

Chapter 2

Topology

2.1 Introduction

Several areas of research in modern mathematics have developed as a result of interaction between two or more specialized areas. For example, the subject of algebraic topology associates with topological spaces various algebraic structures and uses their properties to answer topological questions. An elegant proof of the theorem that \mathbf{R}^m and \mathbf{R}^n with their respective standard topologies, are not homeomorphic for $m \neq n$ is provided by computing the homology of the one point compactification of these spaces. Indeed, the problem of classifying topological spaces up to homeomorphism was fundamental in the creation of algebraic topology. In general, however, the knowledge of these algebraic structures is not enough to decide whether two topological spaces are homeomorphic. The equivalence of algebraic structures follows from a weaker relation among topological spaces, namely, that of homotopy equivalence. In fact, homotopy equivalent spaces have isomorphic homotopy and homology structures. Equivalence of algebraic structures associated to two topological spaces is a necessary but not sufficient condition for their homeomorphism. Thus, one may think of homotopy and homology as providing obstructions to the existence of homeomorphisms. As we impose further structure on a topological space such as piecewise linear, differentiable, or analytic structures other obstructions may arise.

For example, it is well known that \mathbf{R}^n , $n \neq 4$, with the standard topology admits a unique compatible differential structure. On the other hand, as a result of the study of the moduli spaces of instantons by Donaldson and the classification of four-dimensional topological manifolds by Freedman, it follows that \mathbf{R}^4 admits an uncountable number of non-diffeomorphic structures. In the case of the standard sphere $S^n \subset \mathbf{R}^{n+1}$, the generalized Poincaré conjecture states that a compact n -dimensional manifold homotopically equivalent to S^n is homeomorphic to S^n . This conjecture is now known to be true for all n and is one of the most interesting recent results in algebraic

topology. The case $n = 2$ is classical. For $n > 4$ it is due to Stephen Smale. Smale (b. 1930) received a Fields Medal at the ICM 1966 held in Moscow for his contributions to various aspects of differential topology and, in particular, to his novel use of Morse theory, which led him to his solution of the generalized Poincaré conjecture for $n > 4$. Smale has extensive work in the application of dynamical systems to physical processes and to economic equilibria. His discovery of strange attractors led naturally to chaotic dynamical systems. The result for $n = 4$ is due to Michael Hartley Freedman (b. 1951), who received a Fields Medal at ICM 1986¹ held in Berkeley for his complete classification of all compact simply connected topological 4-manifolds, which leads to his proof of the Poincaré conjecture. The original Poincaré conjecture was recently proved by Grigory Yakovlevich Perelman (b. 1966). Perelman received a Fields Medal at ICM 2006 held in Madrid for his fundamental contributions to geometry and for his revolutionary insights into the analytical and geometric structure of the Ricci flow. He studied the geometric topology of 3-manifolds by extending Hamilton’s Ricci flow ideas. While he did not publish his work in a final form, it contains all the essential ingredients of a proof of the Thurston geometrization conjectures and in particular of the original Poincaré conjecture (the case $n = 3$). This problem is one of the seven, million dollar Clay Prize problems. As of this writing, it is not known if and when he will get this prize. We will discuss the topology of 3- and 4-manifolds later in this chapter.

In the category of differentiable manifolds, it was shown by **John Willard Milnor** that S^7 admits an exotic differential structure, i.e., a structure not diffeomorphic to the standard one. This work ushered in the new field of differential topology. Milnor was awarded a Fields Medal at the ICM 1962 held in Stockholm for his fundamental work in differential geometry and topology. Using homotopy theory, Kervaire and Milnor proved the striking result that the number of distinct differentiable structures on S^n is finite for any $n \neq 4$. For $n = 1, 2, 3, 5, 6$, there is a unique differential structure on the standard n -sphere. As of this writing (May 2010) there is no information on the number of distinct differentiable structures on S^4 . The following table gives a partial list of the number of diffeomorphism classes $[S^n]$ of n -spheres.

Table 2.1 Number of diffeomorphism classes of n -spheres

n	7	8	9	10	11	12	13	14
$\#[S^n]$	28	2	8	6	992	1	3	2

¹ The year was the 50th anniversary of the inception of Fields medals. However, several mathematicians including invited speakers were denied U.S. visas. Their papers were read by other mathematicians in a show of solidarity.

We note that the set of these equivalence classes can be given a structure of a group denoted by θ_n . Milnor showed that θ_7 is cyclic group of order 28. Brieskorn has constructed geometric representatives of the elements of θ_7 as 7-dimensional **Brieskorn spheres** $\Sigma(6m-1, 3, 2, 2, 2)$, $1 \leq m \leq 28$, by generalizing the Poincaré homology spheres in three dimensions. Thus the m th sphere is the intersection of $S^9 \subset \mathbf{C}^5$ with the space of solutions of the equation

$$z_1^{6m-1} + z_2^3 + z_3^2 + z_4^2 + z_5^2 = 0, \quad 1 \leq m \leq 28, \quad z_i \in \mathbf{C}, \quad 1 \leq i \leq 5.$$

Until recently, such considerations would have seemed too exotic to be of utility in physical applications. However, topological methods have become increasingly important in classical and quantum field theories. In particular, several invariants associated to homotopy and homology of a manifold have appeared in physical theories as topological quantum numbers. In the remaining sections of this chapter and in the next chapter is a detailed account of some of the most important topics in this area.

2.2 Point Set Topology

Point set topology is one of the core areas in modern mathematics. However, unlike algebraic structures, topological structures are not familiar to physicists. In this appendix we collect some basic definitions and results concerning topological spaces. Topological concepts are playing an increasingly important role in physical applications, some of which are mentioned here.

Let X be a set and $\mathcal{P}(X)$ the **power set** of X , i.e., the class of all subsets of X . $\mathcal{T} \subset \mathcal{P}(X)$ is called a **topology** on X if the following conditions are satisfied:

1. $\{\emptyset, X\} \subset \mathcal{T}$;
2. if $A, B \in \mathcal{T}$, then $A \cap B \in \mathcal{T}$;
3. if $\{U_i \mid i \in I\} \subset \mathcal{T}$, then $\bigcup_{i \in I} U_i \in \mathcal{T}$, where I is an arbitrary indexing set.

The pair (X, \mathcal{T}) is called a **topological space**. It is customary to refer to X as a topological space when the topology \mathcal{T} is understood. An element of \mathcal{T} is called an open set of (X, \mathcal{T}) . If $W \subset X$, then $\mathcal{T}_W := \{W \cap A \mid A \in \mathcal{T}\}$ is a topology on W called the **relative topology** on W induced by the topology \mathcal{T} on X .

Example 2.1 Let $\mathcal{T} = \{\emptyset, X\}$. Then \mathcal{T} is called the **indiscrete topology** on X . If $\mathcal{T} = \mathcal{P}(X)$, then \mathcal{T} is called the **discrete topology** on X . If $\{\mathcal{T}_i \mid i \in I\}$ is a family of topologies on X , then $\bigcap \{\mathcal{T}_i \mid i \in I\}$ is also a topology on X .

The indiscrete and the discrete topologies are frequently called trivial topologies. An important example of a nontrivial topology is given by the metric topology.

Example 2.2 Let \mathbf{R}_+ denote the set of nonnegative real numbers. A metric or a distance function on X , is a function $d : X \times X \rightarrow \mathbf{R}_+$ satisfying, $\forall x, y, z \in X$, the following properties:

1. $d(x, y) = d(y, x)$, **symmetry**;
2. $d(x, y) = 0$ if and only if $x = y$, **non-degeneracy**;
3. $d(x, y) \leq d(x, z) + d(z, y)$, **triangle inequality**.

The pair (X, d) is called a metric space. If (X, d) is a metric space, we can make it into a topological space with topology \mathcal{T}_d defined as follows. \mathcal{T}_d is the class of all subsets $U \subset X$ such that

$$\forall x \in U, \exists \epsilon > 0, \quad \text{such that} \quad B(\epsilon, x) := \{y \in X \mid d(x, y) < \epsilon\} \subset U.$$

The set $B(\epsilon, x)$ is called an ϵ -ball around x and is itself in \mathcal{T}_d . \mathbf{R}^n with the usual Euclidean distance function is a metric space. The corresponding topology is called the standard topology on \mathbf{R}^n . The relative topology on $S^{n-1} \subset \mathbf{R}^n$ is called the standard topology on S^{n-1} .

A topological space (X, \mathcal{T}) is said to be metrizable if there exists a distance function d on X such that $\mathcal{T} = \mathcal{T}_d$. It is well known that Riemannian manifolds are metrizable. It is shown in [256] that pseudo-Riemannian manifolds, and in particular, space-time manifolds, are also metrizable.

Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces. A function $f : X \rightarrow Y$ is said to be continuous if $\forall V \in \mathcal{T}_Y, f^{-1}(V) \in \mathcal{T}_X$. If f is a continuous bijection and f^{-1} is also continuous, then f is called a homeomorphism between X and Y . Homeomorphism is an equivalence relation on the class of topological spaces. A property of topological spaces preserved under homeomorphisms is called a topological property. For example, metrizability is a topological property.

Let $(X_i, \mathcal{T}_i), i \in I$, be a family of topological spaces and let $X = \prod_{i \in I} X_i$ be the Cartesian product of the family of sets $\{X_i \mid i \in I\}$. Let $\pi_i : X \rightarrow X_i$ be the canonical projection. Let $\{\mathcal{S}_j \mid j \in J\}$ be the family of all topologies on X such that π_i is continuous for all $i \in I$. If $\mathcal{S} = \bigcap \{\mathcal{S}_j \mid j \in J\}$, then \mathcal{S} is called the product topology on X . We observe that it is the smallest topology on X such that all the π_i are continuous.

Let (X, \mathcal{T}) be a topological space, Y a set, and $f : X \rightarrow Y$ a surjection. The class \mathcal{T}_f defined by

$$\mathcal{T}_f := \{V \subset Y \mid f^{-1}(V) \in \mathcal{T}\}$$

is a topology on Y called the quotient topology on Y defined by f . \mathcal{T}_f is the largest topology on Y with respect to which f is continuous. We observe that if ρ is an equivalence relation on X , $Y = X/\rho$ is the set of equivalence

classes and $\pi : X \rightarrow Y$ is the canonical projection, then Y with the quotient topology \mathcal{T}_π is called quotient topological space of X by ρ .

Let (X, \mathcal{T}) be a topological space and let A, B, C denote subsets of X . C is said to be closed if $X \setminus C$ is open. The closure \bar{A} or $cl(A)$ of A is defined by

$$\bar{A} := \bigcap \{F \subset X \mid F \text{ is closed and } A \subset F\}.$$

Thus, \bar{A} is the smallest closed set containing A . It follows that C is closed if and only if $C = \bar{C}$. Let $f : X \rightarrow Y$ be a function. We define $\text{supp } f$, the support of f to be the set $\text{cl}\{x \in X \mid f(x) \neq 0\}$. A subset $A \subset X$ is said to be dense in X if $\bar{A} = X$. X is said to be separable if it contains a countable dense subset. A is said to be a neighborhood of $x \in X$ if there exists $U \in \mathcal{T}$ such that $x \in U \subset A$. We denote by \mathcal{N}_x the class of neighborhoods of x . A subclass $\mathcal{B} \subset \mathcal{T} \cap \mathcal{N}_x$ is called a local base at $x \in X$ if for each neighborhood A of x there exists $U \in \mathcal{B}$ such that $U \subset A$. X is said to be first countable if each point in X admits a countable local base. A subclass $\mathcal{B} \subset \mathcal{T}$ is called a base for \mathcal{T} if $\forall A \in \mathcal{T}, \forall x \in A$, there exists $U \in \mathcal{B}$ such that $x \in U \subset A$. X is said to be second countable if its topology has a countable base. A subclass $\mathcal{S} \subset \mathcal{T}$ is called a subbase for \mathcal{T} if the class of finite intersections of elements of \mathcal{S} is a base for \mathcal{T} . Any metric space is first countable but not necessarily second countable. First and second countability are topological properties. We now give some further important topological properties.

X is said to be a **Hausdorff space** if $\forall x, y \in X$, there exist $A, B \in \mathcal{T}$ such that $x \in A, y \in B$ and $A \cap B = \emptyset$. Such a topology is said to separate points and the Hausdorff property is one of a family of separation axioms for topological spaces. The Hausdorff property implies that finite subsets of X are closed. A metric space is a Hausdorff space.

