

2 Modern Stochastic Process Methods for Multi-state System Reliability Assessment

The purpose of this chapter is to describe basic concepts of applying a random process theory to MSS reliability assessment. Here, we do not present the basics of the measure-theoretic framework that are necessary to pure mathematicians. Readers who need this fundamental framework and a more detailed presentation on stochastic processes can find it in Kallenberg (2002), Karlin and Taylor (1981) and Ross (1995). For reliability engineers and analysts, the books of Trivedi (2002), Epstein and Weissman (2008), Aven and Jensen (1999), and Lisnianski and Levitin (2003) are especially recommended. A great impact to stochastic processes application to MSS reliability evaluation was done by Natvig (1985) and Natvig *et al.* (1985).

In this chapter, the MSS system reliability models will be consequently studied based on Markov processes; Markov rewards processes, and semi-Markov processes. The Markov processes are widely used for reliability analysis because the number of failures in arbitrary time intervals in many practical cases can be described as a Poisson process and the time up to the failure and repair time are often exponentially distributed. This chapter presents a detailed description of a discrete-time Markov chain as well as a continuous-time Markov chain in order to provide for readers a basic understanding of the theory and its engineering applications. It will be shown how by using the Markov process theory MSS reliability measures can be determined. It will also be shown how such MSS reliability measures as the mean time to failure, mean number of failures in a time interval, and mean sojourn time in a set of unacceptable states can be found using the Markov reward models. These models are also the basis for reliability-associated cost assessment and life-cycle cost analysis. In practice, basic assumptions about exponential distributions of times between failures and repair times sometimes do not hold. In this case, more complicated mathematical techniques such as semi-Markov processes and embedded Markov chains may be applied. Corresponding issues are also considered in this chapter.

2.1 General Concepts of Stochastic Process Theory

A stochastic or random process is, essentially, a set of random variables where the variables are ordered in a given sequence. For example, the daily maximum temperatures at a weather station form a sequence of random variables, and this ordered sequence can be considered as a stochastic process. Another example is the sequence formed by the continuously changing number of people waiting in line at the ticket window of a railway station.

More formally, the sequence of random variables in a process can be denoted by $X(t)$, where t is the index of the process.

The values assumed by the random variable $X(t)$ are called states, and the set of all possible values forms the state space of the process. So, a stochastic process is a sequence of random variables $\{X(t) | t \in T\}$, defined on a given probability space, indexed by the parameter t , where t varies over an index set T . In this book, we mainly deal with stochastic processes where t represents time.

A random variable X can be considered as the rule for assigning to every outcome ζ of an experiment the value $X(\zeta)$. A stochastic process is a rule for assigning to every ζ the function $X(t, \zeta)$. Thus, a stochastic process is a family of time functions depending on the parameter ζ or, equivalently, a function of t and ζ . The domain of ζ is the set of all the possible experimental outcomes and the domain of t is a set of non-negative real numbers.

For example, the instantaneous speed of a car movement during its trip from point A to point B will be a stochastic process. The speed on each trip can be considered as an experimental outcome ζ , and each trip will have its own speed $X(t, \zeta)$ that characterizes for this case an instantaneous speed of the trip as a function of time. This function will be different from such functions of other trips because of the influence of many random factors (such as wind, broad conditions etc.). In Figure 2.1 one can see three different speed functions for three trips that can be treated as three different realizations of the stochastic process. It should be noticed that the cut of this stochastic process at time instant t_1 will represent the random variable with mean V_m . In real-world systems many parameters such as temperature, voltage, frequency, etc. may be considered stochastic processes.

The time may be discrete or continuous. A discrete time may have a finite or infinite number of values; continuous time obviously has only an infinite number of values. The values taken by the random variables constitute the state space. This state space, in its turn, may be discrete or continuous. Therefore, stochastic processes may be classified into four categories according to whether their state spaces and time are continuous or discrete. If the state space of a stochastic process is discrete, then it is called a discrete-state process, often referred to as a chain.

