

Chapter 2

Dynamic Pricing Models

Abstract In this chapter, some pricing models are presented that are characterized by the following assumptions: (i) the number of potential customers is not limited, and as a consequence, the size of the population is not a parameter of the model, (ii) only one type of item is concerned, (iii) a monopoly situation is considered, and (iv) customers buy items as soon as the price is less than or equal to the price they are prepared to pay (myopic customers). A deterministic model with time-dated items is presented and illustrated first. To build this model, the relationship between the price per item and demand has to be established. Then, the stochastic version of the same model is analyzed. A Poisson process generates customers' arrivals. Finally, a stochastic model with salvage value where the price is a function of inventory level is considered. Detailed algorithms, numerical examples and figures are provided for each model. These models provide practical insights into pricing mechanisms.

2.1 Introduction

Any dynamic pricing model requires establishing how demand responds to changes in price. This chapter is dedicated to mathematical models of monopoly systems. The reader will notice that strong assumptions are made to obtain tractable models. Indeed, such mathematical models can hardly represent real-life situations, but they do illustrate the relationship between price and customers' purchasing behavior.

In this chapter we consider the case of time-dated items, i.e., items that must be sold before a given point in time, say T . Furthermore, there is no supply option before time T . This situation is common in the food industry, the toy business (when toys must be sold before Christmas, for instance), marketing products (products associated with special events like movies, football matches, etc.), fashion apparel

(because the selling period ends when a season finishes), airplane tickets (which are obsolete when the plane takes off), to quote just a few.

The goal is to find a strategy (dynamic pricing, also called yield management or revenue management) that leads to the maximum expected revenue by time T , assuming that the process starts at time 0. This strategy consists in selecting a set of adequate prices for the items that vary according to the number of unsold items and, in some cases, to the time. At a given point in time, we assume that the price of an item is a non-increasing function of the inventory level. For a given inventory level, prices are going down over time.

Indeed, a huge number of models exist, depending of the situation at hand and the assumptions made to reach a working model. For instance, selling airplane tickets requires a pricing strategy that leads to very cheap tickets as takeoff nears, while selling fashion apparel is less constrained since a second market exists, i.e., it is still possible to sell these items at discount after the deadline.

For the models presented in this chapter, we assume that:

- The number of potential customers is infinite. As a consequence, the size of the population does not belong to the set of parameters of the models.
- A single type of item is concerned and its sales are not affected by other types of items.
- We are in a monopoly situation, which means that there is no competition with other companies selling the same type of item. Note that, due to price discrimination, a company can be monopolistic in one segment of the population while other companies sell the same type of item with slight differences to other segments. This requires a sophisticated fencing strategy that prevents customers from moving to a cheaper segment.
- Customers are myopic, which means that they buy as soon as the price is less than the one they are prepared to pay. Strategic customers who optimize their purchasing behavior in response to the pricing strategy of the company are not considered in this chapter; game theory is used when strategic customers are concerned.

To summarize, this chapter provides an insight into mathematical pricing models. Note also that few convenient models exist without the assumptions presented above, that is to say a monopoly situation, an infinite number of potential customers who are myopic and no supply option. The reader will also observe that negative exponential functions are often used to make the model manageable and few persuasive arguments are proposed to justify this choice: this is why we consider that most of these models are more useful to understand dynamic pricing than to treat real-life situations.

2.2 Time-dated Items: a Deterministic Model

2.2.1 Problem Setting

In this model, we know the initial inventory s_0 : it is the maximal quantity that can be sold by time T . We assume that demands appear at times $1, 2, \dots, T$, and x_t represents the demand at time t . The demands are real and positive. The price of one item at time t is denoted by p_t , and this price is a function of the demand and the time: $p_t = p(x_t, t)$.

We assume that there exists a one-to-one relationship between demand and price at any time $t \in \{1, 2, \dots, T\}$. Thus, $x_t = x(p_t, t)$ is the relation that provides the demand when the price is fixed.

We also assume that:

- $x(p, t)$ is continuously differentiable with regard to p .
- $x(p, t)$ is lower and upper bounded and tends to zero as p tends to its maximal value.

Finally, the problem can be expressed as follows:

$$\text{Maximize } \sum_{t=1}^T p_t x_t \quad (2.1)$$

subject to:

$$\sum_{t=1}^T x_t \leq s_0 \quad (2.2)$$

$$x_t \geq 0 \text{ for } t \in \{1, 2, \dots, T\} \quad (2.3)$$

$$x_t \leq x(p_t^{\min}, t) \text{ for } t \in \{1, 2, \dots, T\} \quad (2.4)$$

where p_t^{\min} is the minimal value of p_t .

Criterion 2.1 means that the objective is to maximize the total revenue. Constraint 2.2 guarantees that the total demand at horizon T does not exceed the initial inventory. Constraints 2.3 are introduced to make sure that demands are never less than zero. Finally, Constraints 2.4 provide the upper bound of the demand at any time.

2.2.2 Solving the Problem: Overall Approach

To solve this problem, we use the Kuhn and Tucker approach based on Lagrange multipliers. Since p_t is a function of x_t , then $p_t x_t$ is the function of x_t . Taking into account the constraints of the problem, the Lagrangian is:

$$L(x_1, \dots, x_T, \lambda, \mu_1, \dots, \mu_T, l_1, \dots, l_T) = \sum_{t=1}^T p(x_t, t) x_t - \lambda \left(\sum_{t=1}^T x_t - s_0 \right) + \sum_{t=1}^T \mu_t x_t - \sum_{t=1}^T l_t (x_t - x(p_t^{\min}, t)) \quad (2.5)$$

The goal is to solve the T equations:

$$\frac{\partial L}{\partial x_t} = 0 \text{ for } t \in \{1, 2, \dots, T\} \quad (2.6)$$

Together with the $2T+1$ complementary slackness conditions:

$$\lambda \left(\sum_{t=1}^T x_t - s_0 \right) = 0 \quad (2.7)$$

$$\left. \begin{array}{l} \mu_t x_t = 0 \\ l_t (x_t - x(p_t^{\min}, t)) = 0 \end{array} \right\} \text{ for } t \in \{1, 2, \dots, T\} \quad (2.8)$$

Thus, we have $3T+1$ equations for the $3T+1$ unknowns that are:

$$x_1, \dots, x_T, \lambda, \mu_1, \dots, \mu_T, l_1, \dots, l_T$$

A solution to the system of Equations 2.6–2.8 is admissible if $\lambda \geq 0$, $\mu_t \geq 0$, $l_t \geq 0$ and if Inequalities 2.2–2.4 hold.

Note that, due to Relations 2.7 and 2.8:

$$\lambda = 0 \text{ and/or } \sum_{t=1}^T x_t = s_0$$

$$\mu_t = 0 \text{ and/or } x_t = 0 \text{ for } t \in \{1, 2, \dots, T\}$$

$$l_t = 0 \text{ and/or } x_t = x(p_t^{\min}, t) \text{ for } t \in \{1, 2, \dots, T\}$$

2.2.3 Solving the Problem: Example for a Given Price Function

We consider the case:

$$p(x_t, t) = (A - Bx_t) \frac{D}{D+t}$$

where A , B and D are positive constants.

As a consequence:

$$x(p_t, t) = \frac{1}{B} \left(A - p_t \frac{D+t}{D} \right)$$

As we can see:

- The price is a decreasing function of t .
- The demand must remain less than $\frac{A}{B}$, otherwise the cost would become negative.

The problem to be solved is (see (2.1)–(2.4)):

$$\text{Maximize } \sum_{t=1}^T (A - Bx_t) x_t \frac{D}{D+t}$$

subject to:

$$\sum_{t=1}^T x_t \leq s_0$$

$$\left. \begin{array}{l} x_t \geq 0 \\ x_t \leq \frac{A}{B} \end{array} \right\} \text{ for } t \in \{1, 2, \dots, T\}$$

The last constraints guarantee that prices remain greater than or equal to zero.