A family $\mathcal{U} = \{U_i \mid i \in I\}$ of subsets of X is said to be a **cover** or a **covering** of $A \subset X$ if $A \subset \bigcup \mathcal{U}$. A cover $\{V_j \mid j \in J\}$ of $A \subset X$ is called a **refinement** of \mathcal{U} if, for all $j \in J$, $V_j \subset U_i$ for some $i \in I$. A covering by open sets is called an **open covering**. $A \subset X$ is said to be **compact** if every open covering of A has a finite refinement or, equivalently, if it has a finite subcovering. The continuous image of a compact set is compact. It follows that compactness is a topological property. The **Heine–Borel theorem** asserts that a subset of \mathbf{R}^n is compact if and only if it is closed and bounded. A consequence of this is the **extreme value theorem**, which asserts that every continuous real-valued function on a compact space attains its maximum and minimum values. A Hausdorff space X is said to be **paracompact** if every open covering of X has a **locally finite** open refinement, i.e., each point has a neighborhood that intersects only finitely many sets of the refinement. A family $\mathcal{F} = \{f_i : X \rightarrow \mathbf{R} \mid i \in I\}$ of functions is said to be **locally finite** if each $x \in X$ has a neighborhood U such that $f_i(U) = 0$, for all but a finite subset of I . A family \mathcal{F} of continuous functions is said to be a **partition of unity** if it is a locally finite family of nonnegative functions and

$$\sum_{i \in I} f_i(x) = 1, \quad \forall x \in X.$$

If X is a paracompact space and $\mathcal{U} = \{U_i \mid i \in I\}$ is an open covering of X , then there exists a partition of unity $\mathcal{F} = \{f_i : X \rightarrow \mathbf{R} \mid i \in I\}$ such that $\text{supp } f_i \subset U_i$. \mathcal{F} is called a partition of unity **subordinate** to the cover \mathcal{U} . The existence of such a partition of unity plays a crucial role in showing the existence of a Riemannian metric on a paracompact manifold. The concepts of paracompactness and partition of unity were introduced into topology by Dieudonné. It was shown by the author in [265] that pseudo-Riemannian manifolds are paracompact. In particular, this implies that space-time (a Lorentz manifold) is topologically a metric space. X is said to be **locally compact** if each point has a compact neighborhood.

Let (X, \mathcal{T}) be a topological space and $A \subset X$. $U, V \in \mathcal{T}$ are said to form a **partition** or a **disconnection** of A if the following conditions are satisfied:

1. $A \subset U \cup V$,
2. $A \cap U \neq \emptyset, A \cap V \neq \emptyset$,
3. $A \cap U \cap V = \emptyset$.

The set A is said to be **connected** if there does not exist any disconnection of A . This is equivalent to saying that A is connected as a topological space with the relative topology. It follows that X is connected if and only if the only subsets of X that are both open and closed are \emptyset and X . If X is not connected then it can be partitioned into maximal connected subsets called the **connected components** of X . Each connected component is a closed subset of X . The set of all connected components is denoted by $\pi_0(X)$. The cardinality of $\pi_0(X)$ is a topological invariant. The continuous image of a connected set is connected. Since the connected subsets of \mathbf{R} are intervals, it follows that every real-valued continuous function f on a connected subset of X satisfies the **intermediate value property**, i.e., f takes every value between any two values. X is said to be **locally connected** if its topology has a base consisting of connected sets. The commonly used concepts of path connected and simply connected are discussed in the next section.

2.3 Homotopy Groups

In homotopy theory the algebraic structures (homotopy groups) associated with a topological space X are defined through the concept of homotopy between maps from standard sets (intervals and spheres) to X . The fundamental group or the first homotopy group of a topological space was introduced by H. Poincaré (1895), while the idea of the higher homotopy groups is principally due to W. Hurewicz (1935). All the homotopy groups arise naturally in the mathematical formulation of classical and quantum field theories. To make

our treatment essentially self-contained, we have given more details than are strictly necessary for the physical applications.

Let X and Y be topological spaces and h a map from $\mathcal{D}(h) \subset Y$ to X that can be extended to a continuous map from Y to X . Let $C(Y, X; h)$ be the set

$$C(Y, X; h) = \{f \in C(Y, X) \mid f|_{\mathcal{D}(h)} = h\}$$

where $C(Y, X)$ is the set of continuous maps from Y to X . We say that $f \in C(Y, X)$ is **homotopic** to $g \in C(Y, X)$ **relative** to h and write $f \sim_h g$ if there exists a continuous map $H : Y \times I \rightarrow X$, where $I := [0, 1]$, such that the following conditions hold:

$$H(y, 0) = f(y), \quad H(y, 1) = g(y), \quad \forall y \in Y, \quad (2.1)$$

$$H(y, t) = h(y), \quad \forall y \in \mathcal{D}(h), \quad \forall t \in I. \quad (2.2)$$

H is called a **homotopy relative** to h from f to g . Observe that condition (2.2) implies that $f, g \in C(Y, X; h)$. We may think of H as a family $\{H_t := H(\cdot, t) \mid t \in I\} \subset C(Y, X; h)$ of continuous maps from Y to X parametrized by t , which deforms the map f continuously into the map g , keeping fixed their values on $\mathcal{D}(h)$, i.e., $H_t \in C(Y, X; h), \forall t \in I$. It can be shown that the relation \sim_h is an equivalence relation in $C(Y, X; h)$. We denote the equivalence class of f by $[f]$. If h is the empty map, i.e., $\mathcal{D}(h) = \emptyset$ so that $C(Y, X; h) = C(Y, X)$, then we will simply write $f \sim g$ and say that f and g are **homotopic**. We observe that in this case there is no condition (2.2) but only the condition (2.1). A topological space X is **contractible** if $id_X \sim c_a$, where id_X is the identity map on X and $c_a : X \rightarrow X$ is the constant map defined by $c_a(x) = a, \forall x \in X$ and for some fixed $a \in X$.

Let X be a topological space. A **path** in X from $a \in X$ to $b \in X$ is a map $\alpha \in C(I, X)$ such that $\alpha(0) = a, \alpha(1) = b$. We say that X is **path connected** if there exists a path from a to $b, \forall a, b \in X$. X is **locally path connected** if its topology is generated by path connected open sets. A path connected topological space is connected, but the converse is not true. However, a connected and locally path connected topological space *is* path connected, and hence connected manifolds are path connected. In what follows, we take all topological spaces to be connected manifolds unless otherwise indicated.

Let α be a path in X from a to b ; the **opposite path** of α is the path $\overleftarrow{\alpha}$ in X from b to a such that $\overleftarrow{\alpha}(t) = \alpha(1 - t), \forall t \in I$. A **loop** in X at $a \in X$ is a path in X from a to a . The set of loops in X at a is

$$P(X, a) := C(I, X; h_a),$$

where $\mathcal{D}(h_a) = \partial I = \{0, 1\}$ and $h_a(0) = h_a(1) = a$. Let $[\alpha]$ be the equivalence class of the loops at a that are homotopic to α relative to h_a and let $E_1(X, a)$ be the set of equivalence classes of homotopic loops at a , i.e.,

$$E_1(X, a) := \{[\alpha] \mid \alpha \in P(X, a)\}.$$

If $\alpha, \beta \in P(X, a)$, then we denote by $\alpha * \beta \in P(X, a)$ the loop defined by

$$(\alpha * \beta)(t) = \begin{cases} \alpha(2t), & 0 \leq t \leq 1/2 \\ \beta(2t - 1), & 1/2 \leq t \leq 1. \end{cases} \quad (2.3)$$

The operation $*$ induces an operation on $E_1(X, a)$, which we denote by juxtaposition. This operation makes $E_1(X, a)$ into a group with identity the class $[c_a]$ of the constant loop at a , the class $[\bar{\alpha}]$ being the inverse of $[\alpha]$. This group is called the **fundamental group** or the **first homotopy group** of X at a and is denoted by $\pi_1(X, a)$. A topological space X with a distinguished point a is called a **pointed topological space** and is denoted by (X, a) . Thus, we have associated with every pointed topological space (X, a) a group $\pi_1(X, a)$. Let $(X, a), (Y, b)$ be two pointed topological spaces and $f : X \rightarrow Y$ a morphism of pointed topological spaces; i.e., f is continuous and $f(a) = b$. Then the map

$$\pi_1(f) : \pi_1(X, a) \rightarrow \pi_1(Y, b)$$

defined by $[\alpha] \mapsto [f \circ \alpha]$ is a homomorphism. π_1 turns out to be a covariant functor from the category of pointed topological spaces to the category of groups (see Appendix C). If X is path connected then $\pi_1(X, a) \cong \pi_1(X, b), \forall a, b \in X$ (the isomorphism is induced by a path from a to b and hence is not canonical). In view of this result we sometimes write $\pi_1(X)$ to indicate the fundamental group of a path connected topological space X . A topological space X is said to be **simply connected** if it is path connected and $\pi_1(X)$ is the trivial group consisting of only the identity element. A contractible space is simply connected.

We now introduce the notion of n -connected, which allows us to give an alternative definition of simply connected. Let $X_n := C(S^n, X)$ be the space of continuous maps from the n -sphere S^n to X . We say that X is **n -connected** if the space X_n with its standard (compact open) topology is path connected. Thus **0-connected** is just path connected and **1-connected** is simply connected as defined above. The fundamental group is an important invariant of a topological space, i.e.,

$$X \cong Y \Rightarrow \pi_1(X) \cong \pi_1(Y).$$

A surprising application of the non-triviality of the fundamental group is found in the Bohm–Aharonov effect in Abelian gauge theories. We discuss this application in Chapter 8.

The topological spaces X, Y are said to be **homotopically equivalent** or of the same **homotopy type** if there exist continuous maps $f : X \rightarrow Y$ and $g : Y \rightarrow X$ such that

$$f \circ g \sim id_Y, \quad g \circ f \sim id_X.$$

The relation of homotopy equivalence is, in general, weaker than homeomorphism. The following discussion of the Poincaré conjecture and its generalizations illustrate this.

Poincaré Conjecture: *Every closed (i.e., compact and without boundary) simply connected 3-manifold is homeomorphic to S^3 .*

For $n > 3$ the conjecture is not true, as shown by our discussion after Example 2.6. However, we have the following.

Generalized Poincaré Conjecture: *Every closed n -manifold homotopically equivalent to the n -sphere S^n is homeomorphic to S^n .*

As we remarked in the introduction, this generalized conjecture was proved to be true for $n > 4$ by Smale in 1960. The case $n = 4$ was settled in the affirmative by Freedman [136] in 1981 and the case $n = 3$ by Perelman (see section 6.8 for Perelman's work).

Let $p : E \rightarrow B$ be a continuous surjection. We say that the pair (E, p) is a **covering** of B if each $x \in B$ has a path connected neighborhood U such that each pathwise connected component of $p^{-1}(U)$ is homeomorphic to U . In particular, p is a local homeomorphism. E is called the **covering space**, B the **base space**, and p the **covering projection**. It can be shown that if B is path connected, then the cardinality of the fibers $p^{-1}(x)$, $x \in B$, is the same for all x . If this cardinality is a natural number n , then we say that (E, p) is an n -fold covering of B .

Example 2.3 Let $U(1) := \{z \in \mathbf{C} \mid |z| = 1\}$.

1. Let $q_n : U(1) \rightarrow U(1)$ be the map defined by

$$q_n(z) = z^n,$$

where $z \in U(1) = \{z \in \mathbf{C} \mid |z| = 1\}$. Then $(U(1), q_n)$ is an n -fold covering of $U(1)$.

2. Let $p : \mathbf{R} \rightarrow U(1)$ be the map defined by

$$p(t) = \exp(2\pi it).$$

Then (\mathbf{R}, p) is a simply connected covering of $U(1)$. In this case the fiber $p^{-1}(1)$ is \mathbf{Z} .

3. Let $n > 1$ be a positive integer and $\pi : S^n \rightarrow \mathbf{RP}^n$ be the natural projection

$$x \mapsto [x].$$

This is a 2-fold covering. In the special case $n = 3$, $S^3 \cong SU(2) \cong Spin(3)$ and $\mathbf{RP}^3 \cong SO(3)$. This covering $\pi : Spin(3) \rightarrow SO(3)$ is well known in physics as associating two spin matrices in $Spin(3)$ to the same rotation matrix in $SO(3)$. The distinction between spin and angular momentum is related to this covering map.

A covering space (U, q) with U simply connected is called a **universal covering space** of the base space B . A necessary and sufficient condition

for the existence of a universal covering space of a path connected and locally path connected topological space X is that X be **semi locally simply connected**, i.e., $\forall x \in X$ there should exist an open neighborhood A of x such that any loop in A at x is homotopic in X to the constant loop at x . All connected manifolds are semi locally simply connected. If (E, p) is a covering of B and (U, q) is a universal covering of B and $u \in U, x \in E$ are such that $q(u) = p(x)$, then there exists a unique covering (U, f) of E such that $f(u) = x$ and $p \circ f = q$, i.e., the following diagram commutes.

$$\begin{array}{ccc} U & \xrightarrow{f} & E \\ & \searrow q \quad \swarrow p & \\ & B & \end{array}$$

From this it follows that, if $(U_1, q_1), (U_2, q_2)$ are two universal covering spaces of B and $u_1 \in U_1, u_2 \in U_2$ are such that $q_1(u_1) = q_2(u_2)$, then there exists a unique homeomorphism $f : U_1 \rightarrow U_2$, such that $f(u_1) = u_2$ and $q_2 \circ f = q_1$, i.e., the following diagram commutes.

$$\begin{array}{ccc} U_1 & \xrightarrow{f} & U_2 \\ & \searrow q_1 \quad \swarrow q_2 & \\ & B & \end{array}$$

Thus a universal covering space is essentially unique, i.e., is unique up to homeomorphism. Let (U, q) be a universal covering of B . A **covering** or **deck transformation** f is an automorphism of U such that $q \circ f = q$ or the following diagram commutes:

$$\begin{array}{ccc} U & \xrightarrow{f} & U \\ & \searrow q \quad \swarrow q & \\ & B & \end{array}$$

It can be shown that the set $C(U, q)$ of all covering transformations is a subgroup of $\text{Aut}(U)$ isomorphic to $\pi_1(B)$. This observation is useful in computing fundamental groups of some spaces as indicated in the following example.

