CHAPTER II

A DUALITY IN INTEGRAL GEOMETRY

§1 Homogeneous Spaces in Duality

The inversion formulas in Theorems 3.1, 3.7, 3.8 and 6.2, Ch. I suggest the general problem of determining a function on a manifold by means of its integrals over certain submanifolds. This is essentially the title of Radon's paper. In order to provide a natural framework for such problems we consider the Radon transform $f \to \widehat{f}$ on $\mathbf{R}^n$ and its dual $\varphi \to \check{\varphi}$ from a group-theoretic point of view, motivated by the fact that the isometry group $\mathbf{M}(n)$ acts transitively both on $\mathbf{R}^n$ and on the hyperplane space $\mathbf{P}^n$. Thus

$$\mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,, \quad \mathbf{P}^n = \mathbf{M}(n)/\mathbf{Z}_2 \times \mathbf{M}(n-1)\,, \tag{1}$$

where $\mathbf{O}(n)$ is the orthogonal group fixing the origin $0 \in \mathbf{R}^n$ and $\mathbf{Z}_2 \times \mathbf{M}(n-1)$ is the subgroup of $\mathbf{M}(n)$ leaving a certain hyperplane ξ_0 through 0 stable. ($\mathbf{Z}_2$ consists of the identity and the reflection in this hyperplane.)

We observe now that a point $g_1\mathbf{O}(n)$ in the first coset space above lies on a plane $g_2(\mathbf{Z}_2 \times \mathbf{M}(n-1))$ in the second if and only if these cosets, considered as subsets of $\mathbf{M}(n)$, have a point in common. In fact

$$\begin{aligned} g_1 \cdot 0 \subset g_2 \cdot \xi_0 \quad &\Leftrightarrow \quad g_1 \cdot 0 = g_2 h \cdot 0 \text{ for some } \; h \in \mathbf{Z}_2 \times \mathbf{M}(n-1) \\ &\Leftrightarrow \quad g_1 k = g_2 h \text{ for some } k \in \mathbf{O}(n)\,. \end{aligned}$$

This leads to the following general setup.

Let G be a locally compact group, X and Ξ two left coset spaces of G,

$$X = G/K\,, \quad \Xi = G/H\,, \tag{2}$$

where K and H are closed subgroups of G. Let $L = K \cap H$. We assume that the subset $KH \subset G$ is **closed**. This is automatic if one of the groups K or H is compact.

Two elements $x \in X$, $\xi \in \Xi$ are said to be *incident* if as cosets in G they intersect. We put (see Fig. II.1)

$$\begin{aligned} \check{x} \quad &= \quad \{\xi \in \Xi \,:\, x \text{ and } \xi \text{ incident}\} \\ \widehat{\xi} \quad &= \quad \{x \in X \,:\, x \text{ and } \xi \text{ incident}\}\,. \end{aligned}$$

Let $x_0 = \{K\}$ and $\xi_0 = \{H\}$ denote the origins in X and Ξ, respectively. If $\Pi : G \to G/H$ denotes the natural mapping then since $\check{x}_0 = K \cdot \xi_0$ we have

$$\Pi^{-1}(\Xi - \check{x}_0) = \{g \in G : gH \notin KH\} = G - KH\,.$$

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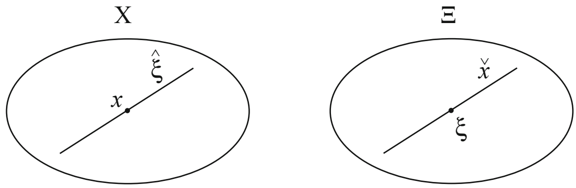

$$x \in \hat{\xi} \Leftrightarrow \xi \in \check{x}$$

FIGURE II.1.

In particular $\Pi(G - KH) = \Xi - \check{x}_0$ so since Π is an open mapping, $\check{x}_0$ is a closed subset of Ξ. This proves the following:

Lemma 1.1. *Each $\check{x}$ and each $\widehat{\xi}$ is closed.*

Using the notation $A^g = gAg^{-1}$ $(g \in G, A \subset G)$ we have the following lemma.

Lemma 1.2. *Let $g, \gamma \in G$, $x = gK$, $\xi = \gamma H$. Then*

$$\check{x} \textit{ is an orbit of } K^g, \ \widehat{\xi} \textit{ is an orbit of } H^\gamma,$$

and

$$\check{x} = K^g/L^g, \quad \widehat{\xi} = H^\gamma/L^\gamma.$$

Proof. By definition

$$\check{x} = \{\delta H : \delta H \cap gK \neq \emptyset\} = \{gkH : k \in K\}, \tag{3}$$

which is the orbit of the point gH under gKg^{-1}. The subgroup fixing gH is $gKg^{-1} \cap gHg^{-1} = L^g$. Also (3) implies

$$\check{x} = g \cdot \check{x}_0, \qquad \widehat{\xi} = \gamma \cdot \widehat{\xi}_0,$$

where the dot $\cdot$ denotes the action of G on X and Ξ.

We often write $\tau(g)$ for the maps $x \to g \cdot x$, $\xi \to g \cdot \xi$ and

$$f^{\tau(g)}(x) = f(g^{-1} \cdot x), \quad S^{\tau(g)}(f) = S(f^{\tau(g^{-1})})$$

for f a function, S a distribution.

Lemma 1.3. *Consider the subgroups*

$$\begin{aligned} K_H &= \{k \in K : kH \cup k^{-1}H \subset HK\}, \\ H_K &= \{h \in H : hK \cup h^{-1}K \subset KH\}. \end{aligned}$$

The following properties are equivalent:

(a) $K \cap H = K_H = H_K$.

(b) *The maps* $x \to \check{x}\,(x \in X)$ *and* $\xi \to \widehat{\xi}\,(\xi \in \Xi)$ *are injective.*

We think of property (a) as a kind of **transversality** of K and H.

Proof. Suppose $x_1 = g_1K$, $x_2 = g_2K$ and $\check{x}_1 = \check{x}_2$. Then by (3) $g_1 \cdot \check{x}_0 = g_1 \cdot \check{x}_0$ so $g \cdot \check{x}_0 = \check{x}_0$ if $g = g_1^{-1}g_2$. In particular $g \cdot \xi_0 \subset \check{x}_0$ so $g \cdot \xi_0 = k \cdot \xi_0$ for some $k \in K$. Hence $k^{-1}g = h \in H$ so $h \cdot \check{x}_0 = \check{x}_0$, that is $hK \cdot \xi_0 = K \cdot \xi_0$. As a relation in G, this means $hKH = KH$. In particular $hK \subset KH$. Since $h \cdot \check{x}_0 = \check{x}_0$ implies $h^{-1} \cdot \check{x}_0 = \check{x}_0$ we have also $h^{-1}K \subset KH$ so by (a) $h \in K$ which gives $x_1 = x_2$.

On the other hand, suppose the map $x \to \check{x}$ is injective and suppose $h \in H$ satisfies $h^{-1}K \cup hK \subset KH$. Then

$$hK \cdot \xi_0 \subset K \cdot \xi_0 \text{ and } h^{-1}K \cdot \xi_0 \subset K \cdot \xi_0 \,.$$

By Lemma 1.2, $h \cdot \check{x}_0 \subset \check{x}_0$ and $h^{-1} \cdot \check{x}_0 \subset \check{x}_0$. Thus $h \cdot \check{x}_0 = \check{x}_0$ whence by the assumption, $h \cdot x_0 = x_0$ so $h \in K$.

Thus we see that under the transversality assumption a) the elements ξ can be viewed as the subsets $\widehat{\xi}$ of X and the elements x as the subsets $\check{x}$ of Ξ. We say X and Ξ are **homogeneous spaces in duality**.

The maps are also conveniently described by means of the following **double fibration**

(4)
$$\begin{tikzcd} & G/L \arrow[dl, "p"'] \arrow[dr, "\pi"] & \\ G/K & & G/H \end{tikzcd}$$

where $p(gL) = gK$, $\pi(\gamma L) = \gamma H$. In fact, by (3) we have

$$\check{x} = \pi(p^{-1}(x)) \qquad \widehat{\xi} = p(\pi^{-1}(\xi))\,.$$

We now prove some group-theoretic properties of the incidence, supplementing Lemma 1.3.

Theorem 1.4. *(i)* *We have the identification*

$$G/L = \{(x, \xi) \in X \times \Xi \,:\, x \text{ and } \xi \text{ incident}\}$$

via the bijection $\tau : gL \to (gK, gH)$.

(ii) *The property*

$$KHK = G$$

is equivalent to the property:

Any two $x_1, x_2 \in X$ *are incident to some* $\xi \in \Xi$. *A similar statement holds for* $HKH = G$.

(iii) The property

$$HK \cap KH = K \cup H$$

is equivalent to the property:

For any two $x_1 \neq x_2$ in X there is at most one $\xi \in \Xi$ incident to both. By symmetry, this is equivalent to the property:

For any $\xi_1 \neq \xi_2$ in Ξ there is at most one $x \in X$ incident to both.

Proof. (i) The map is well-defined and injective. The surjectivity is clear because if $gK \cap \gamma H \neq \emptyset$ then $gk = \gamma h$ and $\tau(gkL) = (gK, \gamma H)$.

(ii) We can take $x_2 = x_0$. Writing $x_1 = gK$, $\xi = \gamma H$ we have

$$\begin{array}{rcll} x_0, \xi \text{ incident } \Leftrightarrow \gamma h & = & k & (\text{some } h \in H, k \in K) \\ x_1, \xi \text{ incident } \Leftrightarrow \gamma h_1 & = & g_1 k_1 & (\text{some } h_1 \in H, k_1 \in K)\,. \end{array}$$

Thus if x_0, x_1 are incident to ξ we have $g_1 = kh^{-1}h_1k_1^{-1}$. Conversely if $g_1 = k'h'k''$ we put $\gamma = k'h'$ and then x_0, x_1 are incident to $\xi = \gamma H$.

(iii) Suppose first $KH \cap HK = K \cup H$. Let $x_1 \neq x_2$ in X. Suppose $\xi_1 \neq \xi_2$ in Ξ are both incident to x_1 and x_2. Let $x_i = g_iK$, $\xi_j = \gamma_jH$. Since x_i is incident to ξ_j there exist $k_{ij} \in K$, $h_{ij} \in H$ such that

$$g_ik_{ij} = \gamma_jh_{ij} \qquad i = 1,2\,;\; j = 1,2\,. \tag{5}$$

Eliminating g_i and γ_j we obtain

$$k_{2\,2}^{-1}k_{2\,1}h_{2\,1}^{-1}h_{1\,1} = h_{2\,2}^{-1}h_{1\,2}k_{1\,2}^{-1}k_{1\,1}\,. \tag{6}$$

This being in $KH \cap HK$ it lies in $K \cup H$. If the left hand side is in K then $h_{2\,1}^{-1}h_{1\,1} \in K$, so

$$g_2K = \gamma_1h_{2\,1}K = \gamma_1h_{1\,1}K = g_1K\,,$$

contradicting $x_2 \neq x_1$. Similarly if expression (6) is in H, then $k_{1\,2}^{-1}k_{1\,1} \in H$, so by (5) we get the contradiction

$$\gamma_2H = g_1k_{1\,2}H = g_1k_{1\,1}H = \gamma_1H\,.$$

Conversely, suppose $KH \cap HK \neq K \cup H$. Then there exist h_1, h_2, k_1, k_2 such that $h_1k_1 = k_2h_2$ and $h_1k_1 \notin K \cup H$. Put $x_1 = h_1K$, $\xi_2 = k_2H$. Then $x_1 \neq x_0$, $\xi_0 \neq \xi_2$, yet both ξ_0 and ξ_2 are incident to both x_0 and x_1.

Examples

(i) **Points outside hyperplanes.** We saw before that if in the coset space representation (1) $\mathbf{O}(n)$ is viewed as the isotropy group of 0 and $\mathbf{Z}_2\mathbf{M}(n-1)$ is viewed as the isotropy group of a hyperplane *through* 0 then the abstract incidence notion is equivalent to the naive one: $x \in \mathbf{R}^n$ is incident to $\xi \in \mathbf{P}^n$ if and only if $x \in \xi$.

On the other hand we can also view $\mathbf{Z}_2\mathbf{M}(n-1)$ as the isotropy group of a hyperplane ξ_δ at a distance $\delta > 0$ from 0. (This amounts to a different embedding of the group $\mathbf{Z}_2\mathbf{M}(n-1)$ into $\mathbf{M}(n)$.) Then we have the following generalization.

Proposition 1.5. *The point $x \in \mathbf{R}^n$ and the hyperplane $\xi \in \mathbf{P}^n$ are incident if and only if distance $(x, \xi) = \delta$.*

Proof. Let $x = gK$, $\xi = \gamma H$ where $K = \mathbf{O}(n), H = \mathbf{Z}_2\mathbf{M}(n-1)$. Then if $gK \cap \gamma H \neq \emptyset$, we have $gk = \gamma h$ for some $k \in K, h \in H$. Now the orbit $H \cdot 0$ consists of the two planes ξ'_δ and ξ''_δ parallel to ξ_δ at a distance δ from ξ_δ. The relation

$$g \cdot 0 = \gamma h \cdot 0 \in \gamma \cdot (\xi'_\delta \cup \xi''_\delta)$$

together with the fact that g and γ are isometries shows that x has distance δ from $\gamma \cdot \xi_\delta = \xi$.

On the other hand if distance $(x, \xi) = \delta$, we have $g \cdot 0 \in \gamma \cdot (\xi'_\delta \cup \xi''_\delta) = \gamma H \cdot 0$, which means $gK \cap \gamma H \neq \emptyset$.

(ii) **Unit spheres.** Let σ_0 be a sphere in $\mathbf{R}^n$ of radius one passing through the origin. Denoting by Σ the set of all *unit* spheres in $\mathbf{R}^n$, we have the dual homogeneous spaces

$$\mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,; \quad \Sigma = \mathbf{M}(n)/\mathbf{O}^*(n) \tag{7}$$

where $\mathbf{O}^*(n)$ is the set of rotations around the center of σ_0. Here a point $x = g\mathbf{O}(n)$ is incident to $\sigma_0 = \gamma\mathbf{O}^*(n)$ if and only if $x \in \sigma$.

§2 The Radon Transform for the Double Fibration

With K, H and L as in §1 we assume now that K/L and H/L have positive measures $d\mu_0 = dk_L$ and $dm_0 = dh_L$ invariant under K and H, respectively. This is for example guaranteed if L is compact.

Lemma 2.1. *Assume the transversality condition (a). Then there exists a measure on each $\check{x}$ coinciding with $d\mu_0$ on $K/L = \check{x}_0$ such that whenever $g \cdot \check{x}_1 = \check{x}_2$ the measures on $\check{x}_1$ and $\check{x}_2$ correspond under g. A similar statement holds for dm on $\widehat{\xi}$.*

Proof. If $\check{x} = g \cdot \check{x}_0$ we transfer the measure $d\mu_0 = dk_L$ over on $\check{x}$ by the map $\xi \to g \cdot \xi$. If $g \cdot \check{x}_0 = g_1 \cdot \check{x}_0$ then $(g \cdot x_0)^\vee = (g_1 \cdot x_0)^\vee$ so by Lemma 1.3, $g \cdot x_0 = g_1 \cdot x_0$ so $g = g_1 k$ with $k \in K$. Since $d\mu_0$ is K-invariant the lemma follows.

The measures defined on each $\check{x}$ and $\widehat{\xi}$ under condition (a) are denoted by $d\mu$ and dm, respectively.

Definition. The Radon transform $f \to \widehat{f}$ and its dual $\varphi \to \check{\varphi}$ are defined by

$$\widehat{f}(\xi) = \int_{\widehat{\xi}} f(x)\, dm(x)\,, \qquad \check{\varphi}(x) = \int_{\check{x}} \varphi(\xi)\, d\mu(\xi)\,, \tag{8}$$

whenever the integrals converge. Because of Lemma 1.1, this will always happen for $f \in C_c(X)$, $\varphi \in C_c(\Xi)$.

In the setup of Proposition 1.5, $\widehat{f}(\xi)$ is the integral of f over the two hyperplanes at distance δ from ξ and $\check{\varphi}(x)$ is the average of φ over the set of hyperplanes at distance δ from x. For $\delta = 0$ we recover the transforms of Ch. I, §1.

