Chapter 2
Convolution and Correlation

2.1 Introduction

In this chapter we will consider two signal analysis concepts, namely convolution and correlation. Signals under consideration are assumed to be real unless otherwise mentioned. *Convolution* operation is basic to linear systems analysis and in determining the probability density function of a sum of two independent random variables. Impulse functions were defined in terms of an integral (see (1.4.4a)) using a test function $\phi(t)$.

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-t_0)dt = \phi(t_0). \qquad (2.1.1)$$

This integral is the convolution of two functions, $\phi(t)$ and the impulse function $\delta(t)$ to be discussed shortly. In a later chapter we will see that the response of a linear time-invariant (LTI) system to an impulse input $\delta(t)$ is described by the convolution of the input signal and the impulse response of the system. Convolution operation lends itself to spectral analysis. There are two ways to present the discussion on convolution, first as a basic mathematical operation and second as a mathematical description of a response of a linear time-invariant system depending on the input and the description of the linear system. The later approach requires knowledge of systems along with Fourier series and transforms. This approach will be considered in Chapter 6. Although we will not be discussing random signals in any detail, convolution is applicable in dealing with random variables.

The process of *correlation* is useful in comparing two deterministic signals and it provides a measure of similarity between the first signal and a time-delayed version of the second signal (or the first signal). A simple way to look at correlation is to consider two signals: $x_1(t)$ and $x_2(t)$. One of these signals could be a delayed, or an advanced, version of the other. In this case we can write $x_2(t) = x_1(t+\tau)$, $-\infty < \tau < \infty$. Multiplying point by point and adding all the products, $x_1(t)x_1(t+\tau)$ will give us a large number for $\tau = 0$, as the product is the square of the function. On the other hand if $\tau \neq 0$, then adding all these numbers will result in an equal or a lower value since a positive number times a negative number results in a negative number and the sum will be less than or equal to the peak value. In terms of continuous functions, this information can be obtained by the following integral, called the autocorrelation function of $x(t)$, as a function of τ *not* t.

$$R_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt = \mathrm{AC}\,[x(t)] \equiv R_x(\tau). \qquad (2.1.2)$$

This gives a comparison of the function $x(t)$ with its shifted version $x(t+\tau)$. Autocorrelation (AC) provides a nice way to determine the spectral content of a random signal. To compare two different functions, we use the *cross-correlation* function defined by

$$R_{xh}(\tau) = x(\tau) ** h(\tau) = \int_{-\infty}^{\infty} x(t)h(t+\tau)dt. \qquad (2.1.3)$$

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Note the symbol (**) for correlation. Correlation is related to the convolution. As in autocorrelation, the cross-correlation in (2.1.3) is a function of τ, the time shift between the function $x(t)$, and the shifted version of the function $h(t)$.

2.1.1 Scalar Product and Norm

The scalar valued function $\langle x(t), y(t)\rangle$ of two signals $x(t)$ and $y(t)$ of the *same* class of signals, i.e., either energy or power signals, is defined by

$$\langle x(t),y(t)\rangle = \begin{cases} \int_{-\infty}^{\infty} x(t)y^*(t)dt, & \\ \text{energy signals.} & (2.1.4a) \\ \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(t)y^*(t)dt, & \\ \text{power signals.} & (2.1.4b) \end{cases}$$

Superscript (*) indicates complex conjugation. Our discussion will be limited to a subclass of power signals, namely periodic signals. In that case, assuming that both the time functions have the same period (2.1.4b) can be written in the symbolic form as follows:

$$\langle x_T(t)y_T(t)\rangle = \frac{1}{T}\int_T x(t)y^*(t)dt. \quad (2.1.4c)$$

Even though our interest is in real functions, for generality, we have used complex conjugates in the above equations. The norm of the function is defined by

$$\|x(t)\| = \langle x(t), x(t)\rangle^{1/2} = \begin{cases} E_x, & \text{energy signals} \\ P_x, & \text{power signals} \end{cases}. \quad (2.1.5)$$

It gives the energy or power in the given energy or the power signal. The two functions, $x(t)$ and $y(t)$, are *orthogonal* if

$$\langle x(t), y(t)\rangle = 0. \quad (2.1.6)$$

In that case,

$$\|x(t) + y(t)\|^2 = \|x(t)\|^2 + \|y(t)\|^2. \quad (2.1.7)$$

If $x(t)$ and $y(t)$ are orthogonal, the energy and power contained in the energy or power signal $z(t) = x(t) + y(t)$ are respectively given by

$$E_z = E_x + E_y \text{ or } P_z = P_x + P_y. \quad (2.1.8)$$

Some of the important properties of the norm are stated as follows:

1. $\|x(t)\| = 0$ if and only if $x(t) = 0$, (2.1.9a)
2. $\|x(t) + y(t)\| \le \|x(t)\| + \|y(t)\|$, triangular inequality (2.1.9b)
3. $\|\alpha x(t)\| = |\alpha|\|x(t)\|$. (2.1.9c)

In (2.1.9c), α is some constant. One measure of *distance*, or *dissimilarity*, between $x(t)$ and $y(t)$ is $\|x(t) - y(t)\|$. A useful inequality is the *Schwarz's inequality* given by

$$|\langle x(t), y(t)\rangle| \le \|x(t)\|\|y(t)\|. \quad (2.1.9d)$$

The two sides are equal when $x(t)$ or $y(t)$ is zero or if $y(t) = \alpha x(t)$ where α is a scalar to be determined. This can be seen by noting that

$$\begin{aligned} \|x(t)+\alpha y(t)\|^2 &= \langle x(t)+\alpha y(t), x(t)+\alpha y(t)\rangle \\ &= \langle x(t)x(t)\rangle + \alpha^*\langle x(t), y(t)\rangle \\ &\quad + \alpha\langle x(t), y(t)\rangle^* + |\alpha|^2\langle y(t), y(t)\rangle \\ &= \|x(t)\|^2 + \alpha^*\langle x(t), y(t)\rangle \\ &\quad + \alpha\langle x(t), y(t)\rangle^* + |\alpha|^2\|y(t)\|^2. \end{aligned} \quad (2.1.10)$$

Since α is arbitrary, select

$$\alpha = -\langle x(t), y(t)\rangle / \|y(t)\|^2. \quad (2.1.11)$$

Substituting this in (2.1.10), the last two terms cancel out, resulting in

$$\|x(t)+\alpha y(t)\|^2=\|x(t)\|^2-\frac{\left|\langle x(t),y(t)\rangle^2\right|}{\|y(t)\|^2}$$
$$\Rightarrow \|x(t)\|^2\|y(t)\|^2-\left|\langle x(t),y(t)\rangle^2\right|\geq 0. \tag{2.1.12}$$

Equality exists in (2.1.9d) only if $x(t)+\alpha y(t)=0$. Another possibility is the trivial case being either one of the functions or both are equal to zero. Ziemer and Tranter (2002) provide important applications on this important topic.

Correlations in terms of time averages: Cross-correlation and autocorrelation functions can be expressed in terms of the time average symbols and

$$R_{xh}(\tau)=\int_{-\infty}^{\infty} x(t)h(t+\tau)dt=\langle x(t)h(t+\tau)\rangle, \tag{2.1.13a}$$

$$R_{T,xh}(\tau)=\frac{1}{T}\int_{-T/2}^{T/2} x_T(t)h_T(t+\tau)dt$$
$$=\frac{1}{T}\int_{T} x_T(t)h_T(t+\tau)dt=\langle x_T(t)h_T(t+\tau)\rangle. \tag{2.1.13b}$$

In the early part of this chapter we will deal with convolution and correlation associated with aperiodic signals. In the later part we will concentrate on convolution and correlation with respect to both periodic and aperiodic signals. Most of the material in this chapter is fairly standard and can be seen in circuits and systems books. For example, see Ambardar (1995), Carlson (1975), Ziemer and Tranter (2002), Simpson and Houts (1971), Peebles (1980), and others.

2.2 Convolution

The convolution of two functions, $x_1(t)$ and $x_2(t)$, is defined by

$$y(t)=x_1(t)*x_2(t)=\int_{-\infty}^{\infty} x_1(\alpha)x_2(t-\alpha)d\alpha$$
$$=\int_{-\infty}^{\infty} x_2(\beta)x_1(t-\beta)d\beta=x_2(t)*x_1(t). \tag{2.2.1}$$

This definition describes a *higher algebra* and allows us to study the response of a linear time-invariant system in terms of a signal and a system response to be discussed in Chapter 6. It should be emphasized that the end result of the convolution operation is a function of time. Coming back to the sifting property of the impulse functions, consider the equation given in (2.1.1). Two special cases are of interest.

$$\phi(t)*\delta(t)=\int_{-\infty}^{\infty}\phi(\alpha)\delta(t-\alpha)d\alpha$$
$$=\int_{-\infty}^{\infty}\phi(t-\beta)\delta(\beta)d\beta=\delta(t)*\phi(t), \tag{2.2.2a}$$

$$\delta(t)*\delta(t)=\int_{-\infty}^{\infty}\delta(\alpha)\delta(t-\alpha)d\alpha=\delta(t). \tag{2.2.2b}$$

2.2.1 Properties of the Convolution Integral

1. Convolution of two functions, $x_1(t)$ and $x_2(t)$, satisfies the *commutative property*,

$$y(t)=x_1(t)*x_2(t)=x_2(t)*x_1(t). \tag{2.2.3}$$

   This equality can be shown by defining a new variable, $\beta=t-\alpha$, in the first integral in (2.2.1) and simplifying the equation.
2. Convolution operation satisfies the *distributive property*, i.e.,

$$x_1(t)*[x_2(t)+x_3(t)]=x_1(t)*x_2(t)+x_1(t)*x_3(t). \tag{2.2.4}$$

3. Convolution operation satisfies the *associative property*, i.e.,

$$x_1(t)*(x_2(t)*x_3(t))=(x_1(t)*x_2(t))*x_3(t). \tag{2.2.5}$$

   The proofs of the last two properties follow from the definition.
4. The *derivative of the convolution* operation can be written in a simple form and

$$y'(t)=\frac{dy(t)}{dt}=\frac{d}{dt}\left[\int_{-\infty}^{\infty}x_1(\alpha)x_2(t-\alpha)d\alpha\right]$$
$$=\int_{-\infty}^{\infty}x_1(\alpha)\frac{dx_2(t-\alpha)}{dt}d\beta=x_1(t)*x_2'(t),$$

$$\Rightarrow\frac{dy(t)}{dt}=\frac{d}{dt}[x_1(t)*x_2(t)]$$
$$=\frac{dx_1(t)}{dt}*x_2(t)=x_1(t)*\frac{dx_2(t)}{dt}. \tag{2.2.6a}$$

Equation (2.2.6a) can be generalized for higher order derivatives. We can then write

$$x_1^{(m)}(t)*x_2(t)=\frac{d^m x_1(t)}{dt^m}*x_2(t)=\frac{d^m y(t)}{dt^m} \tag{2.2.6b}$$
$$=y^{(m)}(t)\left(\text{Note } x_i^{(m)}(t)=\frac{d^i x_i(t)}{dt^i}\right),$$

$$x_1^{(m)}(t)*x_2^{(n)}(t)=\frac{d^m x_1(t)}{dt^m}*\frac{d^n x_2(t)}{dt^n} \tag{2.2.6c}$$
$$=\frac{d^{m+n}y(t)}{dt^{m+n}}=y^{(m+n)}(t).$$

Since the impulse function is the *generalized derivative* of the unit step function $u(t)$ (see Section 1.4.2.), we have

$$y(t)=u(t)*h(t)\Rightarrow y'(t) \tag{2.2.7}$$
$$=u'(t)*h(t)=\delta(t)*h(t)=h(t).$$

5. Convolution is an integral operation and if we know the convolution of two functions and desire to compute its *running integral,* we can use

$$\int_{-\infty}^{t}y(\alpha)d\alpha=\int_{-\infty}^{t}[x_1(\alpha)*x_2(\alpha)]d\alpha$$
$$=\int_{-\infty}^{t}\left[\int_{-\infty}^{\infty}x_1(\beta)x_2(\alpha-\beta)d\beta\right]d\alpha,$$
$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{t}x_2(\alpha-\beta)d\alpha\right]x_1(\beta)d\beta$$
$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{t-\beta}x_2(\lambda)d\lambda\right]x_1(\beta)d\beta,$$
$$=\left[\int_{-\infty}^{t}x_2(\lambda)d\lambda\right]*x_1(t)=\left[\int_{-\infty}^{t}x_1(\beta)d\beta\right]*x_2(t). \tag{2.2.8}$$

Example 2.2.1 Find the convolution of a function $x(t)$ and the unit step function $u(t)$ and show it is a running integral of $x(t)$.

Solution: This can be seen from

$$x(t)*u(t)=\int_{-\infty}^{\infty}x(\beta)u(t-\beta)d\beta$$
$$=\int_{-\infty}^{t}x(\beta)d\beta,\quad\left[u(t-\beta)=\begin{cases}1, & \beta<t\\ 0, & \beta>t\end{cases}\right]. \tag{2.2.9}$$
■

6. *Convolution of two delayed functions* $x_1(t-t_1)$ and $x_2(t-t_2)$ are related to the convolution of $x_1(t)$ and $x_2(t)$.

$$y(t)=x_1(t)*x_2(t)\Rightarrow x_1(t-t_1)*x_2(t-t_2)$$
$$=y(t-(t_1+t_2)). \tag{2.2.10}$$

This can be seen from

$$x_1(t-t_1)*x_2(t-t_2)$$
$$=\int_{-\infty}^{\infty}x_1(\alpha-t_1)x_2(t-\alpha-t_2)d\alpha$$
$$=\int_{-\infty}^{\infty}x_1(\beta)x_2([t-(t_1+t_2)]-\beta)d\beta$$
$$=y(t-(t_1+t_2)). \tag{2.2.11}$$

Example 2.2.2 Derive the expression for $y(t)=x_1(t)*x_2(t)=\delta(t-t_1)*\delta(t-t_2)$.