In this case, the Lagrangian is:

$$\begin{aligned} L(x_1, \dots, x_T, \lambda, \mu_1, \dots, \mu_T, l_1, \dots, l_T) = & \sum_{t=1}^T (A x_t - B x_t^2) \frac{D}{D+t} - \lambda \left(\sum_{t=1}^T x_t - s_0 \right) \\ & + \sum_{t=1}^T \mu_t x_t - \sum_{t=1}^T l_t \left(x_t - \frac{A}{B} \right) \end{aligned}$$

According to Relations 2.6–2.8, the system of equations to solve is:

$$(A - 2Bx_t) \frac{D}{D+t} - \lambda + \mu_t - l_t = 0 \text{ for } t \in \{1, 2, \dots, T\} \quad (2.9)$$

$$\lambda \left(\sum_{t=1}^T x_t - s_0 \right) = 0 \quad (2.10)$$

$$\mu_t x_t = 0 \text{ for } t \in \{1, 2, \dots, T\} \quad (2.11)$$

$$l_t (x_t - A/B) = 0 \text{ for } t \in \{1, 2, \dots, T\} \quad (2.12)$$

Whatever $t \in \{1, 2, \dots, T\}$, x_t is either equal to 0, or to A/B , or belongs to $(0, A/B)$ (which represents the interval without its limits). This third option is justified as follows.

If neither of the first two options holds, then $\mu_t = l_t = 0$ and Relation 2.9 becomes:

$$(A - 2Bx_t) \frac{D}{D+t} - \lambda = 0$$

Let us first assume that $x_t = A/B$. In this case, $\mu_t = 0$ and Equality 2.9 becomes:

$$(A - 2A) \frac{D}{D+t} = \lambda + l_t$$

The first member of this equality is negative, while the second member is greater than or equal to 0 since both λ and l_t must be less than or equal to 0 for the solution to be admissible. As a conclusion, x_t cannot be equal to A/B , and therefore, see (2.12), $l_t = 0$ whatever t .

Let us now assume that $x_t = 0$. In this case, and keeping in mind that $l_t = 0$, Equality 2.9 becomes:

$$A \frac{D}{D+t} - \lambda + \mu_t = 0 \text{ or } \lambda = A \frac{D}{D+t} + \mu_t > 0$$

Thus, according to (2.10), $\sum_{t=1}^T x_t = s_0$

Finally, assume that $x_t \in (0, A/B)$. In this case, $\mu_t = l_t = 0$ and $\lambda = (A - 2Bx_t) \frac{D}{D+t}$.

As a consequence, $x_t \in (0, A/2/B]$ and $x_t = \frac{1}{2B}(A - \lambda \frac{D+t}{D})$.

We have to consider two cases:

1. If $x_t = A/2/B$ then, according to Equations 2.9 and 2.10:

$$\lambda = 0 \text{ and } \sum_{t=1}^T x_t \leq s_0 \quad (2.13)$$

2. If $x_t \in (0, A/2/B)$, then, according to Equations 2.9 and 2.10:

$$\lambda > 0 \text{ and } \sum_{t=1}^T x_t = s_0 \quad (2.14)$$

Let be $Y = \{t \mid t \in \{1, 2, \dots, T\}, x_t > 0\}$ and N_Y the number of elements of Y .

From (2.13) and (2.14) it appears that:

- If $T \times A/2/B \leq s_0$, then $x_t = A/2/B$ for $t \in \{1, 2, \dots, T\}$ is an admissible solution.
- If $N_Y \times A/2/B \geq s_0$, then $\sum_{t=1}^T x_t = s_0$. Since $x_t = \frac{1}{2B}(A - \lambda \frac{D+t}{D})$ when

$x_t > 0$, equality $\sum_{t=1}^T x_t = s_0$ becomes $\sum_{t \in Y} \left\{ \frac{1}{2B} \left(A - \lambda \frac{D+t}{D} \right) \right\} = s_0$ and

$\lambda = \frac{N_Y D A - 2 B D s_0}{N_Y D + \sum_{t \in Y} t}$. Finally:

$$x_t = \frac{1}{2B} \left(A - \frac{N_Y D A - 2 B D s_0}{N_Y D + \sum_{t \in Y} t} \frac{D+t}{D} \right) \text{ for } t \in Y \quad (2.15)$$

We derive an algorithm from the above results.

Algorithm 2.1.

1. If $T \times A/2/B \leq s_0$, then $x_t = A/2/B$ for $t \in \{1, 2, \dots, T\}$, compute the criterion $C^* = \sum_{t=1}^T (A - Bx_t) x_t \frac{D}{D+t}$ and set $x_t^* = x_t$ for $t \in \{1, 2, \dots, T\}$, otherwise set $C^* = 0$.
2. If $T \times A/2/B \geq s_0$, then for all sequences $Y = [y_1, y_2, \dots, y_T]$, where $y_t = 0$ or 1 :
 - 2.1. Set $x_t = 0$ if $y_t = 0$ or compute x_t using (2.15) if $y_t = 1$.
 - 2.2. If $(x_t < 0$ or $x_t > A/2/B)$ for at least one $t \in \{1, 2, \dots, T\}$, then go to the next sequence
 - Y. Otherwise, compute $C = \sum_{t|y_t=1} (A - Bx_t) x_t \frac{D}{D+t}$.
 - 2.3. If $C > C^*$:
 - 2.3.1. Set $C^* = C$.
 - 2.3.2. Set $x_t^* = x_t$ for $t \in \{1, 2, \dots, T\}$.
3. The solution of the problem is $\{x_t^*\}_{t=1, 2, \dots, T}$ and C^* contains the optimal value.

This algorithm consists of computing the value of the criterion for each of the feasible solutions and keeping the solution with the greater value of the criterion. Indeed, this approach can be applied only to problems of reasonable size since the number of feasible solutions is upper bounded by 2^T .

Numerical Illustrations

We present 3 examples. Demands and prices are rounded and $T = 10$. They are listed in the increasing order of time.

Example 1

$A = 200, B = 10$ and $D = 10$

Initial inventory level: 150

Demands: 10, 10, 10, 10, 10, 10, 10, 10, 10, 10

Prices (per item): 90.91, 83.33, 76.92, 71.43, 66.67, 62.5, 58.82, 55.56, 52.63, 50

Total demand: 100

Revenue: 6687.71

Example 2

$A = 500, B = 5$ and $D = 10$

Initial inventory level: 150

Demands: 25.56, 23.33, 21.11, 18.89, 16.67, 14.44, 12.22, 10.0, 7.78, 0

Prices (per item): 338.38, 319.44, 303.42, 289.68, 277.78, 267.36, 258.17, 250.0, 242.69, 0

Total demand: 150

Revenue: 44 013.1

Example 3

$A = 500$, $B = 10$ and $D = 2$

Initial inventory level: 200

Demands: 22.67, 22.56, 22.44, 22.33, 22.22, 22.11, 22.0, 21.89, 21.78, 0

Prices (per item): 260.32, 249.49, 239.61, 230.56, 222.22, 214.53, 207.41, 200.79, 194.64, 0

Total demand: 200

Revenue: 44 933.7

2.2.4 Remarks

Three remarks can be made concerning this model:

- The main difficulty consists in establishing the deterministic relationship between the demand and the price per item. In fact, establishing such a relationship is a nightmare. Several approaches are usually used to reach this objective. One of them is to carry out a survey among a large population, asking customers the price they are prepared to pay for one item. Let n be the size of the population and s_p the number of customers who are prepared to pay p or more for one item, then s_p / n is the proportion of customers who will buy at cost p . Then, evaluating at k the number of customers who demonstrate some interest in the item, we can consider that the demand is $k \times s_p / n$ when the price is p . Another approach is to design a “virtual shop” on the Internet and to play with potential customers to extract the same information as before. This is particularly efficient for products sold via the Internet. Ebay and other auction sites can often provide this initial function for price and demand surveying.
- In the model developed in this section, demands and prices are continuous. The problem becomes much more complicated if demands are discrete. Linear interpolation is usually enough to provide a near-optimal solution.
- In this model, we also assumed that the value of one item equals zero after time T . We express this situation by saying that there is no salvage value.

2.3 Dynamic Pricing for Time-dated Products: a Stochastic Model

In this model, we assume that there is no salvage value, i.e., that the value of an item equals zero at time T .

We are in the case of imperfect competition, which means that the vendor has the monopoly of the items. The monopoly could be the consequence of a specificity of the items that requires a very special know-how, a technological special feature, a novelty or the existence of item differentiation that results in a very large spectrum of similar items.

In the case of imperfect competition, customers respond to the price.

Furthermore, this model is risk-neutral, which means that the objective is only to maximize the expected revenue at time T , without taking into account the risk of poor performance. This kind of model applies when the number of problem instances is large enough to annihilate risk, which is a consequence of the “large number” statistical rule.

These hypotheses are the same as those introduced in the previous model. The differences will appear in the next subsection.

This approach is presented in detail in (Gallego and Van Ryzin, 1994).

2.3.1 Problem Considered

To make the explanation simple, consider that possible customers appear at random. Each customer buys an item, or not, depending on the price and the maximum amount of money they are prepared to pay for it.