Formula (8) can also be written in the group-theoretic terms,

$$\widehat{f}(\gamma H) = \int_{H/L} f(\gamma h K)\, dh_L\,, \quad \check{\varphi}(gK) = \int_{K/L} \varphi(gkH)\, dk_L\,. \tag{9}$$

Note that (9) serves as a definition even if condition (a) in Lemma 1.3 is not satisfied. In this abstract setup the spaces X and Ξ have equal status. The theory in Ch. I, in particular Lemma 2.1, Theorems 2.4, 2.10, 3.1 thus raises the following problems:

Principal Problems:

A. Relate function spaces on X and on Ξ by means of the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$. In particular, determine their ranges and kernels.

B. Invert the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ on suitable function spaces.

C. In the case when G is a Lie group so X and Ξ are manifolds let $\mathbf{D}(X)$ and $\mathbf{D}(\Xi)$ denote the algebras of G-invariant differential operators on X and Ξ, respectively. Is there a map $D \to \widehat{D}$ of $\mathbf{D}(X)$ into $\mathbf{D}(\Xi)$ and a map $E \to \check{E}$ of $\mathbf{D}(\Xi)$ into $\mathbf{D}(X)$ such that

$$(Df)\widehat{\ } = \widehat{D}\widehat{f}\,, \quad (E\varphi)^\vee = \check{E}\check{\varphi}\,?$$

D. Support Property: Does $\widehat{f}$ of compact support imply that f has compact support?

Although weaker assumptions would be sufficient, we assume now that the groups G, K, H and L all have bi-invariant Haar measures dg, dk, dh and $d\ell$. These will then generate invariant measures dg_K, dg_H, dg_L, dk_L, dh_L on G/K, G/H, G/L, K/L, H/L, respectively. This means that

$$\int_G F(g)\,dg = \int_{G/K} \left(\int_K F(gk)\,dk \right) dg_K \tag{10}$$

and similarly dg and dh determine dg_H, etc. Then

$$\int_{G/L} Q(gL)\,dg_L = c \int_{G/K} dg_K \int_{K/L} Q(gkL)\,dk_L \tag{11}$$

for $Q \in C_c(G/L)$ where c is a constant. In fact, the integrals on both sides of (11) constitute invariant measures on G/L and thus must be proportional. However,

$$\int_G F(g)\,dg = \int_{G/L} \left(\int_L F(g\ell)\,d\ell \right) dg_L \tag{12}$$

and

$$\int_K F(k)\,dk = \int_{K/L} \left(\int_L F(k\ell)\,d\ell \right) dk_L\,. \tag{13}$$

We use (13) on (10) and combine with (11) taking $Q(gL) = \int F(g\ell)\,d\ell$. Then we see that from (12) the constant c equals 1.

We shall now prove that $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$ are adjoint operators. We write dx for dg_K and $d\xi$ for dg_H.

Proposition 2.2. *Let $f \in C_c(X)$, $\varphi \in C_c(\Xi)$. Then $\widehat{f}$ and $\check{\varphi}$ are continuous and*

$$\int_X f(x)\check{\varphi}(x)\,dx = \int_\Xi \widehat{f}(\xi)\varphi(\xi)\,d\xi\,.$$

Proof. The continuity statement is immediate from (9). We consider the function

$$P = (f \circ p)(\varphi \circ \pi)$$

on G/L. We integrate it over G/L in two ways using the double fibration (4). This amounts to using (11) and its analog with G/K replaced by G/H with $Q = P$. Since $P(gk\,L) = f(gK)\varphi(gkH)$, the right hand side of (11) becomes

$$\int_{G/K} f(gK)\check{\varphi}(gK)\,dg_K\,.$$

If we treat G/H similarly, the lemma follows.

The result shows how to define the Radon transform and its dual for measures and, in case G is a Lie group, for distributions.

Definition. Let s be a measure on X of compact support. Its Radon transform is the functional $\widehat{s}$ on $C_c(\Xi)$ defined by

$$\widehat{s}(\varphi) = s(\check{\varphi}) . \tag{14}$$

Similarly $\check{\sigma}$ is defined by

$$\check{\sigma}(f) = \sigma(\widehat{f}) , \qquad f \in C_c(X) , \tag{15}$$

if σ is a compactly supported measure on Ξ.

Lemma 2.3. *(i) If s is a compactly supported measure on X, $\widehat{s}$ is a measure on Ξ.*

(ii) If s is a bounded measure on X and if $\check{x}_0$ has finite measure then $\widehat{s}$ as defined by (14) is a bounded measure.

Proof. (i) The measure s can be written as a difference $s = s^+ - s^-$ of two positive measures, each of compact support. Then $\widehat{s} = \widehat{s}^+ - \widehat{s}^-$ is a difference of two positive **functionals** on $C_c(\Xi)$.

Since a positive functional is necessarily a measure, $\widehat{s}$ is a measure.

(ii) We have

$$\sup_x |\check{\varphi}(x)| \leq \sup_\xi |\varphi(\xi)| \, \mu_0(\check{x}_0) ,$$

so for a constant K,

$$|\widehat{s}(\varphi)| = |s(\check{\varphi})| \leq K \sup |\check{\varphi}| \leq K \mu_0(\check{x}_0) \sup |\varphi| ,$$

and the boundedness of $\widehat{s}$ follows.

If G is a Lie group then (14), (15) with $f \in \mathcal{D}(X)$, $\varphi \in \mathcal{D}(\Xi)$ serve to define the Radon transform $s \to \widehat{s}$ and the dual $\sigma \to \check{\sigma}$ for distributions s and σ of compact support. We consider the spaces $\mathcal{D}(X)$ and $\mathcal{E}(X)$ $(= \mathcal{C}^\infty(X))$ with their customary topologies (Chapter VII, §1). The duals $\mathcal{D}'(X)$ and $\mathcal{E}'(X)$ then consist of the distributions on X and the distributions on X of compact support, respectively.

Proposition 2.4. *The mappings*

$$\begin{aligned} f \in \mathcal{D}(X) \quad &\to \quad \widehat{f} \in \mathcal{E}(\Xi) \\ \varphi \in \mathcal{D}(\Xi) \quad &\to \quad \check{\varphi} \in \mathcal{E}(X) \end{aligned}$$

are continuous. In particular,

$$\begin{aligned} s \in \mathcal{E}'(X) \quad &\Rightarrow \quad \widehat{s} \in \mathcal{D}'(\Xi) \\ \sigma \in \mathcal{E}'(\Xi) \quad &\Rightarrow \quad \check{\sigma} \in \mathcal{D}'(X) . \end{aligned}$$

Proof. We have

$$\widehat{f}(g \cdot \xi_0) = \int_{\widehat{\xi}_0} f(g \cdot x)\, dm_0(x)\,. \tag{16}$$

Let g run through a local cross section through e in G over a neighborhood of ξ_0 in Ξ. If $(t_1, \ldots, t_n)$ are coordinates of g and $(x_1, \ldots, x_m)$ the coordinates of $x \in \widehat{\xi}_0$ then (16) can be written in the form

$$\widehat{F}(t_1, \ldots, t_n) = \int F(t_1, \ldots, t_n\,;\, x_1, \ldots, x_m)\, dx_1 \ldots\, dx_m\,.$$

Now it is clear that $\widehat{f} \in \mathcal{E}(\Xi)$ and that $f \to \widehat{f}$ is continuous, proving the proposition.

The result has the following refinement.

Proposition 2.5. *Assume K compact. Then*

(i) $f \to \widehat{f}$ *is a continuous mapping of* $\mathcal{D}(X)$ *into* $\mathcal{D}(\Xi)$.

(ii) $\varphi \to \check{\varphi}$ *is a continuous mapping of* $\mathcal{E}(\Xi)$ *into* $\mathcal{E}(X)$.

A self-contained proof is given in the author's book [1994b], Ch. I, § 3. The result has the following consequence.

Corollary 2.6. *Assume K compact. Then* $\mathcal{E}'(X)\widehat{\ } \subset \mathcal{E}'(\Xi)$, $\mathcal{D}'(\Xi)^\vee \subset \mathcal{D}'(X)$.

Ranges and Kernels. General Features

It is clear from Proposition 2.2 that the range $\mathcal{R}$ of $f \to \widehat{f}$ is orthogonal to the kernel $\mathcal{N}$ of $\varphi \to \check{\varphi}$. When $\mathcal{R}$ is closed one can often conclude $\mathcal{R} = \mathcal{N}^\perp$, also when $\widehat{\ }$ is extended to distributions (Helgason [1994b], Chapter IV, §2, Chapter I, §2). Under fairly general conditions one can also deduce that the range of $\varphi \to \check{\varphi}$ equals the annihilator of the kernel of $T \to \widehat{T}$ for distributions (*loc. cit.*, Ch. I, §3).

In Chapter I we have given solutions to Problems A, B, C, D in some cases. Further examples will be given in § 4 of this chapter and Chapter III will include their solution for the antipodal manifolds for compact two-point homogeneous spaces.

The variety of the results for these examples make it doubtful that the individual results could be captured by a general theory. Our abstract setup in terms of homogeneous spaces in duality is therefore to be regarded as a framework for examples rather than as axioms for a general theory.

Nevertheless, certain general features emerge from the study of these examples. If $\dim X = \dim \Xi$ and $f \to \widehat{f}$ is injective the range consists of functions which are either arbitrary or at least subjected to rather weak conditions. As the difference $\dim \Xi - \dim X$ increases more conditions are imposed on the functions in the range. (See the example of the d-plane transform in $\mathbf{R}^n$.)

In case G is a Lie group there is a group-theoretic explanation for this. Let X be a manifold and Ξ a manifold whose points ξ are submanifolds of X. We assume each $\xi \in \Xi$ to have a measure dm and that the set $\{\xi \in \Xi : \xi \ni x\}$ has a measure $d\mu$. We can then consider the transforms

$$\widehat{f}(\xi) = \int_{\xi} f(x)\,dm(x)\,, \quad \check{\varphi}(x) = \int_{\xi \ni x} \varphi(\xi)\,d\mu(\xi)\,. \tag{17}$$

If G is a Lie transformation group of X permuting the members of Ξ including the measures dm and $d\mu$, the transforms $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ commute with the G-actions on X and Ξ

$$(\widehat{f})^{\tau(g)} = (f^{\tau(g)})^{\widehat{}} \qquad (\varphi^{\tau(g)})^{\vee} = (\check{\varphi})^{\tau(g)}\,. \tag{18}$$

Let λ and Λ be the homomorphisms

$$\lambda : \mathbf{D}(G) \to \mathbf{E}(X)$$
$$\Lambda : \mathbf{D}(G) \to \mathbf{E}(\Xi)$$

in Ch. VIII, §2. Using (13) in Ch. VIII we derive

$$(\lambda(D)f)^{\widehat{}} = \Lambda(D)\widehat{f}\,,\ (\Lambda(D)\varphi)^{\vee} = \lambda(D)\check{\varphi}\,. \tag{19}$$

Therefore $\Lambda(D)$ annihilates the range of $f \to \widehat{f}$ if $\lambda(D) = 0$. In some cases, including the case of the d-plane transform in $\mathbf{R}^n$, the range is characterized as the null space of these operators $\Lambda(D)$ (with $\lambda(D) = 0$). This is illustrated by Theorems 6.5 and 6.8 in Ch. I and even more by theorems of Richter, Gonzalez which characterized the range as the null space of certain explicit invariant operators ([GSS, I, §3]). Much further work in this direction has been done by Gonzalez and Kakehi (see Part I in Ch. II, §4). Examples of (17)–(18) would occur with G a group of isometries of a Riemannian manifold, Ξ a suitable family of geodesics. The framework (8) above fits here too but goes further in that Ξ does not have to consist of subsets of X. We shall see already in the next Theorem 4.1 that this feature is significant.

The Inversion Problem. General Remarks

In Theorem 3.1 and 6.2 in Chapter I as well as in several later results the Radon transform $f \to \widehat{f}$ is inverted by a formula

$$f = D((\widehat{f})^{\vee})\,, \tag{20}$$

where D is a specific operator on X, often a differential operator. Rouvière has in [2001] outlined an effective strategy for producing such a D.

Consider the setup $X = G/K$, $\Xi = G/H$ from §1 and assume G, K and H are unimodular Lie groups and K compact. On G we have a convolution (in the style of Ch. VII),

$$(u * v)(h) = \int_G u(hg^{-1})v(g)\,dg = \int_G u(g)v(g^{-1}h)\,dg\,,$$

provided one of the functions u, v has compact support. Here dg is Haar measure. More generally, if s, t are two distributions on G at least one of compact support the tensor product $s \otimes t$ is a distribution on $G \times G$ given by

$$(s \otimes t)(u(x,y)) = \int_{G\times G} u(x,y)\,ds(x)\,dt(y) \quad u \in \mathcal{D}(G \times G)\,.$$

Note that $s\otimes t = t\otimes s$ because they agree on the space spanned by functions of the type $\varphi(x)\psi(y)$ which is dense in $\mathcal{D}(G \times G)$. The convolution $s * t$ is defined by

$$(s * t)(v) = \int_G \int_G v(xy)\,ds(x)\,dt(y)\,.$$

Lifting a function f on X to G by $\widetilde{f} = f \circ \pi$ where $\pi : G \to G/K$ is the natural map we lift a distribution S on X to a $\widetilde{S} \in \mathcal{D}'(G)$ by $\widetilde{S}(u) = S(\dot{u})$ where

$$\dot{u}(gK) = \int_K u(gk)\,dk\,.$$

Thus $\widetilde{S}(\widetilde{f}) = S(f)$ for $f \in \mathcal{D}(X)$. If $S, T \in \mathcal{D}'(X)$, one of compact support the convolution $\times$ on X is defined by

$$(S \times T)(f) = (\widetilde{S} * \widetilde{T})(\widetilde{f})\,. \tag{21}$$

If one of these is a function f, we have

$$(f \times S)(g \cdot x_0) = \int_G f(gh^{-1} \cdot x_0)\,d\widetilde{S}(h)\,, \tag{22}$$

$$(S \times f)(g \cdot x_0) = \int_G f(h^{-1}g \cdot x_0)\,d\widetilde{S}(h)\,. \tag{23}$$

The first formula can also be written

$$f \times S = \int_G f(g \cdot x_0) S^{\tau(g)}\,dg \tag{24}$$

as distributions. In fact, let $\varphi \in \mathcal{D}(X)$. Then

$$\begin{aligned}(f \times S)(\varphi) &= \int_G (f \times S)(g \cdot x_0)\varphi(g \cdot x_0)\, dg \\ &= \int_G \Big(\int_G f(gh^{-1} \cdot x_0)\, d\widetilde{S}(h)\Big)\varphi(g \cdot x_0)\, dg \\ &= \int_G \Big(\int_G f(g \cdot x_0)\widetilde{\varphi}(gh)\, dg\Big)\, d\widetilde{S}(h) \\ &= \int_G \int_G f(g \cdot x_0)(\varphi^{\tau(g^{-1})})^{\sim}(h)\, dg\, d\widetilde{S}(h) \\ &= \int_G f(g \cdot x_0) S(\varphi^{\tau(g^{-1})})\, dg = \int_G f(g \cdot x_0) S^{\tau(g)}(\varphi)\, dg\,.\end{aligned}$$

Now let D be a G-invariant differential operator on X and D^* its adjoint. It is also G-invariant. If $\varphi = \mathcal{D}(X)$ then the invariance of D^* and (24) imply

$$\begin{aligned}(D(f \times S))(\varphi) = (f \times S)(D^*\varphi) &= \int_G f(g \cdot x_0) S^{\tau(g)}(D^*\varphi)\, dg \\ &= \int_G f(g \cdot x_0) S(D^*(\varphi \circ \tau(g)))\, dg = \int_G f(g \cdot x_0)(DS)^{\tau(g)}(\varphi)\, dg,\end{aligned}$$

so

$$D(f \times S) = f \times DS\,. \tag{25}$$

Let ϵ_D denote the distribution $f \to (D^*f)(x_0)$. Then

$$Df = f \times \epsilon_D\,,$$

because by (24)

$$\begin{aligned}(f \times \epsilon_D)(\varphi) &= \int_G f(g \cdot x_0)\epsilon_D^{\tau(g)}(\varphi) \\ &= \int_G f(g \cdot x_0) D^*(\varphi^{\tau(g^{-1})})(x_0)\, dg = \int_G f(g \cdot x_0)(D^*\varphi)(g \cdot x_0)\, dg \\ &= \int_X f(x)(D^*\varphi)(x)\, dx = \int_X (Df)(x)\varphi(x)\, dx\,.\end{aligned}$$

We consider now the situation where the elements ξ of Ξ are subsets of X (cf. Lemma 1.3).