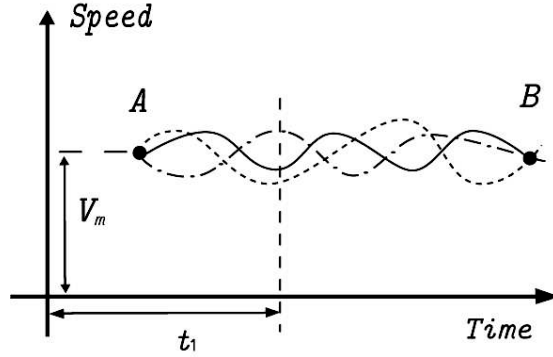


Fig. 2.1 Three realizations of stochastic process $V(t)$

The stochastic process $X(t, \zeta)$ has the following interpretations:

1. It is a family of functions $X(t, \zeta)$, where t and ζ are variables.
2. It is a single time function or a realization (sample) of the given process if t is a variable and ζ is fixed.
3. It is a random variable equal to the state of the given process at time t when t is fixed and ζ is variable.
4. It is a number if t and ζ are fixed.

One can use the notation $X(t)$ to represent a stochastic process omitting, as in the case of random variables, its dependence on ζ .

For a fixed time $t = t_1$, the term $X(t_1)$ is a simple random variable that describes the state of the process at time t_1 . For a fixed number x_1 , the probability of the event $X(t_1) \leq x_1$ gives the CDF of the random variable $X(t_1)$, denoted by

$$F(x_1; t_1) = F_{X(t_1)}(x_1) = \Pr\{X(t_1) \leq x_1\}. \quad (2.1)$$

CDF $F(x_1; t_1)$ is called the first-order distribution of the stochastic process $\{X(t) | t \geq 0\}$. Given two time instants t_1 and t_2 , $X(t_1)$ and $X(t_2)$ are two random variables in the same probability space. Their joint distribution is known as the second-order distribution of the process and is given by

$$F(x_1, x_2; t_1, t_2) = F_{X(t_1)X(t_2)}(x_1, x_2) = \Pr\{X(t_1) \leq x_1, X(t_2) \leq x_2\}. \quad (2.2)$$

In general, the n th-order joint distribution of the stochastic process $\{X(t) | t \geq 0\}$ is defined by

$$\begin{aligned}
F(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) &= F_{X(t_1)X(t_2)\dots X(t_n)}(x_1, x_2, \dots, x_n) = \\
&= \Pr\{X(t_1) \leq x_1, X(t_2) \leq x_2, \dots, X(t_n) \leq x_n\}
\end{aligned} \tag{2.3}$$

for all $t_1 < t_2 < \dots < t_n$.

The last formula represents a complete description of a stochastic process. In practice, to get such a complete description of a stochastic process is a very difficult task. Fortunately, in practice many stochastic processes permit a simpler description.

The simplest form of the joint distribution corresponds to a family of independent random variables. Then the joint distribution is given by the product of individual distributions.

Definition 2.1 A stochastic process $\{X(t) | t \geq 0\}$ is said to be an independent process if its n th-order joint distribution satisfies the condition

$$F(x_1, x_2, \dots, x_n; t_1, t_2, \dots, t_n) = \prod_{i=1}^n F(x_i; t_i) = \prod_{i=1}^n \Pr\{X(t_i) \leq x_i\}. \tag{2.4}$$

The assumption of an independent process considerably simplifies analysis, but it is often unwarranted and we are forced to consider some kind of dependence. The simplest and a very important type of dependence is the first-order dependence or Markov dependence.

Definition 2.2 A stochastic process $\{X(t) | t \geq 0\}$ is called a Markov process if for any $t_0 < t_1 < t_2 < \dots < t_{n-1} < t_n < t$ the conditional distribution of $X(t)$ for given values of $X(t_0), X(t_1), \dots, X(t_n)$ depends only on $X(t_n)$:

$$\begin{aligned}
\Pr\{X(t) \leq x | X(t_n) = x_n, X(t_{n-1}) = x_{n-1}, \dots, X(t_1) = x_1, X(t_0) = x_0\} &= \\
= \Pr\{X(t) \leq x | X(t_n) = x_n\}.
\end{aligned} \tag{2.5}$$

This is a general definition, which applies to Markov processes with a continuous-state space. When MSS reliability is studied, discrete-state Markov processes or Markov chains are mostly involved. In the next sections we will study both discrete-time and continuous-time Markov chains.