Example 2.4 The covering $(U(1), q_n)$ of Example (2.3) above is not a universal covering while the coverings (\mathbf{R}, p) and (S^n, π) discussed there are universal coverings. Every deck transformation of (\mathbf{R}, p) has the form $f_n(t) = t + n$, $n \in \mathbf{Z}$. From this it is easy to deduce the following:

$$\pi_1(\mathbf{RP}^1) \cong \pi_1(S^1) \cong \pi_1(U(1)) \cong \mathbf{Z}.$$

The only deck transformation of (S^n, π) different from the identity is the antipodal map α defined by $\alpha(x) = -x$, $\forall x \in S^n$. It follows that

$$\pi_1(\mathbf{RP}^n) \cong \mathbf{Z}_2, \quad n > 1.$$

The fundamental group can be used to define invariants of geometric structures such as knots and links in 3-manifolds.

Example 2.5 An embedding $k : S^1 \rightarrow \mathbf{R}^3$ is called a **knot** in \mathbf{R}^3 . Two knots k_1, k_2 are said to be **equivalent** if there exists a homeomorphism $h : \mathbf{R}^3 \rightarrow \mathbf{R}^3$ which is the identity on the complement of some disk $D_n = \{x \in \mathbf{R}^3 \mid \|x\| \leq n\}$, $n \in \mathbf{N}$ and such that $h \circ k_1 = k_2$, i.e., the following diagram commutes.

$$\begin{array}{ccc} & S^1 & \\ k_1 \swarrow & & \searrow k_2 \\ \mathbf{R}^3 & \xrightarrow{h} & \mathbf{R}^3 \end{array}$$

We define the **knot group** $\nu(k)$ by

$$\nu(k) := \pi_1(\mathbf{R}^3 \setminus k(S^1)).$$

It is easy to verify that equivalent knots have isomorphic knot groups. An algebraic structure preserved under knot equivalence is called a **knot invariant**. Thus, the fundamental group provides an important example of a knot invariant.

If X, Y are topological spaces, then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$. In particular, if X and Y are simply connected, then $X \times Y$ is simply connected. From this result it follows, for example, that $\pi_1(\mathbf{R}^n) = id$ and

$$\pi_1(T^n) \cong \mathbf{Z}^n \quad \text{where} \quad T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}} \quad \text{is the real } n\text{-torus.}$$

If B is a connected manifold then there exists a universal covering (U, q) of B such that U is also a manifold and q is smooth. If G is a connected Lie group then there exists a universal covering (U, p) of G such that U is a simply connected Lie group and p is a local isomorphism of Lie groups. The pair (U, p) is called the **universal covering group** of the group G . In particular G and all its covering spaces are locally isomorphic Lie groups and hence have the same Lie algebra. This fact has the following application in representation theory. Given a representation r of a Lie algebra L on V , there exists a unique simply connected Lie group U with Lie algebra $u \cong L$ and a representation ρ of U on V such that its induced representation $\hat{\rho}$ of u on V is equivalent to r . If G is a Lie group with Lie algebra $\mathfrak{g} \cong L$, then we get a representation of G on V only if the representation ρ of U is equivariant under

the action of $\pi_1(G)$. Thus, from a representation r of the angular momentum algebra $so(3)$ we get a unique representation of the group $Spin(3) \cong SU(2)$ (spin representation). However, r gives a representation of $SO(3)$ (an angular momentum representation) only for even parity, r in this case, being invariant under the action of $\pi_1(SO(3)) \cong \mathbf{Z}_2$. A similar situation arises for the case of the connected component of the Lorentz group $SO(3,1)$ and its universal covering group $SL(2, \mathbf{C})$. In general, the universal covering group of $SO(r, s)_0$ (the connected component of the identity of the group $SO(r, s)$) is denoted by $Spin(r, s)$ and is called the spinor group (see Chapter 3).

There are several possible ways to generalize the definition of π_1 to obtain the higher homotopy groups. We list three important approaches.

(1) Let

$$I^n = \{t = (t_1, \dots, t_n) \in \mathbf{R}^n \mid 0 \leq t_i \leq 1, \quad 1 \leq i \leq n\}.$$

Define the boundary of I^n by

$$\partial I^n = \{t \in I^n \mid t_i = 0 \text{ or } t_i = 1 \text{ for some } i, \quad 1 \leq i \leq n\}.$$

Consider the homotopy relation in

$$P_n(X, a) := C(I^n, X; h)$$

where $\mathcal{D}(h) = \partial I^n$ and $h(\partial I^n) = \{a\} \subset X$. Let $E_n(X, a)$ be the set of equivalence classes in $P_n(X, a)$. Observe that $P_n(X, a)$ is the generalization of $P(X, a) = P_1(X, a)$ for $n > 1$. We generalize the product $*$ in $P(X, a)$ with the following definition. Let

$$R_1 = \{(t_1, \dots, t_n) \in I^n \mid 0 \leq t_1 \leq 1/2\},$$

$$R_2 = \{(t_1, \dots, t_n) \in I^n \mid 1/2 \leq t_1 \leq 1\}$$

and $j_i : R_i \rightarrow I^n$, $i = 1, 2$, be the maps such that

$$j_1(t) = (2t_1, t_2, \dots, t_n), \quad j_2(t) = (2t_1 - 1, t_2, \dots, t_n).$$

For $\alpha, \beta \in P_n(X, a)$ we define $\alpha * \beta$ by

$$(\alpha * \beta)(t) = \begin{cases} \alpha(j_1(t)), & t \in R_1 \\ \beta(j_2(t)), & t \in R_2. \end{cases}$$

Then $\alpha * \beta \in P_n(X, a)$ and $\alpha \sim_h \alpha_1$, $\beta \sim_h \beta_1$ implies $\alpha * \beta \sim_h \alpha_1 * \beta_1$. This allows us to define a product in $E_n(X, a)$, denoted by juxtaposition, by

$$[\alpha][\beta] = [\alpha * \beta].$$

With this product $E_n(X, a)$ is a group. It is called the **n th homotopy group** of X at a and is denoted by $\pi_n(X, a)$.

(2) The second definition is obtained with S^n in the place of I^n and $e_1 = (1, 0, \dots, 0)$ in the place of ∂I^n . Let us consider the space

$$P'_n(X, a) := C(S^n, X; h_0),$$

where $\mathcal{D}(h_0) = \{e_1\}$ and $h_0(e_1) = a$. Let $q : I^n \rightarrow S^n$ be a continuous map that identifies ∂I^n to e_1 . Then $F_1 = q(R_1), F_2 = q(R_2)$ are hemispheres whose intersection $A = q(R_1 \cap R_2)$ is homeomorphic to S^{n-1} and contains e_1 . The quotient spaces of F_1 and F_2 obtained by identifying A to e_1 are homeomorphic to S^n . Let r_1 (resp., r_2) be a continuous map of F_1 (resp., F_2) to S^n that identifies A to e_1 . One can take q, r_1, r_2 so that

$$q \circ j_i = r_i \circ q|_{R_i}, \quad i = 1, 2.$$

Let us define a product $*'$ in $P'_n(X, a)$ by

$$(\alpha *' \beta)(u) = \begin{cases} \alpha(r_1(u)), & u \in F_1 \\ \beta(r_2(u)), & u \in F_2. \end{cases}$$

Let $E'_n(X, a)$ be the set of equivalence classes of homotopic maps in $P'_n(X, a)$. The operation $*'$ induces a product on $E'_n(X, a)$ which makes $E'_n(X, a)$ into a group, which we denote by $\pi'_n(X, a)$. Let $\phi : P'_n(X, a) \rightarrow P_n(X, a)$ be the map defined by

$$\alpha \mapsto \phi(\alpha) = \alpha \circ q.$$

One can verify that $\alpha \sim \beta \implies \phi(\alpha) \sim \phi(\beta)$ and $\phi(\alpha *' \beta) = \phi(\alpha) * \phi(\beta)$. Then ϕ induces a map $\tilde{\phi} : \pi'_n(X, a) \rightarrow \pi_n(X, a)$ which is an isomorphism. Thus, we can identify $\pi'_n(X, a)$ and $\pi_n(X, a)$.

(3) The third definition considers loops on the space of loops. We give only a brief indication of the construction of $\pi_n(X, a)$ using loop spaces. In order to consider loops in the space $P(X, a)$, we have to define a topology on this set. $P(X, a)$ is a function space and a standard topology on $P(X, a)$ is the **compact-open topology** defined as follows. Let $W(K, U) \subset P(X, a)$ be the set

$$W(K, U) := \{\alpha \in P(X, a) \mid \alpha(K) \subset U, K \subset I, U \subset X\}.$$

The compact-open topology is the topology that has a subbase given by the family of subsets $W(K, U)$, where K varies over the compact subsets of I and U over the open subsets of X . Then we define

$$\pi''_2(X, a) := \pi_1(P(X, a), c_a),$$

where c_a is the constant loop at a . We inductively define

$$\pi''_n(X, a) := \pi''_{n-1}(P(X, a), c_a). \quad (2.4)$$

It can be shown that $\pi_n''(X, a)$ is isomorphic to $\pi_n(X, a)$ and thus can be identified with $\pi_n(X, a)$.

The space $P(X, a)$ with the compact-open topology is called the **first loop space** of the pointed space (X, a) and is denoted by $\Omega(X, a)$, or simply by $\Omega(X)$ when the base point is understood. With the constant loop c_a at a as the base point, the loop space $\Omega(X)$ becomes the pointed space $(\Omega(X), c_a)$. We continue to denote by $\Omega(X)$ this pointed loop space. The n th **loop space** $\Omega^n(X)$ of X is defined inductively by

$$\Omega^n(X) = \Omega(\Omega^{n-1}(X)).$$

From this definition and equation (2.4) it follows that

$$\pi_n(X) = \pi_1(\Omega^{n-1}(X)).$$

Thus one can calculate all the homotopy groups $\pi_n(X)$ of any space if one can calculate just the fundamental group of all spaces. However, very little is known about the topology and geometry of general loop spaces. A loop space carries a natural structure of a Hopf space in the sense of the following definition.

Definition 2.1 *A pointed topological space (X, e) is said to be a **Hopf space** (or simply an **H-space**) if there exists a continuous map*

$$\mu : X \times X \rightarrow X$$

*of pointed spaces called **multiplication** such that the maps defined by $x \mapsto \mu(x, e)$ and $x \mapsto \mu(e, x)$ are homotopic to the identity map of X .*

One can verify that the map μ induced by the operation $*$ defined by equation (2.3) makes the pointed loop space $\Omega(X)$ into an H -space. Iterating this construction leads to the following theorem:

Theorem 2.1 *The loop space $\Omega^n(X)$, $n \geq 1$, is an H -space.*

We note that loop spaces of Lie groups have recently arisen in many mathematical and physical calculations (see Segal and Presley [321]). A general treatment of loop spaces can be found in Adams [4]. For a detailed discussion of the three definitions of homotopy groups and their applications see, for example, Croom [90]. In the following theorem we collect some properties of the groups π_n .

Theorem 2.2 *Let X denote a path connected topological space; then*

1. $\pi_n(X, a) \cong \pi_n(X, b)$, $\forall a, b \in X$. In view of this we will write $\pi_n(X)$ instead of $\pi_n(X, a)$.
2. If X is contractible by a homotopy leaving x_0 fixed, then $\pi_n(X) = id$.
3. $\pi_n(X)$ is Abelian for $n > 1$.
4. If (E, p) is a covering space of X , then p induces an injective homomorphism $p_* : \pi_n(E) \rightarrow \pi_n(X)$ for $n > 1$.

Furthermore, if Y is another path connected topological space, then

$$\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y).$$

The computation of homotopy groups is, in general, a difficult problem. Even in the case of spheres not all the higher homotopy groups are known. The computation of the known groups is facilitated by the following theorem:

Theorem 2.3 (Freudenthal) *There exists a homomorphism*

$$F : \pi_k(S^n) \rightarrow \pi_{k+1}(S^{n+1}),$$

*called the **Freudenthal suspension homomorphism**, with the following properties:*

1. F is surjective for $k = 2n - 1$;
2. F is an isomorphism for $k < 2n - 1$.

The results stated in the above theorems are useful in computing the homotopy groups of some spaces that are commonly encountered in applications.

Example 2.6 *In this example we give the homotopy groups of some important spaces that are useful in physical applications.*

1. $\pi_n(\mathbf{R}^m) = id$.
2. $\pi_k(S^n) = id$, $k < n$.
3. $\pi_n(S^n) \cong \mathbf{Z}$.

If G is a Lie group then $\pi_2(G) = 0$. In many physical applications one needs to compute the homotopy of semi-simple Lie groups such as the groups $SO(n)$, $SU(n)$, $U(n)$. If G is a semi-simple Lie group then $\pi_3(G) \cong \mathbf{Z}$. An element $\alpha \in \pi_3(G)$ often arises in field theories as a **topological quantum number**. It arises in the problem of extending a G -gauge field from \mathbf{R}^4 to its compactification S^4 (see Chapter 8 for details).

From Theorems 2.1 and 2.2 and Example 2.6 it follows that $\pi_2(S^4) = id$ and

$$\pi_2(S^2 \times S^2) \cong \pi_2(S^2) \times \pi_2(S^2) \cong \mathbf{Z} \times \mathbf{Z}.$$

Also $\pi_1(S^4) = id$ and $\pi_1(S^2 \times S^2) \cong \pi_1(S^2) \times \pi_1(S^2) = id$. Thus S^4 and $S^2 \times S^2$ are both closed simply connected manifolds that are not homeomorphic. This illustrates the role that higher homotopy groups play in the generalized Poincaré conjecture.