Solution: Using the integral expression, we have

$$\int_{-\infty}^{\infty} x_1(\alpha)x_2(t-\alpha)d\alpha = \int_{-\infty}^{\infty} \delta(\alpha-t_1)\delta(t-t_2-\alpha)d\alpha$$
$$=\delta(t-t_2-\alpha)|_{\alpha=t_1}=\delta(t-t_1-t_2).$$

Noting $\delta(t)*\delta(t)=\delta(t)$ and using (2.2.11), we have $\delta(t-t_1)*\delta(t-t_2)=\delta(t-t_1-t_2)$. ■

7. The *time scaling property* of the convolution operation is if $y(t)=x_1(t)*x_2(t)$, then

$$x_1(ct)*x_2(ct)=\int_{-\infty}^{\infty} x_1(c\beta)x_2(c(t-\beta))d\beta$$
$$=\frac{1}{|c|}y(ct), c\neq 0. \tag{2.2.12}$$

Assuming $c<0$ and using the change of variables $\alpha=c\beta$, and simplifying, we have

$$x_1(ct)*x_2(ct)=\frac{1}{c}\int_{\infty}^{-\infty} x_1(\alpha)x_2(ct-\alpha)dy$$
$$=\frac{1}{|c|}\int_{-\infty}^{\infty} x_1(\alpha)x_2(ct-\alpha)d\alpha=\frac{1}{|c|}y(ct).$$

A similar argument can be given in the case of $c>0$. Scaling property applies only when both functions are scaled by the *same* constant $c\neq 0$. When $c=-1$, then

$$x_1(-t)*x_2(-t)=y(-t). \tag{2.2.13}$$

This property simplifies the convolution if there are symmetries in the functions. In Chapter 1, even and odd functions were identified by subscripts e for even and 0 for odd (see (1.2.7)). From these

$$x_{ie}(-t)=x_{ie}(t),\ \text{an even function};$$
$$x_{i0}(-t)=-x_{i0}(t),\ \text{an odd function} \tag{2.2.14a}$$

$$x_{1e}(-t)*x_{2e}(-t)=y_{e_1}(t),$$
$$x_{10}(-t)*x_{20}(-t)=y_{e_2}(t),\ \text{even functions} \tag{2.2.14b}$$

$$x_{1e}(-t)*x_{20}(-t)=y_{0_1}(t),$$
$$x_{10}(-t)*x_{2e}(-t)=y_{0_2}(t),\ \text{odd functions.} \tag{2.2.14c}$$

8. The area of a signal was defined in Chapter 1 (see (1.5.1)) by

$$A[x_i(t)]=\int_{-\infty}^{\infty} x_i(\alpha)d\alpha. \tag{2.2.15}$$

Area property of the convolution applies if the areas of the individual functions do *not change* with *a shift in time*. It is given by

$$A[y(t)]=A[x_1(t)*x_2(t)]=A[x_1(t)]A[x_2(t)]. \tag{2.2.16}$$

This can be proved by

$$A[y(t)]=\int_{-\infty}^{\infty} y(\beta)d\beta=\int_{-\infty}^{\infty}[x_1(\beta)*x_2(\beta)]d\beta$$
$$=\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x_1(\alpha)x_2(\beta-\alpha)d\alpha\right]d\beta$$
$$=\int_{-\infty}^{\infty}\left\{x_1(\alpha)\int_{-\infty}^{\infty} x_2(\beta-\alpha)d\beta\right\}d\alpha$$
$$=A[x_2(t)]\int_{-\infty}^{\infty} x_1(\alpha)d\alpha=A[x_2(t)]A[x_1(t)].$$

9. Consider the signals $x_1(t)$ and $x_2(t)$ that are non-zero for the time intervals of t_{x1} and t_{x2}, respectively. That is, we have two time-limited signals, $x_1(t)$ and $x_2(t)$, with time widths t_{x_1} and t_{x_2}. Then, the time width t_y of the signal $y(t)=x_1(t)*x_2(t)$ is the sum of the time widths of the two convolved signals and $t_y=t_{x_1}+t_{x_2}$. This is referred to as the *time duration property of the convolution*. We will come back to some intricacies in this property, as there are some *exceptions* to this property.

Example 2.2.3 Derive the expression for the convolution $y(t)=x_1(t)*x_2(t)$, where $x_i(t),\ i=1,2$ are as follows:

$$x_1(t)=0.5\delta(t-1)+0.5\delta(t-2),$$
$$x_2(t)=0.3\delta(t+1)+0.7\delta(t-3).$$

Solution: Convolution of these two functions is

$$\begin{aligned}
y(t) &= \int_{-\infty}^{\infty} x_1(\alpha)x_2(t-\alpha)d\alpha \\
&= \int_{-\infty}^{\infty} [0.5\delta(\alpha-1)+0.5\delta(\alpha-2)]\times \\
&\quad [0.3\delta(t-\alpha+1)+0.7\delta(t-\alpha-3)]d\alpha \\
&= \int_{-\infty}^{\infty} (0.5)(0.3)\delta(\alpha-1)\delta(t-\alpha+1)d\alpha \\
&\quad + \int_{-\infty}^{\infty} (0.5)(0.7)\delta(\alpha-1)\delta(t-\alpha-3)d\alpha \\
&\quad + \int_{-\infty}^{\infty} (0.5)(0.3)\delta(\alpha-2)\delta(t-\alpha+1)d\alpha \\
&\quad + \int_{-\infty}^{\infty} (0.5)(0.7)\delta(\alpha-2)\delta(t-\alpha-3)d\alpha \\
&= (0.15)\delta(t)+0.35\delta(t-4) \\
&\quad + 0.15\delta(t-1)+0.35\delta(t-5). \qquad \blacksquare
\end{aligned}$$

Notes: If an impulse function is in the integrand of the form $\delta(at-b)$, then use (see (1.4.35), which is

$$\delta(at-b) = (1/|a|)\delta(t-(b/a)).$$

2.2.2 Existence of the Convolution Integral

Convolution of two functions exists if the convolution integral exists. Existence can be given only in terms of sufficient conditions. These are related to signal energy, area, and one sidedness. It is simple to give examples, where the convolution does not exist. Some of these are $a*a$, $a*u(t)$, $cos(t)*u(t)$, $e^{at}*e^{at}$, $a>0$. Convolution of energy signals and the same-sided signals always exist. In Chapter 4 we will be discussing Fourier transforms and the transforms make it convenient to find the convolution.

2.3 Interesting Examples

In the following, the basics of the convolution operation, along with using some of the above properties to simplify the evaluations are illustrated. A few comments are in order before the examples. First, the convolution $y(t) = x_1(t)*x_2(t)$ is an integral operation and can use either one of the integrals in (2.2.1). Note that $y(t)$, $-\infty < t < \infty$ is a time function. The expression for the convolution, say at $t = t_0$, will yield a zero value for those values of t_0 over which $x_1(\beta)$ and $x_2(t_0-\beta)$ do not overlap. The area under the product $[x_1(\beta)x_2(t_0-\beta)]$, i.e., the integral of this product gives the value of the convolution at $t = t_0$. Sketches of the function $x_1(\beta)$ and the time reversed and delayed function $x_2(t_0-\beta)$ on the same figure would be helpful in identifying the limits of integration of the product $[x_1(\beta)x_2(t_0-\beta)]$. As a check, the value of the convolution at end points of each range must match, except in the case of impulses and/or their derivatives in the integrand of the convolution integral. This is referred to as the *consistency check*. The following steps can be used to compute the convolution of two functions $x_1(t)$ and $x_2(t)$.

$$x_2(t) \xrightarrow{\text{New variable}} x_2(\beta) \xrightarrow{\text{Reverse}} x_2(-\beta) \xrightarrow{\text{Shift}} x_2(t-\beta)$$

$$x_1(t) \xrightarrow{\text{New variable}} x_1(\beta) \xrightarrow{\text{Multiply the two functions}} x_1(\beta)x_2(t-\beta).$$

$$\xrightarrow{\text{Integrate}} \int_{-\infty}^{\infty} x_1(\beta)x_2(t-\beta)d\beta = y(t).$$

Example 2.3.1 Derive the expression for the convolution of the two pulse functions shown in Fig. 2.3.1 a,b. These are

$$x_1(t) = \frac{1}{a}\Pi\left[\frac{t-(a/2)}{a}\right] \text{ and}$$
$$x_2(t) = \frac{1}{b}\Pi\left[\frac{t-(b/2)}{b}\right], b \geq a > 0. \qquad (2.3.1)$$

Solution: First

$$y(t) = x_1(t)*x_2(t) = \int_{-\infty}^{\infty} x_1(\beta)x_2(t-\beta)d\beta. \quad (2.3.2)$$

Figure 2.3.1c,d,e,f give the functions $x_1(\beta)$, $x_2(\beta)$, $x_2(-\beta)$, and $x_2(t-\beta)$, respectively. Note that the variable t is some value between $-\infty$ and ∞ on the β axis. Different cases are considered, and

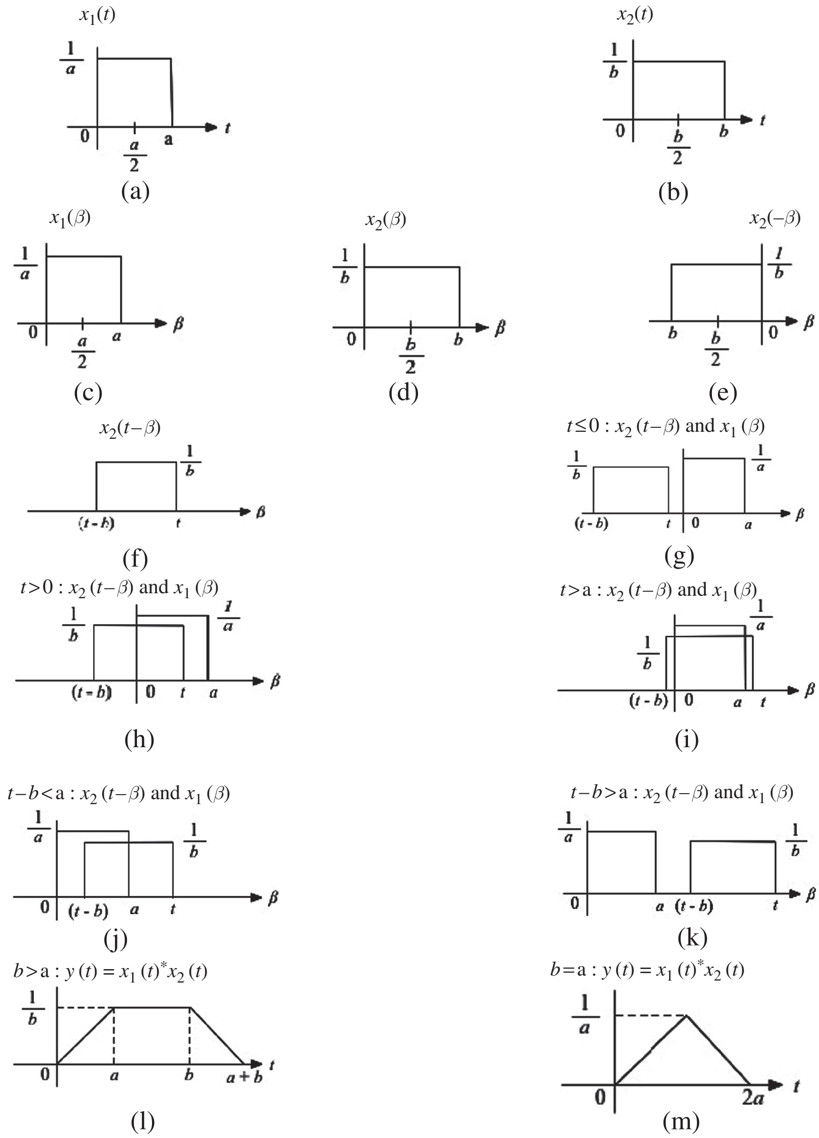

Fig. 2.3.1 Convolution of two rectangular pulses ($b \ge a$)

in each case, we keep the first function $x_1(\beta)$ stationary and move (or shift) the second function $x_2(-\beta)$, resulting in $x_2(t-\beta)$.

Case 1: $t \le 0$. For this case the two functions are sketched in Fig. 2.3.1 g on the same figure. Noting that there is no overlap of these two functions, it follows that

$$y(t) = 0, t \le 0. \tag{2.3.3}$$

Case 2: $0 < t \le a$. The two functions are sketched for this case in Fig. 2.3.1 h. The two functions overlap and the convolution is

$$y(t) = \int_0^t x_1(\beta)x_2(t-\beta)d\beta = \frac{1}{ab}t, 0 < t \le a. \tag{2.3.4}$$

Case 3: $a < t \le b$. This corresponds to the complete overlap of the two functions and the functions are shown in Fig. 2.3.1i. The convolution integral and the area is

$$y(t) = \int_0^a \frac{1}{ab} d\beta = \frac{1}{b}, a < t \le b. \quad (2.3.5)$$

Case 4: $0 < t - b \le a < b$ or $b < t \le (b + a)$. This corresponds to a partial overlap of the two functions and is shown in Fig. 2.3.1j. The convolution integral and the area is

$$y(t) = \int_{t-b}^{a} \frac{1}{ab} d\beta = \frac{(a+b-t)}{ab}, b < t \le (b+a). \quad (2.3.6)$$

Case 5: $t - b > a$ or $t \ge a + b$. The two functions corresponding to this range are sketched in Fig. 2.3.1k and from the sketches we see that the two functions do not overlap and

$$y(t) = 0, \ t \ge (a + b). \quad (2.3.7)$$

Summary:

$$y(t) = \begin{cases} 0, & t \le 0 \\ \dfrac{t}{ab}, & 0 < t < a \\ \dfrac{1}{b}, & a \le t < b \\ \dfrac{a+b-t}{ab}, & b \le t < a+b \\ 0, & t \ge a+b \end{cases}. \quad (2.3.8)$$

This function is sketched in Fig. 2.3.1 l and m for the cases of $b > a$ and $b = a$. There are several interesting aspects in this example that should be noted. First, the two functions we started with have first-order discontinuous and the convolution is an integral operation, which is a smoothing operation. Convolution values at end points of each range must match (*consistency check*) as we do *not* have any *impulse functions or their derivatives* in the functions that are convolved. Some of these are discussed below.

The areas of the two pulses are each equal to 1 and the area of the trapezoid is given by

$$\begin{aligned} \text{Area}[y(t)] = & \ (1/2)a(1/b) + (b - a)(1/b) \\ & + (1/2)a(1/b) = 1 \\ & = \text{Area}[x_1(t)]\text{Area}[x_2(t)]. \end{aligned} \quad (2.3.9)$$

This shows that the area property is satisfied. Peebles (2001) shows the probability density function of the sum of the two independent random variables is also a probability density function. We should note that the probability density function is nonnegative and the area under this function is 1 (see Section 1.7). From the above discussion, it follows that the convolution of two rectangular pulses (these can be considered as uniform probability density functions) results in a nonnegative function and the area under this function is 1. The function $y(t)$ satisfies the conditions of a probability density function.

The time duration of $y(t)$, t_y is $t_y = t_{x_1} + t_{x_2}$ and

$$t_{x_1} = a, t_{x_2} = b \Rightarrow t_y = t_{x_1} + t_{x_2} = a + b. \quad (2.3.10)$$

A special case is when $a = b$ and the function $y(t)$ given in Fig. 2.3.1m, a triangle, is

$$\Pi\left[\frac{t - a/2}{a}\right] * \Pi\left[\frac{t - a/2}{a}\right] = \Lambda\left[\frac{t - a}{a}\right]. \quad (2.3.11) \ \blacksquare$$

Example 2.3.2 Give the expressions for the convolution of the following functions:

$$x_1(t) = u(t) \text{ and } x_2(t) = \sin(\pi t)\Pi\left[\frac{t-1}{2}\right]. \quad (2.3.12)$$

Solution: The convolution integral

$$y(t)=\int_{-\infty}^{\infty} x_2(\beta)x_1(t-\beta)d\alpha=\int_{0}^{2}\sin(\pi\beta)\,u(t-\beta)d\beta$$
$$=\int_{0}^{t}\sin(\pi\beta)d\beta=\begin{cases}0, & t\leq 0\\ (1/\pi)(1-\cos(\pi t)), & 0<t<2,\\ 0, & t\geq 2\end{cases}$$

$$y(t)=\int_{0}^{t}\sin(\pi\beta)d\beta$$
$$=\begin{cases}0, & t\leq 0\\ (1/\pi)(1-\cos(\pi t)), & 0<t<2.\\ 0, & t\geq 2\end{cases} \quad (2.3.13)$$

The time duration of the unit step function is ∞ and the time duration of $x_2(t)$ is 2. The duration of the function $y(t)$ is 2, which illustrates a pathological case where the time duration property of the convolution is not satisfied.