In the Markov process, the probabilities of the random variable at time $t > t_n$ depend on the value of the random variable at t_n but not on the realization of the process prior to t_n . In other words, the state probabilities at a future instant, given the present state of the process, do not depend on the states occupied in the past. Therefore, this process is also called “memoryless.”

In many cases the conditional distribution (2.5) has the property of invariance with respect to the time origin t_n :

$$\Pr\{X(t) \leq x \mid X(t_n) = x_n\} = \Pr\{X(t - t_n) \leq x \mid X(0) = x_n\}. \quad (2.6)$$

Such a Markov process is said to be homogeneous.

In addition we consider here two important stochastic processes that will be used in the future: point and renewal processes.

A *point process* is a set of random points t_i on the time axis. For each point process one can associate a stochastic process $X(t)$ equal to the number of points t_i in the interval $(0, t)$. In reliability theory point processes are widely used to describe the appearance of events in time (e.g., failures, terminations of repair, etc.).

An example of the point processes is the so-called *Poisson process*. The Poisson process is usually introduced using Poisson points. These points are associated with certain events, and the number $N(t_1, t_2)$ of the points in an interval (t_1, t_2) of length $t = t_2 - t_1$ is a Poisson random variable with parameter λt , where λ is the mean occurrence rate of the events:

$$\Pr\{N(t_1, t_2) = k\} = \frac{e^{-\lambda t} (\lambda t)^k}{k!}. \quad (2.7)$$

If the intervals (t_1, t_2) and (t_3, t_4) are not overlapping, then the random variables $N(t_1, t_2)$ and $N(t_3, t_4)$ are independent. Using the points t_i one can form the stochastic process $X(t) = N(0, t)$.

The *Poisson process* plays a special role in reliability analysis, comparable to the role of the normal distribution in probability theory. Many real physical situations can be successfully described with the help of Poisson processes.

A well-known type of point process is the so-called *renewal process*. This process can be described as a sequence of events, the intervals between which are independent and identically distributed random variables. In reliability theory, this kind of mathematical model is used to describe the flow of failures in time.

To every point process t_i one can associate a sequence of random variables y_n such that $y_1 = t_1$, $y_2 = t_2 - t_1$, ..., $y_n = t_n - t_{n-1}$, where t_1 is the first random point to the right of the origin. This sequence is called a renewal process. An example is the life history of items that are replaced as soon as they fail. In this case, y_i is the total time the i th item is in operation and t_i is the time of its failure.

One can see a correspondence among the following three processes:

- a point process t_i ;
- a discrete-state stochastic process $X(t)$ increasing (or decreasing) by 1 at points t_i ; and

- a renewal process consisting of random variables y_i such that $t_n = y_1 + \dots + y_n$.

A generalization of this type of process is the so-called alternating renewal process. This process consists of two types of independent and identically distributed random variables alternating with each other in turn. This type of process is convenient for the description of repairable systems. For such systems, periods of successful operation alternate with periods of idle time.

2.2 Markov Models: Discrete-time Markov Chains

2.2.1 Basic Definitions and Properties

As was described above, a Markov process is a stochastic process whose dynamic behavior is such that the probability distribution for its future development depends only on the present state and not on how the process arrived at that state (Trivedi 2002). Generally Markov technique is very effective in many practical important cases (International Standard IEC 61165 2006).