All the homotopy groups of the circle S^1 except the first one are trivial. A path connected topological space X is said to be an **Eilenberg–MacLane space** for the group π if there exists $n \in \mathbf{N}$ such that

$$\pi_n(X) = \pi \quad \text{and} \quad \pi_k(X) = id, \quad \forall k \neq n.$$

Note that π must be Abelian if $n > 1$. It is customary to denote such a space by $K(\pi, n)$. Thus, S^1 is a $K(\mathbf{Z}, 1)$ space. The construction of Eilenberg–MacLane spaces in the late 1940s is considered a milestone in algebraic topology. In 1955, Postnikov showed how to construct a topological space starting with an Eilenberg–MacLane space as a base and building a succession of fiber spaces with other Eilenberg–MacLane spaces as fibers. This construction is known as the **Postnikov tower** construction and allows us to construct a model topological space having the homotopy type of a given space.

Example 2.7 (Hopf Fibration) *An important example of computation of higher homotopy groups was given by H. Hopf in 1931 in his computation of $\pi_3(S^2)$. Consider the following action $h : U(1) \times \mathbf{C}^2 \rightarrow \mathbf{C}^2$ of $U(1)$ on \mathbf{C}^2 defined by*

$$(z, (z_1, z_2)) \mapsto (zz_1, zz_2).$$

This action leaves the unit sphere $S^3 \subset \mathbf{C}^2$ invariant and hence induces an action on S^3 with fibers isomorphic to S^1 and quotient $\mathbf{CP}^1 \cong S^2$, making S^3 a principal fiber bundle over S^2 . We also denote by $h : S^3 \rightarrow S^2$ the natural projection. The above construction is called the Hopf fibration of S^3 . Hopf showed that $[h] \in \pi_3(S^2)$ is non-trivial, i.e., $[h] \neq \text{id}$, and generates $\pi_3(S^2)$ as an infinite cyclic group, i.e. $\pi_3(S^2) \cong \mathbf{Z}$. This class $[h]$ is essentially the invariant that appears in the Dirac monopole quantization condition (see Chapter 8). The Hopf fibration of S^3 can be extended to the unit sphere $S^{2n-1} \subset \mathbf{C}^n$. The quotient space in this case is the complex projective space \mathbf{CP}^{n-1} and the fibration is called the complex Hopf fibration. This fibration arises in the geometric quantization of the isotropic harmonic oscillator.

One can similarly consider the real, quaternionic and octonionic Hopf fibrations. For example, to study the quaternionic Hopf fibration we begin by observing that

$$SU(2) \cong \{x = x_0 + x_1i + x_2j + x_3k \in \mathbf{H} \mid |x| = 1\}$$

acts as the group of unit quaternions on \mathbf{H}^n on the right by quaternionic multiplication. This action leaves the unit sphere $S^{4n-1} \subset \mathbf{H}^n$ invariant and induces a fibration of S^{4n-1} over the quaternionic projective space \mathbf{HP}^{n-1} . For the case $n = 2$, $\mathbf{HP}^1 \cong S^4$ and the Hopf fibration gives S^7 as a nontrivial principal $SU(2)$ bundle over S^4 . This bundle plays a fundamental role in our discussion of the BPST instanton in Chapter 9.

We conclude this section with a brief discussion of a fundamental result in homotopy theory, namely, the Bott periodicity theorem.

2.3.1 Bott Periodicity

The higher homotopy groups of the classical groups were calculated by Bott [49] in the course of proving his well known periodicity theorem. An excellent account of this proof as well as other applications of Morse theory may be found in Milnor [284]. We comment briefly on the original proof of the Bott periodicity theorem for the special unitary group. Bott considered the space \mathcal{S} of parametrized smooth curves $c : [0, 1] \rightarrow SU(2m)$, joining $-I$ and $+I$ in $SU(2m)$, and applied Morse theory to the total kinetic energy function \mathcal{K} of c defined by

$$\mathcal{K}(c) = \frac{1}{2} \int_0^1 v^2 dt,$$

where $v = \dot{c}$ is the velocity of c . The Euler–Lagrange equations for the functional $\mathcal{K} : \mathcal{S} \rightarrow \mathbf{R}$ are the well known equations of geodesics, which are the auto-parallel curves with respect to the Levi-Civita connection on $SU(2m)$. Now $SU(2m)/SU(m) \times SU(m)$ can be identified with the complex Grassmannian $G_m(2m)$ of m -planes in $2m$ space. The gradient flow of \mathcal{K} is a homotopy equivalence between the loop space on $SU(2m)$ and the Grassmannian up to dimension $2m$, i.e.,

$$\pi_{i+1}SU(2m) = \pi_i(\Omega SU(2m)) = \pi_i G_m(2m), \quad 0 \leq i \leq 2m.$$

This result together with the standard results from algebraic topology on the homotopy groups of fibrations imply the periodicity relation

$$\pi_{i-1}SU(k) = \pi_{i+1}SU(k), \quad i \leq 2m \leq k.$$

We give below a table of the higher homotopy groups of $U(n)$, $SO(n)$, and $SP(n)$ and indicate the **stable range** of values of n in which the periodicity appears.

Table 2.2 Stable homotopy of the classical groups

π_k	$U(n), 2n > k$	$SO(n), n > k + 1$	$SP(n), 4n > k - 2$
π_1	\mathbf{Z}	\mathbf{Z}_2	0
π_2	0	0	0
π_3	\mathbf{Z}	\mathbf{Z}	\mathbf{Z}
π_4	0	0	\mathbf{Z}_2
π_5	\mathbf{Z}	0	\mathbf{Z}_2
π_6	0	0	0
π_7	\mathbf{Z}	\mathbf{Z}	\mathbf{Z}
π_8	0	\mathbf{Z}_2	0
period	2	8	8

Table 2.2 of **stable homotopy groups** of the classical groups we have given is another way of stating the Bott periodicity theorem. More generally, Bott showed that for sufficiently large n the homotopy groups of the n -dimensional unitary group $U(n)$, the rotation group $SO(n)$ and the symplectic group $Sp(n)$ do not depend on n and that they exhibit a certain periodicity relation. To state the precise result we need to define the infinite-dimensional groups $U(\infty)$, $SO(\infty)$, and $Sp(\infty)$. Recall that the natural embedding of \mathbf{C}^n into \mathbf{C}^{n+1} induces the natural embedding of $U(n)$ into $U(n+1)$ and defines the inductive system (see Appendix C)

$$U(1) \subset U(2) \subset \cdots \subset U(n) \subset U(n+1) \subset \cdots$$

of unitary groups. We define the infinite-dimensional unitary group $U(\infty)$ to be the inductive limit of the above system. The groups $SO(\infty)$ and $Sp(\infty)$ are defined similarly. Using these groups we can state the following version of the **Bott periodicity theorem**.

Theorem 2.4 *The homotopy groups of the infinite-dimensional unitary, rotation, and symplectic groups satisfy the following relations:*

1. $\pi_{k+2}(U(\infty)) = \pi_k(U(\infty))$
2. $\pi_{k+8}(SO(\infty)) = \pi_k(SO(\infty))$
3. $\pi_{k+8}(Sp(\infty)) = \pi_k(Sp(\infty))$

We already indicated how the statements of this theorem are related to the periodicity relations of Clifford algebras in Chapter 1. We will give the K -theory version of Bott periodicity in Chapter 5. The Bott periodicity theorem is one of the most important results in mathematics and has surprising connections with several other fundamental results, such as the Atiyah–Singer index theorem. Several of the groups appearing in this theorem have been used in physical theories. Some homotopy groups outside the stable range also arise in gauge theories. For example, $\pi_3(SO(4)) = \mathbf{Z} \oplus \mathbf{Z}$ is closely related to the self-dual and the anti-self-dual solutions of the Yang–Mills equations on S^4 . $\pi_7(SO(8)) = \mathbf{Z} \oplus \mathbf{Z}$ arises in the solution of the Yang–Mills equations on S^8 (see [175, 243] for details). It can be shown that this solution and similar solutions on higher-dimensional spheres satisfy certain generalized duality conditions.

2.4 Singular Homology and Cohomology

In homology theory the algebraic structures (homology modules) associated with a topological space X are defined through the construction of chain complexes (see Appendix D) related to X . If one uses simplexes related to X , one has simplicial homology, which was introduced by Poincaré (see [90] for a very accessible introduction). There are, however, other homology theories

that give rise to isomorphic homology modules under fairly general conditions on X . We will discuss only the singular homology theory, whose introduction is usually attributed to Lefschetz. For other approaches see, for example, Eilenberg, Steenrod [118], and Spanier [357].

Let q be a nonnegative integer and $\Delta^q \subset \mathbf{R}^{q+1}$ be the set

$$\Delta^q := \{(x_0, \dots, x_q) \in \mathbf{R}^{q+1} \mid \sum_{i=0}^q x_i = 1, x_i \geq 0, i = 0, 1, \dots, q\}.$$

The set Δ^q with the relative topology is called the **standard q -simplex**. Let X be a topological space. A **singular q -simplex** in X is a continuous map $s : \Delta^q \rightarrow X$. We denote by $\Sigma_q(X)$ the set of all singular q -simplexes in X . If \mathbf{P} is a principal ideal domain, we denote by $\mathcal{S}_q(X; \mathbf{P})$ the free \mathbf{P} -module generated by $\Sigma_q(X)$ and we will simply write $\mathcal{S}_q(X)$ when the reference to \mathbf{P} is understood. By definition of a free module it follows that every element of $\mathcal{S}_q(X)$ can be regarded as a function $c : \Sigma_q(X) \rightarrow \mathbf{P}$ such that $c(s) = 0$ for all but finitely many singular q -simplexes s in X . An element of $\mathcal{S}_q(X)$ is called a **singular q -chain** and $\mathcal{S}_q(X)$ is called the **q th singular chain module** of X . If $s \in \Sigma_q(X)$, let χ_s denote the singular q -chain defined by

$$\chi_s(s') = \delta_{ss'}, \quad \forall s' \in \Sigma_q(X),$$

where $\delta_{ss'} = 0$ for $s \neq s'$ and $\delta_{ss} = 1$ (1 is the unit element of \mathbf{P}). χ_s is called an **elementary** singular chain. It is customary to write s instead of χ_s . Thus, any element $c \in \mathcal{S}_q(X)$ can be expressed uniquely as

$$c = \sum_{s \in \Sigma_q(X)} g_s s, \quad g_s \in \mathbf{P},$$

where $g_s = 0$ for all but finitely many s .

Let q be a positive integer, $s \in \Sigma_q(X)$ be a singular q -simplex and $i \leq q$ a nonnegative integer. The map

$$s^{(i)} : \Delta^{q-1} \rightarrow X$$

defined by

$$s^{(i)}(x_0, \dots, x_{q-1}) = s(x_0, \dots, x_{i-1}, 0, x_i, \dots, x_{q-1})$$

is a singular $(q-1)$ -simplex, called the **i th face** of s . Let us denote by δ_q the unique linear map

$$\delta_q : \mathcal{S}_q(X) \rightarrow \mathcal{S}_{q-1}(X)$$

such that

$$\delta_q(s) = \sum_{i=0}^q (-1)^i s^{(i)}.$$

One can show that

$$\delta_{q-1} \circ \delta_q = 0.$$

Let $q \in \mathbf{Z}$. For $q < 0$ we define $\mathcal{S}_q(X) = 0$ and for $q \leq 0$ we define $\delta_q = 0$. With these definitions

$$\cdots \longleftarrow \mathcal{S}_{q-1}(X) \xleftarrow{\delta_q} \mathcal{S}_q(X) \xleftarrow{\delta_{q+1}} \mathcal{S}_{q+1}(X) \longleftarrow \cdots$$

is a chain complex, which is also simply denoted by $\mathcal{S}_*(X)$. $\mathcal{S}_*(X)$ is called the **singular chain complex** (with coefficients in \mathbf{P}). Let X, Y be topological spaces and $f : X \rightarrow Y$ a continuous map. For all $q \in \mathbf{Z}$, let us denote by $\mathcal{S}_q(f; \mathbf{P})$, or simply $\mathcal{S}_q(f)$, the unique linear map

$$\mathcal{S}_q(f) : \mathcal{S}_q(X) \rightarrow \mathcal{S}_q(Y)$$

such that $\chi_s \mapsto \chi_{f \circ s}$. One can show that

$$\delta \circ \mathcal{S}_q(f) = \mathcal{S}_{q-1}(f) \circ \delta. \quad (2.5)$$

The family $\mathcal{S}_*(f) := \{\mathcal{S}_q(f) \mid q \in \mathbf{Z}\}$ is a chain morphism such that

1. $\mathcal{S}_*(id_X) = id_{\mathcal{S}_*(X)}$,
2. $\mathcal{S}_*(g \circ f) = \mathcal{S}_*(g) \circ \mathcal{S}_*(f)$, $g \in Mor(Y, Z)$.