Theorem 2.7 (Rouvière). *Under the assumptions above (K compact) there exists a distribution S on X such that*

$$(\widehat{f})^{\vee} = f \times S\,, \quad f \in \mathcal{D}(X)\,. \tag{26}$$

Proof. Define a functional S on $C_c(X)$ by

$$S(f) = (\widehat{f})^{\vee}(x_0) = \int\limits_K \Big(\int\limits_H f(kh \cdot x_0)\, dh \Big)\, dk\,.$$

Then S is a measure because if f has compact support C the set of $h \in H$ for which $kh \cdot x_0 \in C$ for some k is compact. The restriction of S to $\mathcal{D}(X)$ is a distribution which is clearly K-invariant. By (24) we have for $\varphi \in \mathcal{D}(X)$

$$(f \times S)(\varphi) = \int\limits_G f(g \cdot x_0) S(\varphi^{\tau(g^{-1})})\, dg\,,$$

which, since the operations $\widehat{\ }$ and $\vee$ commute with the G action, becomes

$$\int\limits_G f(g \cdot x_0)(\widehat{\varphi})^{\vee}(g \cdot x_0)\, dg = \int\limits_X (\widehat{f})^{\vee}(x)\varphi(x)\, dx\,,$$

because of Proposition 2.2. This proves the theorem.

Corollary 2.8. *If D is a G-invariant differential operator on X such that $DS = \delta$ (delta function at x_0) then we have the inversion formula*

$$f = D((\widehat{f})^{\vee})\,, \quad f \in \mathcal{D}(X)\,. \tag{27}$$

This follows from (26) and $f \times \delta = f$.

§3 Orbital Integrals

As before let $X = G/K$ be a homogeneous space with origin $o = (K)$. Given $x_0 \in X$ let G_{x_0} denote the subgroup of G leaving x_0 fixed, i.e., the isotropy subgroup of G at x_0.

Definition. A **generalized sphere** is an orbit $G_{x_0} \cdot x$ in X of some point $x \in X$ under the isotropy subgroup at some point $x_0 \in X$.

Examples. (i) If $X = \mathbf{R}^n$, $G = \mathbf{M}(n)$ then the generalized spheres are just the spheres.

(ii) Let X be a locally compact subgroup L and G the product group $L \times L$ acting on L on the right and left, the element $(\ell_1, \ell_2) \in L \times L$ inducing action $\ell \to \ell_1 \ell \, \ell_2^{-1}$ on L. Let ΔL denote the diagonal in $L \times L$. If $\ell_0 \in L$ then the isotropy subgroup of ℓ_0 is given by

$$(L \times L)_{\ell_0} = (\ell_0, e) \Delta L (\ell_0^{-1}, e) \tag{28}$$

and the orbit of ℓ under it by

$$(L \times L)_{\ell_0} \cdot \ell = \ell_0 (\ell_0^{-1} \ell)^L \, ,$$

which is the left translate by ℓ_0 of the conjugacy class of the element $\ell_0^{-1}\ell$. Thus the *generalized spheres in the group L are the left (or right) translates of its conjugacy classes.*

Coming back to the general case $X = G/K = G/G_0$ we assume that G_0, and therefore each G_{x_0}, is unimodular. But $G_{x_0} \cdot x = G_{x_0}/(G_{x_0})_x$ so $(G_{x_0})_x$ unimodular implies the orbit $G_{x_0} \cdot x$ has an invariant measure determined up to a constant factor. We can now consider the following general problem (following Problems A, B, C, D above).

E. *Determine a function f on X in terms of its integrals over generalized spheres.*

Remark 3.1. In this problem it is of course significant how the invariant measures on the various orbits are normalized.

(a) If G_0 is compact the problem above is rather trivial because each orbit $G_{x_0} \cdot x$ has finite invariant measure so $f(x_0)$ is given as the limit as $x \to x_0$ of the average of f over $G_{x_0} \cdot x$.

(b) Suppose that for each $x_0 \in X$ there is a G_{x_0}-invariant open set $C_{x_0} \subset X$ containing x_0 in its closure such that for each $x \in C_{x_0}$ the isotropy group $(G_{x_0})_x$ is compact. The invariant measure on the orbit $G_{x_0} \cdot x \; (x_0 \in X, x \in C_{x_0})$ can then be consistently normalized as follows: Fix a Haar measure dg_0 on G_0. If $x_0 = g \cdot o$ we have $G_{x_0} = gG_0g^{-1}$ and can carry dg_0 over to a measure dg_{x_0} on G_{x_0} by means of the conjugation $z \to gzg^{-1} \quad (z \in G_0)$. Since dg_0 is bi-invariant, dg_{x_0} is independent of the choice of g satisfying $x_0 = g \cdot o$, and is bi-invariant. Since $(G_{x_0})_x$ is compact it has a unique Haar measure $dg_{x_0,x}$ with total measure 1 and now dg_{x_0} and $dg_{x_0,x}$ determine canonically an invariant measure μ on the orbit $G_{x_0} \cdot x = G_{x_0}/(G_{x_0})_x$. We can therefore state Problem E in a more specific form.

E′. *Express $f(x_0)$ in terms of integrals*

$$\int\limits_{G_{x_0} \cdot x} f(p) \, d\mu(p) \, , \qquad x \in C_{x_0} \, . \tag{29}$$

For the case when X is an **isotropic Lorentz manifold** the assumptions above are satisfied (with C_{x_0} consisting of the "timelike" rays from x_0) and we shall obtain in Ch. V an explicit solution to Problem E' (Theorem 4.1, Ch. V).

(c) If in Example (ii) above L is a semisimple Lie group Problem E is a basic step (Gelfand–Graev [1955], Harish-Chandra [1954], [1957]) in proving the Plancherel formula for the Fourier transform on L.

§4 Examples of Radon Transforms for Homogeneous Spaces in Duality

In this section we discuss some examples of the abstract formalism and problems set forth in the preceding sections §1–§2.

A. The Funk Transform

This case goes back to Funk [1913], [1916] (preceding Radon's paper [1917]) where he proved, inspired by Minkowski [1911], that a symmetric function on $\mathbf{S}^2$ is determined by its great circle integrals. This is carried out in more detail and in greater generality in Chapter III, §1. Here we state the solution of Problem B for $X = \mathbf{S}^2$, Ξ the set of all great circles, both as homogeneous spaces of $\mathbf{O}(3)$. Given $p \geq 0$ let $\xi_p \in \Xi$ have distance p from the North Pole o, $H_p \subset \mathbf{O}(3)$ the subgroup leaving ξ_p invariant and $K \subset \mathbf{O}(3)$ the subgroup fixing o. Then in the double fibration

$$\begin{tikzcd}[column sep=small] & \mathbf{O}(3)/(K \cap H_p) \arrow[dl] \arrow[dr] & \\ X = \mathbf{O}(3)/K & & \Xi = \mathbf{O}(3)/H_p \end{tikzcd}$$

$x \in X$ and $\xi \in \Xi$ are incident if and only if $d(x, \xi) = p$. The proof is the same as that of Proposition 1.5. We denote by $\widehat{f}_p$ and $\check{\varphi}_p$ the Radon transforms (9) for the double fibration. Then $\widehat{f}_p(\xi)$ the integral of f over two circles at distance p from ξ and $\check{\varphi}_p$ is the average of $\check{\varphi}(x)$ over the great circles ξ that have distance p from x. (See Fig. II.2.) We need $\widehat{f}_p$ only for $p = 0$ and put $\widehat{f} = \widehat{f}_0$. Note that $(\widehat{f})^\vee_p(x)$ is the average of the integrals of f over the great circles ξ at distance p from x (see Figure II.2). As a special case of Theorem 1.22, Chapter III, we have the following inversion.

Theorem 4.1. *The Funk transform $f \to \widehat{f}$ is (for f even) inverted by*

$$f(x) = \frac{1}{2\pi}\left\{\frac{d}{du}\int_0^u (\widehat{f})^\vee_{\cos^{-1}(v)}(x)v(u^2 - v^2)^{-\frac{1}{2}}\,dv\right\}_{u=1}. \tag{30}$$

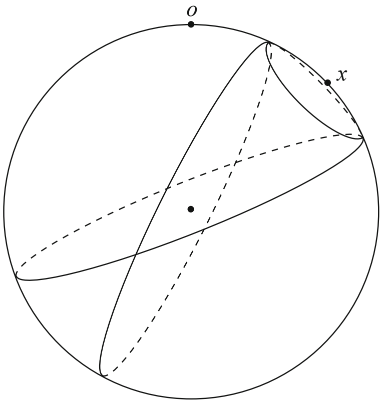

FIGURE II.2.

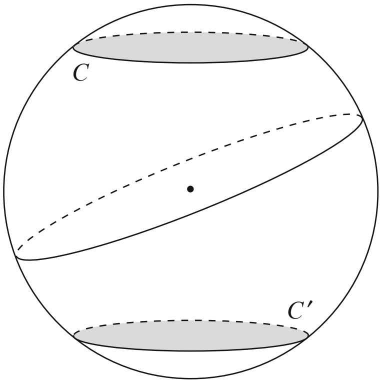

FIGURE II.3.

We shall see later that this formula can also be written

$$(31)\qquad f(x) = \int_{E_x} f(w)\,dw - \frac{1}{2\pi}\int_0^{\frac{\pi}{2}} \frac{d}{dp}\big((\widehat{f})^\vee_p(x)\big)\,\frac{dp}{\sin p}\,,$$

where dw is the normalized measure on the equator E_x corresponding to x. In this form the formula holds in all dimensions.

Also Theorem 1.26, Ch. III shows that if f is even and if all its derivatives vanish on the equator then f vanishes outside the "arctic zones" C and C' if and only if $\widehat{f}(\xi) = 0$ for all great circles ξ disjoint from C and C' (Fig. II.3).

The Hyperbolic Plane $\mathbf{H^2}$

We now introduce the hyperbolic plane. This formulation fits well into **Klein's Erlanger Program** under which geometric properties of a space should be understood in terms of a suitable transformation group of the space.

Theorem 4.2. *On the unit disk $D : |z| < 1$ there exists a Riemannian metric g which is invariant under all conformal transformations of D. Also g is unique up to a constant factor.*

For this consider a point $a \in D$. The mapping $\varphi : z \to \frac{a-z}{1-\bar{a}z}$ is a conformal transformation of D and $\varphi(a) = 0$. The invariance of g requires

$$g_a(u, u) = g_0(d\varphi(u),\, d\varphi(u))$$

for each $u \in D_a$ (the tangent space to D at a) $d\varphi$ denoting the differential of φ. Since g_0 is invariant under rotations around 0, $g_0(z, z) = c|z|^2$, where c is a constant. Here $z \in D_0$ $(= \mathbf{C})$. Let $t \to z(t)$ be a curve with $z(0) = a$,

$z'(0) = u \in \mathbf{C}$. Then $d\varphi(u)$ is the tangent vector

$$\left\{\frac{d}{dt}\varphi(z(t))\right\}_{t=0} = \left(\frac{d\varphi}{dz}\right)_a \left(\frac{dz}{dt}\right)_0 = \left\{\frac{|a|^2-1}{(1-\bar{a}z)^2}\right\}_{z=a} u\,,$$

so

$$g_a(u,u) = c\frac{1}{(1-|a|^2)^2}|u|^2\,,$$

and the proof shows that g is indeed invariant.

Thus we take the hyperbolic plane $\mathbf{H}^2$ as the disk D with the Riemannian structure

$$ds^2 = \frac{|dz|^2}{(1-|z|^2)^2}\,. \tag{32}$$

This remarkable object enters into several fields in mathematics. In particular, it offers at least two interesting cases of Radon transforms. The Laplace-Beltrami operator for (32) is given by

$$L = (1-x^2-y^2)^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\,.$$

The group $G = \mathbf{SU}(1,1)$ of matrices

$$\left\{\begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix} : |a|^2 - |b|^2 = 1\right\}$$

acts transitively on the unit disk by

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a}\end{pmatrix} \cdot z = \frac{az+b}{\bar{b}z+\bar{a}} \tag{33}$$

and leaves the metric (32) invariant. The length of a curve $\gamma(t)$ ($\alpha \leq t \leq \beta$) is defined by

$$L(\gamma) = \int_\alpha^\beta (\langle\gamma'(t), \gamma'(t)\rangle_{\gamma(t)})^{1/2}\,dt\,. \tag{34}$$

In particular take $\gamma(t) = (x(t), y(t))$ such that $\gamma(\alpha) = 0$, $\gamma(\beta) = x$ ($0 < x < 1$), and let $\gamma_0(\tau) = \tau x$, $0 \leq \tau \leq 1$, so γ and γ_0 have the same endpoints. Then

$$L(\gamma) \geq \int_\alpha^\beta \frac{|x'(t)|}{1-x(t)^2}\,dt \geq \int_\alpha^\beta \frac{x'(t)}{1-x(t)^2}\,dt\,,$$

which by $\tau = x(t)/x$, $d\tau/dt = x'(t)/x$ becomes

$$\int_0^1 \frac{x}{1-\tau^2 x^2}\, d\tau = L(\gamma_0)\,.$$

Thus $L(\gamma) \geq L(\gamma_0)$ so γ_0 is a geodesic and the distance d satisfies

$$d(o,x) = \int_0^1 \frac{|x|}{1-t^2x^2}\, dt = \tfrac{1}{2}\log\frac{1+|x|}{1-|x|}\,. \tag{35}$$

Since G acts conformally on D the *geodesics* in $\mathbf{H}^2$ are the circular arcs in $|z| < 1$ perpendicular to the boundary $|z| = 1$.

We consider now the following subgroups of G where $\operatorname{sh} t = \sinh t$ etc.:

$$\begin{aligned}
K &= \{k_\theta = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} : 0 \leq \theta < 2\pi\} \\
M &= \{k_0, k_\pi\}\,, \qquad M' = \{k_0, k_\pi\,, k_{-\frac{\pi}{2}}, k_{\frac{\pi}{2}}\} \\
A &= \{a_t = \begin{pmatrix} \operatorname{ch} t & \operatorname{sh} t \\ \operatorname{sh} t & \operatorname{ch} t \end{pmatrix} \ : \ t \in \mathbf{R}\}, \\
N &= \{n_x = \begin{pmatrix} 1+ix, & -ix \\ ix, & 1-ix \end{pmatrix} \ : \ x \in \mathbf{R}\} \\
\Gamma &= C\mathbf{SL}(2,\mathbf{Z})C^{-1}\,,
\end{aligned}$$

where C is the transformation $w \to (w-i)/(w+i)$ mapping the upper half-plane onto the unit disk.

The orbits of K are the circles around 0. To identify the orbit $A \cdot z$ we use this simple argument by Reid Barton:

$$a_t \cdot z = \frac{\operatorname{ch} t\, z + \operatorname{sh} t}{\operatorname{sh} t\, z + \operatorname{ch} t} = \frac{z + \operatorname{th} t}{\operatorname{th} t\, z + 1}\,.$$

Under the map $w \to \frac{z+w}{zw+1}$ $(w \in \mathbf{C})$ lines go into circles and lines. Taking $w = \operatorname{th} t$ we see that $A \cdot z$ is the circular arc through -1, z and 1. Barton's argument also gives the orbit $n_x \cdot t$ $(x \in \mathbf{R})$ as the image of $i\mathbf{R}$ under the map

$$w \to \frac{w(t-1)+t}{w(t-1)+1}\,.$$

They are circles tangential to $|z| = 1$ at $z = 1$. Clearly $NA \cdot 0$ is the whole disk D so $G = NAK$ (and also $G = KAN$).

B. The X-ray Transform in $\mathbf{H}^2$

The (unoriented) geodesics for the metric (32) were mentioned above. Clearly the group G permutes these geodesics transitively (Fig. II.4). Let

Ξ be the set of all these geodesics. Let o denote the origin in $\mathbf{H}^2$ and ξ_o the horizontal geodesic through o. Then

(36)
$$X = G/K\,, \quad \Xi = G/M'A\,.$$

We can also fix a geodesic ξ_p at distance p from o and write

(37)
$$X = G/K\,, \quad \Xi = G/H_p\,,$$

where H_p is the subgroup of G leaving ξ_p stable. Then for the homogeneous spaces (37), x and ξ are incident if and only if $d(x,\xi) = p$. The transform $f \to \widehat{f}$ is inverted by means of the dual transform $\varphi \to \check{\varphi}_p$ for (37). The inversion below is a special case of Theorem 1.11, Chapter III, and is the analog of (30). Observe also that the metric ds is renormalized by the factor 2 (so curvature is -1).