The integral or the area of a sine or a cosine function over one period is equal to zero. The period of the function $\sin(\pi t)$ is equal to 2 and therefore

$$A[x_2(t)]=A\left[\sin(\pi t).\Pi\left[\frac{t-1}{2}\right]\right]$$
$$=0\Rightarrow A[y(t)]=(1/\pi)\int_{0}^{2}[1-\cos(\pi t)]dt$$
$$=1/\pi\int_{0}^{2}dt=2/\pi.$$

Noting that $A[x_1(t)]=A[u(t)]=\infty$ and $A[y(t)]=2/\pi$, we can see that the area property of the convolution is not satisfied. See Ambardar (1995) for an additional discussion. ■

Example 2.3.3 Derive the expression for the convolution of the following functions shown in Fig. 2.3.2a,b:

$$x(t)=\Pi\left[\frac{t}{2T}\right] \text{ and } h(t)=e^{-at}u(t), a>0. \quad (2.3.14a)$$

Solution:

$$y(t)=x(t)*h(t)=\int_{-\infty}^{\infty} x(\beta)h(t-\beta)d\beta$$
$$=\int_{-\infty}^{\infty} h(\alpha)x(t-\alpha)d\alpha. \quad (2.3.14b)$$

In computing the convolution, we keep one of the functions at one location and the other function is time reversed and then shifted. In this example, since the function $h(t)=0$ for $t<0$, we have a *benchmark* to keep track of the movement of the function $h(t-\beta)$ as t varies. Therefore, the first integral in (2.3.14b) is simpler to use. The functions $x(\beta), h(\beta), h(-\beta)$, and $h(t-\beta)$ are shown in Fig. 2.3.2 c, d, e, and f respectively. As before, we will compute the convolution for different intervals of time.

Case 1: $t\leq -T$: the two functions, $h(t-\beta)$ and $x(\beta)$, are sketched in Fig. 2.3.2 g. Clearly there is no overlap of the two functions and therefore the integral is zero. That is

$$y(t)=0, t\leq -T. \quad (2.3.15)$$

Case 2: $-T<t<T$: The two functions $h(t-\beta)$ and $x(\beta)$ are sketched in Fig. 2.3.2 h in the same figure for this interval. There is a partial overlap of the two functions in the interval $-T>t>T$. The convolution can be expressed by

$$y(t)=\int_{-\infty}^{\infty} x(\beta)h(t-\beta)d\beta=\int_{-T}^{t}(1)e^{-a(t-\beta)}d\beta \quad (2.3.16)$$
$$=e^{-at}\int_{-T}^{t}e^{a\beta}d\beta=\frac{1}{a}\left[1-e^{-a(t+T)}\right], -T<t<T.$$

Case 3: $t>T$: From the sketch of the two functions in Fig. 2.3.2 h, the two functions overlap in this range $-T\leq t\leq T$ and the convolution integral is

$$y(t)=\int_{-T}^{T}e^{-a(t-\beta)}d\beta=\frac{1}{a}\left[e^{aT}-e^{-aT}\right]e^{-at}, \quad t>T. \quad (2.3.17)$$

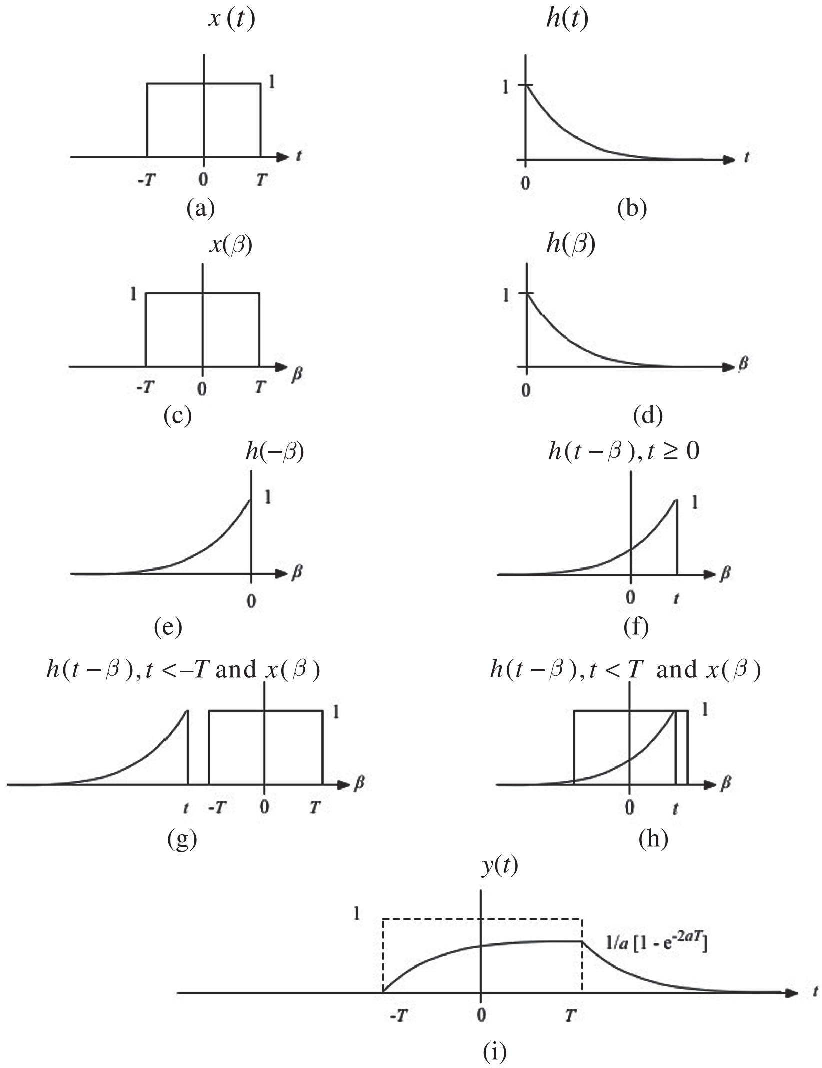

Fig. 2.3.2 Convolution of a rectangular pulse with an exponentially decaying pulse

Summary:

$$y(t) = \begin{cases} 0, & t \le -T \\ \dfrac{1}{a}\left[1 - e^{-a(t+T)}\right], & -T < t \le T \\ \dfrac{1}{a}\left[e^{aT} - e^{-aT}\right]e^{-at}, & t > T \end{cases}. \quad (2.3.18)$$

This function is sketched in Fig. 2.3.2i. Note $y(t)$ is smoother than either of the given functions used in the convolution. Computing the area of $y(t)$ is not as simple as finding the areas of the two functions, $x(t)$ and $h(t)$. Using the area property,

$$A[y(t)] = A[x(t)]A[h(t)] = (2T)(1/a). \quad (2.3.19) \quad \blacksquare$$

Notes: In computing the convolution, one of the sticky points is finding the integral of the product $[x(\beta)h(t-\beta)]$ in (2.3.14b), which requires finding the region of overlap of the two functions. Sketching both functions on the *same figure* allows for an easy determination of this overlap. The *delay property* is quite useful. For example, if $y(t) = x(t) * h(t)$ then it implies $y_1(t) = x(t-T) * h(t) = y(t-T)$. In Example 2.3.3, $x(t) = \Pi[t/2T] = u[t+T] - u[t-T]$. Therefore

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= h(t) * [u(t+T) - x(t-T)] \\ &= h(t) * u(t+T) - h(t) * u(t-T). \end{aligned}$$ ■

Example 2.3.4 Determine the convolution $y(t) = x(t) * x(t)$ with $x(t) = e^{-at}u(t)$, $a > 0$.

Solution: The convolution is

$$\begin{aligned} y(t) &= e^{-at}u(t) * e^{-at}u(t) \\ &= \int_{-\infty}^{\infty} e^{-a\beta}e^{-a(t-\beta)}[u(\beta)u(t-\beta)]d\beta \\ &= e^{-at}\int_{0}^{t} d\beta = te^{-at}u(t). \end{aligned} \quad (2.3.20)$$

In evaluating the integral, the following expression is used (see Fig. 2.3.3a):

$$[u(\beta)u(t-\beta)] = \begin{cases} 0, & \beta < 0 \text{ and } \beta > t \\ 1, & 0 < \beta > t \end{cases}. \quad (2.3.21)$$

The functions $x(t)$ and $y(t)$ are shown in Fig. 2.3.3b,c. Note that the function $x(t)$ has a discontinuity at $t = 0$. The function $y(t)$, obtained by convolving two identically decaying signals, $x(t)$ and $x(t)$ is smoother than either one of the convolved signals. This is to be expected as the convolution operation is a smoothing operation. ■

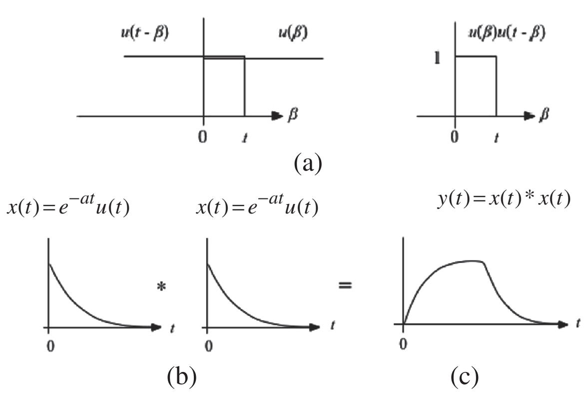

Fig. 2.3.3 Example 2.3.4

Example 2.3.5 Derive the expression $y_i(t) = h(t) * x_i(t)$ for the following two cases:

$$a.x_1(t) = u(t), b.x_2(t) = \delta(t).$$

Solution: *a*. Since $u(t-\alpha) = 0, \alpha > t$, we have the running integral

$$\begin{aligned} y_1(t) = h(t) * u(t) &= \int_{-\infty}^{\infty} h(\alpha)u(t-\alpha)d\alpha \\ &= \int_{-\infty}^{t} h(\alpha)d\alpha. \end{aligned} \quad (2.3.22)$$

b. Noting that the impulse function is the *generalized derivative* of the unit step function, we can compute the convolution

$$y_2(t) = h(t) * \delta(t) = h(t) * \frac{du(t)}{dt} = y'_1(t) = h(t). \quad (2.3.23)$$ ■

Example 2.3.6 Let $h(t) = e^{-at}u(t), a > 0$ *a*. Determine the running integral of $h(t)$.
b. Using (2.3.23), determine $y_2(t)$.

Solution:

$$\begin{aligned} a.\quad y_1(t) &= \int_{-\infty}^{t} h(\beta)d\beta = \int_{-\infty}^{t} e^{-a\beta}u(\beta)d\beta \\ &= \frac{1}{a}(1 - e^{-at})u(t), \end{aligned} \quad (2.3.24)$$

$$\begin{aligned} b.\ y_2(t) &= \frac{dy_1(t)}{dt} = \frac{1}{a}\frac{d}{dt}(1 - e^{-at})u(t) \\ &= \frac{1}{a}(1 - e^{-at})\frac{d}{dt}u(t) + \frac{1}{a}u(t)\frac{d(1 - e^{-at})}{dt} \\ &= (1/a)(1 - e^{-at})\delta(t) + e^{-at}u(t) \\ &= (1/a)[\delta(t) - \delta(t)] + e^{-at}u(t) \\ &= (1/a)e^{-at}u(t). \end{aligned} \quad (2.3.25)$$ ■

In a later chapter this result will be used in dealing with step and impulse inputs to an *RC* circuit with an impulse response $h(t) = e^{-at}u(t)$.

Example 2.3.7 Express the following integral in the form of $x(t) * p(t)$, ($p(t)$ is a pulse function:

$$y(t) = \int_{t-T/2}^{t+T/2} x(\alpha)d\alpha. \tag{2.3.26}$$

Solution:

$$\begin{aligned} y(t) &= \int_{-\infty}^{t+(T/2)} x(\alpha)d\alpha - \int_{-\infty}^{t-(T/2)} x(\alpha)d\alpha \\ &= x(t) * u(t+(T/2)) - x(t) * u(t-(T/2)) \\ &= x(t) * [u(t+(T/2)) - u(t-T/2))] \\ &= x(t) * \Pi\left[\frac{t}{T}\right]. \end{aligned} \tag{2.3.27}$$

The output is the convolution of $x(t)$ with a pulse width of T with unit amplitude and the process is a running average. ■

Example 2.3.8 Find the derivative of the running average of the function in (2.3.27) and express the function $x(t)$ in terms of the derivative of $y(t)$.

Solution: McGillem and Cooper (1991) give an interesting solution for this problem.

$$\begin{aligned} y'(t) &= x(t) * \left[\frac{du(t+(T/2))}{dt} - \frac{du(t-(T/2))}{dt}\right] \\ &= x(t) * \delta\left(t+\frac{T}{2}\right) - x(t) * \delta\left(t-\frac{T}{2}\right) \\ &= x(t+(T/2)) - x(t-(T/2)) \\ &\Rightarrow x(t) = y\prime(t-(T/2)) + x(t-T). \end{aligned} \tag{2.3.28}$$ ■

Example 2.3.9 Derive the expressions $a.\ y_1(t) = u(t) * u(t)$, $b.\ y_2(t) = u(t) * u(-t)$.

Solution:

$$\begin{aligned} a.\ y_1(t) = u(t) * u(t) &= \int_{-\infty}^{\infty} u(\alpha)u(t-\alpha)d\alpha \\ &= \int_0^t (1)dt = \left\{\begin{matrix} 0, & t>0 \\ t, & t<0 \end{matrix}\right\} = tu(t), \end{aligned} \tag{2.3.29}$$

$$\begin{aligned} b.\ y_2(t) &= \int_{-\infty}^{\infty} u(\alpha)u(\alpha+t)d\alpha \\ &= \begin{cases} \int_t^\infty u(\alpha)d\alpha \to \infty, & t \le 0 \\ \int_0^\infty u(\alpha+t)d\alpha \to \infty, & t>0 \end{cases}. \end{aligned} \tag{2.3.30}$$

It follows that $y_2(t) = \infty, -\infty < t < \infty$. In this case, convolution does not exist. ■

2.4 Convolution and Moments

In the examples considered so far, except in the cases of impulses, convolution is found to be a smoothing operation. We like to quantify and compare the results of the convolution of nonimpulse functions to the Gaussian function. In Section 1.7.1, the moments associated with probability density functions were considered.