We assume that a Poisson process generates the arrival of the customers.¹

Let δ be an “infinitely small” increment of time t . The probability that a customer appears during the period $[t, t + \delta)$ is $\lambda \delta$, and at most one customer can appear during this period. In this model, we assume that λ is constant. In particular, λ depends neither on time nor on the number of unsold items. In other words, the arrival process of customers is steady.

After arriving in the system, a customer may buy an item. As mentioned before, this decision depends on the price of the item and the amount of money they are prepared to pay for it. We denote by $f(p)$ the density of probability reflecting the fact that a customer is prepared to pay p for one unit of product. The following characteristics hold:

¹ In this study, a Poisson process of parameter λ generates the arrival of one potential customer during an “infinitely small” period δ with the probability $\lambda \delta$ and does not generate any customer with the probability $1 - \lambda \delta$. In this process, the probability of the arrival of more than one potential customer is $o(\delta)$, which is practically equivalent to zero. Another way to express the Poisson process is to say that the probability that k potential customers arrive during a period $[0, t]$ is:

$$P_k = \exp(-\lambda t) \frac{(\lambda t)^k}{k!}.$$

- The density $f(p)$ is a decreasing function of p , which means that the more expensive the product, the smaller the probability that a customer is prepared to pay this amount of money.
- If the price of an item is p , then any customer who is prepared to pay $p_1 \geq p$ will buy it. Thus, the probability of buying an item when the price is p is:

$$P(p) = \int_{u=p}^{+\infty} f(u) du = 1 - \int_{u=0}^p f(u) du = 1 - F(p)$$

where $F(p)$ is the distribution function of the price.

This probability tends to 0 when p tends to infinity and to 1 when p tends to 0.

A set of n items are available at time 0.

We define the value $v(k, t)$, $k \in [0, n]$ and $t \in [0, T]$, as the maximum expected revenue we can obtain by time T from k items available at time t . We assume that $v(k, t)$ is continuously differentiable with regard to t . Thus, $v(n, 0)$ is the solution to the problem.

Indeed, $v(0, t) = 0$, $\forall t \in [0, T]$ and $v(k, T) = 0$, $\forall k \in [0, n]$. In other words, if the inventory is empty at time t , we cannot expect any further revenue. Also, if the inventory is not empty at time T , it is no longer possible to sell the items that are in inventory.

Assume that k is the number of items available at time t . Three cases should be considered when the system evolves from time t to time $t + \delta$:

- No customer appears during the period $[t, t + \delta)$. The probability of this non-event is $1 - \lambda \delta$ and the value associated with the state $(k, t + \delta)$ at time $t + \delta$ is $v(k, t + \delta)$.
- A customer appears during the period $[t, t + \delta)$ (probability $\lambda \delta$), but does not buy anything (probability $F(p)$). Finally, the probability associated with this case is $\lambda \delta F(p)$ and the value associated with the system is still $v(k, t + \delta)$ at time $t + \delta$.
- A customer appears during the period $[t, t + \delta)$ (probability $\lambda \delta$) and buys one item (probability $1 - F(p)$). The probability associated with this case is $\lambda \delta [1 - F(p)]$ and the value associated with the system at time $t + \delta$ is $v(k - 1, t + \delta) + p - c$. In this expression, p is the price of one item and c is the marginal cost when selling one item (cost to invoice, packaging, transportation, for instance). The cost c depends neither on the inventory level nor on the time. It is assumed to be less than p .

Figure 2.1 represents the evolution of the number of items during the elementary period $[t, t + \delta)$ if k items are available at time t .

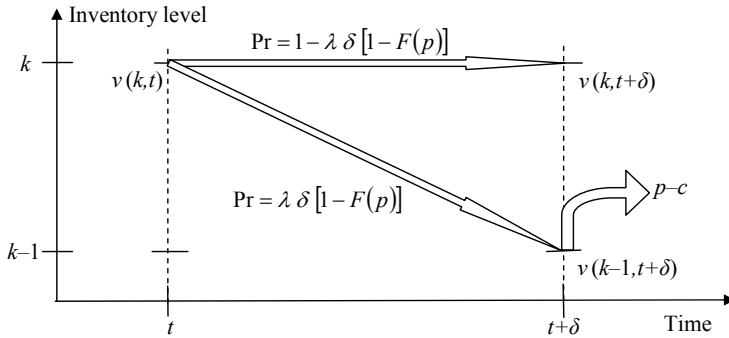


Figure 2.1 Evolution of the number of items

Let p^* be the optimal cost of one item at time t when the inventory level is k , and $v(k, t)$ the maximum expected revenue for the state (k, t) of the system. At time $t + \delta$ the maximum expected revenue becomes either $v(k, t + \delta)$ with the probability $1 - \lambda \delta [1 - F(p^*)]$ or $v(k-1, t + \delta)$ with the probability $\lambda \delta [1 - F(p^*)]$, but, in the later case, some revenue has been taken by the retailer when the item was sold and this amount is $p^* - c$. In terms of flow, we can consider that the flow $p^* - c$ of money exited the system during the elementary period $[t, t + \delta)$. Thus, writing the balance of the maximum expected revenues, we obtain:

$$v(k, t) = [1 - \lambda \delta [1 - F(p^*)]] v(k, t + \delta) + \lambda \delta [1 - F(p^*)] [v(k-1, t + \delta) + p^* - c]$$

As a consequence:

$$v(k, t) = \max_{p \geq 0} \left\{ (1 - \lambda \delta) v(k, t + \delta) + \lambda \delta F(p) v(k, t + \delta) + \lambda \delta [1 - F(p)] [v(k-1, t + \delta) + p - c] \right\}$$

This equality can be rewritten as:

$$v(k, t) = \max_{p \geq 0} \left\{ v(k, t + \delta) - \lambda \delta [1 - F(p)] v(k, t + \delta) + \lambda \delta [1 - F(p)] [v(k-1, t + \delta) + p - c] \right\}$$

This equality leads to:

$$-\frac{v(k, t + \delta) - v(k, t)}{\delta} = \lambda \max_{p \geq 0} \left\{ -[1 - F(p)] v(k, t + \delta) + [1 - F(p)] [v(k-1, t + \delta) + p - c] \right\}$$

If δ tends to 0, the previous equality becomes:

$$v'_t(k, t) = -\lambda \min_{p \geq 0} \{ [1 - F(p)]v(k, t) - [1 - F(p)][v(k-1, t) + p - c] \} \quad (2.16)$$

The solution of the problem (i.e., maximizing the revenue) consists of finding, for each pair (k, t) , the price p^* that minimizes the second member of Equality 2.16, and then solving the differential equation (2.16). Thus, two values will be associated to the pair (k, t) :

- The price $p^*(k, t)$ that should be assigned to a unit of product at time t if the inventory level is k .
- The maximum expected revenue on period $[t, T]$ if the inventory level is k at time t .

Unfortunately, it is impossible to find an analytic solution for a general function $F(p)$. From this point onwards, we assume that:

$$F(p) = 1 - e^{-\alpha p} \quad (2.17)$$

where $\alpha > 0$.

2.3.2 Solution to the Problem

According to Relation 2.17, Equation 2.16 becomes:

$$v'_t(k, t) = -\lambda \min_{p \geq 0} \{ e^{-\alpha p} (v(k, t) - v(k-1, t) - p + c) \} \quad (2.18)$$

Since $e^{-\alpha p} > 0$, the minimum value of the second member of (2.18) is obtained for the value $p^*(k, t)$ of p that makes its derivative with regard to p equal to 0. Thus, $p^*(k, t)$ is the solution of:

$$e^{-\alpha p} [-\alpha v(k, t) + \alpha v(k-1, t) + \alpha p - \alpha c - 1] = 0$$

Finally:

$$p^*(k, t) = v(k, t) - v(k-1, t) + c + \frac{1}{\alpha} \quad (2.19)$$

By replacing p by $p^*(k, t)$ in Equation 2.18 we obtain:

$$v'_t(k, t) = -\frac{\lambda}{\alpha} e^{-\alpha[v(k, t) - v(k-1, t) + c + 1/\alpha]} \quad (2.20)$$

Equation 2.20 holds for $k > 1$.

As we can see, the derivative of v with respect to t is negative. This means that the maximum expected revenue decreases when the time increases, whatever the inventory level. In other words, the closer the deadline, the smaller the maximum expected revenue for any given inventory level.

