Preface

Max-algebra provides mathematical theory and techniques for solving nonlinear problems that can be given the form of linear problems, when arithmetical addition is replaced by the operation of maximum and arithmetical multiplication is replaced by addition. Problems of this kind are sometimes of a managerial nature, arising in areas such as manufacturing, transportation, allocation of resources and information processing technology.

The aim of this book is to present max-algebra as a modern modelling and solution tool. The first five chapters provide the fundamentals of max-algebra, focusing on one-sided max-linear systems, the eigenvalue-eigenvector problem and maxpolynomials. The theory is self-contained and covers both irreducible and reducible matrices. Advanced material is presented from Chap. 6 onwards.

The book is intended for a wide-ranging readership, from undergraduate and postgraduate students to researchers and mathematicians working in industry, commerce or management. No prior knowledge of max-algebra is assumed. We concentrate on linear-algebraic aspects, presenting both classical and new results. Most of the theory is illustrated by numerical examples and complemented by exercises at the end of every chapter.

Chapter 1 presents essential definitions, examples and basic results used throughout the book. It also introduces key max-algebraic tools: the maximum cycle mean, transitive closures, conjugation and the assignment problem, and presents their basic properties and corresponding algorithms. Section 1.3 introduces applications which were the main motivation for this book and towards which it is aimed: feasibility and reachability in multi-machine interactive processes. Many results in Chaps. 6–10 find their use in solving feasibility and reachability problems.

Chapter 2 has a specific aim: to explain two special features of max-algebra particularly useful for its applications. The first is the possibility of efficiently describing the set of *all* solutions to a problem which may otherwise be awkward or even impossible to do. This methodology may be used to find solutions satisfying further requirements. The second feature is the ability of max-algebra to describe a class of problems in combinatorics or combinatorial optimization in algebraic terms. This chapter may be skipped without loss of continuity whilst reading the book.

Most of Chap. 3 contains material on one-sided systems and the geometry of subspaces. It is presented here in full generality with all the proofs. The main results are: a straightforward way of solving one-sided systems of equations and inequalities both algebraically and combinatorially, characterization of bases of max-algebraic subspaces and a proof that finitely generated max-algebraic subspaces have an essentially unique basis. Linear independence is a rather tricky concept in max-algebra and presented dimensional anomalies illustrate the difficulties. Advanced material on linear independence can be found in Chap. 6.

Chapter 4 presents the max-algebraic eigenproblem. It contains probably the first book publication of the complete solution to this problem, that is, characterization and efficient methods for finding all eigenvalues and describing all eigenvectors for any square matrix over $\mathbb{R}\cup\{-\infty\}$ with all the necessary proofs.

The question of factorization of max-algebraic polynomials (briefly, maxpolynomials) is easier than in conventional linear algebra, and it is studied in Chap. 5. A related topic is that of characteristic maxpolynomials, which are linked to the job rotation problem. A classical proof is presented showing that similarly to conventional linear algebra the greatest corner is equal to the principal eigenvalue. The complexity of finding all coefficients of a characteristic maxpolynomial still seems to be an unresolved problem but a polynomial algorithm is presented for finding all essential coefficients.

Chapter 6 provides a unifying overview of the results published in various research papers on linear independence and simple image sets. It is proved that three types of regularity of matrices can be checked in $O(n^3)$ time. Two of them, strong regularity and Gondran–Minoux regularity, are substantially linked to the assignment problem. The chapter includes an application of Gondran–Minoux regularity to the minimal-dimensional realization problem for discrete-event dynamic systems.

Unlike in conventional linear algebra, two-sided max-linear systems are substantially harder to solve than their one-sided counterparts. An account of the existing methodology for solving two-sided systems (homogenous, nonhomogenous, or with separated variables) is given in Chap. 7. The core ideas are those of the Alternating Method and symmetrized semirings. This chapter is concluded by the proof of a result of fundamental theoretical importance, namely that the solution set to a two-sided system is finitely generated.

Following the complete resolution of the eigenproblem, Chap. 8 deals with the problem of reachability of eigenspaces by matrix orbits. First it is shown how matrix scaling can be useful in visualizing spectral properties of matrices. This is followed by presenting the classical theory of the periodic behavior of matrices in max-algebra and then it is shown how the reachability question for irreducible matrices can be answered in polynomial time. Matrices whose orbit from every starting vector reaches an eigenvector are called robust. An efficient characterization of robustness for both irreducible and reducible matrices is presented.

The generalized eigenproblem is a relatively new and hard area of research. Existing methodology is restricted to a few solvability conditions, a number of solvable special cases and an algorithm for narrowing the search for generalized eigenvalues. An account of these results can be found in Chap. 9. Almost all of Sect. 9.3 is original research never published before.

Chapter 10 presents theory and algorithms for solving max-linear programs subject to one or two-sided max-linear constraints (both minimization and maximization). The emphasis is on the two-sided case. We present criteria for the objective function to be bounded and we prove that the bounds are always attained, if they exist. Finally, bisection methods for localizing the optimal value with a given precision are presented. For programs with integer entries these methods turn out to be exact, of pseudopolynomial computational complexity.

The last chapter contains a brief summary of the book and a list of open problems.

In a text of this size, it would be impossible to give a fully comprehensive account of max-algebra. In particular this book does not cover (or does so only marginally) control, discrete-event systems, stochastic systems or case studies; material related to these topics may be found in e.g. [8, 102] and [112]. On the other hand, max-algebra as presented in this book provides the linear-algebraic background to the rapidly developing field of tropical mathematics.

This book is the result of many years of my work in max-algebra. Throughout the years I worked with many colleagues but I would like to highlight my collaboration with Ray Cuninghame-Green, with whom I was privileged to work for almost a quarter of a century and whose mathematical style and elegance I will always admire. Without Ray's encouragement, for which I am extremely grateful, this book would never exist. I am also indebted to Hans Schneider, with whom I worked in recent years, for his advice which played an important role in the preparation of this book. His vast knowledge of linear algebra made it possible to solve a number of problems in max-algebra.

I would like to express gratitude to my teachers, in particular to Ernest Jucovič for his vision and leadership, and to Karel Zimmermann, who in 1974 introduced me to max-algebra and to Miroslav Fiedler who introduced me to numerical linear algebra.

Sections 8.3–8.5 of this book have been prepared in collaboration with my research fellow Sergeĭ Sergeev, whose enthusiasm for max-algebra and achievement of several groundbreaking results in a short span of time make him one of the most promising researchers of his generation. His comments on various parts of the book have helped me to improve the presentation.

Numerical examples and exercises have been checked by my students Abdulhadi Aminu, Kin Po Tam and Vikram Dokka. I am of course taking full responsibility for any outstanding errors or omissions.

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