Thus, $\mathcal{S}_*(\cdot; \mathbf{P})$ is a covariant functor from the category of topological spaces to the category of chain complexes over \mathbf{P} . The q th homology module of the complex $\mathcal{S}_*(X)$ over \mathbf{P} is called the q th **singular homology module** and is denoted by $H_q(X; \mathbf{P})$, or simply $H_q(X)$. An element of $H_q(X)$ is called a q -th homology class of X . In general, computing homology modules is a non-trivial task and requires the use of specialized tools. However, it is easy to show that $H_0(X; \mathbf{P})$ is a free \mathbf{P} -module on as many generators as there are path components of X . In particular, if X is path connected, then

$$H_0(X; \mathbf{P}) \cong \mathbf{P}.$$

Let X, Y be topological spaces and $f : X \rightarrow Y$ a continuous map. By passage to the quotient, $\mathcal{S}_q(f; \mathbf{P})$ induces the map

$$H_q(f; \mathbf{P}) : H_q(X) \rightarrow H_q(Y),$$

which we simply denote also by $H_q(f)$. $H_q(f)$ is a linear map such that

1. $H_q(id_X) = id_{H_q(X)}$,
2. $H_q(g \circ f) = H_q(g) \circ H_q(f)$, $g \in Mor(Y, Z)$.

Thus, $H_q(\cdot; \mathbf{P})$ is a covariant functor from the category of topological spaces to the category of \mathbf{P} -modules. It follows that homeomorphic spaces have isomorphic homology modules. This result is often expressed by saying that homology modules are topological invariants. In fact, one can show that ho-

motopy equivalent spaces have isomorphic homology modules, or that homology modules are homotopy invariants.

Observe that singular 1-simplexes in a topological space X are paths in X . Thus, there exists a natural map

$$\phi : \pi_1(X, x_0) \rightarrow H_1(X; \mathbf{P})$$

such that, if γ is a loop at x_0 , $\phi([\gamma])$ is the homology class of the singular 1-simplex γ . The precise connection between fundamental groups and homology groups of path connected topological spaces is given in the following theorem (see [160] for a proof):

Theorem 2.5 *Let X be a path connected topological space. The map ϕ defined above is a surjective homomorphism whose kernel is the commutator subgroup F of $\pi_1(X)$ (F is the subgroup of $\pi_1(X)$ generated by all the elements of the form $aba^{-1}b^{-1}$). Thus, $H_1(X; \mathbf{Z})$ is isomorphic to $\pi_1(X)/F$. In particular, $H_1(X)$ is isomorphic to $\pi_1(X)$ if and only if $\pi_1(X)$ is Abelian.*

In view of the above theorem the first homology group is sometimes referred to as the “Abelianization” of the fundamental group. For the relation between higher homology and homotopy groups an important result is the following Hurewicz isomorphism theorem, which gives sufficient conditions for isomorphisms between $H_q(X; \mathbf{Z})$ and $\pi_q(X)$ for $q > 1$.

Theorem 2.6 (Hurewicz) *Let X be a simply connected space. If there exists $j \in \mathbf{N}$ such that $\pi_j(X)$ is the first non-trivial higher homotopy group of X , then*

$$\pi_k(X) \cong H_k(X; \mathbf{Z}), \quad \forall k, 1 \leq k \leq j.$$

Thus for a simply connected space the first non-trivial homotopy and homology groups are in the same dimension and are equal.

Let A be a subspace of the topological space X . The pair (X, A) is called a **topological pair**. If (X', A') is another topological pair and $f : X \rightarrow X'$ is a continuous map such that $f(A) \subset A'$, then f is called a morphism of the topological pair (X, A) into (X', A') and is denoted by $f : (X, A) \rightarrow (X', A')$.

Let (X, A) be a topological pair. Then, $\forall q \in \mathbf{Z}$, $\mathcal{S}_q(A)$ can be regarded as a submodule of $\mathcal{S}_q(X)$ and $\delta_q(\mathcal{S}_q(A)) \subset \mathcal{S}_{q-1}(A)$. The quotient chain complex of $(\mathcal{S}_*(X), \delta)$ by $\mathcal{S}_*(A)$ is called the **relative singular chain complex of $X \bmod A$** and is denoted by $\mathcal{S}_*(X, A)$. The q th homology module of this chain complex is denoted by $H_q(X, A)$, or $H_q(X, A; \mathbf{P})$ if one wants to stress the fact that the coefficients are in the principal ideal domain \mathbf{P} . $H_q(X, A)$ is called the q th **relative singular homology module of $X \bmod A$** . Let $Z_q(X, A)$, $B_q(X, A)$ be defined by

$$Z_q(X, A) := \{c \in \mathcal{S}_q(X) \mid \delta c \in \mathcal{S}_{q-1}(A)\},$$

$$B_q(X, A) := \{c \in \mathcal{S}_q(X) \mid c = \delta w + c', \quad w \in \mathcal{S}_{q+1}(X), c' \in \mathcal{S}_q(A)\}.$$

Then one can show that

$$H_q(X, A) \cong Z_q(X, A)/B_q(X, A).$$

Indeed, the above relation is sometimes taken as the definition of $H_q(X, A)$. The elements of $Z_q(X, A)$ (resp. $B_q(X, A)$) are called **q -cycles** (resp., **q -boundaries**) on $X \bmod A$. Let $f : (X, A) \rightarrow (X', A')$ be a morphism of topological pairs. Then the map $\mathcal{S}_q(f)$ sends $\mathcal{S}_q(A)$ into $\mathcal{S}_q(A')$ and hence $\mathcal{S}_q(f)$ induces, by passage to the quotient, the map $\tilde{\mathcal{S}}_q(f) : \mathcal{S}_q(X, A) \rightarrow \mathcal{S}_q(X', A')$. The family $\{\tilde{\mathcal{S}}_q(f) \mid q \in \mathbf{Z}\}$ is denoted by $\tilde{\mathcal{S}}_*(f)$. It is customary to write simply $\mathcal{S}_q(f)$ and $\mathcal{S}_*(f)$ instead of $\tilde{\mathcal{S}}_q(f)$ and $\tilde{\mathcal{S}}_*(f)$, respectively. The map $\mathcal{S}_q(f)$ satisfies equation (2.5). From this it follows that it sends $Z_q(X, A)$ into $Z_q(X', A')$ and $B_q(X, A)$ into $B_q(X', A')$. Thus, $\mathcal{S}_q(f)$ induces, by passage to the quotient, a homomorphism

$$\tilde{H}_q(f; \mathbf{P}) : H_q(X, A) \rightarrow H_q(X', A').$$

It is customary to write $H_q(f; \mathbf{P})$, or simply $H_q(f)$, instead of $\tilde{H}_q(f; \mathbf{P})$. One can show that

$$H_q(id_{(X, A)}) = id_{H_q(X, A)}.$$

Moreover, if $g : (X', A') \rightarrow (X'', A'')$ is a morphism of topological pairs, then

$$H_q(g \circ f) = H_q(g) \circ H_q(f).$$

Thus, $\forall q \in \mathbf{Z}$, $\tilde{H}_q(\cdot; \mathbf{P})$ is a covariant functor from the category of topological pairs to the category of \mathbf{P} -modules.

Let (X, A) be a topological pair, $i : A \rightarrow X$ the natural inclusion map, and $j : (X, \emptyset) \rightarrow (X, A)$ the natural morphism of (X, \emptyset) into (X, A) . Then the sequence induced by these maps

$$0 \longrightarrow \mathcal{S}_*(A) \xrightarrow{\mathcal{S}_*(i)} \mathcal{S}_*(X) \xrightarrow{\mathcal{S}_*(j)} \mathcal{S}_*(X, A) \longrightarrow 0$$

is a short exact sequence of chain complexes. Moreover, one has the related connecting morphism h_* (see Appendix D)

$$h_* = \{h_q : H_q(X, A) \rightarrow H_{q-1}(A) \mid q \in \mathbf{Z}\}.$$

The corresponding long exact sequence

$$\begin{aligned} \cdots \longrightarrow H_{q+1}(X, A) &\xrightarrow{h_{q+1}} H_q(A) \longrightarrow H_q(X) \longrightarrow \\ &\longrightarrow H_q(X, A) \xrightarrow{h_q} H_{q-1}(A) \longrightarrow \cdots \end{aligned}$$

is called the

Relative homology is useful in the evaluation of homology because of the following **excision property**. Let (X, A) be a topological pair and $U \subset A$.

Let $i : (X \setminus U, A \setminus U) \rightarrow (X, A)$ be the natural inclusion. We say that U can be **excised** and that i is an **excision** if

$$H_q(i) : H_q(X \setminus U, A \setminus U) \rightarrow H_q(X, A)$$

is an isomorphism. One can show that, if the closure $\bar{U} \subset A$, then U can be excised. If X is an n -dimensional topological manifold, then, using the excision property, one can show that, $\forall x \in X$,

$$H_n(X, X \setminus \{x\}) \cong \mathbf{P}.$$

Let U be a neighborhood of $x \in X$. If $j_x^U : (X, X \setminus U) \rightarrow (X, X \setminus \{x\})$ denotes the natural inclusion, then we have the homomorphism

$$H_n(j_x^U) : H_n(X, X \setminus U) \rightarrow H_n(X, X \setminus \{x\}).$$

One can show that, $\forall x \in X$, there exists an open neighborhood U of x and $\alpha \in H_n(X, X \setminus U)$ such that $\alpha_y := H_n(j_y^U)(\alpha)$ generates $H_n(X, X \setminus \{y\})$, $\forall y \in U$. Such an element α is called a **local \mathbf{P} -orientation of X along U** . A **\mathbf{P} -orientation system** of X is a set $\{(U_i, \alpha_i) \mid i \in I\}$ such that

1. $\bigcup_{i \in I} U_i = X$;
2. $\forall i \in I, \alpha_i$ is a local \mathbf{P} -orientation of X along U_i ;
3. $\alpha_{i,y} = \alpha_{j,y}, \forall y \in U_i \cap U_j$.

Given the \mathbf{P} -orientation system $\{(U_i, \alpha_i) \mid i \in I\}$ of X , for each $x \in X$, $\exists i \in I$ such that $x \in U_i$ and hence we have a generator α_x of $H_n(X, X \setminus \{x\})$ given by $\alpha_x := \alpha_{i,x}$. Two \mathbf{P} -orientation systems $\{(U_i, \alpha_i) \mid i \in I\}$, $\{(U'_i, \alpha'_i) \mid i \in I'\}$ are said to be **equivalent** if $\alpha_x = \alpha'_x, \forall x \in X$. An equivalence class of \mathbf{P} -orientation systems of X is denoted simply by α and is called a **\mathbf{P} -orientation** of X . One can show that, if X is connected, then two \mathbf{P} -orientations that are equal at one point are equal everywhere. A topological manifold is said to be **\mathbf{P} -orientable** if it admits a \mathbf{P} -orientation. A **\mathbf{P} -oriented** manifold is a \mathbf{P} -orientable manifold with the choice of a fixed \mathbf{P} -orientation α . A manifold is said to be **orientable** (resp., **oriented**) when it is **\mathbf{Z} -orientable** (resp., **\mathbf{Z} -oriented**). We note that homology with integer (resp., rational, real) coefficients is often referred to as the integral (resp. rational, real) homology.

If X is a compact connected n -dimensional, \mathbf{P} -oriented manifold, then

$$H_n(X) \cong \mathbf{P}.$$

This allows us to give the following definition of the fundamental class of a compact connected oriented manifold with orientation α . Let α_x be the local orientation at $x \in X$. Then there exists a unique generator of $H_n(X)$, whose image under the canonical map $H_n(X) \rightarrow H_n(X, X \setminus \{x\})$ is α_x . This generator of $H_n(X)$ is called the **fundamental class** of X with the orientation α and is denoted by $[X]$.

Using integral homology we can define the Betti numbers and Euler characteristic for certain topological spaces. They turn out to be integer-valued topological invariants. In order to define them, let us recall some results from algebra. Let V be a \mathbf{P} -module. An element $v \in V$ is called a **torsion element** if there exists $a \in \mathbf{P} \setminus \{0\}$ such that $av = 0$. The set of torsion elements of V is a submodule of V denoted by V_t and called the **torsion submodule** of V . If $V_t = \{0\}$ then V is said to be **torsion free**. One can show (see Lang [244]) that if V is finitely generated then there exists a free submodule V_f of V such that

$$V = V_t \oplus V_f.$$

The dimension of V_f is called the **rank** of V . Let M be a topological manifold. If the homology modules $H_q(M; \mathbf{Z})$ are finitely generated, then the rank of $H_q(M; \mathbf{Z})$ is called the q th **Betti number** and is denoted by $b_q(M)$. In this case we define the **Euler** (or **Euler–Poincaré**) **characteristic** $\chi(M)$ of M by

$$\chi(M) := \sum_q (-1)^q b_q(M).$$

We observe that if M is compact then the homology modules are finitely generated. Roughly speaking, the Betti numbers count the number of holes of appropriate dimension in the manifold, whereas the torsion part indicates the twisting of these holes.

An example of this is the following. Recall that the **Klein bottle** K is obtained by identifying the two ends of the cylinder $[0, 1] \times S^1$ with an antipodal twist, i.e., by identifying $(0, \theta)$ with $(1, -\theta)$, $\theta \in S^1$. This twist is reflected in the torsion part of homology and we have $H_1(K; \mathbf{Z}) = \mathbf{Z} \oplus \mathbf{Z}_2$, whereas $H_1(K; \mathbf{R}) = \mathbf{R}$. Note that if we use homology with coefficient in \mathbf{Z}_2 then the torsion part also vanishes since \mathbf{Z}_2 has no non-trivial subgroups. If the integral domain \mathbf{P} is taken to be the field \mathbf{Q} or \mathbf{R} , then the Betti numbers remain the same but there is no torsion part in the homology modules.