When the state space, S , is discrete (finite or countably infinite), then the Markov process is known as a Markov chain. Since the state space is discrete and countable, we can assume without loss of generality that $S = \{0, 1, 2, 3, \dots\}$. If the parameter space, T (recall that we usually will consider time as the parameter), is discrete too, then we have a *discrete-time Markov chain*. Since the parameter space is discrete, we will let $T = \{0, 1, 2, 3, \dots\}$. Thus, a Markov chain $\{X(n), n = 0, 1, 2, \dots\}$ is described by a sequence of random variables $X(0) = x_0, X(1) = x_1, X(2) = x_2, \dots$, where x_0, x_1, x_2, \dots are integer numbers. If the state of the system at time step n is j , we denote it as $X(n) = j$. Then X_0 is the initial state of the system at time step 0. By using these designations in analogy with (2.5), the Markov property can be defined as

$$\begin{aligned} \Pr\{X_n = x_n \mid X_0 = x_0, X_1 = x_1, \dots, X_{n-1} = x_{n-1}\} = \\ = \Pr\{X_n = x_n \mid X_{n-1} = x_{n-1}\}. \end{aligned} \quad (2.8)$$

As in the case of a general Markov process, Equation 2.8 implies that chain behavior in the future depends only on its present state and does not depend on its behavior in the past.

We designate the probability that at step n the chain will be in state j as $p_j(n)$. Thus, we can write

$$p_j(n) = \Pr\{X_n = j\}. \quad (2.9)$$

We also define the probability $p_{ij}(m, n)$ that the chain makes a transition to state j at step n if at step m it was in state i . This probability is a conditional probability, and we can write the following

$$p_{ij}(m, n) = \Pr\{X(n) = j \mid X(m) = i\}, \quad 0 \leq m \leq n. \quad (2.10)$$

Conditional probability $p_{ij}(m, n)$ is known as the *transition probability function* of the Markov chain.

Here we will only consider *homogeneous Markov chains* – those in which $p_{ij}(m, n)$ depends only on difference $n-m$. For such chains, the simpler notation

$$p_{ij}(n) = \Pr\{X(m+n) = j \mid X(m) = i\}, \quad 0 \leq m \leq n \quad (2.11)$$

is usually used to denote so-called *n -step transition probabilities*. In words, $p_{ij}(n)$ is the probability that a homogeneous Markov chain will move from state i to state j in exactly n steps.

If $n = 1$, for homogeneous Markov chains we can also write

$$p_{ij}(1) = \Pr\{X(m+1) = j \mid X(m) = i\} = p_{ij} = \text{const.} \quad (2.12)$$

The probabilities p_{ij} are called one-step transition probabilities.

For engineering applications here we will consider only finite and countable state space $S = \{0, 1, 2, \dots, M\}$. The one-step transition probabilities can be condensed into a transition (one-step) probability matrix \mathbf{P} , where

$$\mathbf{P} = [p_{ij}] = \begin{bmatrix} p_{00} & p_{01} & \cdots & p_{0M} \\ p_{10} & p_{11} & \cdots & p_{1M} \\ \cdots & \cdots & \cdots & \cdots \\ p_{M0} & p_{M1} & \cdots & p_{MM} \end{bmatrix}. \quad (2.13)$$

Since for all $i, j \in S$, $0 \leq p_{ij} \leq 1$, and each row in \mathbf{P} adds up to 1, matrix \mathbf{P} is a *stochastic matrix*.

The probability mass function of the random value $X(0)$ is called the initial probability row-vector

$$\mathbf{p}(0) = [p_0(0), p_1(0), \dots, p_M(0)] \quad (2.14)$$

and presents the initial conditions of a Markov chain.

An equivalent description of the Markov chain can be given by a directed graph called the state-transition diagram (or state diagram for short) of the Markov chain. A node labeled i of the state diagram represents state i of the Markov chain and a branch labeled p_{ij} from node i to j represents the corresponding one-step transition probability from state i to state j .