For $k = 0$, the differential equation is useless since we know that:

$$v(0, t) = 0 \quad \forall t \in [0, T]$$

For $k = 1$, the differential equation (2.20) is rewritten as:

$$v'_t(1, t) = -\frac{\lambda}{\alpha} e^{-\alpha[v(1, t) + c + 1/\alpha]}$$

The solution to this differential equation is:

$$v(k, t) = \frac{1}{\alpha} \ln \left\{ \sum_{j=0}^k \frac{\lambda^j e^{-j(1+\alpha c)} (T-t)^j}{j!} \right\} \quad (2.21)$$

Let set:

$$A(k, t) = \sum_{j=0}^k \frac{\lambda^j e^{-j(1+\alpha c)} (T-t)^j}{j!}$$

With this definition, Relation 2.21 can be rewritten as:

$$v(k, t) = \frac{1}{\alpha} \ln [A(k, t)]$$

and Relation 2.19 becomes:

$$p^*(k, t) = \frac{1}{\alpha} \ln \left[\frac{A(k, t)}{A(k-1, t)} \right] + c + \frac{1}{\alpha}$$

or:

$$p^*(k, t) = \frac{1}{\alpha} \ln \left[\frac{e^{1+\alpha c} A(k, t)}{A(k-1, t)} \right] \quad (2.22)$$

Finally, for any pair (k, t) , we can compute $A(k, t)$ and $A(k-1, t)$, and thus we can compute the optimal price associated to this pair by applying Relation 2.22. Note that, when we refer to the state of the system, we refer to the pair (k, t) .

In this model, the system change over time according to a frozen control characterized by parameters α that provides the probability for a customer to buy an item, and λ that defines the probability for a customer to appear in the system during an elementary period. Thus, the behavior of customers, as well as their decision-making process, is frozen as soon as λ and α are selected. As a consequence, this model is not very useful in practice, but it is a good example of helping the reader to understand the objective of dynamic pricing that consists in adjusting dynamically the price of the items to the state of the system.

Example

We illustrate the above presentation using an example defined by the following parameters:

- $\alpha = 0.8$. Remember that the greater α , the faster the probability of buying a product decreases with the cost.
- $\lambda = 1.5$. Remember that the greater λ , the greater the probability that a customer enters the system.
- The initial level of the inventory is 10.
- $T=20$.

The probability of reaching the inventory level k at time t is represented in Figure 2.2. At time 0, the inventory level is equal to 10 with probability 1. When time increases, the set of possible inventory levels with significant probabilities increases and the mean value of the inventory level decreases.

Figure 2.3 represents the optimal price of a product according to the time and the inventory level. For a given inventory level, price decreases with time. Similarly, at a given time, the price increases when the inventory level decreases.

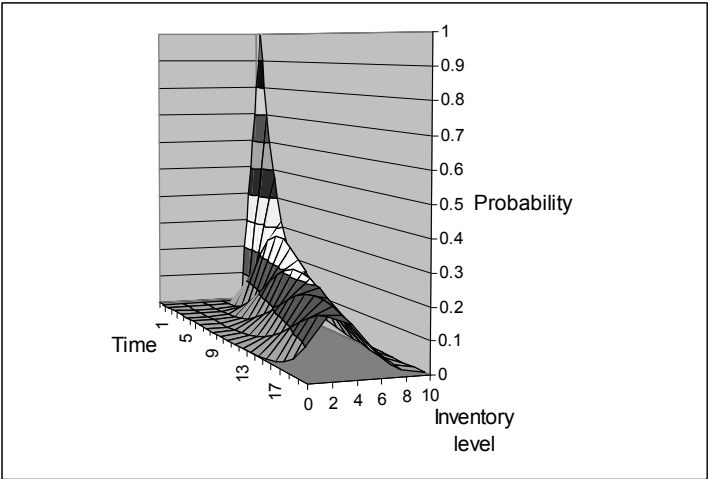


Figure 2.2 Probability versus time and inventory level

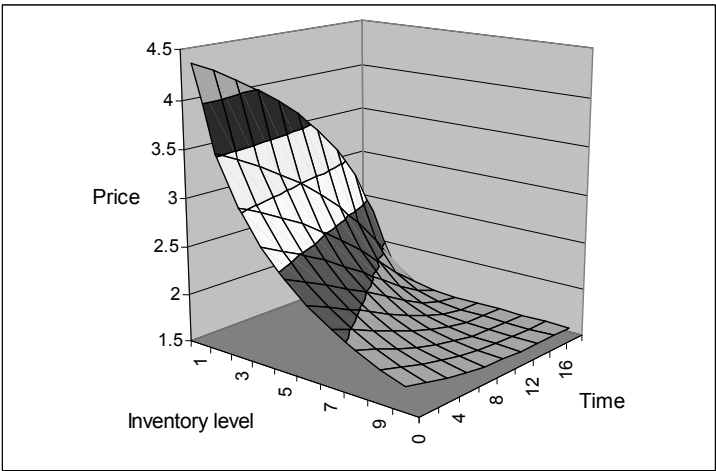


Figure 2.3 Optimal price versus time and inventory level

2.3.3 Probability for the Number of Items at a Given Point in Time

Let n be the number of items available at time 0. We denote by $r(k, t)$ the probability that k items are still available at time t .

Indeed, $r(n, 0) = 1$ since the initial state $(n, 0)$ is given and $r(k, 0) = 0$ whatever the number k of items, $k \in \{0, 1, \dots, n-1\}$.

Result 1

The probability to have k unsold items at time t is:

$$r(k, t) = \frac{[\lambda e^{-(1+\alpha c)} t]^{n-k} A(k, t)}{(n-k)! A(n, 0)} \quad (2.23)$$

Proof

Let dt be an elementary increment of t . To have k items at time $t + dt$ we should be in one of the following cases:

- The number of items at time t was $k + 1$ (this case holds only if $k < n$), a customer appeared on the time interval $[t, t + dt)$ (probability λdt) and this customer bought an item (probability $e^{-\alpha p^*(k+1, t)}$).
- The number of items at time t was k and either no customer appeared on the time interval $[t, t + dt)$ (probability $1 - \lambda dt$) or one customer appeared but he/she didn't buy anything (probability $\lambda dt (1 - e^{-\alpha p^*(k, t)})$).

As a consequence, we obtain the following relation:

$$r(k, t + dt) = r(k + 1, t) \lambda dt e^{-\alpha p^*(k+1, t)} + r(k, t) [1 - \lambda dt e^{-\alpha p^*(k, t)}] \quad (2.24)$$

when $k < n$, and

$$r(n, t + dt) = r(n, t) [1 - \lambda dt e^{-\alpha p^*(n, t)}] \quad (2.25)$$

Let us first consider Relation 2.25. It leads to:

$$r'_t(n, t) = -\lambda r(n, t) e^{-\alpha p^*(n, t)}$$

Using Relation 2.22, we obtain:

$$\ln r(n, t) = \ln A(n, t) + W, \text{ where } W \text{ is constant.}$$

Since $r(n, 0) = 1$, the previous relation leads to:

$$W = -\ln A(n, 0)$$

Finally:

$$r(n, t) = \frac{A(n, t)}{A(n, 0)} \quad (2.26)$$

Thus, Relation 2.23 holds for $k = n$.

From Relation 2.24 and using (2.22), we derive:

$$\begin{aligned} r'_t(k, t) = \\ r(k+1, t) \lambda \frac{e^{-(1+\alpha c)} A(k, t)}{A(k+1, t)} - r(k, t) \lambda \frac{e^{-(1+\alpha c)} A(k-1, t)}{A(k, t)} \end{aligned} \quad (2.27)$$

If we write Equation 2.27 for $k = n-1$, we can use Equation 2.26 to obtain a differential equation in $r(n-1, t)$. Solving this equation leads to:

$$r(n-1, t) = \frac{\lambda e^{-(1+\alpha c)} t A(n-1, t)}{A(n, 0)}$$

In turn, this result used with Relation 2.27 leads to a differential equation in $r(n-2, t)$. As soon as the general form of the solution is recognized, a recursion is applied to verify the result. Q.E.D.

Result 2 concerns the expected number of items sold at the end of period t .

Result 2

The expected number of items sold by time t is

$$E_t = \lambda e^{-(1+\alpha c)} t A(n-1, 0) / A(n, 0)$$

Proof

Taking into account Result 1:

$$E_t = \sum_{k=0}^n (n-k) r(k, t) = \sum_{k=0}^n (n-k) \frac{[\lambda e^{-(1+\alpha c)} t]^{n-k} A(k, t)}{(n-k)! A(n, 0)}$$

This relation can be rewritten as:

$$E_t = \lambda e^{-(1+\alpha c)} t \sum_{k=0}^{n-1} \frac{[\lambda e^{-(1+\alpha c)} t]^{n-k-1} A(k, t)}{(n-k-1)! A(n, 0)}$$

$$E_t = \frac{\lambda e^{-(1+\alpha c) t} A(n-1, 0)}{A(n, 0)} \sum_{k=0}^{n-1} \frac{[\lambda e^{-(1+\alpha c) t}]^{n-k-1} A(k, t)}{(n-k-1)! A(n-1, 0)}$$

The sum in the second member of the equality is equal to 1 since the elements of this sum are the probability $r(k, t)$ assuming that the initial number of items is $n-1$.