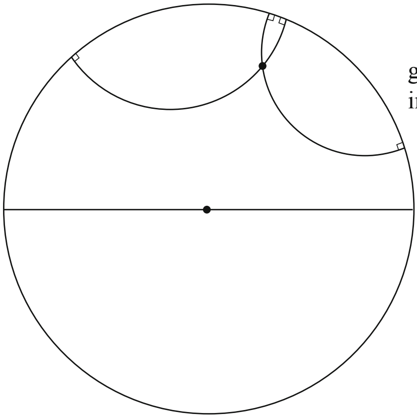

FIGURE II.4.

Theorem 4.3. *The X-ray transform in* $\mathbf{H}^2$ *with the metric*

$$ds^2 = \frac{4|dz|^2}{(1-|z|^2)^2}$$

is inverted by

$$f(z) = -\left\{ \frac{d}{dr} \int_r^\infty (t^2 - r^2)^{-\frac{1}{2}} t (\widehat{f})^\vee_{s(t)}(z)\, dt \right\}_{r=1}, \tag{38}$$

where $s(t) = \cosh^{-1}(t)$.

Another version of this formula is

$$f(z) = -\frac{1}{\pi} \int_0^\infty \frac{d}{dp}\left((\widehat{f})^\vee_p(z) \right) \frac{dp}{\sinh p} \tag{39}$$

and in this form it is valid in all dimensions (Theorem 1.12, Ch. III).

One more inversion formula is

$$f = -\frac{1}{4\pi} LS((\widehat{f})^\vee)\,, \tag{40}$$

where S is the operator of convolution on $\mathbf{H}^2$ with the function $x \to \coth(d(x,o)) - 1$, (Theorem 1.16, Chapter III).

C. The Horocycles in $\mathbf{H}^2$

Consider a family of geodesics with the same limit point on the boundary B. The **horocycles** in $\mathbf{H}^2$ are by definition the orthogonal trajectories of such families of geodesics. Thus the horocycles are the circles tangential to $|z| = 1$ from the inside (Fig. II.5).

One such horocycle is $\xi_0 = N \cdot o$, the orbit of the origin o under the action of N. Now we take $\mathbf{H}^2$ with the metric (32). Since $a_t \cdot \xi$ is the horocycle with diameter $(\tanh t, 1)$ G acts transitively on the set Ξ of horocycles. Since $G = KAN$ it is easy to see that MN is the subgroup leaving ξ_o invariant. Thus we have here

(41)
$$X = G/K\,, \quad \Xi = G/MN\,.$$

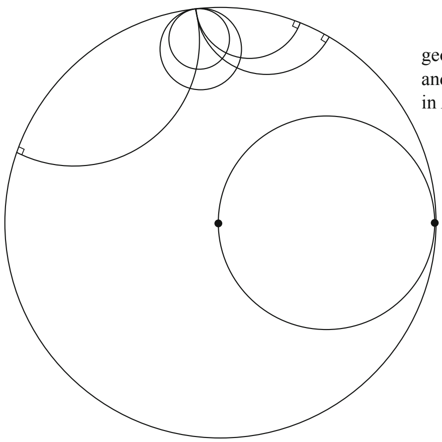

FIGURE II.5.

Furthermore each horocycle has the form $\xi = ka_t \cdot \xi_0$ where $kM \in K/M$ and $t \in \mathbf{R}$ are unique. Thus $\Xi \sim K/M \times A$, which is also evident from the figure.

We observe now that the maps

$$\psi : t \to a_t \cdot o\,, \quad \varphi : x \to n_x \cdot o$$

of $\mathbf{R}$ onto γ_0 and ξ_0, respectively, are isometries. The first statement follows from (35) because

$$d(o, a_t \cdot o) = d(o, \tanh t) = t\,.$$

For the second we note that

$$\varphi(x) = x(x+i)^{-1}\,, \quad \varphi'(x) = i(x+i)^{-2}$$

so

$$\langle \varphi'(x), \varphi'(x) \rangle_{\varphi(x)} = (x^2+1)^{-2}(1 - |x(x+i)^{-1}|^2)^{-2} = 1\,.$$

Thus we give A and N the Haar measures $d(a_t) = dt$ and $d(n_x) = dx$.

Geometrically, the Radon transform on X relative to the horocycles is defined by

$$\widehat{f}(\xi) = \int_\xi f(x)\,dm(x)\,, \tag{42}$$

where dm is the measure on ξ induced by (32). Because of our remarks about φ, (42) becomes

$$\widehat{f}(g \cdot \xi_0) = \int_N f(gn \cdot o)\, dn\,, \tag{43}$$

so the geometric definition (42) coincides with the group-theoretic one in (9). The dual transform is given by

$$\check{\varphi}(g \cdot o) = \int_K \varphi(gk \cdot \xi_o)\, dk\,, \qquad (dk = d\theta/2\pi)\,. \tag{44}$$

In order to invert the transform $f \to \widehat{f}$ we introduce the non-Euclidean analog of the operator Λ in Chapter I, §3. Let T be the distribution on $\mathbf{R}$ given by

$$T\varphi = \tfrac{1}{2} \int_{\mathbf{R}} (\operatorname{sh} t)^{-1} \varphi(t)\, dt\,, \quad \varphi \in \mathcal{D}(\mathbf{R})\,, \tag{45}$$

considered as the Cauchy principal value, and put $T' = dT/dt$. Let Λ be the operator on $\mathcal{D}(\Xi)$ given by

$$(\Lambda\varphi)(ka_t \cdot \xi_0) = \int_{\mathbf{R}} \varphi(ka_{t-s} \cdot \xi_0) e^{-s}\, dT'(s)\,. \tag{46}$$

Theorem 4.4. *The Radon transform $f \to \widehat{f}$ for horocycles in $\mathbf{H}^2$ is inverted by*

$$f = \frac{1}{\pi}(\Lambda \widehat{f})^{\vee}\,, \quad f \in \mathcal{D}(\mathbf{H}^2)\,. \tag{47}$$

We begin with a simple lemma.

Lemma 4.5. *Let τ be a distribution on $\mathbf{R}$. Then the operator $\widetilde{\tau}$ on $\mathcal{D}(\Xi)$ given by the convolution*

$$(\widetilde{\tau}\varphi)(ka_t \cdot \xi_0) = \int_{\mathbf{R}} \varphi(ka_{t-s} \cdot \xi_0)\, d\tau(s)$$

is invariant under the action of G.

Proof. To understand the action of $g \in G$ on $\Xi \sim (K/M) \times A$ we write $gk = k'a_{t'}n'$. Since each $a \in A$ normalizes N we have

$$gka_t \cdot \xi_0 = gka_tN \cdot o = k'a_{t'}n'a_tN \cdot o = k'a_{t+t'} \cdot \xi_0\,.$$

Thus the action of g on $\Xi \simeq (K/M) \times A$ induces this fixed translation $a_t \to a_{t+t'}$ on A. This translation commutes with the convolution by τ, so the lemma follows.

Since the operators Λ, $\widehat{\ }$, $^\vee$ in (47) are all G-invariant, it suffices to prove the formula at the origin o. We first consider the case when f is K-invariant, i.e., $f(k \cdot z) \equiv f(z)$. Then by (43),

$$\widehat{f}(a_t \cdot \xi_0) = \int_{\mathbf{R}} f(a_t n_x \cdot o)\, dx\,. \tag{48}$$

Because of (35) we have

$$|z| = \tanh\, d(o,z)\,, \quad \cosh^2\, d(o,z) = (1 - |z|^2)^{-1}\,. \tag{49}$$

Since

$$a_t n_x \cdot o = (\operatorname{sh} t - ix\, e^t)/(\operatorname{ch} t - ix\, e^t)$$

(49) shows that the distance $s = d(o, a_t n_x \cdot o)$ satisfies

$$\operatorname{ch}^2 s = \operatorname{ch}^2 t + x^2 e^{2t}\,. \tag{50}$$

Thus defining F on $[1, \infty)$ by

$$F(\operatorname{ch}^2 s) = f(\tanh\, s)\,, \tag{51}$$

we have

$$F'(\operatorname{ch}^2 s) = f'(\tanh\, s)(2\operatorname{sh} s \operatorname{ch}^3 s)^{-1}$$

so, since $f'(0) = 0$, $\lim_{u \to 1} F'(u)$ exists. The transform (48) now becomes (with $xe^t = y$)

$$e^t \widehat{f}(a_t \cdot \xi_0) = \int_{\mathbf{R}} F(\operatorname{ch}^2 t + y^2)\, dy\,. \tag{52}$$

We put

$$\varphi(u) = \int_{\mathbf{R}} F(u + y^2)\, dy$$

and invert this as follows:

$$\begin{aligned} \int_{\mathbf{R}} \varphi'(u + z^2)\, dz &= \int_{\mathbf{R}^2} F'(u + y^2 + z^2)\, dy\, dz \\ &= 2\pi \int_0^\infty F'(u + r^2) r\, dr = \pi \int_0^\infty F'(u + \rho)\, d\rho\,, \end{aligned}$$

so

$$-\pi F(u) = \int_{\mathbf{R}} \varphi'(u + z^2)\, dz\,.$$

In particular,

$$\begin{aligned} f(o) &= -\frac{1}{\pi}\int\limits_{\mathbf{R}} \varphi'(1+z^2)\,dz = -\frac{1}{\pi}\int\limits_{\mathbf{R}} \varphi'(\mathrm{ch}^2\tau)\mathrm{ch}\tau\,d\tau\,, \\ &= -\frac{1}{\pi}\int\limits_{\mathbf{R}}\int\limits_{\mathbf{R}} F'(\mathrm{ch}^2 t + y^2)\,dy\,\mathrm{ch}\,t\,dt\,, \end{aligned}$$

so

$$f(o) = -\frac{1}{2\pi}\int\limits_{\mathbf{R}} \frac{d}{dt}(e^t\widehat{f}(a_t\cdot\xi_0))\frac{dt}{\mathrm{sh}\,t}\,.$$

Since $(e^t\widehat{f})(a_t\cdot\xi_0)$ is even (cf. (52)), its derivative vanishes at $t=0$, so the integral is well defined. With T as in (45), the last formula can be written

$$f(o) = \frac{1}{\pi}T_t'(e^t\widehat{f}(a_t\cdot\xi_0))\,, \tag{53}$$

the prime indicating derivative. If f is not necessarily K-invariant we use (53) on the average

$$f^\natural(z) = \int\limits_K f(k\cdot z)\,dk = \frac{1}{2\pi}\int\limits_0^{2\pi} f(k_\theta\cdot z)\,d\theta\,.$$

Since $f^\natural(o) = f(o)$, (53) implies

$$f(o) = \frac{1}{\pi}\int\limits_{\mathbf{R}} [e^t(f^\natural)\widehat{\ }(a_t\cdot\xi_0)]\,dT'(t)\,. \tag{54}$$

This can be written as the convolution at $t=0$ of $(f^\natural)\widehat{\ }(a_t\cdot\xi_0)$ with the image of the distribution e^tT_t' under $t\to -t$. Since T' is even the right hand side of (54) is the convolution at $t=0$ of $\widehat{f^\natural}$ with $e^{-t}T_t'$. Thus by (46),

$$f(o) = \frac{1}{\pi}(\Lambda\widehat{f^\natural})(\xi_0)\,.$$

Since Λ and $\widehat{\ }$ commute with the K action this implies

$$f(o) = \frac{1}{\pi}\int\limits_K (\Lambda\widehat{f})(k\cdot\xi_0) = \frac{1}{\pi}(\Lambda\widehat{f})^\vee(o)$$

and this proves the theorem.

Theorem 4.4 is of course the exact analog to Theorem 3.6 in Chapter I, although we have not specified the decay conditions for f needed in generalizing Theorem 4.4.

D. The Poisson Integral as a Radon Transform

Here we preserve the notation introduced for the hyperbolic plane $\mathbf{H}^2$. Now we consider the homogeneous spaces

$$X = G/MAN\,, \quad \Xi = G/K\,. \tag{55}$$

Then Ξ is the disk $D : |z| < 1$. On the other hand, X is identified with the boundary $B : |z| = 1$, because when G acts on B, MAN is the subgroup fixing the point $z = 1$. Since $G = KAN$, each coset $gMAN$ intersects eK. Thus each $x \in X$ is incident to each $\xi \in \Xi$. Our abstract Radon transform (9) now takes the form

$$\begin{aligned} \widehat{f}(gK) &= \int\limits_{K/M} f(gkMAN)\,dk_M = \int\limits_B f(g\cdot b)\,db\,, \\ &= \int\limits_B f(b)\frac{d(g^{-1}\cdot b)}{db}\,db\,. \end{aligned} \tag{56}$$

Writing g^{-1} in the form

$$g^{-1} : \zeta \to \frac{\zeta - z}{-\overline{z}\zeta + 1}\,, \quad g^{-1}\cdot e^{i\theta} = e^{i\varphi}\,,$$

we have

$$e^{i\varphi} = \frac{e^{i\theta} - z}{-\overline{z}e^{i\theta} + 1}\,, \quad \frac{d\varphi}{d\theta} = \frac{1 - |z|^2}{|z - e^{i\theta}|}\,,$$

and this last expression is the classical Poisson kernel. Since $gK = z$, (56) becomes the classical Poisson integral

$$\widehat{f}(z) = \int\limits_B f(b)\frac{1 - |z|^2}{|z - b|^2}\,db\,. \tag{57}$$

Theorem 4.6. *The Radon transform $f \to \widehat{f}$ for the homogeneous spaces (55) is the classical Poisson integral (57). The inversion is given by the classical Schwarz theorem*

$$f(b) = \lim_{z \to b} \widehat{f}(z)\,, \quad f \in C(B)\,, \tag{58}$$

solving the Dirichlet problem for the disk.

We repeat the geometric proof of (58) from our booklet [1981] since it seems little known and is considerably shorter than the customary solution in textbooks of the Dirichlet problem for the disk. In (58) it suffices to consider the case $b = 1$. Because of (56),

$$\begin{aligned} \widehat{f}(\tanh t) &= \widehat{f}(a_t \cdot 0) = \frac{1}{2\pi} \int_0^{2\pi} f(a_t \cdot e^{i\theta})\, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} f\left(\frac{e^{i\theta} + \tanh t}{\tanh t\, e^{i\theta} + 1} \right) d\theta\,. \end{aligned}$$

Letting $t \to +\infty$, (58) follows by the dominated convergence theorem.

The range question A for $f \to \widehat{f}$ is also answered by classical results for the Poisson integral; for example, the classical characterization of the Poisson integrals of bounded functions now takes the form

$$L^\infty(B)^\wedge = \{\varphi \in L^\infty(\Xi) \,:\, L\varphi = 0\}\,. \tag{59}$$

The range characterization (59) is of course quite analogous to the range characterization for the X-ray transform described in Theorem 6.9, Chapter I. Both are realizations of the general expectations at the end of §2 that when $\dim X < \dim \Xi$ the range of the transform $f \to \widehat{f}$ should be given as the kernel of some differential operators. The analogy between (59) and Theorem 6.9 is even closer if we recall Gonzalez' theorem [1990b] that if we view the X-ray transform as a Radon transform between two homogeneous spaces of $\mathbf{M}(3)$ (see next example) then the range (91) in Theorem 6.9, Ch. I, can be described as the null space of a differential operator which is invariant under $\mathbf{M}(3)$. Furthermore, the dual transform $\varphi \to \check{\varphi}$ maps $\mathcal{E}(\Xi)$ onto $\mathcal{E}(X)$. (See Corollary 4.8 below.)

Furthermore, John's mean value theorem for the X-ray transform (Corollary 6.12, Chapter I) now becomes the exact analog of Gauss' mean-value theorem for harmonic functions.

From a non-Euclidean point of view, Godement's mean-value theorem (Ch. VI, §1) is even closer analog to John's theorem. Because of the special form of the Laplace–Beltrami operator in $\mathbf{H}^2$ non-Euclidean harmonic functions are the same as the usual ones (this fails for $\mathbf{H}^n$ $n > 2$). Also non-Euclidean circles are Euclidean circles (because the map (33) sends circles into circles). However, the mean-value theorem is different, namely,

$$u(z) = \int_S u(\zeta)\, d\mu(\zeta)$$

for a harmonic function u, z being the non-Euclidean center of the circle S and μ being the normalized non-Euclidean arc length measure on X, according to (32). However, this follows readily from the Gauss' mean-value theorem using a conformal map of D.

