldots, N-1$, where $Z^{\mathrm{F}}(k)$ is the DFT of $z(n)$.

2.18 Given

$$\underline{b} = [A]\,\underline{x} = \begin{pmatrix} 1 & 2 & -1 & -2 \\ 2 & 1 & 2 & -1 \\ -1 & 2 & 1 & 2 \\ -2 & -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix} \tag{P2.1}$$

Let $\underline{a} = (1, -2, -1, 2)^T$. To compute (P2.1), we use four-point FFTs as

$$\text{IFFT}[\ \text{FFT}[\underline{a}] \times \text{FFT}\ [\underline{x}]] \tag{P2.2}$$

where $\times$ denotes the element-by-element multiplication of the two vectors. Is this statement true? Explain your answer.

2.19 Using the unitary 1D-DFT, prove the following theorems:
   (a) Convolution theorem
   (b) Multiplication theorem
   (c) Correlation theorem

2.20 Derive (2.26b) from the DFT of (2.26a).

2.21 Regarding the DFT of a permuted sequence
   (a) When $(p, N) = 1$, we can find an integer q such that $0 \leq q \leq N - 1$ and $qp \equiv (1 \,\text{modulo}\, N)$. If $(p, N) \neq 1$, we cannot find such an integer q. Explain the latter with examples.
   (b) Repeat Example 2.1 for $N = 9$.
   (c) Check the DFTs of permuted sequences for $N = 8$ and $N = 9$ using MATLAB.
   (d) Repeat Example 2.1 for $N = 16$.

2.22 Let $[\Lambda] = \text{diag}\,(\lambda_0, \lambda_1, \ldots, \lambda_{N-1})$. Derive (2.45) from (2.43).

2.23 Given the circulant matrix

$$[H] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \\ 3 & 4 & 1 & 2 \\ 2 & 3 & 4 & 1 \end{bmatrix}$$

show that the diagonal elements of the DFT of $[H]$ equal the eigenvalues of the matrix by MATLAB.

2.24 *An orthonormal basis (ONB) for the DFT*
   (a) By using the relation:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_0 \\ x_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} x_0 + \begin{bmatrix} 2 \\ 4 \end{bmatrix} x_1, \ \text{or} \quad [A]\,\underline{x} = \underline{a}_0\, x_0 + \underline{a}_1 x_1 \tag{P2.3}$$

where $[A] = (\underline{a}_0, \underline{a}_1)$, show that

$$\underline{a}_0^T \underline{x}\, \underline{a}_0 + \underline{a}_1^T \underline{x}\, \underline{a}_1 = [A][A]^T \underline{x} \tag{P2.4}$$

   (b) If vector $\underline{a}_k$ represents a column of the normalized inverse DFT matrix, then it is called a *basis vector* for the DFT.

$$\left(\frac{1}{\sqrt{N}}\left([F]^T\right)^*\right) = (\underline{a}_0, \underline{a}_1, \ldots, \underline{a}_{N-1}) \tag{P2.5}$$

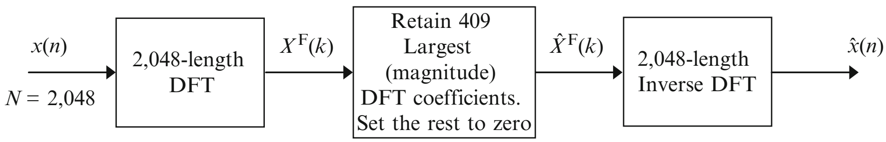


Fig. P2.1 Reconstruct $\hat{x}(n)$ from truncated DFT

where $[F]$ is a DFT matrix. DFT coefficient can be expressed as

$$X_k^{\mathrm{F}} = \langle \underline{a}_k, \underline{x} \rangle = (\underline{a}_k^T)^* \underline{x} \qquad k = 0, 1, \ldots, N-1 \tag{P2.6}$$

Show that

$$\underline{x} = \sum_{k=0}^{N-1} X_k^{\mathrm{F}} \underline{a}_k = \sum_{k=0}^{N-1} \langle \underline{a}_k, \underline{x} \rangle \underline{a}_k = \left(\frac{1}{\sqrt{N}} \left([F]^T \right)^* \right) \left(\frac{1}{\sqrt{N}} [F] \right) \underline{x} \tag{P2.7}$$

where X_k^{F} is a scalar and DFT coefficient.

2.12 Projects

2.1 Access dog ECG data from the Signal Processing Information Base (SPIB) at URL http://spib.rice.edu/spib/data/signals/medical/dog_heart.html ($N = 2{,}048$). Sketch this $x(n)$ versus n. Take DFT of this data and sketch $X^{\mathrm{F}}(k)$ versus k (both magnitude and phase spectra). Retain 409 DFT coefficients (Largest in magnitude) and set the remaining 1,639 DFT coefficients to zeros (truncated DFT). Reconstruct $\hat{x}(n)$ from this truncated DFT (Fig. P2.1).
(1) Sketch $\hat{x}(n)$ versus n.
(2) Compute MSE $= \frac{1}{2048} \sum_{n=0}^{2047} |x(n) - \hat{x}(n)|^2$.
(3) Compute DFT of $[(-1)^n x(n)]$ and sketch the magnitude and phase spectra.
(4) Summarize your conclusions (DFT properties etc.). See Chapter 2 in [B23].

2.2 Let

$$\begin{aligned} x(n) &= \{1, 2, 3, 4, 3, 2, 1\} \\ y(n) &= \{-0.0001, 0.0007, -0.0004, -0.0049, 0.0087, 0.0140, \\ &\quad -0.0441, -0.0174, 0.1287, 0.0005, -0.2840, \\ &\quad -0.0158, 0.5854, 0.6756, 0.3129, 0.0544\} \end{aligned}$$

(1) Compute directly the discrete convolution of the two sequences. Sketch the results.

(2) Use DFT/FFT approach (Fig. 2.9). Show that both give the same results.

http://www.springer.com/978-1-4020-6628-3

Fast Fourier Transform - Algorithms and Applications
Rao, K.R.; Kim, D.N.; Hwang, J.J.
2010, XVIII, 426 p., Hardcover
ISBN: 978-1-4020-6628-3