By duality a homology theory gives a cohomology theory. As an example singular cohomology is defined as the dual of singular homology. The q th **singular cochain module** of a topological space X with coefficients in \mathbf{P} is the dual of $\mathcal{S}_q(X; \mathbf{P})$ and is denoted by $\mathcal{S}^q(X; \mathbf{P})$ or simply $\mathcal{S}^q(X)$. If X, Y are topological spaces and $f : X \rightarrow Y$ is a continuous map, then we denote by $\mathcal{S}^q(f; \mathbf{P})$ or simply by $\mathcal{S}^q(f)$ the map

$$\mathcal{S}^q(f) := {}^t\mathcal{S}_q(f) : \mathcal{S}^q(Y) \rightarrow \mathcal{S}^q(X),$$

where we have used the notation tL for the transpose of the linear map L (here $L = \mathcal{S}^q(f)$). Then it is easy to verify that $\mathcal{S}^q(\cdot; \mathbf{P})$ is a contravariant functor from the category of topological spaces to the category of \mathbf{P} -modules. The q th **singular cohomology \mathbf{P} -module** $H^q(X; \mathbf{P})$ or simply $H^q(X)$ is defined by

$$H^q(X) = \text{Ker } {}^t\delta_{q+1} / \text{Im } {}^t\delta_q,$$

where ${}^t\delta_{q+1} : \mathcal{S}^q(X) \rightarrow \mathcal{S}^{q+1}(X)$ is the q th **coboundary operator**. The module $Z^q(X) := \text{Ker } {}^t\delta_{q+1}$ (resp., $B^q(X) := \text{Im } {}^t\delta_q$) is called the q th singular cohomology module of **cocycles** (resp., **coboundaries**). In particular, the duality of $H^q(X, \mathbf{Z})$ with $H_q(X, \mathbf{Z})$ and the finite dimensionality of $H_q(X, \mathbf{Z})$ implies that $\dim H^q(X, \mathbf{Z}) = \dim H_q(X, \mathbf{Z}) = b_q(X)$, $\forall q \geq 0$, where $b_q(X)$ is the q th Betti number of X . If X, Y are topological spaces and $f : X \rightarrow Y$ is a continuous map, then $\mathcal{S}^q(f)$ sends $Z^q(Y)$ to $Z^q(X)$ and $B^q(Y)$ to $B^q(X)$. Hence, it induces, by passage to the quotient, the homomorphism

$$H^q(f; \mathbf{P}) \equiv H^q(f) : H^q(Y) \rightarrow H^q(X).$$

Then it is easy to verify that $H^q(\cdot; \mathbf{P})$ is a contravariant functor from the category of topological spaces to the category of \mathbf{P} -modules. With an analogous procedure one can define the q th relative singular cohomology modules for a topological pair (X, A) , denoted by $H^q(X, A)$ (see Greenberg [160] for details). A comprehensive introduction to algebraic topology covering both homology and homotopy can be found in Tammo tom Dieck's book [98].

In dealing with noncompact spaces it is useful to consider singular cohomology with compact support that we now define. Let X be a topological manifold. The set \mathcal{K} of compact subsets of X is a directed set with the partial order given by the inclusion relation. Let us consider the direct system

$$D = (\{H^q(X, X \setminus K) \mid K \in \mathcal{K}\}, \{f_K^{K'} \mid (K, K') \in \mathcal{K}_0^2\}),$$

where

$$\mathcal{K}_0^2 := \{(K, K') \in \mathcal{K}^2 \mid K \subset K'\}.$$

The map $f_K^{K'} : H^q(X, X \setminus K) \rightarrow H^q(X, X \setminus K')$ is the homomorphism induced by the inclusion. The q th **singular cohomology \mathbf{P} -module with compact support** is the direct limit of the direct system D and is denoted by $H_c^q(X; \mathbf{P})$, or simply $H_c^q(X)$. Then, by definition

$$H_c^q(X) := \varinjlim H^q(X, X \setminus K).$$

We observe that if X is compact then X is the largest element of \mathcal{K} . Thus, if X is compact we have that $H_c^q(X) = H^q(X)$, $\forall q \in \mathbf{Z}$.

As with homology theories, there are several cohomology theories. An example is given by the **differentiable singular homology** (resp., cohomology) whose difference from singular homology (resp., cohomology) is essentially in the fact that its construction starts with **differentiable singular q -simplexes** instead of (continuous) singular q -simplexes. The differentiable singular homology (resp., cohomology) of X is denoted by ${}_\infty H_*(X; \mathbf{P})$ (resp., $H_\infty^*(X; \mathbf{P})$). Under very general conditions the various cohomology theories are isomorphic (see Warner [396]); for example, $H^*(X; \mathbf{P}) \cong H_\infty^*(X; \mathbf{P})$. In most physical applications we are interested in topological spaces that are dif-

ferentiable manifolds. We now discuss the cohomology theory based on the cochain complex of differential forms on a manifold. This is the well known de Rham cohomology with real coefficients.

2.5 de Rham Cohomology

The **de Rham complex** of an m -dimensional manifold M is the cochain complex $(\Lambda(M), d)$ given by

$$0 \longrightarrow \Lambda^0(M) \xrightarrow{d} \Lambda^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Lambda^n(M) \longrightarrow 0. \quad (2.6)$$

The cohomology $H^*(\Lambda(M), d)$ is called the **de Rham cohomology** of M and is denoted by $H_{deR}^*(M)$. The de Rham cohomology has a natural structure of graded algebra induced by the exterior product. The product on homogeneous elements is given by the map

$$\cup : H^i(M; \mathbf{P}) \times H^j(M; \mathbf{P}) \rightarrow H^{i+j}(M; \mathbf{P})$$

defined by

$$([\alpha], [\beta]) \mapsto [\alpha \wedge \beta].$$

This induced product in cohomology is in fact a special case of a cohomology operation called the cup product (see Spanier [357]).

If M, N are manifolds then we have

$$H_{deR}^*(M \times N) = H_{deR}^*(M) \hat{\otimes} H_{deR}^*(N),$$

where $\hat{\otimes}$ denotes the graded tensor products. In particular, we can express the cohomology of $M \times N$ in terms of the cohomologies of M and N as follows:

$$H_{deR}^k(M \times N) = \bigoplus_{k=i+j} H_{deR}^i(M) \otimes H_{deR}^j(N). \quad (2.7)$$

In fact, the above formula holds more generally and is called the **K nneth formula**.

There is a canonical map ρ called the **de Rham homomorphism**

$$H_{deR}^q(M) \rightarrow (\infty H_q(M; \mathbf{R}))' \cong H_{\infty}^q(M; \mathbf{R})$$

given by the following pairing between de Rham cohomology classes $[\alpha]$ and real differentiable singular homology classes $[c]$

$$([\alpha], [c]) \mapsto \int_c \alpha.$$

One can show that this map ρ is independent of the choice of $\alpha \in [\alpha]$ and $c \in [c]$. The classical de Rham theorem says that the map ρ is an isomorphism. Thus, $\forall q$ we have

$$H_{\text{deR}}^q(M) \cong H_{\infty}^q(M; \mathbf{R}) \cong H^q(M; \mathbf{R}).$$

Let (M, g) be an oriented, closed (i.e., compact and without boundary) Riemannian manifold with metric volume form μ . Recall that the **Hodge-de Rham operator** $\Delta = d \circ \delta + \delta \circ d$ maps $\Lambda^k(M) \rightarrow \Lambda^k(M)$, $\forall k$ and that the Hodge star operator $*$ maps $\Lambda^k(M) \rightarrow \Lambda^{n-k}(M)$, $\forall k$. For further details, see Chapter 3. The map

$$\langle \cdot, \cdot \rangle : \Lambda^k(M) \times \Lambda^k(M) \rightarrow \mathbf{R}$$

defined by

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge * \beta = \int_M g(\alpha, \beta) \mu \quad (2.8)$$

is an inner product on $\Lambda^k(M)$. One can show that, for $\sigma \in \Lambda^{k+1}(M)$, we have

$$\langle d\alpha, \sigma \rangle = \langle \alpha, \delta\sigma \rangle \text{ and } \langle \Delta\alpha, \beta \rangle = \langle \alpha, \Delta\beta \rangle. \quad (2.9)$$

That is, δ is the adjoint of d and Δ is self-adjoint with respect to this inner product. Furthermore, $\Delta\alpha = 0$ if and only if $d\alpha = \delta\alpha = 0$. An element of the set

$$\mathcal{H}^k := \{ \alpha \in \Lambda^k(M) \mid \Delta\alpha = 0 \} \quad (2.10)$$

is called a **harmonic** k -form. It follows that a k -form is harmonic if and only if it is both closed and coclosed. The set \mathcal{H}^k is a subspace of $\Lambda^k(M)$. Using these facts one can prove (see, for example, Warner [396]) the **Hodge decomposition theorem**, which asserts that \mathcal{H}^k is finite-dimensional and $\Lambda^k(M)$ has a direct sum decomposition into the orthogonal subspaces $d(\Lambda^k(M))$, $\delta(\Lambda^k(M))$, and \mathcal{H}^k . Thus any k -form α can be expressed by the formula

$$\alpha = d\beta + \delta\gamma + \theta, \quad (2.11)$$

where $\beta \in \Lambda^{k-1}(M)$, $\gamma \in \Lambda^{k+1}(M)$ and θ is a harmonic k -form. For α in a given cohomology class, the harmonic form θ of equation (2.11) is uniquely determined. Thus, we have an isomorphism of the k th cohomology space $H^k(M; \mathbf{R})$ with the space of harmonic k -forms \mathcal{H}^k . Therefore, the k th Betti number b_k is equal to the $\dim \mathcal{H}^k$. This is an illustration of a relation between physical or analytic data (the solution space of a partial differential operator) on a manifold and its topology. A far-reaching, nonlinear generalization of this idea relating the solution space of Yang-Mills instantons to the topology of 4-manifolds appears in the work of Donaldson (see Chapter 9 for further details).

2.5.1 The Intersection Form

Let M be a closed (i.e., compact, without boundary), connected, oriented manifold of dimension $2n$. Let v denote the volume form on M defining the orientation. We shall use the de Rham cohomology to define ι_M , the intersection form of M as follows. Let $\alpha, \beta \in \Lambda^n(M)$ be two closed n -forms representing the cohomology classes $a, b \in H^n(M; \mathbf{Z}) \subset H^n(M; \mathbf{R})$ respectively, i.e., $a = [\alpha]$ and $b = [\beta]$. Now $\alpha \wedge \beta \in \Lambda^{2n}(M)$ and hence $\int \alpha \wedge \beta$ is well defined with respect to the volume form v . It can be shown that this integral is independent of the choice of forms α, β representing the cohomology classes a, b and takes values that are integral multiples of the volume of M . Thus, we can define the binary operator

$$\iota_M : H^n(M; \mathbf{Z}) \times H^n(M; \mathbf{Z}) \rightarrow \mathbf{Z}$$

by

$$\iota_M(a, b) = \int_M (\alpha \wedge \beta).$$

In what follows we shall use the same letter to denote the cohomology class and an n -form representing that class. It can be shown that ι_M is a symmetric, non-degenerate bilinear form on $H^n(M; \mathbf{Z})$. This symmetric, non-degenerate form ι_M is called the **intersection form** of M . The definition given above works only for smooth manifolds. However, as is the case with de Rham cohomology, the intersection form does not depend on the differential structure and is a topological invariant. In particular, it is defined for topological manifolds. In fact, the intersection form can also be defined for non-orientable manifolds by considering cohomology or homology with coefficients in \mathbf{Z}_2 instead of \mathbf{Z} . Now for a compact manifold M , $H^n(M; \mathbf{Z})$ is a finitely generated free Abelian group of rank b_n (the n th Betti number), i.e., an integral lattice of rank b_n . Thus, the intersection form gives us the map

$$\iota : M \mapsto \iota_M ,$$

which associates to each compact, connected, oriented topological manifold M of dimension $2n$ a symmetric, non-degenerate bilinear form ι_M on a lattice of rank b_n . Let (b^+, b^-) be the signature of the bilinear form ι_M . If $b_n > 0$ and c_j , $1 \leq j \leq b_n$, is a basis of the lattice $H^n(M; \mathbf{Z})$, then the intersection form is completely determined by the matrix of integers $\iota_M(c_j, c_k)$, $1 \leq j, k \leq b_n$. From Poincaré duality it follows that the intersection form is unimodular, i.e.,

$$|\det(\iota_M(c_j, c_k))| = 1.$$

If $b_n = 0$ then we take $\iota_M := \emptyset$ the empty form. Recall that on the abstract level two forms ι_1, ι_2 on lattices L_1, L_2 , respectively, are said to be **equivalent** if there exists an isomorphism of lattices $f : L_1 \rightarrow L_2$ such that $f^* \iota_2 = \iota_1$.

The intersection form plays a fundamental role in Freedman's classification of topological 4-manifolds. We give a brief discussion of this result in the next section.

2.6 Topological Manifolds

In this section we discuss topological manifolds with special attention to low-dimensional manifolds. The following theorem gives some results on the existence of smooth structures on topological manifolds.