What is the dual transform $\varphi \to \check{\varphi}$ for the pair (55)? The invariant measure on $MAN/M = AN$ is the functional

$$\varphi \to \int_{AN} \varphi(an \cdot o)\, da\, dn\,. \tag{60}$$

The right hand side is just $\check{\varphi}(b_0)$ where $b_0 = eMAN$. If $g = a'n'$ the measure (58) is seen to be invariant under g. Thus it is a constant multiple of the surface element $dz = (1-x^2-y^2)^{-2}\, dx\, dy$ defined by (32). Since the maps $t \to a_t \cdot o$ and $x \to n_x \cdot o$ were seen to be isometries, this constant factor is 1. Thus the measure (60) is invariant under each $g \in G$. Writing $\varphi_g(z) = \varphi(g \cdot z)$ we know $(\varphi_g)^\vee = \check{\varphi}_g$ so

$$\check{\varphi}(g \cdot b_0) = \int_{AN} \varphi_g(an)\, da\, dn = \check{\varphi}(b_0)\,.$$

Thus the dual transform $\varphi \to \check{\varphi}$ assigns to each $\varphi \in \mathcal{D}(\Xi)$ its integral over the disk.

Table II.1 summarizes the various results mentioned above about the Poisson integral and the X-ray transform. The inversion formulas and the ranges show subtle analogies as well as strong differences. The last item in the table comes from Corollary 4.8 below for the case $n = 3$, $d = 1$.

E. The d-plane Transform

We now review briefly the d-plane transform from a group theoretic standpoint. As in (1) we write

$$X = \mathbf{R}^n = \mathbf{M}(n)/\mathbf{O}(n)\,, \quad \Xi = \mathbf{G}(d,n) = \mathbf{M}(n)/(\mathbf{M}(d)\times\mathbf{O}(n-d))\,, \tag{61}$$

where $\mathbf{M}(d)\times\mathbf{O}(n-d)$ is the subgroup of $\mathbf{M}(n)$ preserving a certain d-plane ξ_0 through the origin. Since the homogeneous spaces

$$\mathbf{O}(n)/\mathbf{O}(n) \cap (\mathbf{M}(d) \times \mathbf{O}(n-d)) = \mathbf{O}(n)/(\mathbf{O}(d) \times \mathbf{O}(n-d))$$

and

$$(\mathbf{M}(d) \times \mathbf{O}(n-d))/\mathbf{O}(n) \cap (\mathbf{M}(d) \times \mathbf{O}(n-d)) = \mathbf{M}(d)/\mathbf{O}(d)$$

have unique invariant measures the group-theoretic transforms (9) reduce to the transforms (57), (58) in Chapter I. The range of the d-plane transform is described by Theorem 6.3 and the equivalent Theorem 6.5 in Chapter I. It was shown by Richter [1986a] that the differential operators in Theorem 6.5 could be replaced by $\mathbf{M}(n)$-induced second order differential

	Poisson Integral	*X-ray Transform*
Coset spaces	$X = \mathbf{SU}(1,1)/MAN$ $\Xi = \mathbf{SU}(1,1)/K$	$X = \mathbf{M}(3)/\mathbf{O}(3)$ $\Xi = \mathbf{M}(3)/(\mathbf{M}(1) \times \mathbf{O}(2))$
$f \to \widehat{f}$	$\widehat{f}(z) = \int_B f(b) \frac{1-\|z\|^2}{\|z-b\|^2} db$	$\widehat{f}(\ell) = \int_\ell f(p)\, dm(p)$
$\varphi \to \check{\varphi}$	$\check{\varphi}(x) = \int_\Xi \varphi(\xi)\, d\xi$	$\check{\varphi}(x) =$ average of φ over set of ℓ through x
Inversion	$f(b) = \lim_{z \to b} \widehat{f}(z)$	$f = \frac{1}{\pi}(-L)^{1/2}((\widehat{f})^\vee)$
Range of $f \to \widehat{f}$	$L^\infty(X)\widehat{\ } =$ $\{\varphi \in L^\infty(\Xi) : L\varphi = 0\}$	$\mathcal{D}(X)\widehat{\ } =$ $\{\varphi \in \mathcal{D}(\Xi) : \Lambda(\|\xi - \eta\|^{-1}\varphi) = 0\}$
Range characterization	Gauss' mean value theorem	Mean value property for hyperboloids of revolution
Range of $\varphi \to \check{\varphi}$	$\mathcal{E}(\Xi)^\vee = \mathbf{C}$	$\mathcal{E}(\Xi)^\vee = \mathcal{E}(X)$

TABLE II.1. Analogies between the Poisson Integral and the X-ray Transform.

operators and then Gonzalez [1990b] showed that the whole system could be replaced by a single fourth order $\mathbf{M}(n)$-invariant differential operator on Ξ.

Writing (61) for simplicity in the form

$$X = G/K\,, \quad \Xi = G/H \tag{62}$$

we shall now discuss the range question for the dual transform $\varphi \to \check{\varphi}$ by invoking the d-plane transform on $\mathcal{E}'(X)$.

Theorem 4.7. *Let $\mathcal{N}$ denote the kernel of the dual transform on $\mathcal{E}(\Xi)$. Then the range of $S \to \widehat{S}$ on $\mathcal{E}'(X)$ is given by*

$$\mathcal{E}'(X)\widehat{\ } = \{\Sigma \in \mathcal{E}'(\Xi) \,:\, \Sigma(\mathcal{N}) = 0\}\,.$$

The inclusion $\subset$ is clear from the definitions (14),(15) and Proposition 2.5. The converse is proved by the author in [1983a] and [1994b], Ch. I, §2 for $d = n-1$; the proof is also valid for general d.

For Fréchet spaces E and F one has the following classical result. A continuous mapping $\alpha : E \to F$ is surjective if the transpose ${}^t\alpha : F' \to E'$ is injective and has a closed image. Taking $E = \mathcal{E}(\Xi)$, $F = \mathcal{E}(X)$, α as the dual transform $\varphi \to \check{\varphi}$, the transpose ${}^t\alpha$ is the Radon transform on

$\mathcal{E}'(X)$. By Theorem 4.7, ${}^t\alpha$ does have a closed image and by Theorem 5.5, Ch. I (extended to any d) ${}^t\alpha$ is injective. Thus we have the following result (Hertle [1984] for $d = n-1$) expressing the surjectivity of α.

Corollary 4.8. *Every $f \in \mathcal{E}(\mathbf{R}^n)$ is the dual transform $f = \check{\varphi}$ of a smooth d-plane function φ.*

F. Grassmann Manifolds

We consider now the (affine) Grassmann manifolds $\mathbf{G}(p,n)$ and $\mathbf{G}(q,n)$ where $p+q = n-1$. If $p = 0$ we have the original case of points and hyperplanes. Both are homogeneous spaces of the group $\mathbf{M}(n)$ and we represent them accordingly as coset spaces

$$X = \mathbf{M}(n)/H_p\,, \quad \Xi = \mathbf{M}(n)/H_q\,. \tag{63}$$

Here we take H_p as the isotropy group of a p-plane x_0 through the origin $0 \in \mathbf{R}^n$, H_q as the isotropy group of a q-plane ξ_0 through 0, *perpendicular to* x_0. Then

$$H_p = \mathbf{M}(p) \times \mathbf{O}(n-p)\,, \quad H_q = \mathbf{M}(q) \times \mathbf{O}(n-q)\,.$$

Also

$$H_q \cdot x_0 = \{x \in X \,:\, x \perp \xi_0, x \cap \xi_0 \neq \emptyset\}\,,$$

the set of p-planes intersecting ξ_0 orthogonally. It is then easy to see that

$$x \text{ is incident to } \xi \Leftrightarrow x \perp \xi\,, \quad x \cap \xi \neq \emptyset\,.$$

Consider as in Chapter I, §6 the mapping

$$\pi \,:\, \mathbf{G}(p,n) \to \mathbf{G}_{p,n}$$

given by parallel translating a p-plane to one such through the origin. If $\sigma \in \mathbf{G}_{p,n}$, the fiber $F = \pi^{-1}(\sigma)$ is naturally identified with the Euclidean space $\sigma^\perp$. Consider the linear operator $\square_p$ on $\mathcal{E}(\mathbf{G}(p,n))$ given by

$$(\square_p f)|F = L_F(f|F)\,. \tag{64}$$

Here L_F is the Laplacian on F and bar denotes restriction. Then one can prove that $\square_p$ is a differential operator on $\mathbf{G}(p,n)$ which is invariant under the action of $\mathbf{M}(n)$. Let $f \to \widehat{f}$, $\varphi \to \check{\varphi}$ be the Radon transform and its dual corresponding to the pair (61). Then $\widehat{f}(\xi)$ represents the integral of f over all p-planes x intersecting ξ under a right angle. For n odd this is inverted as follows (Gonzalez [1984, 1987]).

Theorem 4.9. *Let $p, q \in \mathbf{Z}^+$ such that $p+q+1 = n$ is odd. Then the transform $f \to \widehat{f}$ from $\mathbf{G}(p,n)$ to $\mathbf{G}(q,n)$ is inverted by the formula*

$$C_{p,q} f = ((\square_q)^{(n-1)/2} \widehat{f})^\vee\,, \quad f \in \mathcal{D}(\mathbf{G}(p,n))$$

where $C_{p,q}$ is a constant.

If $p = 0$ this reduces to Theorem 3.6, Ch. I.

G. Half-lines in a Half-plane

In this example X denotes the half-plane $\{(a,b) \in \mathbf{R}^2 \,:\, a > 0\}$ viewed as a subset of the plane $\{(a,b,1) \in \mathbf{R}^3\}$. The group G of matrices

$$(\alpha,\beta,\gamma) = \begin{pmatrix} \alpha & 0 & 0 \\ \beta & 1 & \gamma \\ 0 & 0 & 1 \end{pmatrix} \in \mathbf{GL}(3,\mathbf{R})\,, \quad \alpha > 0$$

acts transitively on X with the action

$$(\alpha,\beta,\gamma) \odot (a,b) = (\alpha a, \beta a + b + \gamma)\,.$$

This is the restriction of the action of $\mathbf{GL}(3,\mathbf{R})$ on $\mathbf{R}^3$. The isotropy group of the point $x_0 = (1,0)$ is the group

$$K = \{(1,\beta,-\beta) \,:\, \beta \in \mathbf{R}\}\,.$$

Let Ξ denote the set of half-lines in X which end on the boundary $\partial X = 0 \times \mathbf{R}$. These lines are given by

$$\xi_{v,w} = \{(t, v + tw) \,:\, t > 0\}$$

for arbitrary $v, w \in \mathbf{R}$. Thus Ξ can be identified with $\mathbf{R} \times \mathbf{R}$. The action of G on X induces a transitive action of G on Ξ which is given by

$$(\alpha,\beta,\gamma)\Diamond(v,w) = (v+\gamma, \frac{w+\beta}{\alpha})\,.$$

(Here we have for simplicity written (v,w) instead of $\xi_{v,w}$.) The isotropy group of the point $\xi_{(0,0)}$ (the x-axis) is

$$H = \{(\alpha,0,0) \,:\, \alpha > 0\} = \mathbf{R}_+^\times\,,$$

the multiplicative group of the positive real numbers. Thus we have the identifications

$$X = G/K\,, \quad \Xi = G/H\,. \tag{65}$$

The group $K \cap H$ is now trivial so the Radon transform and its dual for the double fibration in (63) are defined by

$$\widehat{f}(gH) = \int_H f(ghK)\,dh\,, \tag{66}$$

$$\check{\varphi}(gK) = \chi(g)\int_K \varphi(gkH)\,dk\,, \tag{67}$$

where χ is the homomorphism $(\alpha, \beta, \gamma) \to \alpha^{-1}$ of G onto $\mathbf{R}_+^\times$. The reason for the presence of χ is that we wish Proposition 2.2 to remain valid even if G is not unimodular. In (66) and (67) we have the Haar measures

$$dk_{(1,\beta-\beta)} = d\beta\,, \quad dh_{(\alpha,0,0)} = d\alpha/\alpha\,. \tag{68}$$

Also, if $g = (\alpha, \beta, \gamma)$, $h = (a, 0, 0)$, $k = (1, b, -b)$ then

$$\begin{aligned} gH &= (\gamma, \beta/\alpha)\,, &\qquad ghK &= (\alpha a, \beta a + \gamma) \\ gK &= (\alpha, \beta + \gamma)\,, &\qquad gkH &= (-b+\gamma, \tfrac{b+\beta}{\alpha}) \end{aligned}$$

so (66)–(67) become

$$\begin{aligned} \widehat{f}(\gamma, \beta/\alpha) &= \int_{\mathbf{R}^+} f(\alpha a, \beta a + \gamma)\frac{da}{a} \\ \check{\varphi}(\alpha, \beta + \gamma) &= \alpha^{-1}\int_{\mathbf{R}} \varphi(-b+\gamma, \tfrac{b+\beta}{\alpha})\, db\,. \end{aligned}$$

Changing variables these can be written

$$\widehat{f}(v, w) = \int_{\mathbf{R}^+} f(a, v + aw)\frac{da}{a}\,, \tag{69}$$

$$\check{\varphi}(a, b) = \int_{\mathbf{R}} \varphi(b - as, s)\, ds \qquad a > 0\,. \tag{70}$$

Note that in (69) the integration takes place over all points on the line $\xi_{v,w}$ and in (70) the integration takes place over the set of lines $\xi_{b-as,s}$ all of which pass through the point (a, b). This is an *a posteriori* verification of the fact that our incidence for the pair (65) amounts to $x \in \xi$.

From (69)–(70) we see that $f \to \widehat{f}, \varphi \to \check{\varphi}$ are adjoint relative to the measures $\frac{da}{a}\, db$ and $dv\, dw$:

$$\int_{\mathbf{R}}\int_{\mathbf{R}_+^\times} f(a, b)\check{\varphi}(a, b)\frac{da}{a}\, db = \int_{\mathbf{R}}\int_{\mathbf{R}} \widehat{f}(v, w)\varphi(v, w)\, dv\, dw\,. \tag{71}$$

The proof is a routine computation.

We recall (Chapter VII) that $(-L)^{1/2}$ is defined on the space of rapidly decreasing functions on $\mathbf{R}$ by

$$((-L)^{1/2}\psi)^{\sim}\ (\tau) = |\tau|\widetilde{\psi}(\tau) \tag{72}$$

and we define Λ on $\mathcal{S}(\Xi)(= \mathcal{S}(\mathbf{R}^2))$ by having $(-L)^{1/2}$ only act on the second variable:

$$(\Lambda\varphi)(v, w) = ((-L)^{1/2}\varphi(v, \cdot))(w)\,. \tag{73}$$

Viewing $(-L)^{1/2}$ as the Riesz potential I^{-1} on $\mathbf{R}$ (Chapter VII, §6) it is easy to see that if $\varphi_c(v,w) = \varphi(v, \frac{w}{c})$ then

$$\Lambda\varphi_c = |c|^{-1}(\Lambda\varphi)_c\,. \tag{74}$$

The Radon transform (66) is now inverted by the following theorem.