This completes the proof.

Q.E.D.

Example

As we can see in Figure 2.4:

- The number of items sold by time T is an increasing function of λ when α is fixed; in other words, the lower the average probability that a customer arrives in the system, the lower the number of items sold at time T .
- The number of products sold by time T is a decreasing function of α when λ is fixed; in other words, the higher the average price of a product, the lower the number of items sold by time T .

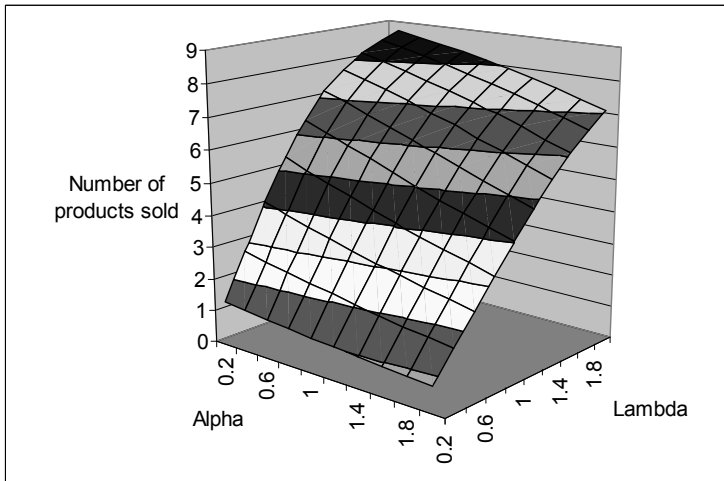


Figure 2.4 Number of products sold by time T with regards to λ (lambda) and α (alpha)

2.3.4 Remarks

In the model presented in this section, it is assumed that the buying activity proceeds in two steps: first, a buyer enters the system with a given probability that

depends on parameter α and, second, he/she decides to buy an item or not, depending on the maximum amount of money he/she is prepared to pay for it, and this buying probability depends upon a parameter denoted by λ . Furthermore, the value of one item is equal to zero at time T , the horizon of the problem; in other words, there is no salvage value.

The biggest drawback with this model is related to its following characteristics:

1. We have to compute the maximum value of $[1 - F(p)][v(k, t) - v(k-1, t) - p + c]$ with respect to p . Let p^* be the value of p that leads to this maximum value. The problem is that, if we want to solve the differential equation (2.18) analytically, we must be able to express p^* as a function of v .
2. We then have to be able to solve the differential equation in which p^* is replaced by its function of v .

These two conditions make the computation of an analytical solution usually impossible, mainly if the problem at hand is a real-life problem, especially if the probability density is not exponential.

2.4 Stochastic Dynamic Pricing for Items with Salvage Values

The difference from the previous model lies not only in the existence of salvage values, but also in the fact that there exists a one-to-one relationship between the demand intensity, denoted by λ , and the price for one item, denoted by p . Thus, the two-stage buying process that was the basis of the previous model vanishes.

We assume that the demand follows a Poisson process of parameter λ . We also assume that only a finite number of prices can be chosen by the retailer and that each price is associated with one demand intensity.

Some additional assumptions will be made. They will be presented in detail in the next section.

2.4.1 Problem Studied

The period available to sell the items is $[0, T]$ and the number of items available at time 0 is n . We denote by $P = \{p_1, p_2, \dots, p_N, p_\infty\}$ the set of prices that can be chosen by the retailer and by $\Lambda = \{\lambda_1, \lambda_2, \dots, \lambda_N, \lambda_\infty\}$ the corresponding demand intensities.

Establishing the relationship between the elements of P and the elements of Λ is not an easy task. We assume that this task has been performed at this point of the process.

The available prices are arranged in their decreasing order, i.e., $p_1 > p_2 > \dots > p_N > p_\infty$, and thus $\lambda_1 < \lambda_2 < \dots < \lambda_N < \lambda_\infty$ since the greater the price, the lower the demand intensity: the price is a decreasing function of the demand intensity.

As mentioned before, salvage values are included in the model, which means that it is still possible to sell the unsold items in a secondary market after time T . We denote by $w(r, T)$ the salvage value of r items unsold at the deadline T . In other words, $w(r, T)$ is the selling price in the second market of the r unsold items at time T . The salvage value $w(r, T)$ is assumed to be non-decreasing and concave in r . This means that:

1. The salvage value of the remaining items increases with the number of items, which is realistic.
2. The average price of one item is a non-increasing function of the number of items. It may also happen that the price per unit does not depend upon the number of remaining items: it is the borderline case. Note that this second hypothesis is quite common.

We denote by $p(r, t)$ the price of one item at time t if the inventory level is r .

We also assume that:

$$\frac{1}{n} w(n, T) \leq w(1, T) \leq p(n, T)$$

where $p(n, T)$ is the price of one item when the inventory level is still full just before the end of the selling period.

Indeed, according to the hypotheses made before, $p(n, T) \leq p(k, t)$ for $k \in \{1, 2, \dots, n\}$ and $t \in [0, T)$. In other words, selling one item for its salvage value is always worse than selling it before time T , whatever the inventory level.

We first consider the case when the price of one item depends on the inventory level only.

2.4.2 Price as a Function of Inventory Levels: General Case

2.4.2.1 Model

In practice, it is rare to assign a different price for each inventory level, except if the items under consideration are very expensive (cars, for instance). This case will be considered in Section 2.4.3. For the time being, we assume that the same

price applies when the inventory level lies between two given limits. In the following, k_i is the rank of the i -th item sold.

We would like to recall the following convention: writing $x \in (a, b]$ means that a does not belong to the interval (i.e., x cannot take the value a) while b does (i.e., x can take the value b). Also, to simplify the notations, we introduce

$N_i = \sum_{j=1}^i n_j$ for $i = 1, 2, \dots, s$, where s is the number of levels, $N_0 = 0$, and n_j is the number of items sold at price p_j .

If n is the inventory level at time 0, we assume that:

- Price p_1 applies to the k_1 -th item sold when $k_1 \in [N_0, N_1)$.
- Price p_2 applies to the k_2 -th item sold when $k_2 \in [N_1, N_2)$.
- ...
- Price p_i applies to the k_i -th item sold when $k_i \in [N_{i-1}, N_i)$.
- ...
- Price p_s applies to the k_s -th item sold when $k_s \in [N_{s-1}, N_s)$.

Indeed,

$$n = \sum_{j=1}^s n_j = N_s$$

At this point of the discussion, the goal does not consist in finding the values of parameters n_i that optimize the mean value of the revenue. We just want to propose a tool that provides the mean value of the revenue when the values of the parameters are given.

Let $k_i \in \{N_{i-1} + 1, \dots, N_i\}$ and let $\Pr(k_i, [a, b] | r)$ be the probability that k_i items are sold during period $[a, b]$ if the inventory level is r at time a . The formulation of the probability has been slightly modified to precisely match the initial value of the inventory, which will be useful in the remainder of the section to avoid confusion. When the initial inventory is n , this information is ignored and we use the previous notation.

In Figure 2.5, we provide the structure that underlies the computation of $\Pr(k_i, [0, T])$.