Theorem 2.7 *Let M be a closed topological manifold of dimension n . Then we have the following results:*

1. For $n \leq 3$ there is a unique compatible smooth structure on M .
2. For $n = 4$ there exist (infinitely many) simply connected manifolds that admit infinitely many distinct smooth structures. It is not known whether there are manifolds that admit only finitely many distinct smooth structures.
3. For $n \geq 5$ there are at most finitely many distinct compatible smooth structures.

Thus, dimension 4 seems very special. This is also true for open topological manifolds. There is a unique smooth structure on \mathbf{R}^n , $n \neq 4$, compatible with its standard topology. However, \mathbf{R}^4 admits uncountably many smooth structures. We do not know at this time if every open topological 4-manifold admits uncountably many smooth structures. For further results in the surprising world of 4-manifolds, see, for example, the book by Scorpan [343].

2.6.1 Topology of 2-Manifolds

The topology of 2-manifolds, or surfaces, was well known in the nineteenth century. Smooth, compact, connected and oriented 2-manifolds are called **Riemann surfaces**. An introduction to compact Riemann surfaces from various points of view and their associated geometric structures may be found in Jost [213]. They are classified by a single non-negative integer, the genus g . The genus counts the number of holes in the surface. There is a standard model Σ_g for a surface of genus g obtained by attaching g handles to a sphere (which has genus zero). Every smooth, compact, oriented surface is diffeomorphic to one and only one Σ_g . The classification is sometimes given in terms of the Euler characteristic of the surface. It is related to the genus by the formula $\chi(\Sigma_g) = 2 - 2g$.

In the classical theory of surfaces, homology classes $a, b \in H^1(M; \mathbf{Z})$ were represented by closed curves, which could be chosen to intersect transversally.

The intersection form was then defined by counting the algebraic number of intersections of these curves. Surfaces are completely classified by their intersection forms. In the orientable case we have the following well known result.

Theorem 2.8 *Let M, N be two closed, connected, oriented surfaces. Then $M \cong N$ (i.e., M is diffeomorphic to N) if and only if the intersection forms ι_M, ι_N are equivalent. Moreover, if $\iota_M = \emptyset$ then $M \cong S^2$ and if $\iota_M \neq \emptyset$ then there exists $k \in \mathbf{N}$ such that $\iota_M \cong k\sigma_1$, where*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

is a Pauli spin matrix, and $k\sigma_1$ is the block diagonal form with k entries of σ_1 , and $M \cong kT^2$, where $T^2 = S^1 \times S^1$ is the standard torus and kT^2 is the connected sum of k copies of T^2 .

The Riemann surface together with a fixed complex structure provides a classical model for one-dimensional algebraic varieties or complex curves. A compact surface corresponds to a projective curve. The genus of such a surface is equal to the dimension of the space of holomorphic one forms on the surface. This way of looking at a Riemann surface is crucial in the Gromov–Witten theory. We will not consider it in this book. The genus has also a topological interpretation as half the first Betti number of the surface. We note that the classification of orientable surfaces given by the above theorem can be extended to include non-orientable surfaces as well. We are interested in extending this theorem to the case of 4-manifolds. This was done by Freedman in 1981. Before discussing his theorem we consider the topology of 3-manifolds where no intersection form is defined.

2.6.2 Topology of 3-Manifolds

The classification of manifolds of dimension 3 or higher is far more difficult than that of surfaces. It was initiated by Poincaré in 1900. The year 1900 is famous for the Paris ICM and Hilbert's lecture on the major open problems in mathematics. The classification of 3-manifolds was not among Hilbert's problems. Armed with newly minted homology groups and his fundamental group Poincaré began his study by trying to characterize the simplest 3-manifold, the sphere S^3 . His first conjecture was the following:

Let M be a compact connected 3-manifold with the same homology groups as the sphere S^3 . Then M is homeomorphic to S^3 .

In attempting to prove this conjecture Poincaré found a 3-manifold P that provided a counterexample to the conjecture. This 3-manifold P is now called the **Poincaré homology sphere**. It is denoted by $\Sigma(2, 3, 5)$ as it can be represented as a special case of the Brieskorn homology 3-spheres,

$$\Sigma(a_1, a_2, a_3) := \{(z_1, z_2, z_3) \mid z_1^{a_1} + z_2^{a_2} + z_3^{a_3} = 0\} \cap S^5, \quad a_1, a_2, a_3 \in \mathbf{N}.$$

We will compute various new invariants of the Brieskorn homology 3-spheres in Chapter 10. Poincaré's original ingenious construction of P can be described via well known geometric figures. It is the space of all regular icosahedra inscribed in the standard unit 2-sphere. Note that each icosahedron is uniquely determined by giving one vertex on the sphere (2 parameters) and a direction to a neighboring vertex (1 parameter). It can be shown that the parameter (or moduli) space P of all icosahedra is a 3-manifold diffeomorphic to $\Sigma(2, 3, 5)$ and that it has the same homology groups as the sphere S^3 . Poincaré showed that $\pi_1(P)$, the fundamental group of P , is non-trivial. Since $\pi_1(S^3)$ is trivial, P cannot be homeomorphic to S^3 . Yet another description of P is obtained by observing that the rotation group $SO(3)$ maps S^2 to itself and the induced action on P is transitive. The isotropy group I_x of a fixed point $x \in P$ can be shown to be a finite group of order 60. Thus, P is homeomorphic to the coset space $SO(3)/I_x$. This fact can be used to show that $\pi_1(P)$ is a perfect group of order 120.

Icosahedron is one of the five **regular polyhedra** or solids known since antiquity. They are commonly referred to as **Platonic solids** (see Appendix B for some interesting properties of Platonic solids). Poincaré's counterexample showed that homology was not enough to characterize S^3 and that one has to take into account the fundamental group. He then made the following conjecture:

Let M be a closed simply connected 3-manifold. Then M is homeomorphic to S^3 .

We give it in an alternative form, which is useful for stating the generalized Poincaré conjecture in any dimension, with 3 replaced by a natural number n .

Poincaré Conjecture: *Let M be a closed connected 3-manifold with the same homotopy type as the sphere S^3 . Then M is homeomorphic to S^3 .*

Definition 2.2 *Let H^n denote the n -dimensional hyperbolic space. A 3-manifold X is said to be a **Thurston 3-dimensional geometry** if it is one of the following eight homogeneous manifolds (i.e., the group of isometries of X acts transitively on X).*

- *Three spaces of constant curvature \mathbf{R}^3, S^3, H^3 .*
- *Two product spaces $\mathbf{R} \times S^2, \mathbf{R} \times H^2$.*
- *Three **twisted product spaces**, each of which is a Lie group with a left invariant metric. They are*
 - *the universal covering space of the group $SL(2, \mathbf{R})$;*
 - *the group of upper triangular matrices in $M_3(\mathbf{R})$ with all diagonal entries 1, called **Nil**;*
 - *the semidirect product of \mathbf{R} and \mathbf{R}^2 , where \mathbf{R} acts on \mathbf{R}^2 through multiplication by the diagonal matrix $\text{diag}(t, t^{-1})$, $t \in \mathbf{R}$, called **Sol**.*

We can now define a geometric 3-manifold in the sense of Thurston.

Definition 2.3 *A 3-manifold M is said to be **geometric** if it is diffeomorphic to X/Γ , where Γ is a discrete group of isometries of X (i.e. $\Gamma < \text{Isom}(X)$) acting freely on X , and X is one of Thurston's eight 3-dimensional geometries.*

We can now state the Thurston geometrization conjecture.

Thurston Geometrization Conjecture: : Let M be a closed 3-manifold that does not contain two-sided projective planes. Then M admits a connected sum decomposition and a decomposition along disjoint incompressible tori and Klein bottles into a finite number of pieces each of which is a geometric manifold.

The Thurston geometrization conjecture was recently proved by Perelman using a generalized form of Hamilton's Ricci flow technique. This result implies the original Poincaré conjecture. We will comment on it in Chapter 6.

2.6.3 Topology of 4-manifolds

The importance of the intersection form for the study of 4-manifolds was already known since 1940 from the following theorem of Whitehead (see Milnor and Husemoller [285]).

Theorem 2.9 *Two closed, 1-connected, 4-manifolds are homotopy equivalent if and only if their intersection forms are equivalent.*

In the category of topological manifolds a complete classification of closed, 1-connected, oriented 4-manifolds has since been carried out by Freedman (see [136]). To state his results we begin by recalling the general scheme of classification of symmetric, non-degenerate, unimodular, bilinear forms (referred to simply as “forms” in the rest of this section) on lattices (see Milnor and Husemoller [285]). The classification of forms has a long history and is an important area of classical mathematics with applications to algebra, number theory, and more recently to topology and geometry. We have already defined two fundamental invariants of a form, namely its rank b_n and its signature (b^+, b^-) . We note that sometimes the signature is defined to be the integer $b^+ - b^- = b_n - 2b^-$. We shall denote this integer by $\sigma(M)$, i.e., $\sigma(M) := b_n - 2b^-$. We say that a form ι on the lattice L is **even** or of **type II** if $\iota(a, a)$ is even for all $a \in L$. Otherwise we say that it is **odd** or of **type I**. It can be shown that for even (type II) forms, 8 divides the signature $\sigma(M)$. In particular, 8 divides the rank of a positive definite even form. The indefinite forms are completely classified by the rank, signature, and type. We have the following result.

Theorem 2.10 *Let ι be an indefinite form of rank r and signature (j, k) , $j > 0, k > 0$. Then we have*

1. $\iota \cong j(1) \oplus k(-1)$ if it is odd (type I),
2. $\iota \cong m\sigma_1 \oplus pE_8$, $m > 0$ if it is even (type II),

where (1) and (-1) are 1×1 matrices representing the two possible forms of rank 1, σ_1 is the Pauli spin matrix defined earlier, and E_8 is the matrix associated to the exceptional Lie group E_8 in Cartan's classification of simple Lie groups, i.e.,

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

The classification of definite forms is much more involved. The number $N(r)$ of equivalence classes of definite forms (which counts the inequivalent forms) grows very rapidly with the rank r of the form, as Table 2.3 illustrates.

Table 2.3 Number of inequivalent definite forms

r	8	16	24	32	40
$N(r)$	1	2	24	$\geq 10^9$	$\geq 10^{51}$

We now give some simple examples of computation of intersection forms that we will use later.

Example 2.8 We denote $H^2(M; \mathbf{Z})$ by L in this example.

1. Let $M = S^4$; then $L = 0$ and hence $\iota_M = \emptyset$.
2. Let $M = S^2 \times S^2$; then L has a basis of cohomology classes α, β dual to the homology cycles represented by $S^2 \times \{(1, 0, 0)\}$ and $\{(1, 0, 0)\} \times S^2$, respectively. With respect to this basis the matrix of ι_M is the Pauli spin matrix

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

3. Let $M = \mathbf{CP}^2$; then $L = \mathbf{Z}$ and hence $\iota_M = (1)$.
4. Let $M = \overline{\mathbf{CP}}^2$, i.e., \mathbf{CP}^2 with the opposite complex structure and orientation. Then $L = \mathbf{Z}$ and hence $\iota_M = (-1)$.

Whitehead's Theorem 2.9 stated above says that the map ι that associates to a closed, 1-connected topological 4-manifold its intersection form induces

an injection from the homotopy equivalence classes of manifolds into the equivalence classes of forms. It is natural to study this map ι in greater detail. One can ask, for example, the following two questions.

1. Is the map ι surjective? I.e., given an intersection form μ , does there exist a manifold M such that $\iota_M = \mu$?
2. Does the injection from the homotopy equivalence classes of manifolds into the equivalence classes of forms extend to other equivalence classes of manifolds?

The first question is an existence question while the second is a uniqueness question. These questions can be restricted to different categories of manifolds such as topological or smooth manifolds. A complete answer to these questions in the topological category is given by the following theorem, proved by Freedman in 1981 [136].

Theorem 2.11 *Let \mathcal{M}_{sp} (resp., \mathcal{M}_{ns}) denote the set of topological equivalence classes (i.e. homeomorphism classes) of closed, 1-connected, oriented, spin (resp., non-spin) 4-manifolds. Let \mathcal{I}_{ev} (resp. \mathcal{I}_{od}) denote the set of equivalence classes of even (resp. odd) forms. Then we have the following:*

1. *the map $\iota : \mathcal{M}_{sp} \rightarrow \mathcal{I}_{ev}$ is bijective;*
2. *the map $\iota : \mathcal{M}_{ns} \rightarrow \mathcal{I}_{od}$ is surjective and is exactly two-to-one. The two classes in the preimage of a given form are distinguished by a cohomology class $\kappa(M) \in H^4(M; \mathbf{Z}_2)$ called the Kirby–Siebenmann invariant.*

We note that the **Kirby–Siebenmann invariant** represents the obstruction to the existence of a piecewise linear structure on a topological manifold of dimension ≥ 5 . Applying the above theorem to the empty rank zero form provides a proof of the Poincaré conjecture for dimension 4. Freedman’s theorem is regarded as one of the fundamental results of modern topology.