A useful result can be determined by considering the center of gravity convolution in terms of the centers of gravity of the factors in the convolution. First, the *moments* $m_n(x)$ of a waveform $x(t)$ and its *center of gravity* η are, respectively, defined as

$$m_n(x) = \int_{-\infty}^{\infty} t^n x(t)dt, \tag{2.4.1}$$

$$\eta = \frac{\int_{-\infty}^{\infty} tx(t)dt}{\int_{-\infty}^{\infty} x(t)dt} = \frac{m_1(x)}{m_0(x)}. \tag{2.4.2}$$

We note that we can define a term like the variance in Section 1.7.1 by

$$\sigma^2(x) = \frac{m_2(x)}{m_0(x)} - \eta^2. \tag{2.4.3}$$

Now consider the expressions for the convolution $y(t) = g(t) * h(t)$. First,

$$m_1(y) = \int_{-\infty}^{\infty} ty(t)dt = \int_{-\infty}^{\infty} \left[t \int_{-\infty}^{\infty} g(\lambda)h(t-\lambda)d\lambda \right] dt$$
$$= \int_{-\infty}^{\infty} g(\lambda) \left[\int_{-\infty}^{\infty} th(t-\lambda)dt \right] d\lambda.$$

Defining a new variable $\xi = t - \lambda$ on the right and rewriting the above equation results in

$$m_1(y) = \left[\int_{-\infty}^{\infty} g(\lambda) \int_{-\infty}^{\infty} (\xi + \lambda)h(\xi)d\xi \right] d\lambda$$
$$= \int_{-\infty}^{\infty} \lambda g(\lambda)d\lambda \int_{-\infty}^{\infty} h(\xi)d\xi + \int_{-\infty}^{\infty} \xi h(\xi)d\xi$$
$$\int_{-\infty}^{\infty} g(\lambda)d\lambda = m_1(g)m_0(h) + m_1(h)m_0(g). \quad (2.4.4)$$

From the area property, it follows that $m_0(y) = m_0(g)m_0(h)$. The center of gravity is

$$\frac{m_1(y)}{m_0(y)} = \frac{m_1(g)}{m_0(g)} + \frac{m_1(h)}{m_0(h)} \Rightarrow \eta_y = \eta_g + \eta_h. \quad (2.4.5)$$

Consider the expression for the squares of the spread of $y(t)$ in terms of the squares of the spreads of $g(t)$ and $h(t)$. The derivation is rather long and only results are presented.

$$\sigma_y^2 = \frac{m_2(y)}{m_0(y)} - \left(\frac{m_1(y)}{m_0(y)} \right)^2. \quad (2.4.6)$$

Using the expressions for $m_0(y)$, $m_1(y)$ and $m_2(y)$ and simplifying the integrals results in

$$\sigma_y^2 = \sigma_g^2 + \sigma_h^2. \quad (2.4.7)$$

That is, the variance of y is equal to the sum of the variances of the two factors. It also verifies that convolution is a broadening operation for pulses. Noting that if $g(t)$ and $h(t)$ are probability density functions then (2.4.7) is valid. In communications theory we are faced with a signal, say $g(t)$ is corrupted by a noise $n(t)$ with the variance, σ_n^2. The signal-to-noise ratio (SNR) is given by

$$\text{Signal-to-noise ratio} = \frac{\text{Average signal power}}{\text{Noise power, } \sigma_n^2}. \quad (2.4.8)$$

Example 2.4.1 Verify the result is true in (2.4.7) using the functions

$$g(t) = h(t) = e^{-t} \text{ and } y(t) = g(t) * h(t).$$

Solution: Using integral tables, it can be shown that

$$m_0(g) = \int_0^{\infty} e^{-t}dt = 1, \; m_1(g) = \int_0^{\infty} te^{-t}dt = 1,$$
$$m_2(g) = \int_0^{\infty} t^2e^{-t}dt = 2,$$
$$\eta_g = \frac{m_1(g)}{m_0(g)} = 1, \sigma_g^2 = \frac{m_2(g)}{m_0(g)} - \eta_g^2 = 1,$$
$$\sigma_h^2 = 1 \text{ (note } g(t) = h(t)),$$
$$y(t) = g(t) * h(t) = te^{-t}u(t)(\text{see Example 2.3.4}).$$
$$m_0(y) = \int_0^{\infty} te^{-t}dt = 1, m_1(y) = \int_0^{\infty} t^2e^{-t}dt = 2,$$
$$m_2(y) = \int_0^{\infty} t^3e^{-t}dt = 6,$$
$$\eta_y = \frac{m_1(y)}{m_0(y)} = 2, \sigma_y^2 = \frac{m_2(y)}{m_0(y)} - \eta_y^2 = 2 \Rightarrow$$
$$\sigma_y^2 = \sigma_g^2 + \sigma_h^2 = 1 + 1 = 2. \quad \blacksquare$$

As an example, consider that we have signal $g(t) = A\cos(\omega_0 t)$ and is corrupted by a noise with a variance equal to σ_n^2. Then, the signal-to-noise ratio is

$$\text{SNR} = \frac{A^2/2}{\sigma_n^2}.$$

In Chapter 10, we will make use of this in quantization methods, wherein A and SNR are given and determine σ_n^2. This, in turn, provides the information on the size of the error that can be tolerated.

Notes: For readers interested in independent random variables, the probability density function of a sum of two independent random variables is the convolution of the density functions of the two factors of the convolution, and the variance of the sum of the two random variables equals the sum of their variances. For a detailed discussion on this, see Peebles (2001).■

2.4.1 Repeated Convolution and the Central Limit Theorem

Convolution operation is an integral operation, which is a smoothing operation. In Example 2.3.1, we have considered the special case of the convolution of two identical rectangular pulses and the convolution of these two pulses resulted in a triangular pulse (see Fig. 2.3.1m). The discontinuities in the functions being convolved are not there in the convolved signal. As more and more pulse functions convolve, the resultant functions become smoother and smoother. Repeated convolution begins to take on the bell-shaped Gaussian function. The generalized version of this phenomenon is called the *central limit theorem*. It is commonly presented in terms of probability density functions. In simple terms, it states that if we convolve N functions and one function does not *dominate* the others, then the convolution of the N functions approaches a Gaussian function as $N \to \infty$. In the general form of the central limit theorem, the means and variances of the individual functions that are convolved are related to the mean and the variance of the Gaussian function (see Peebles (2001)).

Given $x_i(t),\ i = 1, 2, ..., N$, the convolution of these functions is

$$y(t) = x_1(t) * x_2(t) * ... * x_N(t). \qquad (2.4.9)$$

The function $y(t)$ can be approximated using $(m_0)_N$, the sum of the individual means of the functions, and σ_N^2 the sum of the individual variances by

$$y(t) \approx \frac{1}{\sqrt{2\pi\sigma_N^2}} e^{-(t-(m_0)_N)^2/2\sigma_N^2}. \qquad (2.4.10)$$

Example 2.4.2 Illustrate the effects of convolution and compare $y(t)$ to a Gaussian function by considering the convolution

$$\begin{aligned} y(t) &= x_1(t) * x_2(t), \\ x_i(t) &= \frac{1}{a}\Pi\left[\frac{t-a/2}{a}\right],\ i = 1, 2. \end{aligned} \qquad (2.4.11)$$

Solution: $y(t)$ is a triangular function (see Example 2.3.1) given by

$$y(t) = \frac{1}{a}\Lambda\left[\frac{t-a}{a}\right]. \qquad (2.4.12)$$

The mean values of the two rectangular pulses are a/2 (see Section 1.7). The mean value of $y(t)$ is $2(a/2) = a$. The variance of each of the rectangular pulses is

$$\sigma_i^2 = m_2 - m_1^2 = a^2/12, i = 1, 2. \qquad (2.4.13a)$$

The variance is given by $\sigma_y^2 = \sigma_1^2 + \sigma_2^2 = a^2/6$. The Gaussian approximation is

$$(y(t))|_{N=2} \approx \frac{1}{\sqrt{\pi(a^2/3)}} e^{-(t-a)^2/(a^2/3)}. \qquad (2.4.13b)$$

This Gaussian and the triangle functions are symmetric around a. They are sketched in Fig. 2.4.1. Even with $N = 2$, we have a good approximation. ■

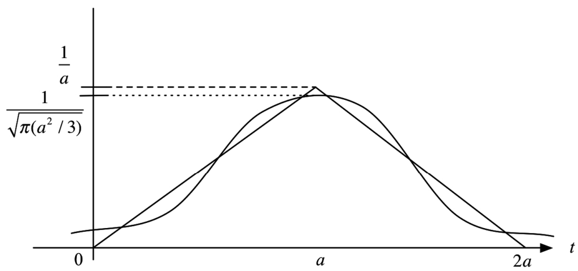

Fig. 2.4.1 Triangle function $y(t)$ in (2.4.12) and the Gaussian function in (2.4.13b)

Example 2.4.3 In Example 2.4.1 we considered two identically exponentially decaying functions: $x_1(t) = e^{-t}u(t) = x_2(t)$. The convolution of these two functions is given by $y_2(t) = te^{-t}u(t)$. Approximate this function using the Gaussian function.

Solution: The Gaussian function approximations of $y_n(t)$, considering $n = 2$ and for n large, are, respectively, given below. Note that $m_0(y) = 2$.

$$\begin{aligned} y_2(t) &\approx \frac{1}{\sqrt{2\pi(2)}} e^{-\left((t-2)^2/2(2)\right)}, \\ y_n(t) &\approx \frac{1}{\sqrt{2\pi(n)}} e^{-\left((t-n)^2/2(n)\right)}. \end{aligned} \qquad (2.4.14)$$

For sketches of these functions for various values of n, see Ambardar (1995). ■

2.4.2 Deconvolution

In this chapter, we have defined the convolution $y(t) = h(t) * x(t)$ as a mathematical operation. If $x(t)$ needs to be recovered from $y(t)$, we use a process called the *deconvolution* defined by

$$x(t) = y(t) * h_{\text{inv}}(t) = x(t) * h(t) * h_{\text{inv}}(t) = x(t) * [h(t) * h_{\text{inv}}(t)], \Rightarrow h(t) * h_{\text{inv}}(t) = \delta(t) \text{ and } x(t) * \delta(t) = x(t). \tag{2.4.15}$$

It is a difficult problem to find $h_{\text{inv}}(t)$, which may not even exist.

Table 2.4.1 Properties of aperiodic convolution

Definition:

$$y(t) = x_1(t)^* x_2(t) = \int_{-\infty}^{\infty} x_1(\alpha)x_2(t-\alpha)d\alpha = \int_{-\infty}^{\infty} x_2(\alpha)x_1(t-\alpha)d\alpha.$$

Amplitude scaling:

$\alpha x_1(t)^* \beta x_2(t) = \alpha\beta(x(t)^* h(t))$, α and β are constants.

Commutative:

$$x_1(t)^* x_2(t) = x_2(t)^* x_1(t).$$

Distributive:

$$x_1(t)^*[x_2(t) + x_3(t)] = x_1(t)^* x_2(t) + x_1(t)^* x_3(t).$$

Associative:

$$x_1(t)^*[x_2(t)^* x_3(t)] = [x_1(t)^* x_2(t)]^* x_3(t).$$

Delay:

$$x_1(t-t_1)^* x_2(t-t_2) = x_1(t-t_2)^* x_2(t-t_1) = y(t-(t_1+t_2)).$$

Impulse response:

$$x(t)^* \delta(t) = x(t).$$

Derivatives:

$$x_1(t)^* x_2'(t) = x_1'(t)^* x_2(t) = y'(t), \quad x_1^{(m)}(t)^* x_2^{(n)}(t) = y^{(m+n)}(t).$$

Step response:

$$y(t) = x(t)^* u(t) = \int_{-\infty}^{t} x(\alpha)d\alpha, \quad y'(t) = x(t)^* \delta(t) = x(t).$$

Area:

$A[x_1(t)^* x_2(t)] = A[y(t)]$, where $A[x(t)] = \int_{-\infty}^{\infty} x(t)dt$.

Duration:

$$t_{x_1} + t_{x_2} = t_y.$$

Symmetry:

$$x_{1e}(t)^* x_{2e}(t) = y_e(t), \quad x_{1e}(t)^* x_{20}(t) = y_0(t), \quad x_{10}(t)^* x_{20}(t) = y_e(t).$$

Time scaling:

$$x_1(ct)^* x_2(ct) = \frac{1}{|c|}y(ct), \; c \neq 0.$$

2.5 Convolution Involving Periodic and Aperiodic Functions

2.5.1 Convolution of a Periodic Function with an Aperiodic Function

Let $h(t)$ be an aperiodic function and $x_T(t)$ be a periodic function with a period T. We desire to find the convolution of these two functions. That is, find $y(t) = x_T(t) * h(t)$.

Example 2.5.1 Derive the expressions for the convolution of the following two functions: $\delta_T(t)$ and $h(t)$ assuming $T = 1.5$ and $T = 2$ and sketch the results for the two cases.

$$\delta_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - nT), \quad h(t) = \Lambda[t]. \tag{2.5.1}$$

Derive the expressions for the convolution of these two functions assuming $T = 1.5$ and $T = 2$ and sketch the results of the convolution for the two cases.

$$\begin{aligned} y(t) &= h(t) * \delta_T(t) = h(t) * \sum_{k=-\infty}^{\infty} \delta(t - kT) \\ &= \sum_{k=-\infty}^{\infty} h(t) * \delta(t - kT). \end{aligned} \tag{2.5.2}$$

Noting that $h(t) * \delta(t - kT) = h(t - kT)$, it follows that

$$y(t) = \sum_{k=-\infty}^{\infty} h(t - kT) = y_T(t). \tag{2.5.3}$$

Figure 2.5.1a,b gives the sketches of the functions $\delta_T(t)$ and $h(t)$. The sketches for the convolution are shown in Fig. 2.5.1c,d. In the first case, there were no overlaps, whereas in the second case there are overlaps. ■

Example 2.5.2 Derive an expression for the convolution $y(t) = h(t) * x_T(t)$,

$$x_T(t) = \cos(\omega_0 t + \theta) \text{ and } h(t) = e^{-at}u(t). \tag{2.5.4}$$

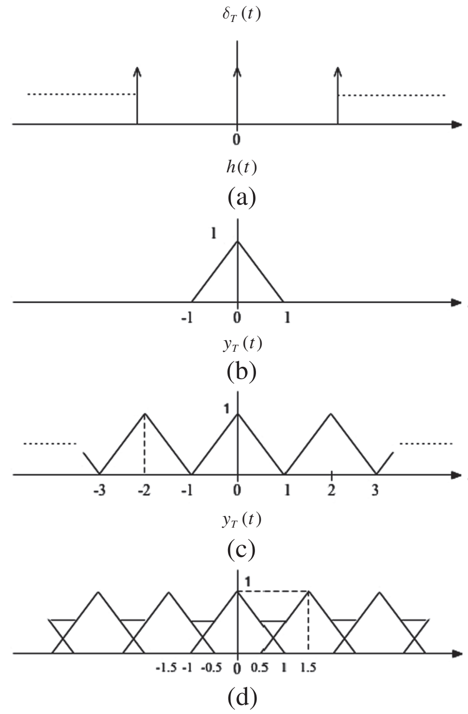

Fig. 2.5.1 (a) Periodic impulse sequence, (b) $\Lambda[t]$, (c) $y_T(t)$, $T = 2$, and (d) $y_T(t)$, $T = 2$

Solution: $y(t) = h(t) * x_T(t) = \int_0^\infty e^{-a\beta}\cos(\omega_0(t-\beta) + \theta)d\beta$

$$\begin{aligned} &= \int_0^\infty e^{-a\beta}[\cos(\omega_0 t + \theta)\cos(\omega_0\beta) \\ &\quad + \sin(\omega_0 t + \theta)\sin(\omega_0\beta)]d\beta \\ &= \left[\int_0^\infty e^{-a\beta}\cos(\omega_0\beta)d\beta\right]\cos(\omega_0 t + \theta) \\ &\quad + \left[\int_0^\infty e^{-a\beta}\sin(\omega_0\beta)d\beta\right]\sin(\omega_0 t + \theta). \end{aligned} \tag{2.5.5}$$