2.2.2 Computation of n -step Transition Probabilities and State Probabilities

The problem being considered here is in obtaining an expression for evaluating the n -step transition probability $p_{ij}(n)$ from the one-step transition probabilities $p_{ij} = p_{ij}(1)$. Recall that for a homogeneous Markov chain according to expression (2.11) we have the following:

$$p_{ij}(n) = \Pr\{X(m+n) = j \mid X(m) = i\}, \quad 0 \leq m \leq n.$$

Let us consider the transition probability $p_{ij}(m+n)$ that the process goes to state j at the $(m+n)$ step, given that at 0 step it is in state i . In order to reach state j at the $(m+n)$ step the process first reaches some intermediate state k at step m with probability $p_{ik}(m)$ and then moves from k and reaches j at step $(m+n)$ with probability $p_{kj}(n)$. The Markov property implies that there are two independent events. Then using the theorem of total probability we obtain

$$p_{ij}(m+n) = \sum_k p_{ik}(m) p_{kj}(n). \quad (2.15)$$

Equation 2.15 is one form of the widely known Chapman–Kolmogorov equation and provides efficient calculation of the n -step transitions probabilities.

We designate as $\mathbf{P}(n)$ the matrix of n -step probabilities or, in other words, the matrix whose (i, j) entry is $p_{ij}(n)$. Then, if in (2.15) we let $m=1$ and replace n by $n-1$, we can rewrite Equation 2.15 in matrix form:

$$\mathbf{P}(n) = \mathbf{P} \cdot \mathbf{P}(n-1) = \mathbf{P}^n, \quad (2.16)$$

where \mathbf{P} is the one-step probabilities of the Markov chain.

In words, the n -step transition probability matrix is the n th power of the one-step transition probability matrix.

Based on the obtained results the unconditional state probabilities $p_j(n)$ can be examined. Their values depend on the initial state probabilities at $n = 0$ and on the number of steps passed since $n = 0$. It can be written as follows:

$$\begin{aligned} p_j(n) &= \Pr\{X(n) = j\} = \\ &= \sum_i \Pr(X(0) = i) \Pr(X(n) = j \mid X(0) = i) = \sum_i p_i(0) p_{ij}(n). \end{aligned} \quad (2.17)$$

In matrix form expression (2.17) can be rewritten as

$$\mathbf{p}(n) = \mathbf{p}(0) \cdot \mathbf{P}^n, \quad (2.18)$$

where $\mathbf{p}(0)$ and $\mathbf{p}(n)$ are the row-vectors of the state probabilities initially (at step $n = 0$) and after the n th' step, respectively.

This implies that unconditional state probabilities of a homogeneous Markov chain are completely determined by the one-step transition probability matrix \mathbf{P} and the initial probability vector $\mathbf{p}(0)$.

To illustrate the presented approach, we consider the following example.

Example 2.1 (Bhat and Miller 2002). Assume a two-state Markov chain with the states denoted by 0 and 1 (Figure 2.2).

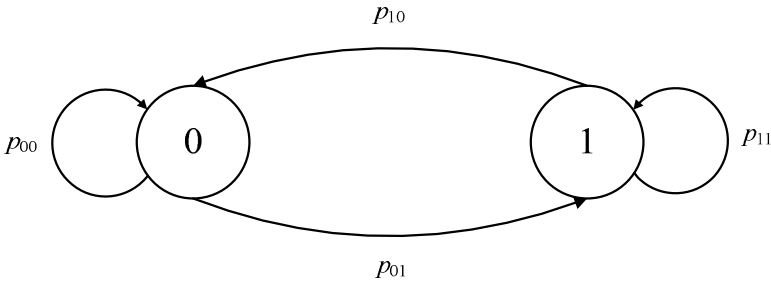


Fig. 2.2 Two-state discrete-time Markov chain

The one-step transition probability matrix will be as follow

$$\mathbf{P} = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix} = \begin{bmatrix} 1 - p_{01} & p_{01} \\ p_{10} & 1 - p_{10} \end{bmatrix},$$

since it must hold that $p_{00} + p_{01} = 1$ and $p_{10} + p_{11} = 1$. Assume that $p_{01} = \alpha$ and $p_{10} = \beta$. Then

$$\mathbf{P} = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}.$$

The initial conditions at step $n=0$ are the following $p_0(0) = a$ and $p_1(0) = 1-a$, so the initial state probability row-vector is given as

$$\mathbf{p}(0) = [p_0(0), p_1(0)] = [a, 1-a].$$

Find the n -step transition probability matrix and unconditional probabilities $p_0(n)$ and $p_1(n)$ of states 0 and 1 at step n , respectively.