We first write that the probability that k_i items are sold during period $[0, T]$ is the integral on $[0, T]$ with regard to t_1 of the product of the two following factors (Bayer's theorem):

- The probability that $n_1 = N_1$ items are sold in period $(0, t_1]$, the last item being sold at time t_1 . This probability is:

$$\Pr(k_i - N_2, (t_2, T] | n - N_2)$$

Thus,

$$\begin{aligned} & \Pr(k_i - N_1, (t_1, T] | n - N_1) \\ &= \int_{t_2=t_1}^T \frac{[\lambda_2(t_2 - t_1)]^{n_2-1}}{(n_2 - 1)!} \exp[-\lambda_2(t_2 - t_1)] \lambda_2 \Pr(k_i - N_2, (t_2, T] | n - N_2) dt_2 \end{aligned}$$

We further extend this approach to the next lower layers. We obtain:

$$\begin{aligned} & \Pr(k_i - N_{i-3}, (t_{i-3}, T] | n - N_{i-3}) \\ &= \int_{t_{i-2}=t_{i-3}}^T \left\{ \frac{[\lambda_{i-2}(t_{i-2} - t_{i-3})]^{n_{i-2}-1}}{(n_{i-2} - 1)!} \exp[-\lambda_{i-2}(t_{i-2} - t_{i-3})] \lambda_{i-2} \right. \\ & \quad \left. \Pr(k_i - N_{i-2}, (t_{i-2}, T] | n - N_{i-2}) \right\} dt_{i-2} \end{aligned}$$

$$\begin{aligned} & \Pr(k_i - N_{i-2}, (t_{i-2}, T] | n - N_{i-2}) \\ &= \int_{t_{i-1}=t_{i-2}}^T \left\{ \frac{[\lambda_{i-1}(t_{i-1} - t_{i-2})]^{n_{i-1}-1}}{(n_{i-1} - 1)!} \exp[-\lambda_{i-1}(t_{i-1} - t_{i-2})] \lambda_{i-1} \right. \\ & \quad \left. \Pr(k_i - N_{i-1}, (t_{i-1}, T] | n - N_{i-1}) \right\} dt_{i-1} \end{aligned}$$

The last echelon of the formulation is slightly different from the previous ones:

$$\begin{aligned} & \Pr(k_i - N_{i-1}, (t_{i-1}, T] | n - N_{i-1}) \\ &= \int_{t_i=t_{i-1}}^T \frac{[\lambda_i(t_i - t_{i-1})]^{k_i - N_{i-1} - 1}}{(k_i - N_{i-1} - 1)!} \exp[-\lambda_i(t_i - t_{i-1})] \lambda_i \Pr(0, (t_i, T] | n - k_i) dt_i \end{aligned}$$

and:

$$\Pr(0, (t_i, T] | n - k_i) = \exp[-\lambda_i(T - t_i)]$$

This sequence of equalities can be rewritten as a unique relation:

$$\begin{aligned} \Pr (k_i, [0, T]) = & \prod_{j=1}^{i-1} \left(\lambda_{n_j} \right) \lambda_{k_i - N_{i-1}} \prod_{j=1}^{i-1} \left(\frac{1}{(n_j - 1)!} \right) \left(\frac{1}{(k_i - N_i - 1)!} \right) \\ & \int_{t_1=0}^T \int_{t_2=t_1}^T \cdots \int_{t_i=t_{i-1}}^T \left\{ \prod_{j=1}^{i-1} \left((t_j - t_{j-1})^{n_j-1} \right) (t_i - t_{i-1})^{k_i - N_{i-1} - 1} \right. \\ & \left. \prod_{j=1}^i \left(\exp \left[-\lambda_j (t_j - t_{j-1}) \right] \right) \exp \left[-\lambda_i (T - t_i) \right] \right\} dt_i dt_{i-1} \cdots dt_1 \end{aligned} \quad (2.28)$$

This relation holds for $k_i = 1, 2, \dots, n-1$.

The probability that any item is sold by horizon T is:

$$\Pr (0, [0, T]) = \exp (-\lambda_1 T)$$

Furthermore, the probability that all the items are sold at time T is:

$$\Pr (n, [0, T]) = 1 - \sum_{k=0}^{n-1} \Pr (k, [0, T])$$

Assuming that the probabilities are known, the mean value of the revenue is:

$$v (n, T) = \sum_{k=0}^n \left\{ \Pr (k, [0, T]) k [p (k) + w (n - k, T)] \right\} \quad (2.29)$$

where $p (k) = p_i$ if $k \in \{ N_{i-1} + 1, \dots, N_i \}$.

2.4.2.2 Computation of the Mean Value of the Revenue

An analytical expression of the integrals of the second member in Relation 2.28 is possible only for very small values of parameters i and n_i since the complexity of the solution increases exponentially. This is why a numerical approach is necessary.

We chose the Monte-Carlo approach. In order to simplify the notations, we denote by q_k the probability $\Pr (k, [0, T])$ and by p_i the cost of one item if the inventory level k belongs to $\{ N_{i-1} + 1, \dots, N_i \}$. Others notations are those introduced in the previous subsection.

Algorithm 2.2.

1. Compute $q_0 = \exp(-\lambda_1 T)$.
2. For $k=1$ to $n-1$ do:
 - 2.1. Compute i such that $N_{i-1} < k \leq N_i$.
 - 2.2. If $i > 1$ set $K_j = n_j$ for $j=1, \dots, i-1$.
 - 2.3. $K_i = k - N_{i-1}$.
 - 2.4. $u = 1$.
 - 2.5. For $j=1, \dots, i$ do $u = u \lambda_j^{K_j}$.

At this point, u contains the term $\prod_{j=1}^{i-1} \left(\lambda_j^{n_j} \right) \lambda_i^{k_i - N_{i-1}}$ of Formulae (2.28).

The Monte-Carlo method starts below.

- 2.6. Set $q_k = 0$.
- 2.7. For $Mc = 1$ to M do:

M is the number of iterations (around 10 000).

 - 2.7.1. Set $t_0 = 0$.
 - 2.7.2. For $j=1, \dots, i$ generate t_j at random on $[t_{j-1}, T]$.
 - 2.7.3. Set $w = 1, s = 0, z = 1$.
 - 2.7.4. For $j=1, \dots, i$ do:
 - 2.7.4.1. Compute $v = \frac{(t_j - t_{j-1})^{K_j-1}}{(K_j - 1)!}$.
 - 2.7.4.2. Compute $v = v (T - t_{j-1})$.
 - 2.7.4.3. Compute $w = w v$.
 - 2.7.4.4. If $(j < i)$, then compute $z = z \exp[-\lambda_j (t_j - t_{j-1})]$.
 - 2.7.4.5. If $[(j = i) \text{ and } (k < N_i)]$ do:
 - 2.7.4.5.1. Compute $z = z \exp[\lambda_j t_{j-1}]$.
 - 2.7.4.5.2. If $(Mc = 1)$, then compute $u = u \exp[-\lambda_j T]$.
 - 2.7.4.6. If $[(j = i) \text{ and } (k = N_i)]$ do:
 - 2.7.4.6.1. Compute $z = z \exp[-\lambda_j (t_j - t_{j-1}) + \lambda_{j+1} t_j]$.
 - 2.7.4.6.2. If $(Mc = 1)$, then compute $u = u \exp[-\lambda_{j+1} T]$.
 - 2.7.5. End of loop j .
 - 2.7.6. Compute $w = w z$.
 - 2.7.7. Compute $q_k = q_k + w / M$.
- 2.8. End of loop Mc .
- 2.9. Compute $q_k = q_k u$.

3. End of loop k .

Computation of q_n .

4. Set $u = 0$.

5. For $k = 0, \dots, n-1$ do $u = u + q_k$.
6. Compute $q_n = 1 - u$.

Computation of the mean value of the revenue denoted by Ct.

7. Set $Ct = 0$.
8. For $k = 0$ to n do:
 - 8.1. Set $cc = 0$.
 - 8.2. Compute i such that $N_{i-1} < k \leq N_i$.
 - 8.3. If $i > 1$ do $cc = cc + p_j n_j$ for $j = 1, \dots, i-1$.
 - 8.4. Compute $cc = cc + (k - N_{i-1}) p_i$.
 - 8.5. Compute $Ct = Ct + q_k [cc + w(n - k, T)]$.
9. End of loop k .

2.4.2.3 Improvement of the Solution

We denote by $n_i, i = 1, 2, \dots, s$ the initial sizes of the layers, from the upper to the lower one, and by λ_i (respectively, p_i) the corresponding demand intensities (respectively, prices). Remember that $\lambda_1 < \lambda_2 < \dots < \lambda_s$ and $p_1 > p_2 > \dots > p_s$.

Since a numerical approach has been used to evaluate the probabilities of the states of the system at time T and the mean value of the revenue knowing the layers, we can also use a numerical approach to reach the layers that maximize the mean value of the revenue. We chose a simulated annealing algorithm to improve a given solution. This method is an iterative approach and some layers may become empty (and thus disappear) during the process. This requires some additional notations.

We denote by r^0 the number of layers, by n_i^0 the size of the i -th layer and by λ_i^0 (respectively, p_i^0) the corresponding demand intensities (respectively, prices) for $i = 1, 2, \dots, r^0$ at the beginning of an iteration of the simulated annealing algorithm or the initial stage. Indeed, $r^0 \leq s$. At the beginning of the first iteration, $r^0 = s$, $n_i^0 = n_i$, $\lambda_i^0 = \lambda_i$ and $p_i^0 = p_i$ for $i = 1, 2, \dots, s$.