Theorem 4.10. *Let $f \in \mathcal{D}(X)$. Then*

$$f = \frac{1}{2\pi}(\Lambda\widehat{f})^{\vee}\,.$$

Proof. In order to use the Fourier transform $F \to \widetilde{F}$ on $\mathbf{R}^2$ and on $\mathbf{R}$ we need functions defined on all of $\mathbf{R}^2$. Thus we define

$$f^*(a,b) = \begin{cases} \frac{1}{a}f(\frac{1}{a}, \frac{-b}{a}) & a > 0\,, \\ 0 & a \leq 0\,. \end{cases}$$

Then

$$\begin{aligned} f(a,b) &= \frac{1}{a}f^*\left(\frac{1}{a}, -\frac{b}{a}\right) \\ &= a^{-1}(2\pi)^{-2}\iint \widetilde{f^*}(\xi,\eta)e^{i(\frac{\xi}{a}-\frac{b\eta}{a})}\,d\xi\,d\eta \\ &= (2\pi)^{-2}\iint \widetilde{f^*}(a\xi+b\eta,\eta)e^{i\xi}\,d\xi\,d\eta \\ &= a(2\pi)^{-2}\iint |\xi|\widetilde{f^*}((a+ab\eta)\xi, a\eta\xi)e^{i\xi}\,d\xi\,d\eta\,. \end{aligned}$$

Next we express the Fourier transform in terms of the Radon transform. We have

$$\begin{aligned} \widetilde{f^*}((a+ab\eta)\xi, a\eta\xi) &= \iint f^*(x,y)e^{-ix(a+ab\eta)\xi}e^{-iya\eta\xi}\,dx\,dy \\ &= \int_{\mathbf{R}}\int_{x\geq 0} \frac{1}{x}f\left(\frac{1}{x}, -\frac{y}{x}\right)e^{-ix(a+ab\eta)\xi}e^{-iya\eta\xi}\,dx\,dy \\ &= \int_{\mathbf{R}}\int_{x\geq 0} f\left(\frac{1}{x}, b+\frac{1}{\eta}+\frac{z}{x}\right)e^{iza\eta\xi}\,\frac{dx}{x}\,dz\,. \end{aligned}$$

This last expression is

$$\int_{\mathbf{R}} \widehat{f}(b+\eta^{-1},z)e^{iza\eta\xi}\,dz = (\widehat{f})^{\sim}(b+\eta^{-1}, -a\eta\xi)\,,$$

where $\sim$ denotes the 1-dimensional Fourier transform (in the second variable). Thus

$$f(a,b) = a(2\pi)^{-2}\iint |\xi|(\widehat{f})^{\sim}(b+\eta^{-1}, -a\eta\xi)e^{i\xi}\,d\xi\,d\eta\,.$$

However $\widetilde{F}(c\xi) = |c|^{-1}(F_c)^{\sim}(\xi)$, so by (74)

$$\begin{aligned} f(a,b) &= a(2\pi)^{-2}\iint |\xi|((\widehat{f})_{a\eta})^{\sim}(b+\eta^{-1},-\xi)e^{i\xi}\,d\xi|a\eta|^{-1}\,d\eta \\ &= (2\pi)^{-1}\int \Lambda((\widehat{f})_{a\eta})(b+\eta^{-1},-1)|\eta|^{-1}\,d\eta \\ &= (2\pi)^{-1}\int |a\eta|^{-1}(\Lambda\widehat{f})_{a\eta}(b+\eta^{-1},-1)|\eta|^{-1}\,d\eta \\ &= a^{-1}(2\pi)^{-1}\int (\Lambda\widehat{f})(b+\eta^{-1},-(a\eta)^{-1})\eta^{-2}\,d\eta\,, \end{aligned}$$

so

$$\begin{aligned} f(a,b) &= (2\pi)^{-1}\int_{\mathbf{R}} (\Lambda\widehat{f})(b-av,v)\,dv \\ &= (2\pi)^{-1}(\Lambda\widehat{f})^{\vee}(a,b)\,. \end{aligned}$$

proving the theorem.

Remark 4.11. It is of interest to compare this theorem with Theorem 3.8, Ch. I. If $f \in \mathcal{D}(X)$ is extended to all of $\mathbf{R}^2$ by defining it 0 in the left half plane then Theorem 3.8 does give a formula expressing f in terms of its integrals over half-lines in a strikingly similar fashion. Note however that while the operators $f \to \widehat{f}, \varphi \to \check{\varphi}$ are in the two cases defined by integration over the same sets (points on a half-line, half-lines through a point) the measures in the two cases are different. Thus it is remarkable that the inversion formulas look exactly the same.

H. Theta Series and Cusp Forms

Let G denote the group $\mathbf{SL}(2,\mathbf{R})$ of 2×2 matrices of determinant one and Γ the *modular group* $\mathbf{SL}(2,\mathbf{Z})$. Let N denote the unipotent group $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ where $n \in \mathbf{R}$ and consider the homogeneous spaces

$$X = G/N\,, \qquad \Xi = G/\Gamma\,. \tag{75}$$

Under the usual action of G on $\mathbf{R}^2$, N is the isotropy subgroup of $(1,0)$ so X can be identified with $\mathbf{R}^2 - (0)$, whereas Ξ is of course 3-dimensional.

In number theory one is interested in decomposing the space $L^2(G/\Gamma)$ into G-invariant irreducible subspaces. We now give a rough description of this by means of the transforms $f \to \widehat{f}$ and $\varphi \to \check{\varphi}$.

As customary we put $\Gamma_\infty = \Gamma \cap N$; our transforms (9) then take the form

$$\widehat{f}(g\Gamma) = \sum_{\Gamma/\Gamma_\infty} f(g\gamma N)\,, \quad \check{\varphi}(gN) = \int_{N/\Gamma_\infty} \varphi(gn\Gamma)\,dn_{\Gamma_\infty}\,.$$

Since N/Γ_∞ is the circle group, $\check{\varphi}(gN)$ is just the constant term in the Fourier expansion of the function $n\Gamma_\infty \to \varphi(gn\Gamma)$. The null space $L_d^2(G/\Gamma)$ in $L^2(G/\Gamma)$ of the operator $\varphi \to \check{\varphi}$ is called the space of **cusp forms** and the series for $\widehat{f}$ is called **theta series**. According to Prop. 2.2 they constitute the orthogonal complement of the image $C_c(X)\widehat{}$.

We have now the G-invariant decomposition

$$L^2(G/\Gamma) = L_c^2(G/\Gamma) \oplus L_d^2(G/\Gamma)\,, \tag{76}$$

where ($-$ denoting closure)

$$L_c^2(G/\Gamma) = (C_c(X)\widehat{}\,)^- \tag{77}$$

and as mentioned above,

$$L_d^2(G/\Gamma) = (C_c(X)\widehat{}\,)^\perp\,. \tag{78}$$

It is known (cf. Selberg [1962], Godement [1966]) that the representation of G on $L_c^2(G/\Gamma)$ is the *continuous* direct sum of the irreducible representations of G from the principal series whereas the representation of G on $L_d^2(G/\Gamma)$ is the *discrete* direct sum of irreducible representations each occurring with finite multiplicity.

I. The Plane-to-Line Transform in $\mathbf{R}^3$. The Range

Now we consider the set $\mathbf{G}(2,3)$ of planes in $\mathbf{R}^3$ and the set $\mathbf{G}(1,3)$ of lines. The group $G = \mathbf{M}^+(3)$ of orientation preserving isometries of $\mathbf{R}^3$ acts transitively on both $\mathbf{G}(2,3)$ and $\mathbf{G}(1,3)$. The group $\mathbf{M}^+(3)$ can be viewed as the group of 4×4 matrices

$$\left(\begin{array}{c|c} & x_1 \\ \mathbf{SO}(3) & x_2 \\ & x_3 \\ \hline & 1 \end{array}\right),$$

whose Lie algebra $\mathfrak{g}$ has basis

$$E_i = E_{i4}\ (1 \leq i \leq 3)\,, \quad X_{ij} = E_{ij} - E_{ji}\,, \quad 1 \leq i \leq j \leq 3\,.$$

We have bracket relations

$$[E_i, X_{jk}] = 0 \text{ if } i \neq j,k\,,\ [E_i, X_{ij}] = E_j - E_i\,, \tag{79}$$

$$[X_{ij}, X_{k\ell}] = -\delta_{ik}X_{j\ell} + \delta_{jk}X_{i\ell} + \delta_{i\ell}X_{jk} - \delta_{j\ell}X_{ik}\,. \tag{80}$$

We represent $\mathbf{G}(2,3)$ and $\mathbf{G}(1,3)$ as coset spaces

$$\mathbf{G}(2,3) = G/H\,, \quad \mathbf{G}(1,3) = G/K\,, \tag{81}$$

where

$$H = \text{ stability of group of } \tau_0 \ (x_1, x_2\text{-plane}),$$
$$K = \text{ stability group of } \sigma_0 \ (x_1\text{-axis}).$$

We have $\mathbf{G} = \mathbf{SO}(3)\,\mathbf{R}^3$, $H = \mathbf{SO}_3(2)\,\mathbf{R}^2$, $K = \mathbf{SO}_1(2) \times \mathbf{R}$ the first two being semi-direct products. The subscripts indicate fixing of the x_3-axis and x_1-axis, respectively. The intersection $L = H \cap K = \mathbf{R}$ (the translations along the x_1-axis).

The elements $\tau_0 = eH$ and $\sigma_0 = eK$ are incident for the pair G/H, G/K and $\sigma_0 \subset \tau_0$. Since the inclusion notion is preserved by G we see that

$$\tau = \gamma H \text{ and } \sigma = gK \text{ are incident } \Leftrightarrow \sigma \subset \tau\,.$$

In the double fibration

(82)
$$\begin{tikzcd} & G/L = \{(\sigma,\tau) | \sigma \subset \tau\} \arrow[dl] \arrow[dr] & \\ \mathbf{G}(2,3) = G/H & & G/K = \mathbf{G}(1,3) \end{tikzcd}$$

we see that the transform $\varphi \to \check{\varphi}$ in (9) (Chapter II,§2) is the plane-to-line transform which sends a function on $\mathbf{G}(2,3)$ into a function on lines:

(83)
$$\check{\varphi}(\sigma) = \int_{\tau \ni \sigma} \varphi(\tau)\, d\mu(\tau)\,,$$

the measure $d\mu$ being the normalized measure on the circle.

For the study of the range of (83) it turns out to be simpler to replace G/L by another homogeneous space of G, namely the space of unit vectors $\omega \in \mathbf{S}^2$ with an initial point $x \in \mathbf{R}^3$. We denote this pair by ω_x. The action of G on this space $\mathbf{S}^2 \times \mathbf{R}^3$ is the obvious geometric action of $(u, y) \in \mathbf{SO}(3)\mathbf{R}^3$ on ω_x:

(84)
$$(u, y) \cdot \omega_x = (u \cdot \omega)_{(u \cdot x + y)}\,.$$

The subgroup fixing the North Pole ω_0 on $\mathbf{S}^2$ equals $\mathbf{SO}_3(2)$ so $\mathbf{S}^2 \times \mathbf{R}^3 = G/\mathbf{SO}(2)$. Instead of (82) we consider

$$\begin{tikzcd} & \mathbf{S}^2 \times \mathbf{R}^3 \arrow[dl, "\pi''"'] \arrow[dr, "\pi'"] & \\ \mathbf{G}(2,3) & & \mathbf{G}(1,3) \end{tikzcd}$$

the maps π' and π'' being given by

$$\pi'(\omega_x) = \mathbf{R}\omega + x \quad (\text{line through } x \text{ in direction } \omega),$$
$$\pi''(\omega_x) = \omega^{\perp} + x \quad (\text{plane through } x \perp \omega).$$

The geometric nature of the action (84) shows that π' and π'' commute with the action of G.

For analysis on $\mathbf{S}^2 \times \mathbf{R}^3$ it will be convenient to write ω_x as the pair (ω, x). Note that

$$(\pi')^{-1}(\mathbf{R}\omega + x) = \{(\omega, y) : y - x \in \mathbf{R}\omega\} \tag{85}$$

or equivalently, the set of translates of ω along the line $x + \mathbf{R}\omega$. Also

$$(\pi'')^{-1}(\omega^\perp + x) = \{(\omega, z) : x - z \in \omega^\perp\}\,, \tag{86}$$

the set of translates of ω with initial point on the plane through x perpendicular to ω.

Let ∇_x denote the gradient $(\partial/\partial x_1, \partial/\partial x_2, \partial/\partial x_3)$. Let $F \in \mathcal{E}(\mathbf{S}^2 \times \mathbf{R}^3)$. Then if θ is a unit vector and $\langle\, ,\, \rangle$ the standard inner product on $\mathbf{R}^3$,

$$\frac{d}{dt} F(\omega, x + t\theta) = \langle(\nabla_x F(\omega, x + t\theta)), \theta\rangle\,. \tag{87}$$

Thus for $\Psi \in \mathcal{E}(\mathbf{S}^2 \times \mathbf{R}^3)$,

$$\Psi(\omega, x + t\omega) = \Psi(\omega, x) \quad (t \in \mathbf{R}) \Leftrightarrow (\nabla_x \Psi)(\omega, x) \perp \omega\,. \tag{88}$$

Lemma 4.12. *A function $\Psi \in \mathcal{E}(\mathbf{S}^2 \times \mathbf{R}^3)$ has the form $\Psi = \psi \circ \pi'$ with $\psi \in \mathcal{E}(\mathbf{G}(1,3))$ if and only if*

$$\Psi(\omega, x) = \Psi(-\omega, x)\,, \quad \nabla_x \Psi(\omega, x) \perp \omega\,. \tag{89}$$

Proof. Clearly, if $\psi \in \mathcal{E}(\mathbf{G}(1,3))$ then Ψ has the property stated. Conversely, if Ψ satisfies the conditions (89) it is constant on each set (85).

Lemma 4.13. *A function $\Phi \in \mathcal{E}(\mathbf{S}^2 \times \mathbf{R}^3)$ has the form $\Phi = \varphi \circ \pi''$ with $\varphi \in \mathcal{E}(\mathbf{G}(2,3))$ if and only if*

$$\Phi(\omega, x) = \Phi(-\omega, x)\,, \quad \nabla_x \Phi(\omega, x) \in \mathbf{R}\omega\,. \tag{90}$$

Proof. If $\varphi \in \mathcal{E}(\mathbf{G}(2,3))$ then (87) for $F = \Phi$ implies

$$\frac{d}{dt} \Phi(\omega, x + t\theta) = 0 \text{ for each } \theta \in \omega^\perp$$

so (90) holds. Conversely, if Φ satisfies (90) then by (87) for $F = \Phi$, Φ is constant on each set (86).

We consider now the action of G on $\mathbf{S}^2 \times \mathbf{R}^3$. The Lie algebra $\mathfrak{g}$ is $\mathfrak{so}(3) + \mathbf{R}^3$, where $\mathfrak{so}(3)$ consists of the 3×3 real skew-symmetric matrices.

For $X \in \mathfrak{so}(3)$ and $\Psi \in \mathcal{E}(\mathbf{S}^2 \times \mathbf{R}^3)$ we have by Ch. VIII, (12),

$$\begin{aligned}(\lambda(X)\Psi)(\omega, x) &= \left\{\frac{d}{dt}\Psi(\exp(-tX)\cdot\omega, \exp(-tX)\cdot\omega)\right\}_{t=0} \\ &= \left\{\frac{d}{dt}\Psi(\exp(-tX)\cdot\omega, x)\right\}_{t=0} \\ &\qquad + \left\{\frac{d}{dt}\Psi(\omega, \exp(-tX)\cdot x)\right\}_{t=0}\end{aligned}$$

so

$$\lambda(X)\Psi(\omega, x) = X_\omega\Psi(\omega, x) + X_x\Psi(\omega, x)\,, \tag{91}$$

where X_ω and X_x are tangent vectors to the circles $\exp(-tX)\cdot\omega$ and $\exp(-tX)\cdot x$ in $\mathbf{S}^2$ and $\mathbf{R}^3$, respectively.

For $v \in \mathbf{R}^3$ acting on $\mathbf{S}^2 \times \mathbf{R}^3$ we have

$$(\lambda(v)\Psi)(\omega, x) = \left\{\frac{d}{dt}\Psi(\omega, x - tv)\right\}_{t=0} = -\langle\nabla_x\Psi(\omega, x), v\rangle\,. \tag{92}$$

For $X_{12} = E_{12} - E_{21}$ we have

$$\exp tX_{12} = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ etc.}$$

so if $f \in \mathcal{E}(\mathbf{R}^3)$

$$(\lambda(X_{ij})f)(x) = \left\{\frac{d}{dt}f(\exp(-tX_{ij})\cdot x)\right\} = x_i\frac{\partial f}{\partial x_j} - x_j\frac{\partial f}{\partial x_i}\,. \tag{93}$$

Given $\ell \in \mathbf{G}(1,3)$ let $\check{\ell}$ denote the set of 2-planes in $\mathbf{R}^3$ containing it. If $\ell = \pi'(\sigma, x)$ then $\check{\ell} = \{\pi''(\omega, x) : \omega \in \mathbf{S}^2, \omega \perp \sigma\}$, which is identified with the great circle $A(\sigma) = \sigma^\perp \cap \mathbf{S}^2$. We give $\check{\ell}$ the measure μ_ℓ corresponding to the arc-length measure on $A(\sigma)$. In this framework, the plane-to-line transform (83) becomes

$$(R\varphi)(\ell) = \int_{\xi\in\check{\ell}} \varphi(\xi)\, d\mu_\ell(\xi) \tag{94}$$

for $\varphi \in \mathcal{E}(\mathbf{G}(2,3))$, $\ell \in \mathbf{G}(1,3)$. Expressing this on $\mathbf{S}^2 \times \mathbf{R}^3$ we have with $\Phi = \varphi \circ \pi''$

$$(R\varphi \circ \pi')(\sigma, x) = \frac{1}{2\pi}\int_{A(\sigma)} \Phi(\omega, x)\, d_\sigma(\omega)\,, \tag{95}$$

where d_σ represents the arc-length measure on $A(\sigma)$.

