Chapter 2
Decision Systems

> The correct statement of the laws of physics involves some very unfamiliar ideas which require advanced mathematics for their description. Therefore one needs a considerable amount of preparatory training even to learn what the words mean.
>
> Richard Feynman

2.1 Preliminaries

Decision theory emerged from the requirements of diverse fields of human activity such as medicine, gambling, politics, warfare, economics and finance, and engineering. Perhaps this is the reason for the terminological diversity that sometimes impedes not only mutual understanding between specialists in different fields but also the development of decision theory itself. In this sense, control theory has been more fortunate, for its terminology turned out to be common to many spheres of its application.

There are two points of view on the relationship between decision theory and control theory. According to one such view, they have nothing in common. According to the other, these theories are gradually converging because the differences between them are not fundamental [14, 18]. The author of this book is an adherent of the latter viewpoint, and proposes the following motivation.

In control theory one studies a *control system* that consists of a pair of objects: a *plant* and a *controller*. In decision theory one studies a pair consisting of a *decision situation* and a *decision-maker*. It is natural to call such a pair a *decision system*. A *control system* is defined similarly.

The problem of choice of a decision or a control—an action that produces some consequence—is a problem common to both systems. In both systems, one may encounter two basic difficulties in the process of making this choice: dynamics and uncertainty. The development of control theory began in engineer-

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ing, and the dynamics of plants became its central problem. The development of decision theory began in economics, and uncertainty became its central problem. While this dichotomy still exists, more and more attention is now being devoted to uncertainty in control theory [53] and to dynamics in decision theory [14].

But there is still an essential difference. Whereas the choice of decision criterion is at the center of decision theory, it is still on the periphery in control theory.

A systematic mathematical study of a control system becomes possible only if we define mathematical models of its components: the controlled plant, the controller, and the experiment (observation) the controller can perform over the plant. The same must be true about a decision system. Therefore, in this chapter we introduce the notion of a decision system and mathematical models of its components: the decision situation, the decision-maker, and the experiment (observation) the decision-maker can perform over the decision situation. An attempt to define a model of the second component of a decision system (the decision-maker) may seem surprising if we do not mention that our model concerns only the sequence of specific operations any decision-maker performs in the process of decision-making.

The existence of two types of real decision situations—nonparametric and parametric—leads to the existence of two types of model of these situations, called respectively the *lottery model* and the *matrix model*.

The *lottery model* has the form

$$S_l = (Z_l, I_l),$$

where

- $Z_l = (U, C, \psi(\cdot))$ is called a *lottery scheme*;
- U is the set of possible decisions or actions;
- C is the set of all possible consequences;
- $\psi : U \to 2^C$ is a model of the cause–effect mechanism of the decision situation in the form of a multivalued mapping;
- I_l is some datum or *regularity* of this mechanism, available to the decision-maker by the moment at which the decision is made.

The *matrix model* has the form

$$S_m = (Z_m, I_m),$$

where

- $Z_m = (\Theta, U, C, g(\cdot,\cdot))$ is called a *matrix scheme*;
- U and C have the same meaning as in the lottery model;
- Θ is the set of values of the parameter that affects a decision consequence;
- $g : \Theta \times U \to C$ is a model of the cause–effect mechanism of the decision situation in the form of a function of two variables;

- I_m is some datum or *regularity* of this mechanism, available to the decision-maker by the moment at which the decision is made.

It turns out (Theorem 2.1) that the matrix and lottery models are equivalent, that is, each of them can be used to describe parametric as well as nonparametric situations.

In real parametric situations, two types of experiments (observations) are conceivable: first, observation of the parameter $\theta \in \Theta$ if it is physically possible, and second, the observation of the consequence $c \in C$ of the previous decision. In nonparametric situations a decision-maker can observe only the consequences $c \in C$ of previous decisions. In real life, parametric situations are less frequent than nonparametric situations, but their natural (matrix) model happens to be more convenient for the purposes of accounting and processing the data obtained in the experiment.

So the model of experiment in a nonparametric situation has the form

$$h_l : C \to Y_{h_l},$$

while in a parametric situation it has the form

$$h_m^\theta : C \to Y_{h_m^\theta}$$

or

$$h_m^c : \Theta \to Y_{h_m^c},$$

where the Y's are the sets in which the results of the experiments take values.

In describing a nonparametric situation in terms of the matrix model, the set Θ of values of the parameter θ is introduced artificially. That is why the parameter θ may not be observed explicitly. But due to the equivalence of the matrix and lottery models—due to their one-to-one correspondence—the artificial parameter θ characterizes the cause–effect mechanism of the nonparametric situation.[1]

We assume further that any decision-maker that happens to be in a situation demanding a decision has her own preference regarding the consequences of her decisions. This preference, or more precisely, *preference relation*[2] that the decision-maker establishes on the set C of consequences, is denoted by β_C. It turns out that the decision situation scheme Z and the preference relation β_C determine together whether there is uncertainty in the decision system. Theorem 2.2 establishes necessary and sufficient conditions for the existence of uncertainty in a decision system. In this way, uncertainty reveals itself as a system notion: there is no uncertainty without a definite decision-maker, without his

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[1] Observation of consequences allows one to construct an estimate $\hat{\theta}(y,u)$ of this parameter. This is discussed in Chapter 8.

[2] See Appendix A.1.

personal preference relation β_C. In two different decision systems that share the same decision situation but have different decision-makers (different preference relations β_C), uncertainty can exist in one system and be absent in the other. It turns out that uncertainty in a decision system is what creates the problem of choice of decision criterion, or in other words, the problem of choice of a decision, a problem with a strong psychological accent.

2.2 The Structure of Decision Systems

Let us begin by considering some examples.

Example 2.1. In many folktales there appears the following type of episode: a hero, together with his beloved and his treasure, is riding his horse along a road and eventually comes to a crossroads, where there stands a stone with the following inscription:

If you go left you may lose your beloved
If you go right you may lose your horse
If you go straight you may lose your wealth

The hero cannot turn back. After all, he is the hero! He must choose one of the three roads on which to continue his travels. He is keenly aware of the possible consequences of each of the three decisions. Our hero, preferring his beloved and his horse to his wealth, with little deliberation goes straight. But he eventually reaches another crossroads, and another stone, with the following inscription:

If you go left you may lose your beloved or your wealth
If you go right you may lose your horse or your wealth
If you go straight you may lose your horse or your beloved

This choice is more difficult. Being a typical hero, he will most likely prefer his beloved to his horse and therefore turn to the right. After all, typical heroes value their beloved more than anything else. But eventually he is confronted with a new crossroads and a new stone:

If you go left you may lose your horse or your wealth or your beloved
If you go right you may lose your horse or your wealth or your beloved
If you go straight may lose your horse or your wealth or your beloved

And now the hero has a problem: wherever he goes, he is in danger of losing his beloved. His preferences (or priorities)—the beloved is more valuable than the horse and the horse is more valuable than wealth—do not help him to choose the optimal road. A conscious choice, a reasonable preference of one road over another, is impossible in this situation: the hero's preference for one or another of the possible consequences of his choice do not apply at this crossroads. The hero can, of course, throw a die, thereby submitting himself to the whims of chance. But if he wishes to justify his decision, he needs some additional data as to which direction offers the least likelihood that he will lose his beloved and a good chance of losing nothing. In order to obtain such data he addresses an old raven sitting on a branch in a nearby oak who has seen many travelers at this crossroads. It turns out that indeed, the probabilities are different: on the left-hand road, there is a greater chance of losing nothing and less chance of keeping his beloved than on the right-hand road. Now the hero's decision depends on his personality: a self-confident hero—perhaps he views himself as an invincible fighter—may turn to the left, while the more cautious, more prudent hero turns right, for he is ready to lose his horse or his treasure if he can keep his beloved. Which of the two is our hero? Which type is more "rational"?

Example 2.2. A college graduate wants to choose a direction for further studies: humanities, sciences, or engineering. He may be more apt at one of the three, but he does not know which one. If his choice coincides with his abilities, his life will be more satisfying. But how is he to make a choice if he does not know his abilities? Again, like the hero in the previous example, our graduate must obtain further data about his state, where we assume that he is in one of only three possible states, that is, that he possesses the greatest ability in only one of the three areas. Our student may take some aptitude tests, but there are no tests that can rule out the possibility of a mistake. So the uncertainty of the situation cannot be eliminated.