In the smooth category the situation is much more complicated. It is well known that the map ι is not surjective in this case. In fact, we have the following theorem:

Theorem 2.12 (Rochlin) *Let M be a smooth, closed, 1-connected, oriented, spin manifold of dimension 4. Then $\sigma(M)$, the signature of M is divisible by 16.*

Now, as we observed earlier, 8 always divides the signature of an even form, but 16 need not divide the form. Thus, we can define the **Rochlin invariant** $\rho(\mu)$ of an even form μ by

$$\rho(\mu) := \frac{1}{8}\sigma(\mu) \pmod{2}.$$

We note that the Rochlin invariant and the Kirby–Siebenmann invariant are equal in this case, but for non-spin manifolds the Kirby–Siebenmann invariant is not related to the intersection form and thus provides a further obstruction to smoothability. From Freedman’s classification and Rochlin’s

theorem it follows that a topological manifold with nonzero Rochlin invariant is not smoothable. For example, the topological manifold $|E_8| := \iota^{-1}(E_8)$ corresponds to the equivalence class of the form E_8 and has signature 8 (Rochlin invariant 1) and hence is not smoothable.

For several years very little progress was made beyond the result of the above theorem in the smooth category. Then, in 1982, through his study of the topology and geometry of the moduli space of instantons on 4-manifolds Donaldson discovered the following, unexpected, result. The theorem has led to a number of important results including the existence of uncountably many exotic differentiable structures on the standard Euclidean topological space \mathbf{R}^4 .

Theorem 2.13 (Donaldson) *Let M be a smooth closed 1-connected oriented manifold of dimension 4 with positive definite intersection form ι_M . Then $\iota_M \cong b_2(1)$, the diagonal form of rank b_2 , the second Betti number of M .*

Donaldson's work uses in an essential way the solution space of the Yang–Mills field equations for $SU(2)$ gauge theories and has already had profound influence on the applications of physical theories to mathematical problems. In 1990 Donaldson obtained more invariants of 4-manifolds by using the topology of the moduli space of instantons. Donaldson theory led to a number of new results for the topology of 4-manifolds, but it was technically a difficult theory to work with. In fact, Atiyah announced Donaldson's new results at a conference at Duke University in 1987, but checking all the technical details delayed the publication of his paper until 1990. The matters simplified greatly when the Seiberg–Witten equations appeared in 1994. We discuss the Donaldson invariants of 4-manifolds in more detail in Chapter 9. It is reasonable to say that at that time a new branch of mathematics which may be called “Physical Mathematics” was created.

In spite of these impressive new developments, there is at present no analogue of the geometrization conjecture in the case of 4-manifolds. Here geometric topologists are studying the variational problems on the space of metrics on a closed oriented 4-manifold M for one of the classical curvature functionals such as the square of the L^2 norm of the Riemann curvature Rm , Weyl conformal curvature W , and its self-dual and anti-dual parts W_+ and W_- , respectively, and Ric , the Ricci curvature. The Hilbert–Einstein variational principle based on the scalar curvature functional and its variants are important in the study of gravitational field equations. Einstein metrics, i.e., metrics satisfying the equation

$$K := Ric - \frac{1}{4}Rg = 0$$

are critical points of all of the functionals listed above. Here K is the trace-free part of the Ricci tensor. In many cases the Einstein metrics are minimizers, but there are large classes of minimizers that are not Einstein metrics. A well-known obstruction to the existence of Einstein metrics is the Hitchin–Thorpe

inequality $\chi(M) \geq \frac{3}{2}|\tau(M)|$, where $\chi(M)$ is the Euler characteristic and $\tau(M)$ is the signature of M . A number of new obstructions are now known. Some of these indicate that their existence may depend on the smooth structure of M as opposed to just the topological structure. These obstructions can be interpreted as implying a coupling of matter fields to gravity (see [269, 270, 94]). The basic problem is to understand the existence and moduli spaces of these metrics on a given manifold and perhaps to find a geometric decomposition of M with respect to a special functional. One of the most important tools for developing such a theory is the Chern–Gauss–Bonnet theorem which states that

$$\chi(M) = \frac{1}{8\pi^2} \int (|Rm|^2 - |K|^2) dv = \frac{1}{8\pi^2} \int \left(|W|^2 - \frac{1}{2}|K|^2 + \frac{1}{24}R^2 \right) dv.$$

This result allows one to control the full Riemann curvature in terms of the Ricci curvature Ric . It is interesting to note that in [242], Lanczos had arrived at the same result while searching for Lagrangians to generalize Einstein's gravitational field equations. He noted the curious property of the Euler class that it contains no dynamics (or is an invariant). He had thus obtained the first topological gravity invariant (without realizing it). Chern's fundamental paper [74] appeared in the same journal seven years later. Chern–Weil theory and Hirzebruch's signature theorem give the following expression for the signature $\tau(M)$:

$$\tau(M) = \frac{1}{12\pi^2} \int (|W_+|^2 - |W_-|^2) dv.$$

The Hitchin–Thorpe inequality follows from this result and the Chern–Gauss–Bonnet theorem. In dimension 4, all the classical functionals are conformally (or scale) invariant, so it is customary to work with the space of unit volume metrics on M .

2.7 The Hopf Invariant

As we remarked in the preface, mathematicians and physicists have often developed the same ideas from different perspectives. The Hopf fibration and Dirac's monopole construction provide an example of this. Each is based on the observation that S^2 (the base of the Hopf fibration) is a deformation retract of $\mathbf{R}^3 \setminus \{0\}$ (the base of the Dirac monopole field). Thus non-triviality of $\pi_3(S^2)$ can be interpreted as Dirac's monopole quantization condition. Other Hopf fibrations and Hopf invariants also arise in physical theories, as we indicate later in this section.

Let $f : S^{2n-1} \rightarrow S^n, n > 1$, be a continuous map. Let X denote the quotient space of the disjoint union $D^{2n} \sqcup S^n$ under the identification of

$x \in S^{2n-1} \subset D^{2n}$ with $f(x) \in S^n$. The map f is called the **attaching map**. The space X is called the **adjunction space** obtained by attaching the $2n$ -cell $e^{2n} := (D^{2n}, S^{2n-1})$ to S^n by f and is denoted by $S^n \cup_f e^{2n}$. The cohomology $H^*(X; \mathbf{Z})$ of X is easy to calculate and is given by

$$H^i(X; \mathbf{Z}) = \begin{cases} \mathbf{Z} & \text{if } i = n \text{ or } 2n, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\alpha \in H^n(X; \mathbf{Z})$ and $\beta \in H^{2n}(X; \mathbf{Z})$ be the generators of the respective cohomology groups. We note that in de Rham cohomology α, β can be identified with closed differential forms. It follows that α^2 is an integral multiple of β . The multiplier $h(f)$ is completely determined by the map f and is called the **Hopf invariant** of f . Thus we have

$$\alpha^2 = h(f)\beta.$$

It can be shown that $h(f)$ is, in fact, a homotopy invariant and hence defines a map (also denoted by h)

$$[f] \mapsto h(f) \text{ of } \pi_{2n-1}(S^n) \rightarrow \mathbf{Z}.$$

To include the case $n = 1$ we note that the double covering $c_2 : S^1 \rightarrow S^1$ has adjunction space \mathbf{RP}^2 and Hopf invariant 1. The following theorem is due to H. Hopf:

Theorem 2.14 *Let $\mathcal{F}_n := C(S^{2n-1}, S^n)$, $n > 0$, denote the space of continuous functions from S^{2n-1} to S^n . Then we have the following:*

1. *If $n > 1$ is odd, then $h(f) = 0$, $\forall f \in \mathcal{F}_n$.*
2. *If n is even, then for each $k \in \mathbf{Z}$ there exists a map $f_k \in \mathcal{F}_n$ such that $h(f_k) = 2k$.*
3. *If there exists $g \in \mathcal{F}_n$ such that $h(g)$ is odd, then $n = 2^m$, where m is a nonnegative integer.*
4. *Let $\pi \in \mathcal{F}_1$ (resp., \mathcal{F}_2) be the real (resp., complex) Hopf fibration. Then $h(\pi) = 1$.*

We note that the real Hopf fibration $\pi : S^1 \rightarrow \mathbf{RP}^1 \cong S^1$ occurs in the geometric quantization of the harmonic oscillator [271, 272] while the complex Hopf fibration $\pi : S^3 \rightarrow \mathbf{CP}^1 \cong S^2$ occurs in the geometric construction of the Dirac monopole. It can be shown that the last result in the above theorem can be extended to include the quaternionic and octonionic Hopf fibrations (which arise in the solution of Yang–Mills equations on S^4 and S^8 , respectively) and that this extended list exhausts all \mathcal{F}_n that contain a map with Hopf invariant 1. This result is part of the following extraordinary theorem, which links several specific structures from algebra, topology, and geometry.

Theorem 2.15 *Let $S^{n-1} \subset \mathbf{R}^n$, $n > 0$ denote the standard $(n-1)$ -sphere in the real Euclidean n -space with the convention that $S^0 := \{-1, 1\} \subset \mathbf{R}$ and $\mathbf{R}^0 := \{0\}$. Then the following statements are equivalent:*

1. *The integer $n \in \{1, 2, 4, 8\}$.*
2. *\mathbf{R}^n has the structure of a normed algebra.*
3. *\mathbf{R}^n has the structure of a division algebra.*
4. *\mathbf{R}^{n-1} admits a cross product (or a vector product).*
5. *S^{n-1} is an H -space.*
6. *S^{n-1} is parallelizable (i.e., its tangent bundle is trivializable).*
7. *There exists a map $f : S^{2n-1} \rightarrow S^n$ with Hopf invariant $h(f) = 1$.*

The relation of conditions (1) and (3) with condition (4) and its generalizations have been considered in [126]. It is well known that complex numbers have applications to 2-dimensional geometry and its ring of **Gaussian integers** (i.e., numbers of the form $m + ni$, where $m, n \in \mathbf{Z}$) is used in many classical questions in arithmetic. Similarly, the quaternions are related to 3-dimensional and 4-dimensional geometry and they contain rings of integers with many properties similar to those of Gaussian integers. The octonions have applications to 7- and 8-dimensional geometry. In elementary algebra one encounters the construction of the complex numbers in terms of certain real matrices of order 2. This **doubling procedure** of constructing \mathbf{C} from \mathbf{R} was generalized by Dixon to construct \mathbf{H} from \mathbf{C} and the octonions \mathbf{O} from \mathbf{H} . This procedure leads to identities expressing the product of two sums of 2^n squares as another such sum. The familiar identity from high school

$$(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$$

is the special case $n = 1$. Many interesting further developments of these ideas can be found in the book [83] by Conway and Smith.

2.7.1 Kervaire invariant

In 1960 Kervaire defined a geometric topological invariant of a framed differential manifold M of dimension $m = 4n + 2$ generalizing the Arf invariant for surfaces. Ten years earlier Pontryagin had used the Arf invariant of surfaces embedded in S^{k+2} with trivialized normal bundle to compute the homotopy groups $\pi_{k+2}(S^k)$ for $k > 1$. This group can be identified with the cobordism group of such surfaces. The cobordism of manifolds and the corresponding cobordism groups were defined by Thom in 1952. Algebraically the Arf invariant is defined for any quadratic form over \mathbf{Z}_2 . Kervaire defined a quadratic form q on the homology group $H_{2n+1}(M; \mathbf{Z}_2)$ by using the framing and the Steenrod squares. The **Kervaire invariant** is the Arf invariant of q . A general reference for this section is Snaith [356]. Kervaire used his invariant to

obtain the first example of a non-smoothable 10-dimensional PL-manifold. In the smooth category the first three examples of manifolds with Kervaire invariant 1 are $S^1 \times S^1$, $S^3 \times S^3$, and $S^7 \times S^7$. In these three cases the Kervaire invariant is related to the Hopf invariant of certain maps of spheres. No further examples of manifolds with Kervaire invariant 1 were known for many years. The problem of finding the dimensions of framed manifolds for which the Kervaire invariant is 1 came to be known as the **Kervaire invariant 1 problem**.

In 1969 Bill Browder proved that the Kervaire invariant is 0 for a manifold M if its dimension is different from $2^{j+1} - 2$, $j \in \mathbf{N}$. By 1984, it was known that there exist manifolds of dimensions 30 and 62 with the Kervaire invariant 1. Then on April 21, 2009, during the Atiyah 80 conference at Edinburgh, Mike Hopkins announced that he, Mike Hill, and Doug Ravenel had proved that there are no framed manifolds of dimension greater than 126 with Kervaire invariant 1. The case $n = 126$ was open as of January 2010. Mike Hopkins gave a very nice review of the problem and indicated key steps in the proof at the Strings, Fields and Topology workshop at Oberwolfach (June, 2009). This section is based in part on that review. The proof makes essential use of ideas from a generalized cohomology theory, called **topological modular forms**, or **tmf** theory. It was developed by Hopkins and collaborators. Witten has introduced a homomorphism from the string bordism ring to the ring of modular forms, called the Witten genus. This can be interpreted in terms of the theory of **topological modular forms** or **tmf**. We have already seen in Theorem 2.15, how the spheres S^1, S^3, S^7 enter from various perspectives in it. We have also discussed their relation to the Hopf fibration and to different physical theories. The relation of the other manifolds with other parts of mathematics and with physics is unclear at this time.

The discussion of homotopy and cohomology given in this chapter forms a small part of an area of mathematics called algebraic topology, where these and other related concepts are developed for general topological spaces. Standard references for this material and other topics in algebraic topology are Bott and Tu [55], Massey [279], and Spanier [357]. A very readable introduction is given in Croom [90].



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