We consider now the basis E_i, X_{jk} of the Lie algebra $\mathfrak{g}$. For simplicity we drop the tilde in $\widetilde{E}_i$ and $\widetilde{X}_{jk}$.

Lemma 4.14. (Richter.) *The operator*

$$D = E_1 X_{23} - E_2 X_{13} + E_3 X_{12}$$

belongs to the center $\mathbf{Z}(G)$ *of* $\mathbf{D}(G)$.

Proof. First note by the above commutation relation that the factors in each summand commute. Thus D commutes with E_i. The commutation with each X_{ij} follows from the above commutation relations (79)–(80).

Because of Propositions 1.7 and 2.3 in Ch. VIII, D induces G-invariant operators $\lambda(D)$, $\lambda'(D)$ and $\lambda''(D)$ on $\mathbf{S}^2 \times \mathbf{R}^3$, $\mathbf{G}(1,3)$ and $\mathbf{G}(2,3)$, respectively.

Lemma 4.15. *(i)* $\lambda(D) = 0$ *on* $\mathcal{E}(\mathbf{R}^3)$.

(ii) $\lambda''(D) = 0$ *on* $\mathcal{E}(\mathbf{G}(2,3))$.

Proof. Part (i) follows from $(\lambda(E_i)f)(x) = -\partial f/\partial x_i$ and the formula (93). For (ii) we take $\varphi \in \mathcal{E}(\mathbf{G}(2,3))$ and put $\Phi = \varphi \circ \pi''$. Since π'' commutes with the G-action, we have

$$\Phi(g \cdot (\omega, x)) = \varphi(g \cdot \pi''(\omega, x))$$

so by (13) in Ch.VIII,

$$\lambda(D)\Phi = \lambda''(D)\varphi \circ \pi'' . \tag{96}$$

By (91)–(92) we have

$$\begin{aligned} \lambda(D)\Phi(\omega, x) &= (\lambda(E_1 X_{23} - E_2 X_{13} + E_3 X_{12}))_x \Phi(\omega, x) \\ &+ \big[\lambda(E_1)_x \lambda(X_{23})_\omega - \lambda(E_2)_x \lambda(X_{13})_\omega + \lambda(E_3)_x \lambda(X_{12})_\omega\big] \Phi(\omega, x) . \end{aligned} \tag{97}$$

By Part (i) the first of the two terms vanishes. In the second term we exchange E_i and X_{jk}. Recalling that $\nabla_x \Phi(\omega, x)$ equals $h(\omega, x)\,\omega$ (h a scalar) we have

$$\lambda(E_i)_x \Phi(\omega, x) = h(\omega, x)\,\omega_i , \quad 1 \leq i \leq 3 .$$

Since $\exp tX_{23}$ fixes ω_1 we have $\lambda(X_{23})\omega_1 = 0$ etc. Putting this together we deduce

$$\begin{aligned} \lambda(D)\Phi(\omega, x) = &-\omega_1 \lambda(X_{23})_\omega\, h(\omega, x) + \omega_2 \lambda(X_{13})_\omega\, h(\omega, x) \\ &- \omega_3 \lambda(X_{12})_\omega\, h(\omega, x) . \end{aligned} \tag{98}$$

Part (ii) will now follow from the following.

Lemma 4.16. *Let $u \in \mathcal{E}(\mathbf{S}^2)$. Let $\mu(X_{ij})$ denote the restriction of the vector field $\lambda(X_{ij})$ to the sphere. Then*

$$(\omega_1\, \mu(X_{23}) - \omega_2\, \mu(X_{13}) + \omega_3\, \mu(X_{12}))u = 0\,.$$

Proof. For a fixed $\epsilon > 0$ extend u to a smooth function $\widetilde{u}$ on the shell $S^2_\epsilon : 1-\epsilon < \|x\| < 1+\epsilon$ in $\mathbf{R}^3$. The group $\mathbf{SO}(3)$ acts on S^2_ϵ by rotation so by (12), Ch. VIII, the vector fields $\mu(X_{ij})$ extend to vector fields $\widetilde{\mu}(X_{ij})$ on S^2_ϵ. But these are just the restrictions of the vector fields $x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$ to S^2_ϵ. These vector fields satisfy

$$x_1\left(x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2}\right) - x_2\left(x_1\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_1}\right) + x_3\left(x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1}\right) = 0,$$

so the lemma holds.

We can now state Gonzalez's main theorem describing the range of R.

Theorem 4.17. *The plane-to-line transform R maps $\mathcal{E}(\mathbf{G}(2,3))$ onto the kernel of D:*

$$R\big(\mathcal{E}(\mathbf{G}(2,3))\big) = \big\{\psi \in \mathcal{E}(\mathbf{G}(1,3)) : \lambda'(D)\psi = 0\big\}\,.$$

Proof. The operator R obviously commutes with the action of G. Thus by (13) in Ch. VIII, we have for each $E \subset \mathbf{D}(G)$,

$$R(\lambda''(E)\varphi) = 0 = \lambda'(E)R\varphi \qquad \varphi \in \mathcal{E}(\mathbf{G}(2,3))\,. \tag{99}$$

In particular, Lemma 4.15 implies

$$\lambda'(D)(R\varphi) = 0 \qquad \text{for } \varphi \in \mathcal{E}(\mathbf{G}(2,3))\,.$$

For the converse assume $\psi \in \mathcal{E}(\mathbf{G}(1,3))$ satisfies

$$\lambda'(D)\psi = 0\,.$$

Put $\Psi = \psi \circ \pi'$. Then by the analog of (96) $\lambda(D)\Psi = 0$. In analogy with the formula (97) for $\lambda(D)\Phi$ (where the first term vanished) we get for each $(\sigma, x) \in \mathbf{S}^2 \times \mathbf{R}^3$,

(100)
$$0 = \lambda(D)\Psi = \big[\lambda(E_1)_x\lambda(X_{23})_\sigma - \lambda(E_2)_x\lambda(X_{13})_\sigma + \lambda(E_3)_x\lambda(X_{12})_\sigma\big]\Psi(\sigma, x)\,.$$

Now $\Psi(\sigma, x) = \Psi(-\sigma, x)$ so by the surjectivity of the great circle transform (which is contained in Theorem 2.2 in Ch. III) there exists a unique even smooth function $\omega \to \Phi_x(\omega)$ on $\mathbf{S}^2$ such that

$$\Psi(\sigma, x) = \frac{1}{2\pi}\int\limits_{A(\sigma)} \Phi_x(\omega)\, d_\sigma(\omega)\,. \tag{101}$$

We put $\Phi(\omega, x) = \Phi_x(\omega)$. The task is now to prove that $\nabla_x\Phi(\omega, x)$ is a multiple of ω, because then by Lemma 4.13, $\Phi = \varphi \circ \pi''$ for some $\varphi \in \mathcal{E}(\mathbf{G}(1,3))$. Then we would in fact have by (95), $R\varphi \circ \pi' = \Psi = \psi \circ \pi'$ so $R\varphi = \psi$.

Applying the formula (100) above and differentiating (101) under the integral sign, we deduce

(102)
$$0 = \int\limits_{A(\sigma)} \left[\lambda(X_{23})_\omega \lambda(E_1)_x - \lambda(X_{13})_\omega \lambda(E_2)_x + \lambda(X_{12})_\omega (E_3)_x\right] \Phi(\omega, x)\, d_\sigma(\omega)\,.$$

For x fixed the integrand is even in ω, so by injectivity of the great circle transform, the integrand vanishes. Consider the $\mathbf{R}^3$-valued vector field on $\mathbf{S}^2$ given by

$$\begin{aligned}\vec{G}(\omega) &= -\nabla_x\Phi(\omega, x) = (\lambda(E_1)_x\Phi(\omega, x), \lambda(E_2)_x\Phi(\omega, x), \lambda(E_3)_x\Phi(\omega, x))\\ &= (G_1(\omega), G_2(\omega), G_3(\omega))\,,\end{aligned}$$

where each $G_i(\omega)$ is even. By the vanishing of the integrand in (102) we have

$$\lambda(X_{23})G_1 - \lambda(X_{13})G_2 + \lambda(X_{12})G_3 = 0\,. \tag{103}$$

We decompose $\vec{G}(\omega)$ into tangential and normal components, respectively, $\vec{G}(\omega) = \vec{T}(\omega) + \vec{N}(\omega)$, with components $T_i(\omega)$, $N_i(\omega)$, $1 \le i \le 3$. We wish to show that $\vec{G}(\omega)$ proportional to ω, or equivalently, $\vec{T}(\omega) = 0$. We substitute $G_i = T_i + N_i$ into (103) and observe that

$$\lambda(X_{23})(N_1) - \lambda(X_{13})(N_2) + \lambda(X_{12})(N_3) = 0\,, \tag{104}$$

because writing $\vec{N}(\omega) = n(\omega)\omega$, n is an odd function on $\mathbf{S}^2$ and (104) equals

$$\begin{aligned}&\omega_1\lambda(X_{23})n(\omega) - \omega_2\lambda(X_{13})n(\omega) + \omega_3\lambda(X_{12})n(\omega)\\ &+n(\omega)(\lambda(X_{23})(\omega_1) - \lambda(X_{13})\omega_2 + \lambda(X_{12})(\omega_3)) = 0\end{aligned}$$

by Lemma 4.16 and $\lambda(X_{jk})\omega_i = 0$, $(i \neq j, k)$. Thus we have the equation

$$\lambda(X_{23})T_1 - \lambda(X_{13})T_2 + \lambda(X_{12})T_3 = 0\,. \tag{105}$$

From Lemma 4.12 $\langle\sigma, \nabla_x\Psi(\sigma, x)\rangle = 0$ and by (101) we get

$$\begin{aligned}0 &= \int\limits_{A(\sigma)} \langle\sigma, \nabla_x\Phi(\omega, x)\, d_\sigma(\omega) = -\int\limits_{A(\sigma)} \langle\sigma, \vec{G}(\omega)\rangle\, d_\sigma\omega\\ &= -\int\limits_{A(\sigma)} \langle\sigma, \vec{T}(\omega)\rangle\, d_\sigma(\omega) - \int\limits_{A(\sigma)} \langle\sigma, \vec{N}(\omega)\rangle\, d_\sigma\omega\\ &= -\int\limits_{A(\sigma)} \langle\sigma, \vec{T}(\omega)\rangle\, d_\sigma\omega\,,\end{aligned}$$

since $\sigma \perp \vec{N}(\omega)$ on $A(\sigma)$. Thus $\vec{T}(\omega)$ is an even vector field on $\mathbf{S}^2$ satisfying (105) and

$$\int_{A(\sigma)} \langle \sigma, \vec{T}(\omega) \rangle \, d_\sigma \omega = 0 \,. \tag{106}$$

We claim that $\vec{T}(\omega) = \operatorname{grad}_{S^2} t(\omega)$, where $\operatorname{grad}_{\mathbf{S}^2}$ denotes the gradient on $\mathbf{S}^2$ and t is an odd function on $\mathbf{S}^2$. To see this, we extend $T(\omega)$ to a smooth vector field $\widetilde{T}$ on a shell $S^2_\epsilon : 1 - \epsilon < \|x\| < 1 + \epsilon$ in $\mathbf{R}^3$ by $\vec{T}(r\omega) = \vec{T}(\omega)$ for $r \in (1 - \epsilon, 1 + \epsilon)$. Again the $\mathbf{SO}(3)$ action on S^2_ϵ induces vector fields $\widetilde{\mu}(X_{ij})$ on S^2_ϵ, which are just $x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$. Thus (105) becomes

$$\langle \operatorname{curl} \widetilde{T}(x), x \rangle = 0 \text{ on } S^2_\epsilon \,.$$

By the classical Stokes' theorem for $\mathbf{S}^2$ this implies that the line integral

$$\int_\gamma T_1 \, dx_1 + T_2 \, dx_2 + T_3 \, dx_3 = 0$$

for each simple closed curve γ on $\mathbf{S}^2$. Let τ be the pull back of the form $\sum_i T_i \, dx_i$ to $\mathbf{S}^2$. By the Stokes' theorem for τ on $\mathbf{S}^2$ we deduce $d\tau = 0$ on $\mathbf{S}^2$, i.e., τ is closed. Since $\mathbf{S}^2$ is simply connected, τ is exact, i.e., $\tau = dt$, $t \in \mathcal{E}(\mathbf{S}^2)$. (This is an elementary case of deRham's theorem; t can be constructed as in complex variable theory.) For any vector field Z on $\mathbf{S}^2$ $dt(Z) = \langle \operatorname{grad}_{\mathbf{S}^2} t, Z \rangle$ so $T(\omega) = \operatorname{grad}_{\mathbf{S}^2} t(\omega)$. Decomposing $t(\omega)$ into odd and even components we see that the even component is constant so we can take $t(\omega)$ odd.

Let $H(\sigma)$ denote the hemisphere on the side of $A(\sigma)$ away from σ. Note that σ located at points of $A(\sigma)$ form the outward pointing normals of the boundary $A(\sigma)$ of $H(\sigma)$. With $\vec{T}(\omega) = \operatorname{grad}_{\mathbf{S}^2} t(\omega)$ the integral (106) equals

$$\int_{H(\sigma)} (L_{\mathbf{S}^2} t)(\omega) \, d\omega \,, \quad \sigma \in \mathbf{S}^2 \,,$$

by the divergence theorem on $\mathbf{S}^2$. Since $L_{\mathbf{S}^2} t$ is odd the next lemma implies that $L_{\mathbf{S}^2} t = 0$ so t is a constant, hence $t \equiv 0$ (because t is odd).

Lemma 4.18. *Let τ denote the hemisphere transform on $\mathbf{S}^2$, $\tau(h) = \int_{H(\sigma)} h(\omega) \, d\omega$ for $h \in \mathcal{E}(\mathbf{S}^2)$. If $\tau(h) = 0$ then h is an even function.*

Proof. Let H_m denote the space of degree m spherical harmonics on $\mathbf{S}^2$ ($m = 0, 1, 2, \ldots$). Then $\mathbf{SO}(3)$ acts irreducibly on H_m. Since τ commutes with the action of $\mathbf{SO}(3)$ it must (by Schur's lemma) be a scalar operator

c_m on H_m. The value can be obtained by integrating a zonal harmonic P_m over the hemisphere

$$c_m = 2\pi \int_0^{\frac{\pi}{2}} P_m(\cos\theta)\sin\theta\, d\theta = 2\pi \int_0^1 P_m(x)\, dx\,.$$

According to Erdelyi et al. [1953], I p. 312, this equals

$$c_m = 4\pi^{\frac{1}{2}} \frac{\Gamma(1+\frac{m}{2})}{\Gamma\left(\frac{m+1}{2}\right)} \frac{1}{m(m+1)} \sin\frac{m\pi}{2}\,, \tag{107}$$

which equals 2π for $m = 0$, is 0 for m even, and is $\neq 0$ for m odd. Since each $h \in \mathcal{E}(\mathbf{S}^2)$ has an expansion $h = \sum_0^\infty h_m$ with $h_m \in H_m$, $\tau(h) = 0$ implies $c_m = 0$ (so m is even) if $h_m \neq 0$. Thus h is even as claimed.

Remark. The value of c_m in (107) appears in an exercise in Whittaker–Watson [1927], p. 306, attributed to Clare, 1902.

J. Noncompact Symmetric Space and Its Family of Horocycles

This example belongs to the realm of the theory of semisimple Lie groups G. See Chapter IX, §2 for orientation. To such a group with finite center is associated a coset space $X = G/K$ (a Riemannian symmetric space) where K is a maximal compact subgroup (unique up to conjugacy). The group G has an Iwasawa decomposition $G = NAK$ (generalizing the one in Example C for $\mathbf{H}^2$.) Here N is nilpotent and A abelian. The orbits in X of the conjugates gNg^{-1} to N are called **horocycles**. These are closed submanifolds of X and are permuted transitively by G. The set Ξ of those horocycles ξ is thus a coset space of G, in fact $\Xi = G/MN$, where M is the centralizer of A in K. To this pair

$$X = G/K\,, \qquad \Xi = G/MN$$

are associated a Radon transform $f \to \widehat{f}$ and its dual $\varphi \to \check{\varphi}$ as in formula (9). More explicitly,

$$\widehat{f}(\xi) = \int_\xi f(x)\, dm(x)\,, \qquad \check{\varphi}(x) = \int_{\xi \ni x} \varphi(\xi)\, d\mu(\xi)\,, \tag{108}$$

where dm is the Riemannian measure on the submanifold ξ and $d\mu$ is the average over the (compact) set of horocycles passing through x.