Example 2.3. The expression "Hobson's choice" refers to a situation in which there is an appearance of choice, but in fact no choice at all. Thomas Hobson (1544–1610) was an entrepreneur in Cambridge, England, where he operated a livery stable with horses for hire. Having found that customers preferred the best horses, which were becoming overworked, Hobson instituted the rule that the customer had to accept the first horse in line at the stable gate. Thus the customer was unable to indulge any preference regarding the horse's color or temperament. Today we might say that by paying the fee to hire a horse, the customer has paid for the right to participate in a lottery with the horses as the prizes (or the consequences): the customer will receive one horse from the set of horses held in the stable. In fact, there is no choice (or decision-making) here. The choice has been offered to the client earlier: knowing the rules of the stable, he could agree to hire a horse or refuse to do so. Here we would say that he chooses between two alternatives:

(1) participation in the lottery with "any of Hobson's horses" as a consequence;
(2) participation in the lottery with "none of Hobson's horses" as a consequence.

Perhaps it was not so easy to refrain from participating in the Hobson's horse lottery. Different ethical (aesthetic, moral, social) considerations could favor the decision to hire an arbitrary Hobson's horse. For example, it is possible that those who frequented the stables were members of a particular group, or a club, to which it was important to belong, and thus it was worthwhile to risk getting a bad horse.

Decision theory today considers only ethically neutral alternative acts. A lottery, in which the consequences are monetary gains and the preference relation regarding the consequences is determined by the gain size, is the traditional example of such neutrality. Let \$10 be the price for the right to participate in any of the two following lotteries: the first lottery offers the gains \$1,000,000, \$10, \$0, while the second offers \$10,000, \$100, and \$0. Which lottery do we prefer? Most likely, we will not be able to make an informed choice, for the situation is rather uncertain. Of course, the desire for riches might make some of us choose the first lottery. But such a choice would be opposed to the neutrality of the alternatives. Therefore, we would first wish to get some complementary data regarding the probabilities of gains in each of the two lotteries. Here we can say that the uncertainty in the choice of the best alternative appears due to the uncertainty of the situation: the same act can result in different consequences, and the same consequence can result from different acts.

One should not think that such an ambiguity in the consequences of our actions appears only in situations in which a human being deals with systems created by human beings, i.e., those that function not according to the laws of nature, but according to other laws devised by people. The influences of our actions on various physical or chemical systems and their responses (or the consequences of our actions) have been studied in control theory and its applications, a discipline close to decision theory and its applications. Ambiguity in the aforementioned responses also leads to decision-making under uncertainty. Here is an example.

Example 2.4. Consider the construction of some part or subassembly that will go into a certain technological device. During construction, the part will be tested by subjecting it to a certain influence (a temperature field, a vacuum, ionic/molecular bombardment, etc.) for a certain period of time t. Each part requires its own processing time t^*. But time t^* is unknown at the moment that processing time t is chosen. However, we know that $t^* \in [T_1, T_2]$, and in order for a part to be certified, it is necessary that time t not deviate from t^* by more than Δt, that is, t should be chosen in such a way that $|t^* - t| < \Delta t$. If this condition is not satisfied, the part is declared unfit. It is clear that if $\Delta t < |T_1 - T_2|/2$, there is no chance for all the parts to be certified. Thus, the problem is to choose a value for t that minimizes the quantity of unfit parts. So far, what is important for us is to choose a certain value of t. But when t^* is not known in advance, the result (or the consequence) of the operation is uncertain: for any choice of t at least two outcomes are possible: the part is good or it is bad. The set of possible alternatives is the set of all possible time values $t \in |T_1 - T_2|$. So the question is, what value of t shall we choose?

This problem resembles the game "guess the number," in which a player is to guess a number chosen from $[1, \dots, 10]$ by another person. If he guesses correctly, he wins a prize, and if not, he loses something. This game is artificial, whereas in our technological situation we are dealing with the laws of physics and chemistry. Here ambiguity appears, due to the fact that the system is open to an external influence about which we know nothing. In this sense, there is no principal difference between physical and socioeconomic systems: both can be under the influence of factors "chosen by us" or "by the decision-maker" as well as "chosen by something (somebody) else," and hence our decisions and corresponding actions do not determine the consequence uniquely.[3]

It is easy to see that all our examples have much in common. Indeed, everywhere in the above examples, someone—a decision-maker—happens to be in a situation demanding that he make a decision (alternative, action) out of a certain set of possible decisions. In the first example, it is the choice of the road; in the second, it is the choice of the educational profile; in the third, it is the choice of the lottery; in the fourth, it is the choice of the processing time. In all these examples, the choice—making a decision—implies an action leading to certain consequences. We call such situations *decision situations*.

In all our examples, at least one action may have several possible consequences, only one of which will take place. In all the examples, the consequence of interest to the decision-maker can be the result of more than one of his actions, and the decision-maker does not know for certain which action generates this consequence. It is for this reason that such actions are called "decisions under uncertainty."

In all the examples, the decision-maker may have some data about "probabilities" in favor of one or another consequence. Sometimes, before making a decision, a decision-maker can execute some experiment (observation) that would provide additional data about these "probabilities." Therefore, data about the decision situation are usually divided into a priori data, those that the decision-maker has before the experiment, and a posteriori data, those that he has after the experiment.

Note that in all our examples there is no dependency between consequences of different actions: whatever the consequence of an action, it in no way influences the consequences of other actions. We limit ourselves in this book to this independence of consequences. As we shall see later, this independence, induced by the structure of the cause–effect mechanism of a situation, increases in a certain sense the complexity of the decision situation.

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[3] This situation is similar to the situation in which at each moment of time, a portfolio manager or a trader makes a decision to change or not to change the structure of his investment portfolio.

Finally, note that our examples naturally form two groups. The first group includes Examples 2.1 and 2.3, while the second includes Example 2.2 and 2.4. Namely, in situations belonging to the second group, there is an explicitly present "physical"[4] parameter that along with the decision-maker's actions influences the consequences. Thus in Example 2.2, this parameter is the true but unknown vocation of a college graduate; in Example 2.4, it is the true, but unknown to the decision-maker, processing time of a machine part. In situations of the first group, there is no such explicit parameter. Nevertheless, the sets of possible outcomes (consequences) that correspond to a decision-maker's actions still are such that he does not know which of his actions will lead to the required outcome.

We call the situations of the first type (first group) *nonparametric situations*, while the situations of the second type (second group) are called *parametric situations*. In this connection there are two types of experiments, or observations, that may be provided by a decision-maker. In situations of nonparametric type, a decision-maker can observe only consequences (probably distorted) of his previous actions (decisions). So, in Example 2.1, the hero obtains such observations from the raven. In the situations of parametric type, the decision-maker also may observe consequences of his previous actions. But sometimes, before making a decision he may observe the value (which may be distorted) of the parameter that affects the consequences. So in Example 2.2, the graduate may observe the results of some special tests. Note also that in parametric situations, both types of observation may sometimes be provided.

Finally, we note that all our examples have the same structure, a sketch of which is presented in Figure 2.1:

In the figure, DM = decision-maker, DMS = decision-making situation. They are connected by channels 1 and 2. Through channel 1 the decision-maker receives the data $S = (Z, I)$ about the situation before making a decision. Through channel 2 the decision-maker makes decisions $u \in U$, which result in consequences $c \in C$ in channel 3. Channel 4 is an experiment of the first type; channel 5 is an experiment of the second type. Finally, $\theta \in \Theta$ is the unknown parameter in the parametric decision situation.

We call the structure consisting of a decision-making situation and a decision-maker a *decision system*, in analogy to control theory, where the pair consisting of a controlled plant and a controller is called a control system. If an experiment is provided, we shall say that we are dealing with a *decision system with experiment*.

Our goal is to study decision systems. For such a study we need mathematical models of all parts of the decision system: the decision situation, the decision-maker, and the experiment. Decision systems with experiment will be studied in Chapters 6, 7, and 8.

[4] We use the word "physical" in the widest possible sense, meaning "actually existing," following Bellman [7].

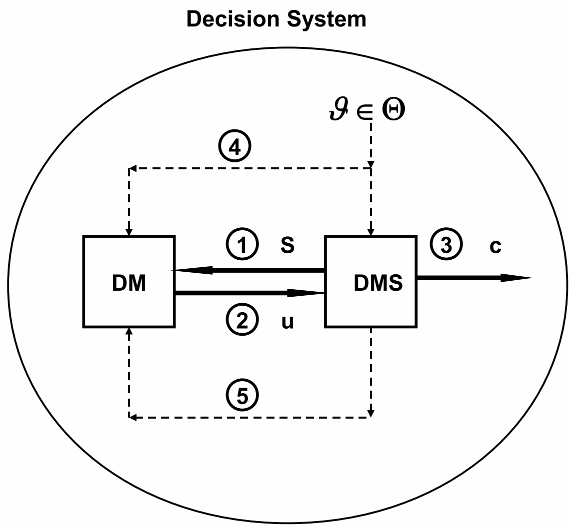

Fig. 2.1 The structure of a decision system.