Solution. According to the given one-step transition probability matrix \mathbf{P} we can write

$$\begin{aligned} p_{00}(1) &= p_{00} = 1-\alpha, & p_{01}(1) &= p_{01} = \alpha, \\ p_{10}(1) &= p_{10} = \beta, & p_{11}(1) &= p_{11} = 1-\beta. \end{aligned}$$

For $n > 1$, using Equation 2.16, we obtain

$$\begin{aligned} p_{00}(n) &= p_{00}(1)p_{00}(n-1) + p_{10}(1)p_{01}(n-1) = \\ &= (1-\alpha)p_{00}(n-1) + \beta p_{01}(n-1). \end{aligned}$$

Now since the row sums of matrix \mathbf{P}^{n-1} are unity, we have

$$p_{01}(n-1) = 1 - p_{00}(n-1).$$

Substituting $p_{01}(n-1)$ into the previous equation we obtain for $n > 1$

$$p_{00}(n) = (1-\alpha)p_{00}(n-1) + \beta[1 - p_{00}(n-1)] = \beta + (1-\alpha-\beta)p_{00}(n-1).$$

By using the last recurrent equation we can write the following:

$$\begin{aligned}
p_{00}(1) &= 1 - \alpha, \\
p_{00}(2) &= \beta + (1 - \alpha - \beta)(1 - \alpha), \\
p_{00}(3) &= \beta + \beta(1 - \alpha - \beta) + (1 - \alpha - \beta)^2(1 - \alpha), \\
&\dots \\
p_{00}(n) &= \beta + \beta(1 - \alpha - \beta) + \beta(1 - \alpha - \beta)^2 + \dots + \\
&\quad + \beta(1 - \alpha - \beta)^{n-2} + (1 - \alpha - \beta)^{n-1}(1 - \alpha) \\
&= \beta \left[\sum_{k=0}^{n-2} (1 - \alpha - \beta)^k \right] + (1 - \alpha - \beta)^{n-1}(1 - \alpha).
\end{aligned}$$

Based on the formula for the sum of a finite geometric series, we can write:

$$\sum_{k=0}^{n-2} (1 - \alpha - \beta)^k = \frac{1 - (1 - \alpha - \beta)^{n-1}}{1 - (1 - \alpha - \beta)} = \frac{1 - (1 - \alpha - \beta)^{n-1}}{\alpha + \beta}.$$

Therefore, the expression for $p_{00}(n)$ can be rewritten in the following form:

$$p_{00}(n) = \frac{\beta}{\alpha + \beta} + \frac{\alpha(1 - \alpha - \beta)^n}{\alpha + \beta}.$$

Now $p_{01}(n)$ can be found:

$$p_{01}(n) = 1 - p_{00}(n) = \frac{\alpha}{\alpha + \beta} - \frac{\alpha(1 - \alpha - \beta)^n}{\alpha + \beta}.$$

Expressions for the two remaining entries $p_{10}(n)$ and $p_{11}(n)$ can be found in a similar way. (Readers can do it themselves as an exercise.)

Thus, the n -step transition probability matrix can be written as

$$\mathbf{P}(n) = \mathbf{P}^n = \begin{bmatrix} \frac{\beta + \alpha(1 - \alpha - \beta)^n}{\alpha + \beta} & \frac{\alpha - \alpha(1 - \alpha - \beta)^n}{\alpha + \beta} \\ \frac{\beta - \beta(1 - \alpha - \beta)^n}{\alpha + \beta} & \frac{\alpha + \beta(1 - \alpha - \beta)^n}{\alpha + \beta} \end{bmatrix}.$$