Using the identities given below (see (2.5.7 a, b, and c.)), (2.5.5) can be simplified.

$$\begin{aligned} y(t) &= \left[a/(a^2 + \omega_0^2)\right]\cos(\omega_0 t + \theta) \\ &\quad + \left[\omega_0/(a^2 + \omega_0^2)\right]\sin(\omega_0 t + \theta) \\ &= \frac{1}{\sqrt{a^2 + \omega_0^2}}\cos(\omega_0 t + \theta - \tan^{-1}(\omega_0/a)) \\ &\equiv y_T(t), \end{aligned} \tag{2.5.6}$$

$$\int_0^\infty e^{-a\beta}\sin(\omega_0\beta)d\beta = \frac{e^{-a\beta}}{a^2+\omega_0^2}[-a\sin(\omega_0\beta) - \omega_0\cos(\omega_0\beta)]\Big|_0^\infty = \frac{\omega_0}{a^2+\omega_0^2}, \tag{2.5.7a}$$

$$\int_0^\infty e^{-a\beta}\cos(\omega_0\beta)d\beta = \frac{e^{-a\beta}}{a^2+\omega_0^2}[-a\cos(\omega_0\beta) + \omega_0\sin(\omega_0\beta)]\Big|_0^\infty = \frac{a}{a^2+\omega_0^2}, \tag{2.5.7b}$$

$$\alpha\cos(\omega_0 t+\theta) + \beta\sin(\omega_0 t+\theta) = c\cos(\omega_0 t+\phi),$$
$$c = \sqrt{\alpha^2+\beta^2}, \phi = [-\tan^{-1}(\beta/\alpha)+\theta]. \tag{2.5.7c}$$

The functions $y(t) = y_T(t)$ and $x_T(t)$ are sinusoids at the same frequency ω_0. The amplitude and the phase of $y(t)$ are *different* compared to that of $x_T(t)$. ■

The derivation given above can be generalized for a periodic function

$$x_T(t) = X_s[0] + \sum_{k=0}^{\infty} c[k]\cos(k\omega_0 t + \theta[k]), \omega_0 = 2\pi/T, \tag{2.5.8a}$$

$$y(t) = x_T(t) * h(t) = X_s[0] * h(t) + \sum_{k=0}^{\infty} c[k][\cos(k\omega_0 t + \theta[k]) * h(t)], \omega_0 = 2\pi/T. \tag{2.5.8b}$$

2.5.2 Convolution of Two Periodic Functions

In Section 1.5 energy and power signals were considered. The energy in a periodic function is infinity and its average power is finite. One period of a periodic function has all its information. In the same vein, the *average convolution* is a useful measure of periodic convolution. Such averaging process is called *periodic* or *cyclic convolution*. The convolution of two periodic functions with *different* periods is very difficult and is limited here to the convolution of two periodic functions, each with *the same period.*

The periodic convolution of two periodic functions, $x_T(t)$ and $h_T(t)$, is defined by

$$\begin{aligned} y_T(t) &= x_T(t) \otimes h_T(t) = \frac{1}{T}\int_{t_0}^{t_0+T} x_T(\alpha)h_T(t-\alpha)d\alpha \\ &= \frac{1}{T}\int_T x_T(\alpha)h_T(t-\alpha)d\alpha = \frac{1}{T}\int_T x_T(t-\alpha)h_T(\alpha)d\alpha \\ &= h_T(t)\otimes x_T(t). \end{aligned} \tag{2.5.9a}$$

Note that the symbol $\otimes$ used for the periodic convolution and the constant (T) in the denominator in (2.5.9a) indicates that it is an average periodic convolution. $y_T(t)$ is periodic since

$$h_T(t+T-\alpha) = h_T(t-\alpha) \text{ and}$$
$$x_T(t+T-\alpha) = x_T(t-\alpha). \tag{2.5.9b}$$

Also, periodic convolution is *commutative*. Many of the aperiodic convolution properties discussed earlier are applicable for periodic convolution with some modifications. The expression for the periodic convolution can be obtained by considering aperiodic convolution for one period of each of the two functions.

Consider the periodic functions in the form

$$x_T(t) = \sum_{n=-\infty}^{\infty} x(t-nT) \text{ and } h_T(t) = \sum_{n=-\infty}^{\infty} h(t-nT), \tag{2.5.10a}$$

$$x(t) = \begin{cases} x_T(t), & t_0 \le t < t_0+T \\ 0, & \text{otherwise} \end{cases},$$
$$h(t) = \begin{cases} h_T(t), & t_0 \le t < t_0+T \\ 0, & \text{otherwise.} \end{cases} \tag{2.5.10b}$$

Note that the time-limited functions, $x(t)$ and $h(t)$, are defined from the periodic functions $x_T(t)$ and $h_T(t)$. Using (2.5.10b) the periodic convolution is

$$\begin{aligned} y_T(t) &= \frac{1}{T}\int_T x_T(\alpha)h_T(t-\alpha)d\alpha \\ &= \frac{1}{T}\int_T x_T(\alpha)\sum_{n=-\infty}^{\infty} h(t-\alpha-nT)d\alpha \end{aligned}$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty}\int_{T} x(\alpha)h(t-\alpha-nT)d\alpha$$

$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} x(t) * h(t-nT), \tag{2.5.11a}$$

$$y_T(t) = x_T(t) \otimes h_T(t)$$
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} y(t-nT), y(t) = x(t) * h(t). \tag{2.5.11b}$$

That is, $y_T(t)$ can be determined by considering one period of each of the two functions and finding the aperiodic convolution.

Example 2.5.3 *a*. Determine and sketch the aperiodic convolution $y(t) = h(t) * x(t)$.

$$x(t) = \frac{1}{2}\Pi\left[\frac{t-1}{2}\right], \quad h(t) = \frac{1}{3}\Pi\left[\frac{t-1.5}{3}\right]. \tag{2.5.12}$$

b. Determine and sketch the periodic convolution $y_T(t) = x_T(t) \otimes h_T(t)$ for periods $T = 6$ and 4.

$$x_T(t) = \sum_{k=-\infty}^{\infty} x(t-kT) \text{ and } h_T(t) = \sum_{k=-\infty}^{\infty} h(t-kT). \tag{2.5.13}$$

Solution: *a*. From (2.5.13), the results for the aperiodic convolution can be derived. The sketches of the two functions and the result of the convolution are shown in Fig. 2.5.2a. The periodic convolutions for the two different periods are shown in Fig. 2.5.2b,c. There are no overlaps of the functions from one period to the next in Fig. 2.5.2b, whereas in Fig. 2.5.2c, the pulses overlap. ■

Convolution of *almost periodic* or *random signals*, $x(t)$ and $h(t)$, is defined by

$$y(t) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(\alpha)h(t-\alpha)d\alpha. \tag{2.5.14}$$

This reduces to the periodic convolution if $x(t)$ and $h(t)$ are periodic with the same period.

2.6 Correlation

Equation (2.1.3) gives the cross-correlation of $x(t)$ and $h(t)$ as the integral of the product of two functions, one displaced by the other by τ between the interval $a < t < b$ and is given by

$$R_{xh}(\tau) = x(\tau) ** h(\tau) = \int_a^b x(t)h(t+\tau)dt = \langle x(t)h(t+\tau)\rangle.$$

Cross-correlation function gives the *similarity* between the two functions: $x(t)$ and $h(t+\tau)$. Many a times the second function $h(t)$ may be a corrupted version of $x(t)$, such as $h(t) = x(t) + n(t)$, where $n(t)$ is a noise signal. In the case of $x(t) = h(t)$, cross-correlation reduces to autocorrelation. In this case, at $\tau = 0$, the autocorrelation integral gives the highest value at $\tau = 0$. Comparison of two functions appears in many identification situations. For example, to identify an individual based upon his speech pattern, we can store his speech

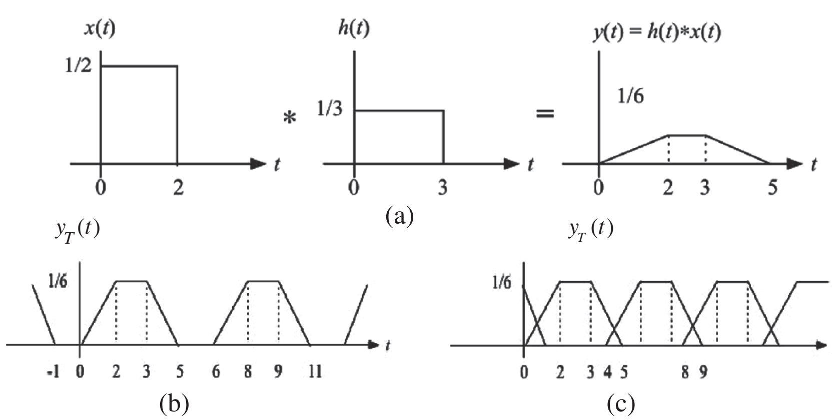

Fig. 2.5.2 Example 2.5.1 (a) Aperiodic convolution; (b) periodic convolution $T = 6$; (c) periodic convolution, $T = 4$

segment in a computer. When he enters, say a secure area, we can request him to speak and compute the cross-correlation between the stored and the recorded. Then decide on the individual's identification based on the cross-correlation function. Generally, an individual is identified if the peak of the cross-correlation is close to the possible peak autocorrelation value. Allowance is necessary since the speech is a function of the individual's physical and mental status of the day the test is made. Quantitative measures on the cross-correlation will be considered a bit later.

The order of the subscripts on the cross-correlation function $R_{xh}(\tau)$ is important and will get to it shortly. In the case of $x(t) = h(t)$, we have the autocorrelation and the function is referred to as $R_x(\tau)$ with a *single subscript*. The cross- and autocorrelation functions are functions of τ and not t. Correlation is applicable to periodic, aperiodic, and random signals. In the case of periodic functions, we assume that both are periodic with *the same period*.

Cross-correlation: Aperiodic:

$$R_{xh}(\tau) = \int_{-\infty}^{\infty} x(t)h(t+\tau)dt \qquad (2.6.1a)$$

Cross-correlation: Periodic:

$$R_{T,}xh(\tau) = \frac{1}{T}\int_{T} x_T(t)h_T(t+\tau)dt \qquad (2.6.1b)$$

Autocorrelation: Aperiodic:

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt \qquad (2.6.1c)$$

Autocorrelation: Periodic:

$$R_{T,x}(\tau) = \frac{1}{T}\int_{T} x_T(t)x_T(t+\tau)dt. \qquad (2.6.1d)$$

Notes: Cross- and autocorrelations of periodic functions and random signals are referred to as *average periodic cross- and autocorrelation functions*. In the case of random or noise signals, the average cross-correlation function is defined by

$$R_{a,xh}(\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)h(t+\tau)dt. \qquad (2.6.1e)$$

For periodic functions, (2.6.1e) reduces to (2.6.1b).

2.6.1 Basic Properties of Cross-Correlation Functions

Folding relationship between the two cross-correlation functions is

$$R_{xh}(\tau) = R_{hx}(-\tau), \qquad (2.6.2)$$

$$\Rightarrow R_{xh}(\tau) = \int_{-\infty}^{\infty} x(t)h(t+\tau)dt$$
$$= \int_{-\infty}^{\infty} x(\alpha-\tau)h(\alpha)d\alpha = R_{hx}(-\tau). \qquad (2.6.3)$$

2.6.2 Cross-Correlation and Convolution

The cross-correlation function is related to the convolution. From (2.6.3) we have

$$R_{xh}(\tau) = x(\tau) ** h(\tau) = x(-\tau) * h(\tau), \qquad (2.6.4a)$$

$$R_{hx}(\tau) = h(\tau) ** x(\tau) = h(-\tau) * x(\tau). \qquad (2.6.4b)$$

Equation (2.6.4a) can be seen by first rewriting the first integral in (2.6.3) using a new variable $t = -\alpha$, and then simplifying it. That is,

$$R_{xh}(\tau) = \int_{-\infty}^{\infty} x(t)h(t+\tau)dt = \int_{-\infty}^{\infty} x(-\alpha)h(\tau-\alpha)d\alpha$$
$$= x(-\tau) * h(\tau). \qquad (2.6.4c)$$

Equation (2.6.4b) can be similarly shown. Noting the explicit relation between correlation and convolution, many of the convolution properties are applicable to the correlation. To compute the cross–

correlation, $R_{xh}(\tau)$, one can use either of the integral in (2.6.3) or the integral in (2.6.4c). $R_{xh}(\tau)$ is not always equal to $R_{hx}(\tau)$. In case, if one of the functions is *symmetric*, say $x(t) = x(-t)$, then

$$R_{xh}(\tau) = x(-\tau) * h(\tau) = x(\tau) * h(\tau). \quad (2.6.5)$$

Example 2.6.1 illustrates the use of this property. In particular, the area and duration properties for convolution also apply to the correlation. We should note that the correlations *are functions of* τ and *not* t, where τ is the time shift between $x(t)$ and $h(t+\tau)$. In the case of energy signals, the energies in the real signals, $g(t)$ and $h(t)$, are

$$R_g(0) = \int_{-\infty}^{\infty} g^2(t)dt = E_g, \quad R_h(0) = \int_{-\infty}^{\infty} h^2(t)dt = E_h. \quad (2.6.6)$$ ■

2.6.3 Bounds on the Cross-Correlation Functions

Consider the integral

$$\int_{-\infty}^{\infty} [x(t) \pm h(t+\tau)]^2 dt = \int_{-\infty}^{\infty} x^2(t)dt + \int_{-\infty}^{\infty} h^2(t+\tau)dt \pm 2\int_{-\infty}^{\infty} x(t)h(t+\tau)dt$$
$$= R_x(0) + R_h(0) \pm 2R_{xh}(\tau) \geq 0. \quad (2.6.7a)$$

This follows since the integrand in (2.6.7a) is nonnegative and

$$|R_{xh}(\tau)| \leq (R_x(0) + R_h(0))/2. \quad (2.6.7b)$$

An interesting bound can be derived using the *Schwarz's inequality*. See (2.1.9d).

$$\langle x(t)h(t+\tau)\rangle^2 = \left[\int_{-\infty}^{\infty} x(t)h(t+\tau)dt\right]^2$$
$$\leq \left(\int_{-\infty}^{\infty} |x(t)|^2 dt\right)\left(\int_{-\infty}^{\infty} |h(t+\tau)|^2 dt\right), \quad (2.6.8)$$

$$\Rightarrow |R_{xh}(\tau)|^2 \leq \left(\int_{-\infty}^{\infty} x^2(t)dt\right)\left(\int_{-\infty}^{\infty} h^2(t)dt\right)$$
$$= R_x(0)R_h(0), \quad (2.6.9a)$$

$$|R_{xh}(\tau)| \leq \sqrt{R_x(0)R_h(0)}. \quad (2.6.9b)$$

Equation (2.6.9b) represents a tighter bound compared to the one in (2.6.7b), as the geometric mean cannot exceed the arithmetic mean. That is,

$$\sqrt{R_x(0)R_h(0)} \leq (R_x(0) + R_h(0))/2. \quad (2.6.9c)$$

Another way to prove (2.6.9b) is as follows. Start with the inequality below. Expand the function and identify the auto- and cross-correlation terms.

$$\int_{-\infty}^{\infty} [x(t) + \alpha h(t+\tau)]^2 dt \geq 0. \quad (2.6.10)$$

Write the resulting equation in a quadratic form in terms of α. In order for the equation in (2.6.10) to be true, the roots of the quadratic equation have to be real and equal or the roots have to be complex conjugates. The proof is left as a homework problem.