2.3 Decision Situations

We suppose that everything that is unknown to the decision-maker is introduced into the decision system only by the decision situation. We shall construct a mathematical model of the decision situation in the form of the pair $S = (Z, I)$, where Z is the scheme of the decision situation and I is the data about the regularity of the cause–effect mechanism. The above-mentioned existence of two classes of decision situations—nonparametric and parametric—engenders two different models of decision situations.

2.3.1 The Scheme of a Decision Situation

For any situation we admit the following terminology and notation. We say that a decision-maker has to choose a decision u from some set U of all admissible decisions for a given situation, i.e., $u \in U$.[5] We shall identify a decision u with an action that may generate some consequence c from the set of all possible consequences C_u for this decision.

[5] Henceforth, if it is not specified, we mean arbitrary sets.

Let

$$\bigcup_{u \in U} C_u = C$$

denote the set of all possible consequences of the situation.

We call the triple

$$Z_l = (U, C, \psi(\cdot)) \tag{2.1}$$

a *lottery scheme of a decision situation*, or simply a *lottery scheme* for short. The mapping $\psi(\cdot)$ will sometimes be written in the form of a parametric set $(C_u,\ u \in U)$, and the lottery scheme analogously in the form of the triple

$$Z_l = (U, C, (C_u,\ u \in U)).$$

We call the quadruple

$$Z_m = (\Theta, U, C, g(\cdot,\cdot)) \tag{2.2}$$

a *matrix scheme*. Here Θ is the set of all possible values of the unknown parameter θ that may be chosen by someone other than a decision-maker.

In the lottery scheme (2.1), the multivalued mapping

$$\psi\colon U \to 2^C \in \left\{2^C\right\}^U$$

describes the cause–effect mechanism of generation, or generator, of the consequences in a decision situation. In the matrix scheme (2.2), the single-valued mapping

$$g : \Theta \times U \to C$$

is another description of the cause–effect mechanism of generation of consequences.

If the decision-maker knows nothing about the regularity of the cause–effect mechanism, then the model of the situation is reduced to, or becomes exhausted by, its scheme.

It seems natural to use the lottery scheme Z_l in modeling a nonparametric situation, and the matrix scheme Z_m in modeling a parametric situation. However, it is not difficult to see that a parametric situation can be described in terms of a lottery scheme Z_l, and a nonparametric situation in terms of a matrix scheme Z_m.

Indeed, let the nonparametric situation be described in terms of the lottery scheme

$$Z_l = (U, C, \psi(\cdot)).$$

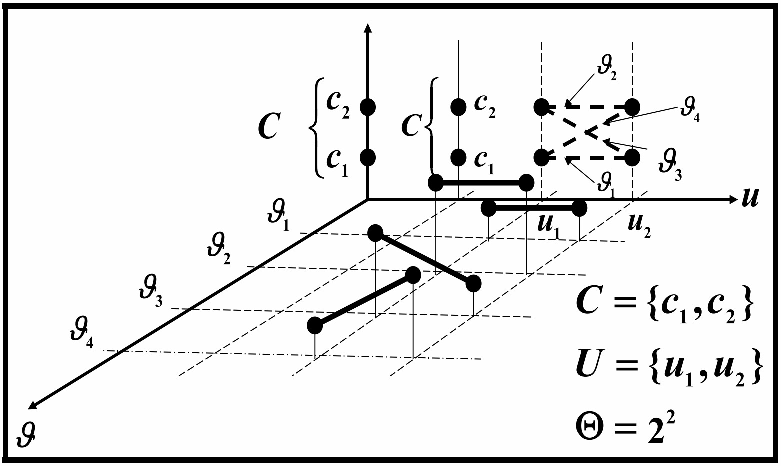

Fig. 2.2 Consistency between a lottery and a matrix scheme.

In order to describe this situation in terms of the matrix scheme Z_m, it is necessary to construct the set Θ and the mapping $g(\cdot,\cdot)$ in such a way that will conserve the "complexity" of the original lottery scheme.

Having noted that in response to the chosen action $u \in U$, the cause–effect mechanism generates only one consequence $c \in C_u$, we describe the *complexity* of the original nonparametric situation in terms of the set of all complex events

$$\hat{\Theta} = \left\{ \hat{\theta} \in (U \to C) : \hat{\theta}(u) \in C_u, \ \forall u \in U \right\}. \tag{2.3}$$

Here $\hat{\theta}(\cdot)$ is a single-valued function with the domain of definition U (see Figure 2.2).

In addition, it is reasonable to evaluate the complexity of the situation described in terms of the lottery scheme (2.1) by the capacity m of the set $\hat{\Theta}$, assuming $m(\hat{\Theta}) = m(C^U)$.

In the case of finite sets U and C, it is obvious that the complexity is given by the number

$$\mathrm{Card}(\hat{\Theta}) = \mathrm{Card}(C^U). \tag{2.4}$$

Setting

$$\hat{\theta}(u) = \hat{g}(\hat{\theta}, u), \tag{2.5}$$

we obtain a matrix scheme $\hat{Z}_m = (\hat{\Theta}, U, C, \hat{g}(\cdot,\cdot))$ in which, in contrast to the matrix scheme $Z_m = (\Theta, U, C, g(\cdot,\cdot))$ describing some parametric situation, the parameter $\widehat{\theta}$ is artificial and may not have physical sense. Following Savage [73], one may call this artificially constructed parameter $\hat{\theta}$ *a state of nature*.

The procedure defined by the operations (2.3) and (2.5) is called *parameterization*. Parameterization defines the mapping

$$\tau : \mathbb{Z}_l = \{Z_l = (U, C, \psi(\cdot))\} \to \hat{\mathbb{Z}}_m \left\{\hat{Z}_m = (\hat{\Theta}, U, C, \hat{g}(\cdot,\cdot))\right\}, \tag{2.6}$$

where $\mathbb{Z}_l$ and $\hat{\mathbb{Z}}_m$ are classes of all lottery schemes and matrix schemes equivalent to them in capacity.

It is clear that the matrix scheme $\widehat{Z}_m$ can always serve to restore the original lottery scheme Z_l. Indeed, define the multivalued mapping $\psi(\cdot)$ according to the following rule:

$$\psi(u) = \left\{\hat{g}(\hat{\theta}, u) : \hat{\theta} \in \hat{\Theta}\right\} \quad \forall u \in U. \tag{2.7}$$

We call this procedure *compression*. It defines the inverse mapping

$$\chi : \hat{\mathbb{Z}}_m = \left\{\hat{Z}_m = \left(\hat{\Theta}, U, C, \hat{g}(\cdot,\cdot)\right)\right\} \to \mathbb{Z}_l = \{Z_l = (U, C, \psi(\cdot))\}. \tag{2.8}$$

The mappings (2.6) and (2.8) are bijective, and therefore the set $\mathbb{Z}_l$ and $\hat{\mathbb{Z}}_m$ are equivalent (have equal capacities).

Then taking into account (2.6) and (2.8), we have

$$\begin{aligned}
&\chi[\tau(Z_l)] \\
&\quad = \chi[\tau(U, C, \psi(\cdot))] \\
&\quad = \chi[\{\theta \in (U \to C) : \theta(u) \in \psi(u), \forall u \in U\}, U, C, \{g(\cdot,\cdot) : \\
&\qquad\qquad g(\theta, u) = \theta(u), \forall \theta \in \Theta, \forall u \in U\}] \\
&\quad = (U, C, (\{\theta(u) : \theta \in \Theta\}, \forall u \in U)) = (U, C, \psi(\cdot)) = Z_l, \forall Z_l \in \mathbb{Z}_l.
\end{aligned} \tag{2.9}$$

This result may be written as our next theorem.

Theorem 2.1. *The class of decision situations that can be represented by the matrix scheme Z_m coincides with the class of situations that can be represented by the lottery scheme Z_l, i.e.,*

$$\mathbb{Z}_m = \mathbb{Z}_l. \tag{2.10}$$

Thus we can use either of the two schemes Z_l and Z_m to describe any decision situation, parametric or nonparametric.

The majority of real-life decision situations seem to be of nonparametric type. However, in many real parametric situations the capacity of the set Θ of possible values of the parameter θ is less than the capacity of the set C^U: some of the complex events $\theta(u)$ may be absent. This makes the parametric situation less complex. So in our Example 2.2, the set Θ (the set of all possible abilities of the graduate student) consists of three elements, whereas the set C^U consists of twenty-seven elements! In the degenerate case, in which it is known that the graduate is talented in only one type of activity (for example, engineering), the uncertainty of choice disappears; the complexity of the situation is minimal (zero?).