Problems A, B, C, D all have solutions here (with some open questions); there is injectivity of $f \to \widehat{f}$ (with inversion formulas), surjectivity of $\varphi \to \check{\varphi}$, determinations of ranges and kernels of these maps, support theorems and applications to differential equations and group representations.

The transform $f \to \widehat{f}$ has the following inversion

$$f = (\Lambda \widehat{f})^{\vee},$$

where Λ is a G-invariant pseudo-differential operator on Ξ. In the case when G has all Cartan subgroups conjugate one has a better formula

$$f = \square((\widehat{f})^{\vee}),$$

where $\square$ is an explicit differential operator on X.

The support theorem for $f \to \widehat{f}$ states informally, $B \subset X$ being any ball:

$$\widehat{f}(\xi) = 0 \ \text{ for } \ \xi \cap B = \emptyset \Rightarrow f(x) = 0 \qquad \text{for} \ \ x \notin B .$$

Here f is assumed "rapidly decreasing" in a certain technical sense.

Thus the conjugacy of the Cartan subgroups corresponds to the case of odd dimension for the Radon transform on $\mathbf{R}^n$. For complete proofs of these results, with documentation, see my book [1994b] or [2008].

Exercises and Further Results

1. The Discrete Case.

For a discrete group G, Proposition 2.2 (via diagram (4)) takes the form (# denoting incidence):

$$\sum_{x \in X} f(x)\check{\varphi}(x) = \sum_{(x,\xi) \in X \times \Xi, x \# \xi} f(x)\varphi(\xi) = \sum_{\xi \in X} \widehat{f}(\xi)\varphi(\xi) .$$

2. Linear Codes. (Boguslavsky [2001])

Let $\mathbf{F}_q$ be a finite field and $\mathbf{F}_q^n$ the n-dimensional vector space with its natural basis. The *Hamming metric* is the distance d given by $d(x, y) =$ number of distinct coordinate positions in x and y.

A linear $[n, k, d]$-code C is a k-dimensional subspace of $\mathbf{F}_k^n$ such that $d(x, y) \geq d$ for all $x, y \in C$. Let $\mathbf{P}C$ be the projectivization of C on which the projective group $G = \mathbf{PGL}(k-1, \mathbf{F}_q)$ acts transitively. Let $\ell \in \mathbf{P}C$ be fixed and π a hyperplane containing ℓ. Let K and H be the corresponding isotropy groups. Then $X = G/K$, and $\Xi = G/H$ satisfy Lemma 1.3 and the transforms

$$\widehat{f}(\xi) = \sum_{x \in \xi} f(x) , \quad \check{\varphi}(x) = \sum_{\xi \ni x} \varphi(\xi)$$

are well defined. They are inverted as follows. Put

$$s(\varphi) = \sum_{\xi \in \Xi} \varphi(\xi) , \quad \sigma(f) = \sum_{x \in X} f(x) .$$

The projective space $\mathbf{P}^m$ over $\mathbf{F}_q$ has a number of points equal to $p_m = \frac{q^{m+1}-1}{q-1}$. Here $m = k-1$ and we consider the operators D and Δ given by

$$(D\varphi)(\xi) = \varphi(\xi) - \frac{q^{k-2}-1}{(q^{k-1}-1)^2} s(\varphi)\,, \quad (\Delta f)(x) = f(x) - \frac{q^{k-2}-1}{(q^{k-1}-1)^2}\sigma(f)\,.$$

Then

$$f(x) = \frac{1}{q^{k-2}}(D\widehat{f})^{\vee}(x)\,, \quad \varphi(\xi) = \frac{1}{q^{k-2}}(\Delta\check{\varphi})\widehat{\ }(\xi)\,.$$

3. Radon Transform on Loops. (Brylinski [1996])

Let M be a manifold and LM the free loop space in M. Fix a 1-form α on M. Consider the functional I_α on LM given by $I_\alpha(\gamma) = \int_\gamma \alpha$.

With the standard $\mathcal{C}^\infty$ structure on LM

$$(dI_\alpha, v)_\gamma = \int_0^1 d\alpha(v(t), \dot{\gamma}(t))\, dt$$

for $v \in (LM)_\gamma$. Clearly $I_\alpha = 0$ if and only if α is exact.

Inversion, support theorem and range description of this transform are established in the cited reference. Actually I_α satisfies differential equations reminiscent of John's equations in Theorem 6.5, Ch. I.

4. Theta Series and Cusp Forms.

This concerns Ch. II, §4, Example **H**. For the following results see Godement [1966].

(i) In the identification $G/N \approx \mathbf{R}^2 - (0)$ (via $gN \to g\binom{1}{0}$), let $f \in \mathcal{D}(G/N)$ satisfy $f(x) = f(-x)$. Then, in the notation of Example **H**,

$$\tfrac{1}{2}(\widehat{f})^{\vee}(gN) = f(gN) + \sum_{(\gamma)} \int_N f(gn\gamma N)\, dn\,,$$

where $\sum_{(\gamma)}$ denotes summation over the nontrivial double cosets $\pm\Gamma_\infty\gamma\Gamma_\infty$ (γ and $-\gamma$ in Γ identified).

(ii) Let $A = \left\{\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right) : t > 0\right\}$ be the diagonal subgroup of G and $\beta(h) = t^2$ if $h = \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1}\end{smallmatrix}\right)$. Consider the Mellin transform

$$\widetilde{f}(gN, 2s) = \int_A f(ghN)\beta(h)^s\, dh$$

and (viewing G/N as $\mathbf{R}^2 - 0$) the twisted Fourier transform

$$f^*(x) = \int_{\mathbf{R}^2} f(y) e^{-2\pi i B(x,y)}\, dy\,,$$

where $B(x,y) = x_2y_1 - x_1y_2$ for $x = (x_1, x_2)$, $y = (y_1, y_2)$. The *Eisenstein series* is defined by

$$E_f(g,s) = \sum_{\gamma \in \Gamma/\Gamma_\infty} \widetilde{f}(g\gamma N, 2s)\,, \quad (\text{convergent for } \operatorname{Re} s > 1)\,.$$

Theorem. *Assuming $f^*(0) = 0$, the function $s \to \zeta(2s)E_f(g,s)$ extends to an entire function on $\mathbf{C}$ and does not change under $s \to (1-s)$, $f \to f^*$.*

5. Radon Transform on Minkowski Space. (Kumahara and Wakayama [1993], see Figure II,6)

Let X be an $(n+1)$-dimensional real vector space with inner product $\langle\ ,\ \rangle$ of signature $(1,n)$. Let $e_0, e_1, \ldots, e_n$ be a basis such that $\langle e_i, e_j\rangle = -1$ for $i = j = 0$ and 1 if $i = j > 0$ and 0 if $i \neq j$. Then if $x = \sum_0^n x_ie_i$, a hyperplane in X is given by

$$\sum_0^n a_ix_i = c\,, \qquad a \in \mathbf{R}^{n+1}\,, \ a \neq 0$$

and $c \in \mathbf{R}$. We put

$$\omega_0 = -a_0/|\langle a,a\rangle|^{\frac{1}{2}}\,, \quad \omega_j = a_j/|\langle a,a\rangle|^{\frac{1}{2}}\,, \ j > 0\,, \quad p = c/|\langle a,a\rangle|^{\frac{1}{2}}$$

if $\langle a,a\rangle \neq 0$ and if $\langle a,a\rangle = 0$

$$\omega_0 = -a_0/|a_0|\,, \ \omega_j = a_j/|a_0|\ j > 0\,, \quad p = c/|a_0|\,.$$

The hyperplane above is thus

$$\langle x, \omega\rangle = -x_0\omega_0 + x_1\omega_1 + \cdots + x_n\omega_n = p\,,$$

written $\xi(\omega,p)$. The semidirect product $\mathbf{M}(1,n)$ of the translations of X with the connected Lorentz group $G = \mathbf{SO}_0(1,n)$ acts transitively on X and $\mathbf{M}(1,n)/\mathbf{SO}(1,n) \cong X$.

To indicate how the light cone $\langle\omega,\omega\rangle = 0$ splits X we make the following definitions (see Figure II,6).

$$\begin{aligned}
X_-^+ &= \{\omega \in X : \langle\omega,\omega\rangle = -1\,, \ \omega_0 > 0\}\\
X_-^- &= \{\omega \in X : \langle\omega,\omega\rangle = -1\,, \ \omega_0 < 0\}\\
X_+ &= \{\omega \in X : \langle\omega,\omega\rangle = +1\}\\
X_0^+ &= \{\omega \in X : \langle\omega,\omega\rangle = 0\,, \ \omega_0 > 0\}\\
X_0^- &= \{\omega \in X : \langle\omega,\omega\rangle = 0\,, \ \omega_0 < 0\}\\
S_\pm &= \{\omega \in X : \langle\omega,\omega\rangle = 0\,, \ \omega_0 = \pm 1\}\,.
\end{aligned}$$

The scalar multiples of the X_i fill up X. The group $\mathbf{M}(1,n)$ acts on X as follows: If $(g,z) \in \mathbf{M}(1,n)$, $g \in G$, $z \in X$ then

$$(g,z)\cdot x = z + g\cdot x\,.$$

The action on the space Ξ of hyperplanes $\xi(\omega, p)$ is

$$(g, z) \cdot \xi(\omega, p) = \xi(g \cdot \omega, p + \langle z, g \cdot \omega \rangle)\,.$$

Then Ξ has the $\mathbf{M}(1, n)$ orbit decomposition

$$\Xi = \mathbf{M}(1, n)\xi(e_0, 0) \cup \mathbf{M}(1, n)\xi(e_1, 0) \cup M(1, n)\xi(e_0 + e_1, 0)$$

into three homogeneous spaces of $\mathbf{M}(1, n)$.

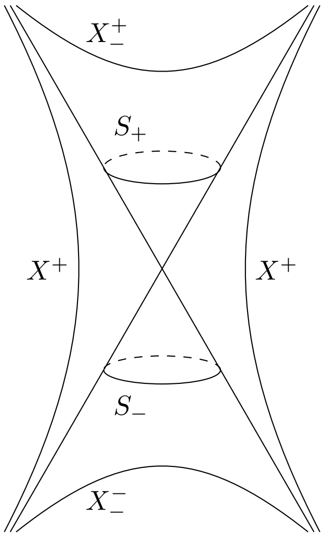

FIGURE II.6.
Minkowski space for dimension 3

Note that $\xi(-\omega, -p) = \xi(\omega, p)$ so in the definition of the Radon transform we assume $\omega_0 > 0$. The sets X_-^+, X_+ and X_0^+ have natural G-invariant measures $d\mu_-(\omega)$ and $d\mu_+(\omega)$ on $X_-^+ \cup X_-^-$ and X_+, respectively; in fact

$$d\mu_\pm = \frac{1}{|\omega_i|} \prod_{j \neq i} d\omega_j \qquad \text{where} \;\; \omega_i \neq 0\,.$$

Viewing $X_-^+ \cup X_-^- \cup X_+ \cup S_+ \cup S_-$ as a substitute for a "boundary" ∂X of X we define

$$\int_{\partial X} \psi(\omega)\, d\mu(\omega) = \int_{X_-^+ \cup X_-^-} \psi(\omega)\, d\mu_-(\omega) + \int_{X+} \psi(\omega)\, d\mu_+(\omega)$$

for $\psi \in C_c(X)$. (S_+ and S_- have lower dimension.) The Radon transform $f \to \widehat{f}$ and its dual $\varphi \to \check{\varphi}$ are now defined by

$$\widehat{f}(\xi) = \int_{\xi} f(x)\, dm(x) = \int_{\langle x,\omega\rangle = p} f(x)\, dm(x) = \widehat{f}(\omega, p)\,,$$

$$\check{\varphi}(x) = \int_{\xi \ni x} \varphi(\xi)\, d\sigma_x(\xi) = \int_{\partial X} \varphi(\omega, \langle x, \omega\rangle)\, d\mu(\xi)\,.$$

Here dm is the Euclidean measure on the hyperplane ξ and a function φ on Ξ is identified with an even function $\varphi(\omega, p)$ on $\partial X \times \mathbf{R}$. The measure $d\sigma_x$ is defined by the last relation.

There are natural analogs $\mathcal{S}(X)$, $\mathcal{S}(\Xi)$ and $\mathcal{S}_H(\Xi)$ of the spaces $\mathcal{S}(\mathbf{R}^n)$, $\mathcal{S}(\mathbf{P}^n)$ and $\mathcal{S}_H(\mathbf{P}^n)$ defined in Ch. I, §2. The following analogs of the $\mathbf{R}^n$ theorems hold.

Theorem. *$f \to \widehat{f}$ is a bijection of $\mathcal{S}(X)$ onto $\mathcal{S}_H(\Xi)$.*

Theorem. *For $f \in \mathcal{S}(X)$,*

$$f = (\Lambda \widehat{f})^{\vee}\,,$$

where

$$(\Lambda\varphi)(\omega, p) = \begin{cases} \frac{1}{2(2\pi)^2}\left(\frac{\partial}{i\partial p}\right)^n \varphi(\omega, p) & n \ \text{even} \\ \frac{1}{2(2\pi)^n}\mathcal{H}_p\left(\frac{\partial}{i\partial p}\right)^n \varphi(\omega, p) & n \ \text{odd} \end{cases}\,.$$

6. John's Equation for the X-ray transform on $\mathbf{R}^3$.

According to Richter [1986b] the equation $\lambda'(D)\psi = 0$ in Gonzalez' Theorem 4.17 characterizes the range of the X-ray transform on $\mathbf{R}^3$. Relate this to John's equation $\Lambda\psi = 0$ in Theorem 6.9, Ch. I.

Bibliographical Notes

The Radon transform and its dual for a double fibration

$$(1) \qquad \begin{array}{ccc} & Z = G/(K \cap H) & \\ \swarrow & & \searrow \\ X = G/K & & \Xi = G/H \end{array}$$

was introduced in the author's paper [1966a]. The results of §1–§2 are from there and from [1994b]. The definition uses the concept of *incidence* for

$X = G/K$ and $\Xi = G/H$ which goes back to Chern [1942]. Even when the elements of Ξ can be viewed as subsets of X and vice versa (Lemma 1.3) it can be essential for the inversion of $f \to \widehat{f}$ not to restrict the incidence to the naive one $x \in \xi$. (See for example the classical case $X = \mathbf{S}^2, \Xi =$ set of great circles where in Theorem 4.1 a more general incidence is essential.) The double fibration in (1) was generalized in Gelfand, Graev and Shapiro [1969], by relaxing the homogeneity assumption.

For the case of geodesics in constant curvature spaces (Examples A, B in §4) see notes to Ch. III.

The proof of Theorem 4.4 (a special case of the author's inversion formula in [1964], [1965b]) makes use of a method by Godement [1957] in another context. Another version of the inversion (47) for $\mathbf{H}^2$ (and $\mathbf{H}^n$) is given in Gelfand-Graev-Vilenkin [1966]. A further inversion of the horocycle transform in $\mathbf{H}^2$ (and $\mathbf{H}^n$), somewhat analogous to (38) for the X-ray transform, is given by Berenstein and Tarabusi [1994].

The analogy suggested above between the X-ray transform and the horocycle transform in $\mathbf{H}^2$ goes even further in $\mathbf{H}^3$. There the 2-dimensional transform for totally geodesic submanifolds has *the same* inversion formula as the horocycle transform (Helgason [1994b], p. 209).

For a treatment of the horocycle transform on a Riemannian symmetric space see the author's paper [1963] and monograph [1994b], Chapter II, where Problems A, B, C, D in §2 are discussed in detail along with some applications to differential equations and group representations. See also Gelfand–Graev [1964] for a discussion and inversion for the case of complex G. See also Quinto [1993a] and Gonzalez and Quinto [1994] for new proofs of the support theorem.

Example G is from Hilgert's paper [1994], where a related Fourier transform theory is also established. It has a formal analogy to the Fourier analysis on $\mathbf{H}^2$ developed by the author in [1965b] and [1972].

Example I is from Gonzalez's beautiful paper [2001]. Higher dimensional versions have been proved by Gonzalez and Kakehi [2004]. The relationship between the operator D and John's operator Λ in Ch. I, §6 was established by Richter [1986b].

In conclusion we note that the determination of a function in $\mathbf{R}^n$ in terms of its integrals over unit spheres (John [1955]) can be regarded as a solution to the first half of Problem B in §2 for the double fibration (4) and (7). See Exercise 5 in Ch. VI.

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