Example 2.6.1 Determine the cross-correlation of the functions given in Fig. 2.3.2.

$$x(t) = \Pi\left[\frac{t}{2T}\right], \; h(t) = e^{-at}u(t), a > 0. \quad (2.6.11a)$$

Solution: Example 2.3.3 dealt with computing the convolution of these two functions. The cross-correlation functions are as follows:

$$R_{hx}(\tau) = \int_{-\infty}^{\infty} h(t)x(t+\tau)dt = h(-\tau) * x(\tau),$$
$$R_{xh}(\tau) = \int_{-\infty}^{\infty} x(t)h(t+\tau)dt = x(-\tau) * h(\tau). \quad (2.6.11b)$$

Note that we have $x(-\tau) = x(\tau)$, and therefore the cross-correlation $R_{xh}(\tau) = x(\tau) * h(\tau)$ is the convolution determined before (see (2.3.18).), except the cross-correlation is a function of τ rather than t. It is given below. The two cross-correlation functions are sketched in Fig. 2.6.1a,b. Note $R_{hx}(\tau) = R_{xh}(-\tau)$

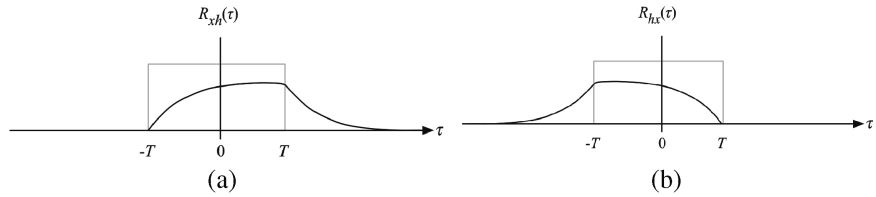

Fig. 2.6.1 Cross-correlations (a)$R_{xh}(\tau)$, (b) $R_{hx}(\tau)(R_{xh}(T) = \frac{1}{a}\left[1 - e^{-2aT}\right] = R_{hx}(-T))$

$$R_{xh}(\tau) = \begin{cases} 0, & \tau \leq -T \\ \frac{1}{a}\left[1 - e^{-a(\tau+T)}\right], & -T < \tau \leq T \\ \frac{1}{a}\left[e^{aT} - e^{-aT}\right]e^{-a\tau}, \tau > T \end{cases} \tag{2.6.11c}$$ ■

2.6.4 Quantitative Measures of Cross-Correlation

The amplitudes of $R_{xh}(\tau)$ (and $R_{hx}(\tau)$) vary. It is appropriate to consider the *normalized correlation coefficient* (or correlation coefficient) of two energy signals defined by

$$\rho_{xh}(\tau) = \frac{R_{xh}(\tau)}{\sqrt{\left[\int\limits_{-\infty}^{\infty} x^2(t)dt\right]\left[\int\limits_{-\infty}^{\infty} h^2(t)dt\right]}} = \frac{R_{xh}(\tau)}{\sqrt{E_x E_h}}, \tag{2.6.12a}$$

$$\Rightarrow |\rho_{xh}(\tau)| \leq 1. \tag{2.6.12b}$$

Equation (2.6.12b) can be shown as follows. From (2.1.13a) and using the Schwarz's inequality (see (2.1.9d)), we have

$$R_{xh}(\tau) = \langle x(t)h(t+\tau)\rangle \leq \|x(t)\|\|h(t+\tau)\| = \sqrt{E_x E_h}$$

It should be noted that the case of $x(t) = h(t)$, the correlation coefficient reduces to

$$\rho_{xx}(\tau) = \frac{R_x(\tau)}{R_x(0)}. \tag{2.6.13}$$

Correlation measures are very useful in statistical analysis. See Yates and Goodman (1999), Cooper and McGillem (1999) and others.

The significance of $\rho_{xh}(\tau)$ can be seen by considering some extreme cases. When $x(t) = \alpha h(t)$, $\alpha > 0$, we have the correlation coefficient $\rho_{xh}(\tau) = 1$. In the case of $x(t) = \alpha h(t)$, $\alpha < 0$ and $\rho_{xh}(\tau) = -1$. In communication theory, we will be interested in signals that are corrupted by *noise*, usually identified by $n(t)$, which can be defined only in *statistical terms*. In the following, we will consider the analysis without going through statistical analysis. *Noise signal* $n(t)$ is assumed to have a *zero average* value. That is,

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_{-T/2}^{T/2} n(t)dt = 0. \tag{2.6.14}$$

Cross-correlation function can be used to compare two signals. The signals $x(t)$ and $h(t)$ are *uncorrelated* if the average cross-correlation satisfies the relation

$$\begin{aligned} R_{a,xh}(\tau) &= \lim_{T\to\infty} \frac{1}{T} \int\limits_{T/2}^{T/2} x(t)h(t+\tau)dt \\ &= \left[\lim_{T\to\infty} \frac{1}{T} \int\limits_{T/2}^{T/2} x(t)dt\right]\left[\lim_{T\to\infty} \frac{1}{T} \int\limits_{T/2}^{T/2} h(t)dt\right]. \end{aligned} \tag{2.6.15}$$

Example 2.6.2 If the signals $x(t)$ and a zero average noise signal $n(t)$ are uncorrelated, then show

$$\lim_{T\to\infty} \frac{1}{T} \int\limits_{-T/2}^{T/2} x(t)n(t-\tau)dt = 0 \text{ for all } \tau. \tag{2.6.16}$$

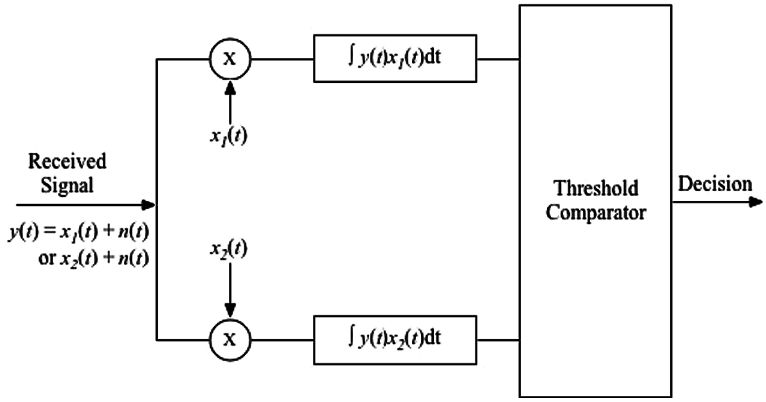

Fig. 2.6.2 Correlation detector

Solution: Using (2.6.14) and (2.6.15), we have

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} x(t)n(t+\tau)dt$$
$$=\left[\lim_{T\to\infty}\frac{1}{T}\int_{T/2}^{T/2} x(t)dt\right]\left[\lim_{T\to\infty}\frac{1}{T}\int_{T/2}^{T/2} n(t+\tau)dt\right]=0. \tag{2.6.17}$$

Cross-correlation function can be used to estimate the *delay* caused by a system. Suppose we know that a finite duration signal $x(t)$ is passed through an ideal transmission line resulting in the output function $y(t) = x(t - t_0)$. The delay t_0 caused by the transmission line is unknown and can be estimated using the cross-correlation function $R_{xy}(\tau)$. At $\tau = t_0$, $R_{xy}(t_0)$ gives a maximum value. Then determine τ corresponding to the maximum value of $R_{xy}(\tau)$. ■

Example 2.6.3 Consider the transmitted signals $x_1(t)$ and $x_2(t)$ in the interval $0 < t < T_s$ and zero otherwise. Use the cross-correlation function to determine which signal was transmitted out of the two. They are assumed to be mutually *orthogonal* (see Section 2.1.1) over the interval and satisfy

$$\int_0^{T_s} x_i(t)x_j(t)dt = \begin{cases} E_{x_i} = E_x, & i = j \\ 0, & i \neq j, i = 1,2 \end{cases}. \tag{2.6.18}$$

E_x is the energy contained in each signal. The two signals to be transmitted are assumed to be available at the receiver. A simple receiver is the *binary correlation detector* (or receiver) shown in Fig. 2.6.2. The received signals $y_i(t)$ are assumed to be of the form in (2.6.19). Decide which signal has been transmitted using the cross-correlation function.

$$y_i(t) = x_i(t) + \text{noise},\ i = 1 \text{ or } 2. \tag{2.6.19}$$

Solution: Let the transmitted signal be $x_1(t)$. Using the top path in Fig. 2.6.2, we have

$$\int_0^{T_s}[x_1(t)+n(t)]x_1(t)dt = A\int_0^{T_s} x_1^2(t)dt. \tag{2.6.20}$$

Using the bottom path, with the transmitted signal equal to $x_1(t)$, we have

$$\int_0^{T_s}[x_1(t)+n(t)]x_2(t)dt = \int_0^{T_s}[x_1(t)x_2(t)+x_1(t)n(t)]dt$$
$$= \int_0^{T_s} x_1(t)n(t)dt = B. \tag{2.6.21}$$

Since the noise signal has no relation to $x_1(t)$, B will be near zero and $A \gg B$, implying $x_1(t)$ was transmitted. If $x_2(t)$ was transmitted, the roles are reversed and $B \gg A$. The *correlation method of detection* is based on the following:

1. If $A > B \Rightarrow$ transmitted signal is $x_1(t)$.
2. If $B > A \Rightarrow$ transmitted signal is $x_2(t)$.
3. If $B = A \Rightarrow$ no decision can be made as noise swamped the transmitted signal. ■

Example 2.6.4 Derive the expressions for the cross-correlation $R_{xh}(\tau)$ and $R_{hx}(\tau)$ assuming $x(t) = e^{-t}u(t),\ h(t) = e^{-2t}u(t)$.

Solution: Using the expression in (2.6.3), we have

$$R_{xh}(\tau) = \int_{-\infty}^{\infty} x(\alpha - \tau)h(\alpha)d\alpha$$
$$= \int_{-\infty}^{\infty} e^{-(\alpha-\tau)}e^{-2\alpha}[u(\alpha-\tau)u(\alpha)]d\alpha \quad (2.6.22a)$$

Consider the following and then the corresponding correlations:

$$\tau \geq 0: \quad [u(\alpha)u(\alpha-\tau)] = \begin{cases} 1, & \alpha \geq \tau \\ 0, & \text{Otherwise} \end{cases},$$

$$\tau < 0: \quad [u(\alpha)u(\alpha-\tau)] = \begin{cases} 1, & \alpha \geq 0 \\ 0, & \text{Otherwise} \end{cases},$$

$$\tau \geq 0: \ R_{xh}(\tau) = e^{\tau}\int_{\tau}^{\infty} e^{-3\alpha}d\alpha$$
$$= e^{\tau}\frac{1}{-3}e^{-3\alpha}\Big|_{\tau}^{\infty} = \frac{1}{3}e^{-2\tau}u(\tau), \quad (2.6.22b)$$

$$\tau < 0: \ R_{xh}(\tau) = e^{\tau}\int_{0}^{\infty} e^{-3\alpha}d\alpha = \frac{1}{(-3)}e^{\tau}e^{-3\alpha}\Big|_{0}^{\infty} = \frac{e^{\tau}}{3}. \quad (2.6.22c)$$

$R_{xh}(\tau)$ is shown in Fig. 2.6.3. Note that $R_{hx}(\tau) = R_{xh}(-\tau)$. ■

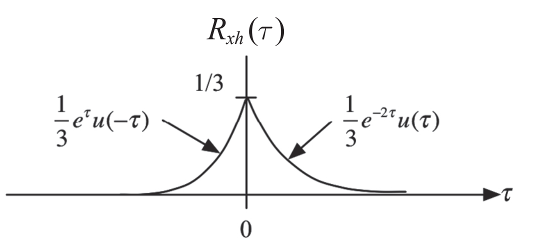

Fig. 2.6.3 $R_{xh}(\tau)$

Example 2.6.5 Derive the cross–correlation $R_{xh}(\tau)$ for the following functions:

$$x(t) = \Pi[t - .5],\ h(t) = t\Pi\left[\frac{t-1}{2}\right], R_{xh}(\tau)$$
$$= \int_{-\infty}^{\infty} x(t)h(t+\tau)dt. \quad (2.6.23)$$

Solution: See Fig. 2.6.4c for $h(t+\tau)$ for an arbitrary τ. The function $h(t+\tau)$ starts at $t = -\tau$ and ends at $t = 2-\tau$. As τ varies from $-\infty$ to ∞, there are five possible regions we need to consider. These are sketched in Fig. 2.6.4 d,e,f,g,h. In each of these cases both the functions are sketched in the same figure, which allows us to find the regions of overlap. The regions of overlap are listed in Table 2.6.1.