It is clear that when $m(\Theta) \leq m(C^U)$, one should describe the parametric decision situation in terms of the matrix scheme. In other words, compression of this scheme into the lottery scheme can lead only to an unjustified complication of the situation model, or to the loss of data about the simplicity of the parametric situation. Indeed, if the real set Θ of the parametric situation is preserved (is not forgotten) in the operation of compression, then the lottery scheme will consist of the matrix scheme completed by the useless multivalued mapping. If, on the other hand, after compression χ one does not preserve, or forgets, the data about the set Θ, then such a compression turns out to be a surjection of the set $\mathbb{Z}_m$ onto the set $\mathbb{Z}_l$: a single lottery scheme (the image) corresponds to more than one matrix scheme (the preimage).

2.3.2 Data about the Unknown

Looking back at the examples of decision situations from Section 2.2 and fitting them to the decision schemes Z_l and Z_m, one can see that a scheme by itself is insufficient to model a situation. In Examples 2.1 and 2.2, not only does the decision-maker know the schemes Z_l and Z_m, but he receives as well some additional data about peculiarities of the appearance of consequences of every decision. We call these data the *data about the unknown*.

When we say “unknown,” we mean that the decision-maker does not know in advance which consequence $c \in C_u$ will be the result of the chosen action $u \in U$. The notion of the unknown, with all its richness, can scarcely be formulated in precise mathematical language. In this book, we shall confine our treatment of the unknown to following, to a certain extent, R. Bellman [7]. Namely, we shall assume that the consequence $c \in C_u$ is induced by some cause–effect mechanism, or generator of consequences, which is switched on by the chosen action $u \in U$ in a way shown in Figure 2.3 for nonparametric situations and in Figure 2.4 for parametric situations. It is precisely this cause–effect mechanism that represents in our model the dependence, specific to each decision situation, of consequences on actions. Clearly, the range of these dependencies is very wide, from functional dependence to complete independence.

It is natural to suppose that the source of unknowns is the actual environment, or the reality of the physical—in the widest possible sense of this word—world,

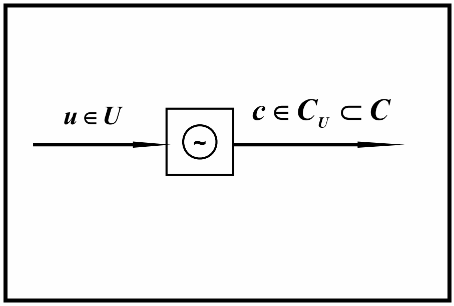

Fig. 2.3 A nonparametric situation.

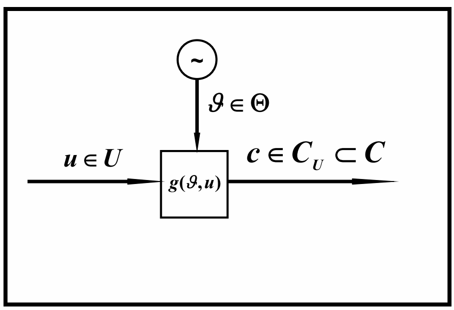

Fig. 2.4 A parametric situation.

in which the decision situation takes place. So, for example, the profit from operations on the stock exchange depends on the totality of financial markets; the outcome of a disease depends on the entire state of medicine; the length of a human life depends on social conditions, but of course not on that only.

It is natural to consider the unknown consequence $c \in C_u$ as random, and the cause–effect mechanism as the generator of such randomness. The specificity of a decision situation consists then in some regularity of the considered random phenomenon. Since these regularities are proposed by modern probability theory, it would seem that there should be no new difficulties. But this is not the case. Already Borel in [4] remarks that the world of random phenomena is much wider than that of its parts, which is described in terms of probability theory and remarks on the absence of scientific means of studying this world. In [47], Kolmogorov says in this respect the following: “Speaking of randomness in the usual sense of this word, we mean those phenomena in which we do not find regularities, allowing us to predict their behavior. Generally speaking, there are no reasons to assume that phenomena random in this sense are subjected to some probabilistic laws. Hence, it is necessary to distinguish the randomness in this *broad sense* and *stochastic* randomness (which is the subject of probability theory).”

However, what do the words “we do not find regularities, allowing us to predict their behavior” mean? We should scarcely understand them in the sense that such regularities do not exist, that in the range between the determinism of functional dependence and complete uncertainty there is only the regularity of stochastic randomness. More likely, they express the need to find the regularities of nonstochastic random phenomena. We shall return to this topic later.

So far, independent of the form—perhaps even a nonmathematical one—in which the regularity of the cause–effect mechanism is expressed, we shall denote it by the symbol I. Denote the lottery and the matrix models of a decision-making situation by $S_l = (Z_l, I_l)$ and $S_m = (Z_m, I_m)$ respectively.

We have already seen that the matrix and lottery schemes, Z_m and Z_l, are equivalent ways of describing decision situations: each of them can be used for the description of a parametric as well as a nonparametric situation. It seems that the models S_l and S_m are equivalent in this sense as well: making use of the operations of parameterization τ and compression χ, the regularity I_m could be rewritten in terms of I_l and vice versa. However, so far one can see this only for a concretely given regularity I.

In particular, let us demonstrate this equivalence of the models S_m and S_l for situations with finite sets of decisions U and consequences C in the case of stochastic regularity of the cause–effect mechanism.

For a certain situation, let its lottery model be $S_l = (Z_l, I_l)$, where

$$\begin{aligned} Z_l &= (U, C, \{C_k, k = \overline{1,d}\}), \\ U &= (u_1, u_2, \ldots, u_d), \\ C &= (c_1, c_2, \ldots, c_t) = \bigcup_{k=1}^{d} C_k, \end{aligned}$$

and the stochastic regularity I_l has the form of a family of probability distributions

$$I_l = Q = \{Q_k, k = \overline{1,d}\}, \quad Q_k = (q_k(c_1), q_k(c_2), \ldots, q_k(c_t)).$$

The matrix scheme Z_m, constructed on the basis of the lottery scheme Z_l according to the parameterization τ (2.6), has the form $Z_m = (U, C, \hat{\Theta}, g(\hat{\cdot}, \cdot))$, where the artificial parameter $\hat{\theta}$ takes values in the set $\hat{\Theta}$, $\mathrm{Card}(\hat{\Theta}) = n = t^d$, i.e.,

$$\hat{\Theta} = (\hat{\theta}_1, \hat{\theta}_2, \dots, \hat{\theta}_n), \quad \hat{\theta}_\mu : U \xrightarrow{\mu} C,$$
$$\mu = \overline{1,n}, \quad \hat{\theta}_\mu(u_k) = g(\hat{\theta}_\mu, u_k), \quad \forall k = \overline{1,d}.$$

Here $\hat{\theta}_\mu$ is a "complex" event, consisting of the "simple" events $c_{\nu_\mu k}$, $\nu = \overline{1,t}$ that constitute $\hat{\theta}_\mu$.

In order not to complicate our example, suppose that there is no dependency between the consequences of different actions, in other words, that consequences are overall independent. Then the regularity I_m in terms of the probability distribution over the set $\hat{\Theta}$ is given as

$$I_m = P(\hat{\Theta}) = \big(p(\hat{\theta}_1), p(\hat{\theta}_2), \dots, p(\hat{\theta}_n)\big), \tag{2.11}$$

where

$$p(\hat{\theta}_\mu) = \prod_{k=1}^{d} q_k(c_{\nu_\mu}), \quad \mu = \overline{1,n}. \tag{2.12}$$

The matrix model $S_m = (Z_m, I_m)$ obtained is equivalent to the original (primitive) lottery model $S_l = (Z_l, I_l)$. Indeed, we make the inverse transformation from S_m to S_l, the compression (2.8). To do this we need to reconstruct I_l from I_m in the form of the family

$$Q'_k = (q'_k(c_1), q'_k(c_2), \dots, q'_k(c_t)), \quad k = \overline{1,d}, \tag{2.13}$$

where

$$q'_k(c_\nu) = p\left\{\hat{\theta}_\mu, \mu = \overline{1,n} : g(\cdot, u_k) = c_\nu\right\}, \quad \nu = \overline{1,t}. \tag{2.14}$$

The model $S_l = (U, C, \left\{C_k, Q'_k, k = \overline{1,d}\right\})$ constructed in this way coincides with the original model S_l.[6]

To prove this equivalence in the case of arbitrary sets C and U is technically more complicated. Nevertheless, in what follows we shall everywhere, unless otherwise specified, write down the model of the decision-making situation omitting the indices, i.e., as $S = (Z, I)$.