We denote by r^1 the number of layers at the end of the first iteration, by n_i^1 the size of the i -th layer, and by λ_i^1 (respectively, p_i^1) the corresponding demand intensities (respectively, prices) for $i = 1, 2, \dots, r^1$. Indeed, $r^1 \leq s$. Furthermore, the corresponding mean value of the revenue is $Copt^1$.

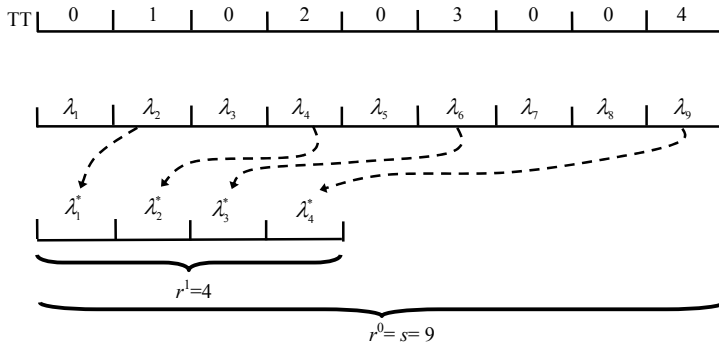


Figure 2.6 Links between the second and the initial iteration

We introduce a vector TT to link the initial demand intensities (and thus the initial price), with the current demand intensities and prices. This linkage is illustrated in Figure 2.6.

Algorithm 2.3 describes the simulated annealing mechanism that we apply to our problem. Note: Algorithm 2.3 contains Algorithm 2.2.

Algorithm 2.3.

1. Introduce s, n, λ_i, p_i for $i = 1, 2, \dots, s$ and the salvage costs $w(k, T)$ for $k = 1, 2, \dots, n$.
2. Generate at random n_i for $i = 1, 2, \dots, s$ such that $\sum_{i=1}^s n_i = n$, the value generated being integer and positive.

The first two steps of the algorithm provide the initial data.

3. Introduce KK .
 KK is the number of iterations that will be made. For instance, we may assign the value 2000 or 3000 to this variable.

4. Set $r^0 = s$, $\lambda_i^0 = \lambda_i$, $p_i^0 = p_i$ and $n_i^0 = n_i$ for $i = 1, 2, \dots, s$.

This set of values represents the initial solution that is called S^0 .

5. Compute the mean value of the revenue corresponding to solution S^0 . We denote this value $Copt^0$.

This is obtained by applying Algorithm 2.2.

6. We set $S^* \equiv S^0$ and $Copt^* = Copt^0$.

For each iteration, S^ contains the best solution and $Copt^*$ the greatest mean value of the revenue obtained since the beginning of the algorithm.*

The simulated annealing process starts at this point.

7. For $k = 1$ to KK do:

7.1. For $i = 1$ to s set $TT_i = 0$.

7.2. Set $i = 1$ and $j = 1$.

7.3. While ($i \leq r^0$)

7.3.1. If ($\lambda_i^0 = \lambda_i$) do:

7.3.1.2. Set $TT_j = i$.

7.3.1.3. Set $i = i + 1$.

7.3.1.4. Set $j = j + 1$.

7.3.2. If ($\lambda_i^0 \neq \lambda_i$) do: $j = j + 1$.

The instructions of Stage 7 lead to vector TT.

In the following steps, we modify the layers.

7.4. Generate the integer i at random on $\{1, 2, \dots, r^0\}$.

7.5. Generate the integer j at random on $\{1, 2, \dots, s\}$.

7.6. If ($TT_j \neq 0$) do:

In this case, one item will be added to the i -th layer that is not empty.

7.6.1. Set $i_1 = TT_j$.

7.6.2. Set $n_{i_1}^1 = n_{i_1}^0 + 1$.

7.7. If ($TT_j = 0$) do:

In this case, one item will be added to an empty layer that corresponds to the j -th initial layer. This layer is temporarily set at the last position in the current solution.

7.7.1. Set $i_1 = r^0 + 1$.

7.7.2. Set $r^1 = r^0 + 1$.

7.7.3. Set $n_{i_1}^1 = 1, p_{i_1}^1 = p_j, \lambda_{i_1}^1 = \lambda_j$.

7.8. Set $n_i^1 = n_i^0 - 1$.

7.9. If ($n_i^1 = 0$) do:

In this case, one layer becomes empty and disappears.

7.9.1. If ($i < r^0$) do:

7.9.1.1. For $k = i$ to $r^0 - 1$ set $n_k^1 = n_{k+1}^1, p_k^1 = p_{k+1}^1, \lambda_k^1 = \lambda_{k+1}^1$.

7.9.1.2. Set $r^1 = r^0 - 1$.

The next stage consists in putting the parameters into the increasing order of the demand intensities.

7.10. For $i = 1$ to $r^1 - 1$ do

7.10.1. For $j = i + 1$ to r^1

7.10.1.1. If ($\lambda_j^1 < \lambda_i^1$) permute λ_j^1 and λ_i^1, p_j^1 and p_i^1, n_j^1 and n_i^1 .

At this stage of the computation, a new solution S^1 is available.

7.11. Compute the mean value of the revenue corresponding to solution S^1 . We denote this value by Copt^1 . This value is obtained by applying Algorithm 2.2.

7.12. If $\text{Copt}^1 \geq \text{Copt}^0$ do:

7.12.1. If $\text{Copt}^1 > \text{Copt}^*$ set $S^* = S^1$ and $\text{Copt}^* = \text{Copt}^1$.

7.12.2. Set $S^0 = S^1$.

7.13. If $\text{Copt}^1 < \text{Copt}^0$ do:

7.13.1. Compute $y = \exp[- (\text{Copt}^0 - \text{Copt}^1) / \text{kkt}]$.

7.13.2. Generate x at random on $[0, 1]$ (probability density 1).

7.13.3. If $(x \leq y)$ set $S^0 = S^1$.

2.4.2.4 Numerical Example

In the case presented hereafter, the number of items to sold before time T is 25 ($n = 25$). A one-to-one relationship has been established between five prices and five demand intensities. These data are presented in Table 2.1.

Table 2.1 Price versus demand intensity

Price	20	14	10	7	5
Demand intensity	0.2	0.4	0.6	0.8	1

The salvage value is linear: each item can be sold on the second market for 2 monetary units. The computation starts with five layers numbered from the upper to the lower layer. Each of them is initially made with 5 consecutive inventory levels. The number of iterations made in the simulated annealing process is 5000.

Remark: A large number of choices (see Appendix A) are available when applying simulated annealing for defining, in particular:

- the number of iterations;
- the evolution of the “temperature” that affects the selection of the next state;
- the “neighborhood” of a solution.

In Table 2.2, we give some intermediate results provided by the simulated annealing algorithm. The last one is the near-optimal solution.

Table 2.2 Some intermediate steps of the simulated annealing process

Iteration number	Solution						Mean value of the revenue (rounded)
1	Layer size	5	5	5	5	5	174
	Price	20	14	10	7	5	
	Demand intensity	0.2	0.4	0.6	0.8	1	
16	Layer size	2	4	6	7	6	184
	Price	20	14	10	7	6	
	Demand intensity	0.2	0.4	0.6	0.8	1	
993	Layer size	0	11	1	10	3	199
	Price	20	14	10	7	6	
	Demand intensity	0.2	0.4	0.6	0.8	1	
1243	Layer size	0	13	1	10	1	202
	Price	20	14	10	7	6	
	Demand intensity	0.2	0.4	0.6	0.8	1	
3642	Layer size	0	1	15	1	8	205
	Price	20	14	10	7	6	
	Demand intensity	0.2	0.4	0.6	0.8	1	
4006	Layer size	0	1	17	7	0	211
	Price	20	14	10	7	6	
	Demand intensity	0.2	0.4	0.6	0.8	1	

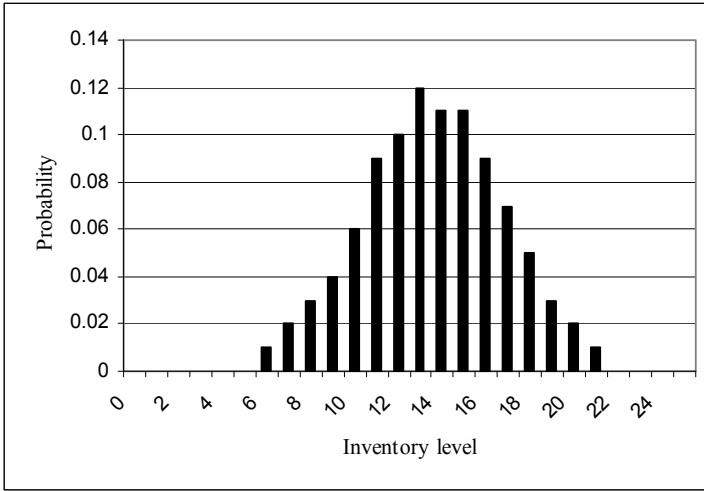


Figure 2.7 Probabilities at time T for the last structure of layers

Figure 2.7 provides the probabilities of the different inventory levels at time T for the last structure of layers.