Case 1: $\tau \geq -1$ **:** See Fig. 2.6.4d. There is no overlap between $x(t)$ and $h(t+\tau)$ and

$$R_{xh}(\tau) = 0, -\tau > 1 \text{ or } \tau \leq -1. \quad (2.6.24)$$

Case 2: $0 < -\tau \leq 1$ or $-1 < \tau \leq 0$: See Fig. 2.6.4e. Using Table 2.6.1

$$R_{xh}(\tau) = \int_{-\tau}^{1}(t+\tau)dt = \frac{t^2}{2} + \tau t\Big|_{t=-\tau}^{t=1}$$
$$= \frac{(\tau+1)^2}{2}, -1 \leq \tau < 0. \quad (2.6.25)$$

Case 3: $0 < \tau \leq 1$: See Fig. 2.6.4 f. Using Table 2.6.1, we have

$$R_{xh}(\tau) = \int_{0}^{1}(t+\tau)dt = \frac{t^2}{2} + \tau t\Big|_{t=0}^{t=1} = \frac{1+2\tau}{2}, 0 < \tau \leq 1. \quad (2.6.26)$$

Case 4: $1 < \tau \leq 2$: See Fig. 2.6.3 g. Using Table 2.6.1 we have

$$R_{xh}(\tau) = \int_{0}^{2-\tau}(t+\tau)dt = \frac{t^2}{2} + \tau t\Big|_{t=0}^{t=2-\tau}$$
$$= \frac{4-\tau^2}{2}, 1 < \tau \leq 2. \quad (2.6.27)$$

Case 5: $2 < \tau$: See Fig. 2.6.4 h. There is no overlap and

Fig. 2.6.4 (a) $x(t)$, (b) $h(t)$, (c) $h(t+\tau)$, (d) $x(t)$ and $h(t+\tau)$, $-\tau > 1$(or $\tau \le -1$), (e) $x(t)$ and $h(t+\tau)$, $-1<\tau\le 0$, (f) $x(t)$ and $h(t+\tau)$, $0<\tau\le 1$, (g) $x(t)$ and $h(t+\tau)$, $1<\tau<2$, (h) $x(t)$ and $h(t+\tau)$, $\tau>2$, (i) $R_{xh}(\tau)$

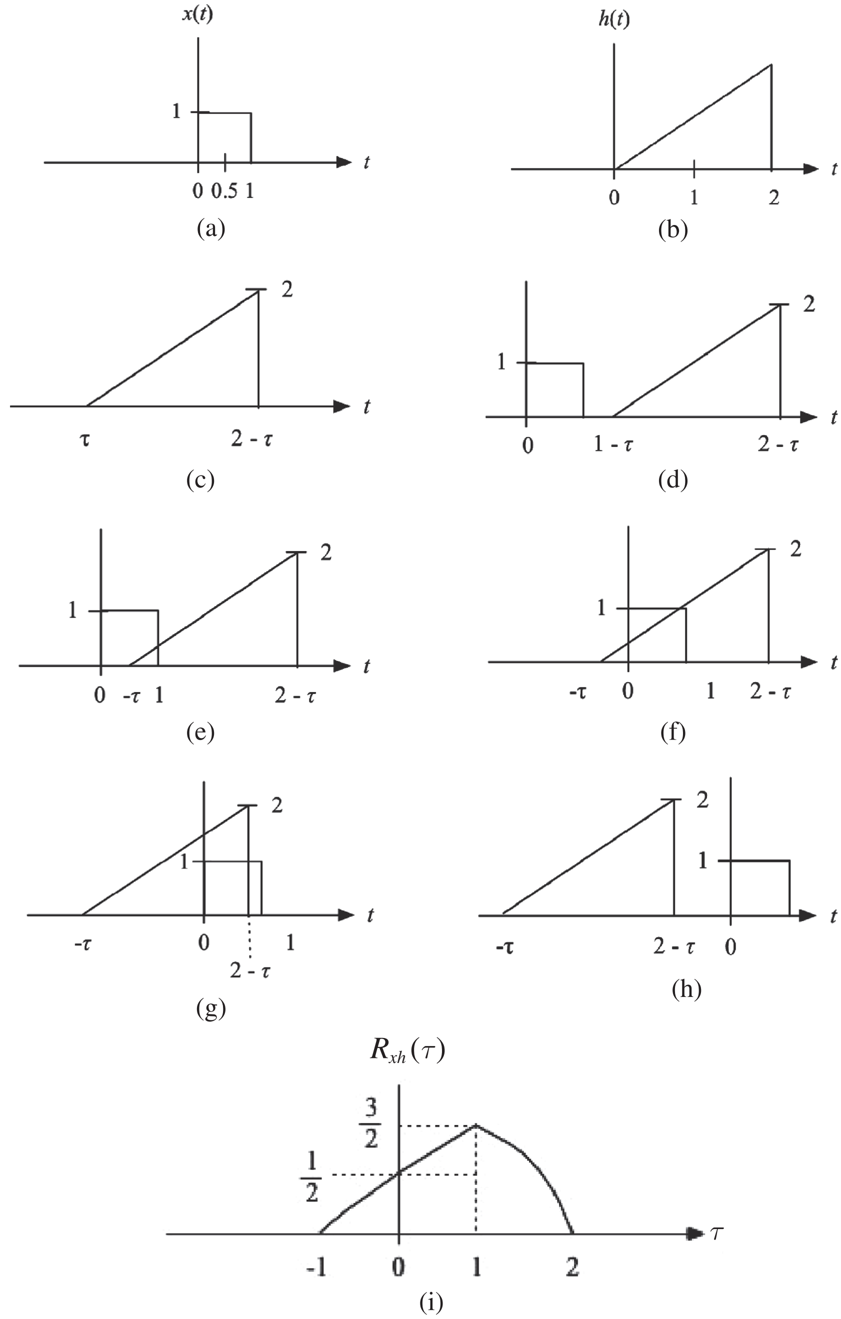

Table 2.6.1 Example 2.6.4

Case	τ	Range of overlap/ integration range
1	$\tau \le -1$	No over lap
2	$-1 < \tau \le 0$	$-\tau < t < 1$
3	$0 < \tau \le 1$	$0 < t < 1$
4	$1 < \tau \le 2$	$1 < t < 2 - \tau$
5	$\tau > 2$	No over lap

$$R_{xh}(\tau) = 0, \tau > 2. \tag{2.6.28}$$

See Fig. 2.6.4i for the cross-correlation $R_{xh}(\tau)$ sketch. There are no impulses in either of the two functions and therefore the cross-correlation function is continuous. ■

2.7 Autocorrelation Functions of Energy Signals

Autocorrelation function describes *the similarity* or *coherence* between the given function $x(t)$ and its delayed or its advanced version $x(t \pm \tau)$. It is an even function. The autocorrelation (AC) function of an aperiodic signal $x(t)$ was defined by

$$\text{AC}\{x(t)\} = R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t+\tau)dt$$

$$= \int_{-\infty}^{\infty} x(t)x(t-\tau)dt = \langle x(t)x(t-\tau)\rangle. \quad (2.7.1)$$

$$R_x(-\tau) = R_x(\tau).(\text{even function}) \quad (2.7.2)$$

Second, the maximum value of the autocorrelation function occurs at $\tau = 0$. That is

$$|R_x(\tau)| \le R_x(0). \quad (2.7.3)$$

The proof of the symmetry property in (2.7.2) can be shown by changing the variable $\beta = t + \tau$ and simplifying the integral. The proof on the upper bound on the autocorrelation function is shown below by noting the integral with a nonnegative integrand is nonnegative.

$$\int_{-\infty}^{\infty} [x(t) \pm x(t-\tau)][x(t) \pm x(t-\tau)]dt \ge 0. \quad (2.7.4)$$

$$\int_{-\infty}^{\infty} x^2(t)dt + \int_{-\infty}^{\infty} x^2(t-\tau)dt \pm 2\int_{-\infty}^{\infty} x(t)x(t-\tau)dt$$

$$= 2[\int_{-\infty}^{\infty} x^2(t)dt \pm \int_{-\infty}^{\infty} x(t)x(t-\tau)dt] \ge 0$$

$$\Rightarrow R_x(0) = \int_{-\infty}^{\infty} x^2(t)dt \ge |R_x(\tau)|$$

$$= \left|\int_{-\infty}^{\infty} x(t)x(t-\tau)dt\right|. \quad (2.7.5)$$

Third,

$$E_x = R_x(0) = \int_{-\infty}^{\infty} x^2(t)dt.(\text{energy in } x(t)). \quad (2.7.6)$$

In addition, if $y(t) = x(t \pm \alpha)$, then

$$R_x(\tau) = R_y(\tau). \quad (2.7.7)$$

This can be seen first for $\tau > 0$ from

$$R_y(\tau) = \int_{-\infty}^{\infty} y(t)y(t-\tau)dt = \int_{-\infty}^{\infty} x(t-\alpha)x(t-\alpha-\tau)dt$$

$$= \int_{-\infty}^{\infty} x(\beta)x(\beta-\tau)d\beta = R_x(\tau). \quad (2.7.8)$$

Change of a variable $\beta = (t - \alpha)$ was made in the above integral and then simplified. Since the autocorrelation function is even, the result follows for $\tau < 0$.

Example 2.7.1 Find the AC of $x(t) = e^{-at}u(t), a > 0$ by first computing the AC for $\tau > 0$ and then use the symmetry property to find the other half of the autocorrelation function.

Solution: First,

$$u(t)u(t-\tau) = u(t-\tau) = \begin{cases} 1, & t > \tau \\ 0, & \text{otherwise} \end{cases}, \quad (2.7.9)$$

$$\tau > 0 : R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt$$

$$= \int_{-\infty}^{\infty} e^{-at}u(t)e^{-a(t-\tau)}u(t-\tau)dt$$

$$= e^{a\tau}\int_{\tau}^{\infty} e^{-2at}dt = \frac{e^{-a\tau}}{2a}.$$

Using the symmetry property of the AC, we have

$$R_x(\tau) = (1/2a)e^{-a|\tau|}. \quad (2.7.10)$$

The energy contained in the exponentially decaying pulse is $E = R_x(0) = (1/2a)$. The autocorrelation function is sketched in Fig. 2.7.1. ■

Example 2.7.2 Consider the function $x(t) = \Pi[t - 1/2]$. Determine its autocorrelation function and its energy using this function.

Solution: The AC function for $\tau \ge 0$ is

$$R_x(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt = \int_{-\infty}^{\infty} \Pi[t-.5]\Pi[t-\tau-.5]dt.$$

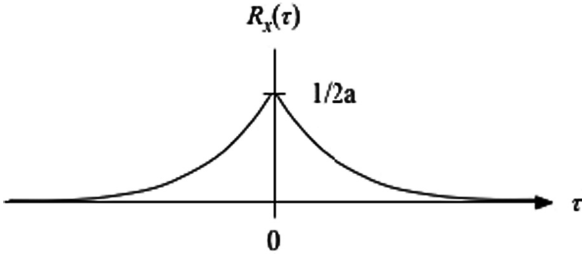

Fig. 2.7.1 Example 2.7.1

The function $\Pi[t-1/2]$ is a rectangular pulse centered at $t=1/2$ with a width of 1, and $\Pi[t-(\tau+(1/2))]$ is a rectangular pulse centered at $(\tau+0.5)$ with a width of 1. See Fig. (2.7.2a) for the case $0<\tau<1$. In the case of $\tau \geq 1$, there is no overlap indicating that $R_x(\tau)=0, \tau \geq 1$.

$$R_x(\tau)=\int_{\tau}^{1} dt=(1-\tau), 0 \leq \tau<1.$$

Using the symmetry property, we have

$$R_x(\tau)=R_x(-\tau)=\left\{\begin{array}{ll}(1-|\tau|), 0 \leq|\tau| \leq 1 \\ 0, \quad \text{Otherwise}\end{array}\right\}=\Lambda[\tau]. \tag{2.7.11a}$$

This is sketched in Fig. 2.7.2b indicating that there is correlation for $|\tau|<1$ and no correlation for $|\tau| \geq 1$. The peak value of the autocorrelation is when $\tau=0$ and is $R_x(0)=1$. The energy contained in the unit rectangular pulse is equal to 1 and by using the autocorrelation function, i.e., $R_x(0)=1$, the same by both the methods. Noting that the autocorrelation function of a given function and its delayed or advanced version are the same, the AC function is much easier to compute using this property. The AC of the pulse function $\Pi[t-.5]$ can be computed by ignoring the delay. That is, $AC\{\Pi[t-.5]\}=AC\{\Pi[t]\}$. Interestingly,

$$AC\left\{\Pi\left[\frac{t}{T}\right]\right\}=T\Lambda\left[\frac{\tau}{T}\right]. \tag{2.7.11b}$$

The AC function of a rectangular pulse of width T is a triangular pulse of width $2T$ and its amplitude at $t=0$ is T. We can verify the last part by noting

$$AC\left\{\Pi\left[\frac{t}{T}\right]\right\}\Big|_{\tau=0}=T\Lambda\left[\frac{\tau}{T}\right]\Big|_{\tau=0}=T. \qquad \blacksquare$$

Note $x(t)\Pi\left[\frac{t}{T}\right]$ extracts $x(t)$ for the time $-T/2<t<T/2$. That is,

$$x(t)\Pi\left[\frac{t}{T}\right]=\left\{\begin{array}{ll}x(t), -T/2<t<T/2 \\ 0, \quad \text{otherwise}\end{array}\right.. \tag{2.7.12}$$

Example 2.7.3 Find the autocorrelation of the function $y(t)=\cos(\omega_0 t)\Pi[t/T]$.

Solution:

$$\begin{aligned} R_y(\tau)&=\int_{-\infty}^{\infty}\Pi\left[\frac{t}{T}\right]\Pi\left[\frac{t-\tau}{T}\right]\cos(\omega_0 t)\cos(\omega_0(t-\tau))dt \\ &=\frac{\cos(\omega_0\tau)}{2}\int_{-\infty}^{\infty}\Pi\left[\frac{t}{T}\right]\Pi\left[\frac{t-\tau}{T}\right]dt \\ &+\frac{1}{2}\int_{-\infty}^{\infty}\Pi\left[\frac{t}{T}\right]\Pi\left[\frac{t-\tau}{T}\right]\cos(2\omega_0 t-\tau)dt. \end{aligned}$$

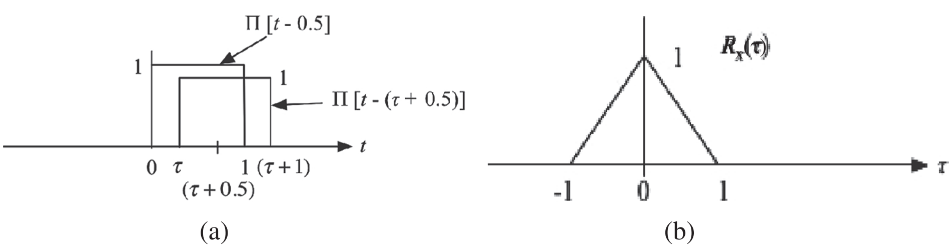

Fig. 2.7.2 Example 2.7.2 Autocorrelation of a rectangular pulse

$$= \begin{cases} (1/2)T\Lambda\left[\frac{\tau}{T}\right]\cos(\omega_0\tau) + B, |\tau| \le T \\ 0, \ |\tau| > T \end{cases}. \quad (2.7.13)$$

Now consider the evaluation of B. For $\tau \ge 0$,

$$B = \frac{1}{2}\int_{-\infty}^{\infty} \Pi\left[\frac{t}{T}\right]\Pi\left[\frac{t-\tau}{T}\right]\cos(2\omega_0 t - \tau)dt$$

$$= \frac{1}{2}\int_{\tau}^{T/2} \cos(2\omega_0 t - \omega_0\tau)dt$$
$$= \frac{1}{4\omega_0}[\sin(\omega_0 T - \omega_0\tau) - \sin(\omega_0\tau)]. \quad (2.7.14)$$

If ω_0 is large, $R_y(\tau)$ in (2.7.13) can be *approximated* by the first term and

$$R_Y(\tau) \simeq \frac{1}{2}T\Lambda\left[\frac{t}{T}\right]\cos(\omega_0\tau). \quad (2.7.15)$$

The envelope of the autocorrelation function in (2.7.16) is a triangular function, which follows since the correlation of the two identical rectangular functions is a triangular function. Noting that the cosine function oscillates between ± 1, the envelope of the autocorrelation function in (2.7.15) is shown in Fig. 2.7.3. ∎

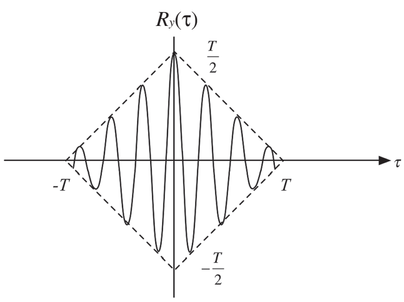

Fig. 2.7.3 Sketch of $R_y(\tau)$

Notes: Conditions for the *existence* of an *aperiodic autocorrelation* are similar to those of convolution (see Section 2.2.3). But there are a few exceptions. For example, the autocorrelation of the unit step function does not exist.