[6] For situations with stochastic regularity in the case of an arbitrary set C and finite set U, the equivalence of models is proved in [42]. In Chapter 6 we show the equivalence of the matrix and lottery models for nonstochastic regularity.

2.4 Experiments in Decision Systems

The models of experiment in decision systems discussed in this section will be used only in the last three chapters of this book. Therefore, the impatient reader may skip this section and proceed to the following one, returning here only before reading those chapters.

Suppose the decision-maker is in the situation $S = (Z, I)$. His knowledge about the regularity of the cause–effect mechanism may be enriched by means of an observation, or an experiment, over the decision situation.[7] A description of the regularity before observation is called *a priori*. We have already denoted it by the symbol I. The regularity enriched by the observation is called *a posteriori*, and we denote it by the symbol I_h. Suppose that I and I_h are linked by the relation

$$I_h = f(I, w_h), \quad w_h \in W_h, \tag{2.15}$$

where f is some algorithm of transition from the a priori description of the regularity I to the a posteriori description I_h under the observation w_h from some set W_h.

To limit the definition of experiment and make it more precise, we return again to our examples in Section 2.2. Obviously there are two types of experiments conducted by the decision-maker. An experiment of the first type is conducted by the hero in Example 2.1: the raven informs the hero of the consequences that followed the decisions of the hero's predecessors. An experiment of the second type is conducted in Example 2.2 when the young man explores his inclinations toward different professional occupations.

Generalizing what we have said, we define the model of the experiment of the first type as the mapping

$$h_l : C \to W_{h_l}. \tag{2.16}$$

This corresponds to the structure of the decision system in Figure 2.5.

We define the model of an experiment of the second type as the mapping

$$h_m : \Theta \to W_{h_m} \tag{2.17}$$

corresponding to the structure of the decision system in Figure 2.6. These figures make it clear that an experiment of the first type may be provided both in parametric and in nonparametric situations. An experiment of the second type may be provided only in parametric situations and only if the parameter θ can be observed. Statistical studies consider primarily this type of experiment [14].

There is a profound difference between the two types of experiment. While observation w_m depends only on the parameter $\theta \in \Theta$, observation w_l is more complicated. According to (2.16), the observation w_l depends on the consequence $c \in C$, and hence depends on the decision $u \in U$ that resulted in this

[7] We shall use the words "observation" and "experiment" as synonyms.

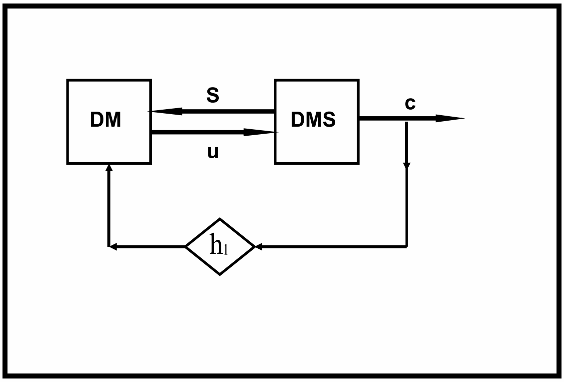

Fig. 2.5 The structure of a decision system with an experiment of the first type.

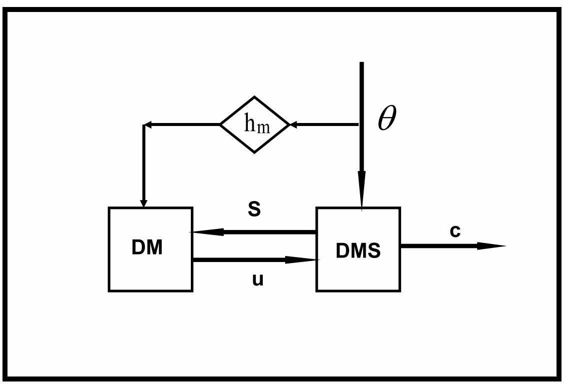

Fig. 2.6 The structure of a decision system with an experiment of the second type.

consequence. The decision-maker uses this observation in order to construct an a posteriori description of the regularity. Then this description is used for future decision-making. Thus a dependence arises between sequential decisions. Such dependencies produce some "information dynamics," particularly important in multistep decision systems, i.e., those in which the decision-maker makes sequential decisions, remaining in the same situation and accumulating, step by step, data about the regularity of the cause–effect mechanism.

Note finally that refusing to consider random functions of θ in our model of experiments in no way restricts the generality of this model. Indeed, let, for example, an experiment be written in the form of a random function of θ, i.e., as

$$h_m : \Theta \times \Omega \to W_m, \tag{2.18}$$

where Ω is the set of primitive events. This model can always be reduced to the model (2.17) by an obvious change of variables: it is enough to consider the direct product $\Theta \times \Omega$ as a new set Θ' of the unknown parameter.

2.5 The Decision-Maker

It follows from the examples of Section 2.2 that every decision-maker that happens to be in a decision situation has the goal of achieving what is in his opinion the best outcome or consequence. To fulfill this task, the decision-maker, equipped with the mathematical model of the situation $S = (Z, I)$, has to choose the action generating this outcome. We call this choice the *decision problem* or the *problem of choice*. To solve this problem, the decision-maker must perform, roughly speaking, the following sequence of steps:

1. Establish his personal preference relation[8] (or in other words, a criterion) on the set of all possible outcomes C. Denote this preference relation by β_C.
2. Determine the best, according to β_C, consequence $c^0 \in C$.
3. Establish some preference relation (criterion) on the set of all admissible actions U. Denote this preference relation by β_U.
4. Determine the best, according to β_U, action $u^0 \in U$.

These four steps constitute a *compound decision problem*, compound because each step in this sequence is a certain mathematical problem. Even the first step—establishing a personal preference relation β_C—is a mathematical problem: this relation must satisfy certain conditions if one wants to remain on stable ground.

The second and the fourth steps—the search for c^0 and the search for u^0—are optimization problems that, generally speaking, can be solved by appropriate mathematical methods. These two steps are not problems specific to decision-making only. The problem specific to decision-making is that of the third step, namely, the problem of construction of the preference relation β_U. This problem is the kernel of the compound decision problem, and therefore, in speaking of such problems we sometimes will mean only the problem of the third step.

[8] See Appendix A.1.

It is natural here to note that the data on the regularity I of the situation can only simplify the decision problem. Therefore, following our model of the decision situation, we say that if we know nothing of the situation but the scheme Z, then we limit the model S of the situation to its scheme, i.e., $S = Z$. Nevertheless, the situation cannot be considered separately from the decision system, or more accurately, independently from β_C. Is it not obvious that some situation can be simple for one decision-maker, i.e., for his preference relation β_C, and be complicated for another decision-maker that has another preference relation β_C? Suppose, for example, that the first is a colorblind person in a situation in which the consequences are colors.

But the model S (or just the scheme Z) of the situation and the preference relation β_C do not define the unique preference relation β_U. Indeed, it was already in Example 2.1, at the third crossroads, where we saw that two different heroes in the same situation, having the same preference β_C, later chose two different preferences β_U and correspondingly made two different decisions, i.e., chose two different roads. Each one justifies his choice in his own way, and each one is right—in his own way. But then we are tempted to classify these choices as arbitrary! To eliminate such arbitrariness it is necessary to transform the third step into a precise mathematical problem.

Before doing this, let us turn again to our examples. In particular, in Example 2.1, at the first crossroads, the hero of the story, having his preference β_C with respect to consequences, makes a decision without any speculation. In this decision situation, the choice of the road is uniquely determined by the hero's preference β_C, because a single consequence corresponds to only one road. We generalize this episode in the following way. Consider the situation in which the mapping $\psi(\cdot)$ of the decision set U in the set C of consequences is single-valued. Denote such a mapping by $\hat{\psi}$. The inverse mapping $\hat{\psi}^{-1} : C \to U$ is not necessarily single-valued, and the consequence $c \in C$ can be generated by any element of some set $U_{c\in C} \subset U$. Note that the sets U_{c_i} and U_{c_j} are disjoint for all $c_i, c_j, i \neq j$. Thus the mapping $\hat{\psi}^{-1}$ decomposes the decision set U into a system $\hat{U}$ of disjoint subsets $\{U_c, c \in C\} = \hat{U}$ in such a way that $\hat{\psi}^{-1}(c_i)\beta_C\hat{\psi}^{-1}(c_j)$ follows from $c_i\beta_C c_j$. Thus if the preference β_C is satisfied for the set of images C, then the same β_C is satisfied for the set of preimages $\hat{U}$. But we denoted the preference relation on the set U by the symbol β_U. Thus the mapping $\hat{\psi}^{-1}$ *transports*, or *projects*, the preference relation from the set C onto the set U in such a way that if $c_i\beta_C c_j$ and $u_{c_i} \in U_{c_i}, u_{c_j} \in U_{c_j}$, then $\hat{\psi}(u_{c_i})\beta_C\hat{\psi}(u_{c_j})$. Clearly, there exists a unique preference relation β_U that does not contradict the preference relation β_C, or to put it briefly, retains (or supports) β_C. We say that the third step of such a decision problem is *simple* or *degenerate* simply because there exists only one such preference relation β_U. Indeed, in this case one must solve only the optimization problem of the fourth step. In other words, the whole compound decision problem degenerates into the optimization problem.