2.4.2.5 How to Use the Approach?

The previous approach is used on a periodic basis. This strategy corresponds to the usual behavior of vendors: they choose a pricing policy on a given period (one or two weeks for instance) and they reconsider the pricing policy for the next period according to the inventory level at the end of the previous period, and so on. In other words, they work on a rolling-horizon basis.

2.4.3 Price as a Function of Inventory Levels: a Special Case

We assume that the demand intensity, and thus the price, is different from one inventory level to the next. This kind of situation happens when the items are expensive (cars, for instance). In this particular case, it is possible to express analytically the probability that k items are sold at the end of period T .

We denote by λ_i the demand intensity when the inventory level is i and the price of the next item is p_i . The initial inventory level is n . Indeed $\lambda_n < \lambda_{n-1} < \dots < \lambda_2 < \lambda_1$ and, as mentioned earlier, $p_n > p_{n-1} > \dots > p_2 > p_1$.

As in the previous section, $\Pr(k, [t_1, t_2])$ refers to the probability that k items are sold in period $[t_1, t_2]$.

Since at each inventory level the demand is generated by a Poisson process:

- $\Pr(0, [0, T]) = \exp(-\lambda_n T)$
- $\Pr(1, [0, T]) = \int_{t=0}^T \exp(-\lambda_n t) \lambda_n \exp[-\lambda_{n-1}(T-t)] dt$

$$= -\frac{\lambda_n}{\lambda_n - \lambda_{n-1}} \exp(-\lambda_n T) - \frac{\lambda_n}{\lambda_{n-1} - \lambda_n} \exp(-\lambda_{n-1} T)$$
- $\Pr(2, [0, T]) = \int_{t=0}^T \Pr(1, [0, t]) \lambda_{n-1} \Pr(0, [t, T]) dt$

$$= -\frac{\lambda_n \lambda_{n-1}}{\lambda_n - \lambda_{n-1}} \int_{t=0}^T \exp(-\lambda_n t) \exp[-\lambda_{n-2}(T-t)] dt$$

$$- \frac{\lambda_n \lambda_{n-1}}{\lambda_{n-1} - \lambda_n} \int_{t=0}^T \exp(-\lambda_{n-1} t) \exp[-\lambda_{n-2}(T-t)] dt$$

$$= \lambda_n \lambda_{n-1} \left\{ \frac{\exp(-\lambda_n T)}{(\lambda_n - \lambda_{n-1})(\lambda_n - \lambda_{n-2})} + \frac{\exp(-\lambda_{n-1} T)}{(\lambda_{n-1} - \lambda_n)(\lambda_{n-1} - \lambda_{n-2})} \right.$$

$$\left. + \frac{\exp(-\lambda_{n-2} T)}{(\lambda_{n-2} - \lambda_n)(\lambda_{n-2} - \lambda_{n-1})} \right\}$$

At this level of the computation, it appears that the formula could be:

$$\Pr(k, [0, T]) = (-1)^k \prod_{i=0}^{k-1} (\lambda_{n-i}) \sum_{i=0}^k \left\{ \frac{\exp(-\lambda_{n-i} T)}{\prod_{\substack{j=0 \\ j \neq i}}^k (\lambda_{n-i} - \lambda_{n-j})} \right\}$$

for $k = 1, \dots, n-1$ (2.30)

To complete the proof, we will show that if (2.30) holds for k , then it also holds for $k+1$. If we express $\Pr(k+1, [0, T])$ according to $\Pr(k, [0, t])$ (Bayes' theorem), we obtain:

$$\begin{aligned}
\Pr (k+1, [0, T]) &= \int_{t=0}^T \Pr (k, [0, t]) \lambda_{n-k} \Pr (0, [t, T]) dt \\
&= (-1)^k \prod_{i=0}^{k-1} (\lambda_{n-i}) \int_{t=0}^T \sum_{i=0}^k \left\{ \frac{\exp(-\lambda_{n-i} t)}{\prod_{\substack{j=0 \\ j \neq i}}^k (\lambda_{n-i} - \lambda_{n-j})} \right\} \lambda_{n-k} \exp[-\lambda_{n-k-1} (T-t)] dt
\end{aligned}$$

Developing this expression, we reach the following equality:

$$\begin{aligned}
\Pr (k+1, [0, T]) &= (-1)^{k+1} \prod_{i=0}^k (\lambda_{n-i}) \sum_{i=0}^k \left\{ \frac{\exp(-\lambda_{n-i} T)}{\prod_{\substack{j=0 \\ j \neq i}}^{k+1} (\lambda_{n-i} - \lambda_{n-j})} \right\} \\
&\quad - (-1)^{k+1} \prod_{i=0}^k (\lambda_{n-i}) \exp(-\lambda_{n-k-1} T) \sum_{i=0}^k \left\{ \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^{k+1} (\lambda_{n-i} - \lambda_{n-j})} \right\}
\end{aligned} \tag{2.31}$$

Expanding the left side of the following equality, we obtain:

$$\sum_{i=0}^{k+1} \left\{ \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^{k+1} (\lambda_{n-i} - \lambda_{n-j})} \right\} = 0$$

This equality can be rewritten as:

$$-\sum_{i=0}^k \left\{ \frac{1}{\prod_{\substack{j=0 \\ j \neq i}}^{k+1} (\lambda_{n-i} - \lambda_{n-j})} \right\} = \frac{1}{\prod_{\substack{j=0 \\ j \neq k+1}}^{k+1} (\lambda_{n-k-1} - \lambda_{n-j})}$$

Thus, Equation 2.31 becomes:

$$\begin{aligned} \Pr(k+1, [0, T]) &= (-1)^{k+1} \prod_{i=0}^k (\lambda_{n-i}) \sum_{i=0}^k \left\{ \frac{\exp(-\lambda_{n-i} T)}{\prod_{\substack{j=0 \\ j \neq i}}^{k+1} (\lambda_{n-i} - \lambda_{n-j})} \right\} \\ &+ (-1)^{k+1} \prod_{i=0}^k (\lambda_{n-i}) \exp(-\lambda_{n-k-1} T) \frac{1}{\prod_{\substack{j=0 \\ j \neq k+1}}^{k+1} (\lambda_{n-k-1} - \lambda_{n-j})} \\ &= (-1)^{k+1} \prod_{i=0}^k (\lambda_{n-i}) \sum_{i=0}^{k+1} \left\{ \frac{\exp(-\lambda_{n-i} T)}{\prod_{\substack{j=0 \\ j \neq i}}^{k+1} (\lambda_{n-i} - \lambda_{n-j})} \right\} \end{aligned}$$

This completes the computation.

Result 3 is derived from the above development. In this result, according to the usual mathematical convention, we assume that if no factor remains in a product,

then the product equals 1. For instance, $\prod_{i=n}^m a_i = 1$ if $m < n$.

Result 3

The probability that k items are sold in period $[0, T]$ is given by (2.30) for $k \in \{1, \dots, n-1\}$. Furthermore, $\Pr(n, [0, T]) = 1 - \sum_{k=0}^{n-1} \Pr(k, [0, T])$. Then the mean value of the revenue can be obtained applying (2.29), Section 2.4.2.1.

2.5 Concluding Remarks

The goal of this chapter was to provide an insight into the domain of pricing models. We limited ourselves to time-dated items with no supply option in a monopolistic environment with myopic customers. Although these assumptions drastically simplify the problem, many additional restrictive assumptions are required to obtain a mathematical model that is easy to analyze.

Nevertheless, the numerical development of the stochastic dynamic pricing model with salvage values is an interesting tool when integrated in a rolling-horizon approach since it allows prices to be adjusted periodically according to the inventory level and time, as required in the case of sales. Unfortunately, the one-to-one relationship between price and demand intensity remains under the responsibility of the user, and is not a risk-free task.

In our opinion, the pricing models are simply tools to help better understand what dynamic pricing is rather than something to solve real-life problems.

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