2.8 Cross- and Autocorrelation of Periodic Functions

The cross- and the autocorrelation functions of periodic functions of $x_T(t)$ and $h_T(t)$ are

$$R_{T,}xh(\tau) = \frac{1}{T}\int_{T} x_T(t)h_T(t+\tau)dt = \langle x_T(t)h_T(t+\tau)\rangle\ , \quad (2.8.1a)$$

$$R_{T,x}(\tau) = \frac{1}{T}\int_{T} x_T(t)x_T(t+\tau)dt = \langle x_T(t)x_T(t+\tau)\rangle\ . \quad (2.8.1b)$$

Note that the periods of the functions, $x_T(t)$ and $h_T(t)$, are assumed to be the *same* and the constant $(1/T)$ before the integrals in (2.8.1a and b). If they have different periods, computation of (2.8.1a) is difficult and these cases will not be discussed here. Many of the cross-correlation and AC function properties derived earlier for the aperiodic case apply for the periodic functions with some modifications. Note that

$$\begin{aligned} &R_{T,xh}(\tau) = R_{T,hx}(-\tau), \\ &R_{T,x}(0) + R_{T,h}(0) \ge 2\left|R_{T,xh}(\tau)\right|. \end{aligned} \quad (2.8.2)$$

In Section 2.5.1, aperiodic convolution was used to find periodic convolution. The same type of analysis can be used to determine periodic cross-correlations using aperiodic cross-correlations. Furthermore, as discussed before, correlation is related to convolution. First define two finite duration functions, $x(t)$ and $h(t)$, over the interval $t_0 \le t < t_0 + T$. Assume that they are zero outside this interval. Now create two periodic functions:

$$x_T(t) = \sum_{n=-\infty}^{\infty} x(t-nT), \quad h_T(t) = \sum_{n=-\infty}^{\infty} h(t-nT). \quad (2.8.3a)$$

The periodic cross-correlation function is defined by

$$R_{T,xh}(\tau) = \frac{1}{T}\int_{t_0}^{t_0+T} x_T(t)h_T(t+\tau)dt,\ h_T(t+\tau)$$
$$= \sum_{n=-\infty}^{\infty} h(t+\tau-nT). \tag{2.8.3b}$$

The expression for periodic convolution is given in terms of aperiodic convolution and

$$R_{T,xh}(\tau) = \frac{1}{T}\sum_{n=-\infty}^{\infty} R_{xh}(\tau-nT),$$
$$R_{xh}(\tau-nT) = \int_{t_0}^{t_0+T} x(t)h(t+\tau-nT). \tag{2.8.3c}$$

The details of the derivation are left as an exercise. Copies of $R_{xh}(\tau)$ will overlap if the width of $R_{xh}(\tau)$ is wider than T.

Example 2.8.1 Give the lower bound on the period T so that there are no overlaps in the cross-correlation of the functions $x_T(t)$ and $h_T(t)$ given below. See Example 2.6.5.

$$x(t) = \Pi[t-.5], h(t) = t\Pi\left[\frac{t-1}{2}\right],$$
$$x_T(t) = \sum_{n=-\infty}^{\infty} x(t+nT), h_T(t) = \sum_{n=-\infty}^{\infty} h(t+nT).$$

Solution: If the period T is larger than 3, then there are no overlaps in the periodic cross-correlation function. In that case, one period of the cross-correlation function can be obtained from the aperiodic cross-correlation in that example and dividing it by the period T. If the period is less than 3, then there will be overlaps. ■

Example 2.8.2 Consider the periodic functions

$$x_{T,1}(t) = X_s[0], \quad x_{T,2}(t) = c[k]\cos(k\omega_0 t + \theta[k]). \tag{2.8.4}$$

a. Find the AC functions for the functions in (2.8.4). *b*. Find the cross-correlation of the two functions.

Solution:

$$a.\ R_{T,x_{T,1}}(\tau) = \frac{1}{T}\int_T X_s^2[0]dt = X_s^2[0], \tag{2.8.5a}$$

$$R_{T,x_{T,2}}(\tau) = \frac{1}{T}\int_T x_{T,2}(t)x_{T,2}(t+\tau)dt$$
$$= \frac{c^2[k]}{2T}\int_T \cos(k\omega_0\tau)dt$$
$$+ \frac{c^2[k]}{2T}\int_T \cos(k\omega_0(2t+\tau)+2\theta[k])dt$$
$$= \frac{c^2[k]\cos(k\omega_0\tau)}{2T}\int_0^T dt = \frac{c^2[k]\cos(k\omega_0\tau)}{2}. \tag{2.8.5b}$$

Note that the integral of a cosine function over any integer number of periods is zero.
b. The cross-correlation of a constant and a cosine function over one period is zero. Also note that the two functions are orthogonal. That is $\langle x_{T_1}(t), x_{T_2}(t)\rangle = 0$. ■

Example 2.8.3 Find the AC of $x_T(t)$ given below with $k \neq m, k$and$m$are integers.

$$x_T(t) = x_{T,1}(t) + x_{T,2}(t), x_{T,1}(t)$$
$$= c[k]\cos(k\omega_0 t + \theta[k]), x_{T,2}(t)$$
$$= c[m]\cos(m\omega_0 t + \theta[m]).$$

Solution: The periodic autocorrelations are determined as follows:

$$R_{T,x}(\tau) = \frac{1}{T}\int_T [x_{T,1}(t) + x_{T,2}(t)][x_{T,1}(t+\tau)$$
$$+ x_{T,2}(t+\tau)]dt \ = \frac{1}{T}\int_T x_{T,1}(t)x_{T,1}(t+\tau)dt$$
$$+ \frac{1}{T}\int_T x_{T,2}(t)x_{T,2}(t+\tau)dt$$
$$+ \frac{1}{T}\int_T x_{T,1}(t)x_{T,2}(t+\tau)dt$$
$$+ \frac{1}{T}\int_T x_{T,2}(t)x_{T,1}(t+\tau)dt. \tag{2.8.6}$$

Note

$$\frac{1}{T}\int_{T} x_{T,1}(t)x_{T,2}(t+\tau)dt$$
$$=\frac{1}{2T}\int_{T} c[k]c[m]\cos[(k+m)\omega_0 t+m\omega_0\tau+\theta[k]$$
$$+\theta[m])dt+\frac{1}{2T}\int_{T} h[k]h[m]\cos[(k-m)\omega_0 t-m\omega_0\tau)$$
$$+(\theta[k]-\theta[m])]dt=0.$$

Similarly the fourth term in (2.8.6) goes to zero. From the last example,

$$R_{T,x}(\tau)=\frac{c^2[k]}{2}\cos(k\omega_0\tau)+\frac{c^2[m]}{2}\cos(m\omega_0\tau), k\neq m. \quad (2.8.7)$$ ■

These results can be generalized using the last two examples and the autocorrelation of a periodic function $x_T(t)$ is given as follows:

$$x_T(t)=X_s[0]+\sum_{k=1}^{\infty} c[k]\cos(k\omega_0 t+\theta[k]), \quad (2.8.8)$$

$$\Rightarrow R_{T,x}(\tau)=X_s^2[0]+\frac{1}{2}\sum_{k=1}^{\infty} c^2[k]\cos(k\omega_0\tau), \omega_0=2\pi/T. \quad (2.8.9)$$

AC function of a periodic function is also a periodic function with the same period. It is independent of $\theta[k]$. It does not have the phase information contained in (2.8.8). In the next chapter, (2.8.8) will be derived for an arbitrary periodic function and will be referred to as the harmonic form of Fourier series of a periodic function $x_T(t)$. ■

Notes: The AC function of a constant $X_s[0]$ is $X_s^2[0]$. The AC of the sinusoid $c[k]\cos(k\omega_0 t+\theta[k])$ is $(c^2[k]/2)\cos(k\omega_0\tau)$. That is, it *loses* the phase information in the function in the sinusoid. The power contained in the periodic function $x_T(t)$ in (2.8.8) can be computed from the autocorrelation function evaluated at $\tau=0$. That is,

$$P=X_s^2[0]+\frac{1}{2}\sum_{k=1}^{\infty} c^2[k]. \quad (2.8.10)$$

The difference between the total power and the dc power is the variance and is given by

$$\text{Variance} = \frac{1}{2}\sum_{k=1}^{\infty} c^2[k]. \quad (2.8.11)$$ ■

Example 2.8.4 Consider the corrupted signal $y(t)=x(t)+n(t)$, where $n(t)$ is assumed to be noise. Assuming the signal $x(t)$ and noise $n(t)$ are uncorrelated, derive an expression for the autocorrelation function of $y(t)$.

Solution:

$$R_{yy}(\tau)=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} y(t)y(t+\tau)dt$$
$$=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} [x(t)+n(t)][x(t+\tau)n(t+\tau)]dt,$$
$$=\lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt+\lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)n(t+\tau)dt$$
$$+\lim_{T\to\infty}\int_{-T/2}^{T/2} n(t)x(t+\tau)dt+\lim_{T\to\infty}\int_{-T/2}^{T/2} x(t)x(t+\tau)dt. \quad (2.8.12)$$

Noting that the signal and the noise are *uncorrelated*, i.e., $R_{xn}(\tau)=R_{nx}(\tau)=0$, we have

$$R_{yy}(\tau)=R_{xx}(\tau)+R_{nn}(\tau). \quad (2.8.13)$$ ■

The average power contained in the signal and the noise is given by

$$P_y=P_x+P_n=R_x(0)+R_n(0)=R_x(0)+\sigma_n^2. \quad (2.8.14)$$

The signal-to-noise ratio (SNR), P_x/P_n, can be computed. It is normally identified in terms of decibels. See Section 1.9.

2.9 Summary

We have introduced the basics associated with the two important signal analysis concepts: convolution and correlation. Specific principal topics that were included are

- Convolution integral: its computations and its properties
- Moments associated with functions
- Central limit theorem
- Periodic convolutions
- Auto- and cross-correlations
- Examples of correlations involving noise without going into probability theory
- Quantitative measures of cross-correlation functions and the correlation coefficient
- Auto- and cross-correlation functions of energy and periodic signals
- Signal-to-noise ratios

Problems

2.1.1 Consider the following functions defined over $0 < t < 1$. Using (2.1.3), identify the two functions that give the maximum cross-correlation at $\tau = 0$.
$x_1(t) = e^{-t}$, $x_2(t) = \sin(t), x_3(t) = (1/t)$.

2.2.1 Prove the commutative, distributive, and the associate properties of the convolution.

2.2.2 Find the convolution $y(t) = h(t) * x(t)$ for the following functions:

$$a.\ x(t) = .5\delta(t-1) + .5\delta(t-2),\quad h(t) = .5\delta(t-2) + .5\delta(t-3)$$

$$b.\ x(t) = (t-1)\Pi[t-1],\quad h(t) = x(t),$$

$$c.\ x(t) = (1-t^2),\ -1 \le t \le 1,\quad h(t) = \Pi[t],$$

$$d.\ x(t) = e^{-at}u(t),\quad h(t) = e^{-bt}u(t)$$
for cases : $1.a > 0,\ b > 0, \quad 2.a = 0, b > 0$

$$e.\ x(t) = \Pi[t/2],\quad h(t) = \Pi[t-.5] - \Pi[t-1.5]$$

$$f.\ x(t) = \delta(t-1),\quad h(t) = e^{-t}u(t)$$

$$g.\ x(t) = \cos(\pi t)\Pi[t],\quad h(t) = e^{-t}u(t).$$

2.2.3 Use the area property of convolution to find the integrals of $y(t)$ in Problem 2.2.2.

2.3.1 a. Derive the expression for the convolution of two pulse functions given by $x(t) = \Pi[t-1]$ and $h[t] = \Pi[t-2]$. Compute this directly first and then verify your result by using the delay property of convolution.
b. Verify the time duration property of the convolution using the above problems.

2.3.2 Determine the area of $y(t)$ in (2.3.18) using the area property of the convolution.

2.4.1 Approximate the function $y(t)$ in Example 2.3.1 using the Gaussian function.

2.4.2 Use the derivative property of the convolution to derive the convolution of the two functions given below using the results in Example 2.5.2.

$$x_T(t) = \sin(\omega_0 t), h(t) = e^{-at}u(t), a > 0.$$

2.4.3 Use the delay property of the convolution to determine

$$x(t) = e^{-at}u(t) * u(t-1).$$

2.5.1 Derive the expressions for the periodic convolution of the two periodic functions

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT), \quad h(t) = \sum_{n=-\infty}^{\infty} \Pi\left[\frac{t-nT}{T/2}\right].$$

2.6.1 Find the cross-correlation of the functions $x(t)$ and $h(t)$ given in (2.6.11a) by directly deriving the result and verify the result using the results in Example 2.6.1.

2.6.2 Show the bounds given in (2.6.7a and b) and (2.6.9b) are valid. Use (2.6.11a).

2.6.3 Show (2.6.9b) using (2.6.10).

2.7.1 Find the autocorrelations of the following functions:

$$a.x_1(t) = \Pi[t-.5] - \Pi[t-1.5],$$
$$b.x_2(t) = u(t-.5) - u(t+.5), \quad c.x_3(t) = t\Pi[t].$$

Compute the energies contained in the functions directly and then verify the results using the autocorrelation functions derived in the first part.

2.7.2 Verify the result in (2.7.3) using the results in Example 2.7.1.

2.7.3 Show the identity

$$AC[x(t-t_0)] = AC[x(t)].$$

2.7.4 Derive the AC function step by step for the function $x(t) = \cos(\omega_0 t)\Pi[t/T]$.

Use the integral formula by assuming $\omega_0 = \pi$ and $T = 4$. Verify the results in Example 2.7.3 using the information provided in this problem. Give the appropriate bounds.

2.7.5 Show that the autocorrelations of the function $x_2(t) = e^{at}u(t)$ for $a \geq 0$ do not exist.

2.8.1 *a*. Derive the time-average periodic autocorrelation function $R_{x,T}(\tau)$ for the following periodic function using the integral formula.

$$x_T(t) = A_1 \cos(\omega_0 t + \theta_1) + A_2 \cos(2\omega_0 t + \theta_2).$$

b. Verify the result using (2.8.8) and (2.8.9).

c. Compute the average power contained in the function directly and by evaluating the autocorrelation function at $\tau = 0$. Sketch the function $x(t)$ by assuming the values $A_1 = 5, A_2 = 2, \theta_1 = 20^0, \theta_2 = 120^0$. Sketch the autocorrelation function using these constants. Suppose we are interested in determining the period T from these two sketches, which function is better, the given function or its autocorrelation? Why?

2.8.2 Let $y_T(t) = A + x_T(t),\ A -$ constant. Repeat the last problem, except for the plots.

2.8.3 *a*. Show that the following functions are orthogonal over a period:
$x_T(t) = \cos(\omega_0 t + \theta),\ y(t) = A$ *b*. Show the functions $x(t) = \Pi[t],\ y[t] = t$ are orthogonal.

2.8.4 Consider the signal $z(t) = x(t) + y(t)$. Show that the AC of this function is given by

$$R_z(\tau) = R_x(\tau) + R_y(\tau) + R_{xy}(\tau) + R_{yx}(\tau).$$

Simplify the expression for $R_z(\tau)$ by assuming that $x(t)$ is orthogonal to $y(t)$ for all τ.

2.8.5 Complete the details in deriving the periodic cross-correlation function in terms of the aperiodic convolution leading up to Equation (2.8.3c).

2.8.6 Show (2.8.3c) using (2.6.5).

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