So perhaps the decision problem does not become degenerate in the case that the mapping $\psi(\cdot)$ is multivalued, and we do not know what the result of our action will be. That is frequently the case, but not always. Here is an example.

Let $C_{u_1} = \{c_1, c_2\}$ and $C_{u_2} = \{c_3, c_4\}$, and according to β_C let $c_3\beta_C c_1$, $c_3\beta_C c_2$, $c_4\beta_C c_1$, $c_4\beta_C c_2$, i.e., all results of the action u_2 are preferred to all results of the action u_1. So we have here the multivalued function $\psi(\cdot)$. But there exists a unique β_U that does not contradict β_C. This β_U is produced by a specific combination of the inverse mapping $\psi^{-1}(\cdot)$ (which is a surjection) and the preference relation β_C.

But the crossroads in the second and the third episodes from Example 2.1 are different. Here, at every crossroads, it is not known which of the several consequences will occur. In terms of the lottery model, in this situation the mapping $\psi(\cdot)$ is many-valued, and it is such that the inverse mapping $\psi^{-1}(\cdot)$ does not exist. Therefore, the hero's preference β_C does not determine the unique preference β_U. We say that this is the effect of uncertainty inherent in the situation.

And nevertheless, our hero is making his decision according to his personal, i.e., arbitrary, preference relation β_U.

We now resume our attempt to model the third step of the compound decision problem. Denote by $\mathbb{S}$ the set of decision situations. For any decision situation $S \in \mathbb{S}$ let $\mathbb{B}_C$ denote the set of all preference relations β_C on C, B_C the subset of $\mathbb{B}_C$ available to the given decision-maker, $\mathbb{B}_U$ the set of all preference relations $\mathbb{B}_U$ on U, and B_U the subset of $\mathbb{B}_U$ available for the decision-maker.

Definition 2.1. We call the mapping

$$\pi : \mathbb{S} \times B_C \to B_U \tag{2.19}$$

a *projector* or *criterion choice rule*.

Denote by Π the set of all possible projectors or criterion choice rules π. Then it follows from the above that any decision-maker in a decision system reacts in his own way to the information about what is unknown, executing in his own way the operation of projection. In other words, any decision-maker in the decision system is a certain projector π, or equivalently, has his own criterion choice rule. This allows us to represent a decision-maker as a triple

$$\Phi = (B_C, B_U, \pi), \quad B_C \subseteq \mathbb{B}_C, \quad B_U \subseteq \mathbb{B}_U, \quad \text{with } \pi \subseteq \Pi. \tag{2.20}$$

This model allows us to describe the aforementioned arbitrariness in the following way: in the same situation S, with the same preference relations β_C, two different decision-makers π_1 and π_2 can establish two different preference relations β_U^1 and β_U^2 that can define two different best actions. A natural question is then, which one of them has chosen the best action? And what does it mean, "the best," if each decision-maker, π_1 or π_2, has his own opinion, his own fourth step “optimization” problem according to which each of them determines his own “best”?

The only exit from this arbitrariness is, most likely, to find and to put in the same group all decision-makers that have the same criterion choice rule. This will eliminate the arbitrariness inside the chosen group, or class: in the same decision situation S, all representatives of this class, having the same preference relation β_C, will establish the same preference relation β_U, and finally will make the same decision. However, is there such a criterion choice rule? And if it exists, then what kind of criterion β_U does it generate?

We are not ready to answer these questions. The problem of choice arises, as we saw, in decision systems with uncertainty, but we still do not have a precise notion of what it means for uncertainty to exist in a decision system.

2.6 Existence of Uncertainty in Decision Systems

We have already mentioned a few times the word “uncertainty,” but we have thus far never tried to make it precise. The sense of this word in everyday speech does not need any refinement. However, in everyday speech the word “probably” also does not need to be refined. Nevertheless, the quantitative characterization of the probable event demands a precise mathematical method, namely probability theory. Similarly, the notion of uncertainty demands at least a more precise definition. To begin with, note that in decision theory one has to be able to divide the whole class of decision problems into two subclasses: decision problems with and without uncertainty. For such a classification we need a criterion of existence of uncertainty in a decision system. Before we introduce such a criterion, we shall make some remarks.

There are two parts of a decision system that can be sources of uncertainty: the decision-maker and the decision situation. We suppose that the decision-maker can reveal uncertainty only in the choice of the preference relation β_C. So let us restrict decision systems to those for which any decision-maker can have a single personal preference relation β_C for the given situation. In this case, when β_C is fixed, the decision situation becomes the only source of uncertainty in the decision system, and thus in the corresponding decision problem.

Note that the data I can only decrease the uncertainty of the situation. Under these assumptions, uncertainty in the decision system comes only from the scheme Z of the situation.

To simplify our reasoning we limit the set $\mathbb{B}_C$ to linear preferences (linear ordering) and define

$$\beta_C = (C, \succeq).$$

Let

$$Z = Z_l = (C, U, \psi(\cdot)).$$

Set

$$\mathfrak{K} = (2^C)^U,$$

that is, the set of all mappings $\psi : U \to 2^C$.

Denote the class of mappings that bring uncertainty into the decision system by $K \subseteq \mathfrak{K}$.

Definition 2.2. We say that decision u_1 *dominates* decision u_2 relative to $\beta_C = (C, \succeq)$ if $C_{u_1} \succ C_{u_2}$, i.e., $u_1 \succ u_2$ if $c_1 \succeq c_2$ $\forall c_1 \in C_{u_1}$ and $\forall c_2 \in C_{u_2}$, we have $\mathrm{Card}(C_{u_1} \bigcap C_{u_2}) \leq 1, C_{u_1} \neq C_{u_2}$.

Definition 2.3. The procedure of constructing the preference relation $\beta_U = (U, \succeq)$ in accordance with the following conditions is called *projecting*:

Condition 1

$$C_{u_1} \succ C_{u_2} \Rightarrow u_1 \succ u_2, \quad \forall u_1, u_2 \in U;$$

Condition 2

$$(C_{u_1} = C_{u_2}, \mathrm{Card}(C_{u_1}) = 1) \Rightarrow u_1 \sim u_2, \quad \forall u_1, u_2 \in U.$$

Definition 2.4. A decision system contains uncertainty if the projecting procedure is not single-valued.

Obviously the multivaluedness of the mapping $\psi(\cdot)$ is a necessary condition for the existence of uncertainty, i.e., $K \neq (2^C)^U$. But it is clear that this condition is not sufficient, i.e., $K \neq (2^C)^U \setminus C^U$.

Intuitively, one can easily propose some version of a sufficient condition. Suppose for simplicity that the sets U and C are finite, and the mapping ψ is single-valued everywhere but on one action $u^* \in U$. At this point, let the set of consequences be given by $C_{u^*} = \{c_1, c_2\}$, $c_1 \prec c_2$. Suppose there exists the action u^{**} with the consequence $C_{u^{**}} = \{c_3\}$ such that $c_1 \prec c_3 \prec c_2$. Then there exist two variants of the preference relation β_U: the first $\beta_U^{(1)}$ with $u^* \succ u^{**}$, and the second $\beta_U^{(2)}$ with $u^* \prec u^{**}$. So a sufficient condition can be formulated as follows: uncertainty exists if there is a situation $S = Z$ and if there is a pair of decisions u^*, u^{**} with consequences $C_{u^*} = \{c_1, c_2\}$ and $C_{u^{**}} = \{c_3\}$ such that $c_1 \prec c_3 \prec c_2$.

But it turns out that one can prove a much stronger statement. In order to define the class K we need to prove our next lemma.

Lemma 2.1. *We have $\psi(\cdot) \in K$ if there exist a preference relation $(C, \succeq)$ and distinct $u_1, u_2 \in U$ and consequences $c_1, c_2 \in \psi(u_1), c_3, c_4 \in \psi(u_2)$ such that $c_1 \prec c_3$, $c_2 \succ c_4$.*

Proof. Necessity. Let some relation $(C, \succeq)$ generate different projections on the set U. Assume the converse. This means that for all $u_1, u_2 \in U, u_1 \neq u_2$, precisely one of the following conditions holds:

$$\begin{aligned}
&\psi(u_1) \prec \psi(u_2), \quad \mathrm{Card}(\psi(u_1) \cap \psi(u_2)) \leq 1,\\
&\psi(u_1) \succ \psi(u_2), \quad \mathrm{Card}(\psi(u_1) \cap \psi(u_2)) \leq 1,\\
&\psi(u_1) = \psi(u_2), \quad \mathrm{Card}(\psi(u_1)) = 1.
\end{aligned}$$

Then for the chosen $(C, \succ)$, according to Conditions 1 and 2, there exists a unique $(U, \succ)$, namely $u_1 \succ u_2 \Longleftrightarrow \psi(u_1) > \psi(u_2)$, for all $u_1, u_2 \in U$. *This contradicts the assumption that the relation $(C, \succeq)$ generates different projections on the set U.* In other words, this contradicts the ambiguity of the choice of the preference relations on the set of decisions U.

Sufficiency. Let the preference relation $(C, \succ)$ satisfy the conditions of Lemma 2.1. Then there exist $u', u'' \in U$, $u' \neq u''$, $c_1, c_2 \in \psi(u')$, $c_3, c_4 \in \psi(u'')$ such that $c_1 \prec c_3$, $c_2 \succ c_4$. Let $(U, \succeq')$ be some preference relation on U (for example it may be a lexicographic relation with respect to $(C, \succeq)$). Then one can take another preference relation $(U, \succeq'')$ such that

$$u_1 \succeq'' u_2 \Longleftrightarrow \begin{cases} u_1 \succeq' u_2, \\ u_1 \preceq' u_2,\ u_1 \preceq u_2. \end{cases}$$

The lemma is proved.

Next we need the following lemma.

Lemma 2.2. *The necessary and sufficient conditions are equivalent to the following conditions: there must exist distinct $u_1, u_2 \in U$ and consequences $c_1, c_2 \in \psi(u_1)$, $c_3, c_4 \in \psi(u_2)$ such that either $c_1 \prec c_3 \prec c_2$ or $c_1 = c_3$, $c_2 = c_4$, $c_1 \neq c_2$.*

Proof. Necessity. Suppose the conditions of Lemma 2.1 are satisfied for u_1, u_2, c_1, c_2, c_3, c_4. Then without loss of generality, one may assume that $c_1 \preceq c_2$. If $c_1 = c_2$, then setting $c_1' = c_4$, $c_3' = c_4' = c_1$, $c_2' = c_3$, we obtain for c_1', c_2', c_3', c_4' the first condition of Lemma 2.2. If $c_1 \prec c_2$, then when $c_3 \succ c_2$, for $c_1' = c_4$, $c_2' = c_3$, $c_3' = c_2$, $c_4' = c_1$ we obtain again the first condition of Lemma 2.2. The same holds if $c_3 \prec c_2$. In the case $c_3 = c_2$, if $c_1 = c_4$ we have at $c_1' = c_1$, $c_2' = c_2$, $c_3' = c_3$, $c_4' = c_4$ the second condition of Lemma 2.2. And finally, if $c_1 \neq c_4$, for example at $c_4 \succ c_1$, then at $c_1' = c_1$, $c_2' = c_2$, $c_3' = c_4$, $c_4' = c_3$ we have the second condition of Lemma 2.2

Sufficiency. Let the conditions of Lemma 2.2 be used for $u_1, u_2, c_1, c_2, c_3, c_4$. Then if $c_1 \prec c_3 \prec c_4$, the condition of Lemma 2.1 holds for $c_1' = c_1$, $c_2' = c_2$, $c_3' = c_4'$, $c_4' = c_3$, and if $c_1 = c_3 \succ c_2 = c_4$, then the condition of Lemma 2.1 holds for $c_1' = c_2$, $c_2' = c_1$, $c_3' = c_3$, $c_4' = c_4$. The lemma is proved.

From these two lemmas we obtain the following theorem.

Theorem 2.2. *A decision scheme Z_l contains uncertainty if there is $\psi(\cdot)$ such that one can determine distinct $u_1, u_2 \in \mathrm{Dom}(\psi)$ and c_1, c_2 such that either $\psi(u_1) = \psi(u_2) = \{c_1, c_2\}$ or there is also c_3 different from c_1 and c_2 such that $c_1, c_2 \in \psi(u_1)$ and $c_3 \in \psi(u_2)$.*

Now let $Z = Z_m = (\Theta, U, C, g(\cdot,\cdot))$, and this Z_m is equivalent to Z_l in the sense of Theorem 2.1. Then Theorem 2.2 can be reformulated for Z_m as follows.

Theorem 2.3. *A decision scheme Z_m contains uncertainty if and only if there is $g(\cdot,\cdot)$ such that there exist distinct $u_1, u_2 \in U$ for which either $g(\Theta, u_1) = g(\Theta, u_2)$ and $\mathrm{Card}(g(\Theta, u_1)) = 2$ or there are $\theta_1, \theta_2 \in \Theta$ such that $g(\theta_1, u_1) \neq g(\theta_2, u_1) \neq g(\theta_1, u_2) = g(\theta_2, u_2)$.*

In other words, these conditions define the class K of such schemes Z for which there exists $\beta_C \in \mathfrak{B}_C$ such that one can project the pair (β_C, Z) into more then one preference relation β_U, i.e., more than one criterion of choice of the best action.

2.7 Criterion Choice Rule

Now we can return to the central, third, step of the compound decision problem from Section 2.5, namely to the construction of the preference relation β_U, or in other words, of the criterion of choice of the best decision. We already know that this problem becomes nontrivial if the decision situation contains uncertainty. We have already seen that in any decision system uncertainty exists if for a given preference relation β_C there is a scheme Z of the decision situation such that there exists more than one preference relation β_U, i.e., there is more than one criterion of choice of optimal decision. In other words, in this case the mapping (2.19) is multivalued, and the decision-maker can choose one of several criteria following his personal tastes. As is known [58], the complement of the decision scheme Z by the regularity I of the cause–effect mechanism does not, in general, remove uncertainty and the resulting arbitrariness from the decision system. To remove this arbitrariness one has to subordinate the projector π to some conditions. In some sense, these conditions, or axioms, may be considered as an axiomatic description, or a model, of the decision-maker. On the other hand, these conditions define a class of projectors, or decision-makers, that in the same decision situation S, having the same preference relation β_C, project the pair (S, β_C) in their common preference relation β_U. Thus at the fourth step of the compound decision problem, they will choose the same optimal decision.

To proceed further with this task it is necessary to make the set B_C and the set B_U common to all decision-makers that we are going to put in the same class. To this end, we may assume, for example, that $B_C = \mathbb{B}_C$ and $B_U = \mathbb{B}_U$, where $\mathbb{B}_C$ and $\mathbb{B}_U$ are all possible preference relations on the sets C and U respectively.

As to the conditions imposed on the criterion choice rule, or on the projector π, it is not clear now whether such conditions may be found in our quite general formulation of the decision problem, i.e., in terms of preference relations. However, for some constraints on the class of decision situations, significant results have been obtained in this direction. These results constitute today's armory of decision theory.

The first significant constraint is that instead of the preference relations B, real-valued functions are used as the criterion for the ordering of the sets C and U. In economics these functions are called utility functions, and in engineering they are called loss functions. This restriction was sufficient to suggest an axiomatic description of four classes of projectors that brought forth four criteria (by Wald, Savage, Hurwicz, and Laplace [20, 58]) uniquely for the matrix scheme Z_m, i.e., for the matrix situation model S_m with strict uncertainty.

However, the central result, the foundation of modern decision theory and its applications, particularly in economics, is the expected utility theorem [69, 61]. This theorem has played a principal role in microeconomics since the 1940s. Therefore, we consider it here with some comments but without proof (see the proof, for example, in [21, 61, 82].

Suppose that in a decision system there is a nonparametric stochastic situation $S_l = (Z_l, I_l)$, where $Z_l = (C, U, \{C_u \quad \forall u \in U\})$ and the regularity of the cause–effect mechanism I_l is the family of stochastic distributions $\{Q_u, \forall u \in U\}$ on the set C. Let the decision-maker choose some complete, reflexive, and transitive binary preference relation $\beta_C = (C, \succeq) \in B_C$ that orders the set of consequences C.

Definition 2.5. A real-valued function $\mathfrak{U}$ defined on the ordered set C is a *utility function* if it is monotonic, i.e., if for all pairs (c_i, c_j),

$$c_i \succeq c_j \Longleftrightarrow \mathfrak{U}(c_i) \geq \mathfrak{U}(c_j). \tag{2.21}$$

In order not to complicate this discussion we limit the presentation of this result to decision situation models with finite sets of decisions U and consequences C. Then, for example, the indices i of the ordered consequences $c_1 \succeq c_2 \succeq \cdots \succeq c_t$, $\mathfrak{U}(c_i) = i, i = 1, 2, \ldots, t$, can serve as a utility function. Thus instead of the pair (S, β_C) we have the pair $(S, \mathfrak{U}(\cdot))$. But we need to construct a criterion of choice of optimal decision, that is, a utility function of decisions $\hat{\mathfrak{U}}(\cdot)$. In order to do this, first note that the set of probability distributions $Q = \{Q_u, u \in U\}$ is homeomorphic to the set of decisions U, i.e., $Q_{u_i} \succ Q_{u_j}$ if and only if $u_i \succ u_j$.[9] Therefore, the set of probability distributions Q, or the set of lotteries, can substitute the set U. Suppose that the decision-maker would like her utility function on decisions to be linear on the set Q (i.e., on the set U). To satisfy this demand, the preference relation on Q should satisfy the following conditions:

Condition 3 *For any $Q_1, Q_2, Q_3 \in Q$ the sets*

$$\{\alpha : \alpha Q_1 + (1-\alpha)Q_2 \succeq Q_3\} \quad \textit{and} \quad \{\alpha : Q_3 \succeq \alpha Q_1 + (1-\alpha)Q_2\}$$

for $\alpha \in [0,1]$ are closed.

Condition 4 *For any $Q_1, Q_2, Q_3 \in Q$, $Q_1 \sim Q_2$, and for any $\alpha, 0 \leq \alpha \leq 1$,*

$$\alpha Q_1 + (1-\alpha)Q_3 \simeq \alpha Q_2 + (1-\alpha)Q_3.$$

[9] In economics, the set Q is called the set of lotteries.

The first condition is the condition of continuity of the preference relation on Q. The second is the condition of independence of the preference relation on Q. Both of these conditions, conveying the wish of the decision-maker to have a linear decision utility function, are nevertheless purely mathematical, and could be common to many decision-makers. But what conditions can be characterized as specific to a certain group of decision-makers? Intuitively, it is clear that the preference relation on the set of lotteries Q should be consistent with the preference relation on the set of consequences C. One such consistency condition can be written, roughly speaking, in the following form:

Condition 5 *If $c_1 \succeq c_2$, then*

$$Q = \left(\binom{c_1}{q_1}, \binom{c_2}{q_2} \right) \succeq Q' = \left(\binom{c_1}{q'_1}, \binom{c_2}{q'_2} \right) \tag{2.22}$$

if $q_1 > q'_1$.

That is, the decision-maker prefers the lottery in which the best consequence has the highest probability.

We now have the following theorem.

Theorem 2.4. *In order that a linear utility function $\hat{\mathfrak{U}}(u)$ exist in the form*

$$\hat{\mathfrak{U}}(u) = \sum_{c \in C} \mathfrak{U}(c) q_u(c), \tag{2.23}$$

it is necessary and sufficient that on the set U, Conditions 3–5 be satisfied. The utility function (2.23) *is unique to within an increasing linear transformation.*

The utility (2.23) is called the *expected utility*. One can reformulate Theorem 2.4 in terms of a matrix situation. The expected utility function in this case is

$$\hat{\mathfrak{U}}(u) = \sum_{\theta \in \Theta} \mathfrak{U}(g(\theta, u)) p(\theta), \tag{2.24}$$

where $p(\theta)$ and $g(\theta, u)$ are found according to formulas (2.12) and (2.5) respectively.

In other words, in our framework, Conditions (or axioms) 3–5 describe a specific criterion choice rule or the class Π_0 of projectors (or decision-makers) π. All decision-makers from this class would in the identical situation make decisions assuming that

$$\sum_{c \in C} \mathfrak{U}(c) q_{u_1}(c) \geq \sum_{c \in C} \mathfrak{U}(c) q_{u_2}(c) \Leftrightarrow u_1 \succeq u_2. \tag{2.25}$$

Note also that all conditions (axioms) of Theorem 2.4 presuppose that the regularity of the cause–effect mechanism of the decision situation is a stochastic regularity in the form of a family of probability distributions $I = \{Q_u, u \in U\}$. Thus the preference relation (2.25) makes sense only for multiple, or mass, decisions under stochastic regularity.

Recall, though, that (2.25) takes place only over a very large number (infinite) of decisions in the same situation. When a single decision is made, the consequence is random with the probability distribution that corresponds to the chosen decision. This means that the consequence of a single choice of decision u_1 can be worse than the consequence of a single choice of decision u_2. The same is true for all criteria belonging to the so-called nonexpected utility criteria group (see [59] and the literature therein), where the probability distribution on consequences is used in one way or another.

It seems that the only decision-maker who can avoid this is the decision-maker who makes decisions according to the criterion of Wald that sometimes is called *the principle of guaranteed result*.[10] In this case, the best decision has the form

$$u^0 = \arg\max_u \min_\theta L. \tag{2.26}$$

In other words, the decision-maker does not use any information about the regularity of the cause–effect mechanism. However, when in this situation decisions are made multiply (or massively), i.e., by many decision-makers but only once, or by one decision-maker but many times, this behavior, or attitude, seems less reasonable.[11]

The mass character of decisions in the axiomatics of the expected utility is presupposed by the regularity of the cause–effect mechanism in the form of a probability distribution. Namely, the third axiom (Condition 5) reflects this psychological element, demanding a decision-maker to prefer that decision where the most "useful" consequence has the greatest probability. In [68, p. 4] the following is said on this occasion: "Once you have introduced probabilities into the definition and measurement of utilities, you have made a bargain with the devil, and you can't get rid of them again."

We can ask whether we might be able to conjure up such an axiomatics of a criterion choice rule whereby the mass character of decisions does not rely on the regularity of the cause–effect mechanism. Perhaps in this way we can obtain a criterion that is good for decision-making in situations with regularities of mass phenomena different from stochastic ones. But are there such regularities, and what do we know about them?

To answer this question let us try—at first intuitively—to understand what phenomena it is natural to consider as random. The foremost difference between random and nonrandom phenomena consists in the fact that when we call some

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[10] The axiomatics of this principle can be found, for example, in [20, 82].

[11] The author knew a professor who had the privilege of flying free of charge but never did so, fearing an accident.

phenomenon random, it always means that we do not know the regularities (call them local) that would allow us to predict the behavior of the phenomenon. The study of a random phenomenon can be twofold: one can reduce it to the nonrandom, looking for its local regularities, and one can, if it is a mass random phenomenon, try to find its global, statistical regularities, that is, regularities of the asymptotic behavior of the average values of different characteristics of the phenomenon. For example, this could be frequencies of certain consequences, arithmetic averages of certain functionals, etc.

If with the increase of the number of decisions and, correspondingly, of the number of appearances of their consequences, all these averages tend to certain limits (and some other similar conditions are satisfied as well; see details in [47]), then this phenomenon is called *statistically stable* or *stochastic*. And as is well known, the study of such phenomena is the subject matter of probability theory. At the same time, we can call all such phenomena where the aforementioned sample averages do not tend to unique limits *statistically unstable* random phenomena. It is reasonable to call these random phenomena *nonstochastic*. This takes place, for example, for the increasing average lifetime of a human being: for a newborn child, the probability of reaching the age of 60 has a tendency to increase, due to successes of medicine and hygiene.[12] Today such examples may be found in excess in economics and finance: time series of equity price indexes, interest rates, commodities, foreign exchange rates, etc.

It is natural to consider as *random in a broad sense* all mass phenomena that are studied only to within their statistical regularities. Once again, if a decision-maker happens to be in a situation with nonstochastic randomness, she cannot make use of the optimality criterion from Theorem 2.4. There are two reasons for this. First, the axioms of this theorem, i.e., the definition of the class Π_0 of projectors, substantially use the regularity of stochastic randomness. Second, she does not know whether there exists any regularity of a nonstochastic random phenomenon.

Therefore two new problems appear. First, it is necessary to establish the existence of regularity of phenomena that are random in a broad sense. Second, it is necessary to find an axiomatic description of the class Π_1 of decision-makers that would take into account only the mass character of a random phenomenon but not its regularity.

These two problems make up the core of the remainder of this book.

[12] This example is due to Emile Borel [4].

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