Based on this n -step transition probability matrix and on the given initial state probability row-vector $\mathbf{p}(0)$, one can find state probabilities after the n th step by using Equation 2.18

$$\begin{aligned}
\mathbf{p}(n) &= \mathbf{p}(0) \mathbf{P}^n = [a, 1-a] \times \begin{bmatrix} \frac{\beta + \alpha(1-\alpha-\beta)^n}{\alpha + \beta} & \frac{\alpha - \alpha(1-\alpha-\beta)^n}{\alpha + \beta} \\ \frac{\beta - \beta(1-\alpha-\beta)^n}{\alpha + \beta} & \frac{\alpha + \beta(1-\alpha-\beta)^n}{\alpha + \beta} \end{bmatrix} \\
&= \left[\frac{\beta + (1-\alpha-\beta)^n}{\alpha + \beta} [a(\alpha + \beta) - \beta], \quad \frac{\alpha - (1-\alpha-\beta)^n}{\alpha + \beta} [a(\alpha + \beta) - \beta] \right].
\end{aligned}$$

Therefore, the state probabilities after the n th step are as follows:

$$\begin{aligned}
p_0(n) &= \frac{\beta + (1-\alpha-\beta)^n}{\alpha + \beta} [a(\alpha + \beta) - \beta], \\
p_1(n) &= \frac{\alpha - (1-\alpha-\beta)^n}{\alpha + \beta} [a(\alpha + \beta) - \beta].
\end{aligned}$$

2.3 Markov Models: Continuous-time Markov Chains

2.3.1 Basic Definitions and Properties

As in the previous section we confine our attention to discrete-state Markov stochastic processes or Markov chains. The continuous-time Markov chain is similar to that of the discrete-time case, except that the transitions from any given state to another state can take place at any instant of time. Therefore, for a discrete-state continuous-time Markov chain the set of values $X(t)$ is discrete, $X(t) \in \{1, 2, \dots\}$, and parameter t has a continuous range of values, $t \in [0, \infty)$. In reliability applications the set S of states is usually finite, $S = \{1, 2, \dots, K\}$, and so $X(t) \in \{1, 2, \dots, K\}$.

A discrete-state continuous-time stochastic process $\{X(t) | t \geq 0\}$ is called a Markov chain if for $t_0 < t_1 < \dots < t_{n-1} < t_n$ its conditional probability mass function satisfies the relation

$$\begin{aligned}
&\Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}, \dots, X(t_1) = x_1, X(t_0) = x_0\} \\
&= \Pr\{X(t_n) = x_n \mid X(t_{n-1}) = x_{n-1}\}.
\end{aligned} \tag{2.19}$$

Introducing the notations $t = t_{n-1}$ and $t_n = t_{n-1} + \Delta t$ the expression (2.19) simplifies to

$$\Pr\{X(t + \Delta t) = i \mid X(t) = j\} = \pi_{ji}(t, t + \Delta t). \quad (2.20)$$

The following designation is often used for the simplification:

$$\pi_{ji}(t, t + \Delta t) = \pi_{ji}(t, \Delta t).$$

These conditional probabilities are called *transition probabilities*. If the probabilities $\pi_{ji}(t, \Delta t)$ do not depend on t , but only on the time difference Δt , the Markov process is said to be (*time-*) *homogeneous*. $\pi_{ji}(t, \Delta t)$ is the probability that no change in the state will occur in a time interval of length Δt given that the process is in state j at the beginning of the interval. Note that

$$\pi_{ji}(t, t) = \begin{cases} 1, & \text{if } j = i, \\ 0, & \text{otherwise.} \end{cases} \quad (2.21)$$

Taking into account (2.21) one can define for each j a non-negative continuous function $a_j(t)$:

$$a_j(t) = \lim_{\Delta t \rightarrow 0} \frac{\pi_{jj}(t, t) - \pi_{jj}(t, t + \Delta t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{1 - \pi_{jj}(t, t + \Delta t)}{\Delta t} \quad (2.22)$$

and for each j and $i \neq j$ a non-negative continuous function $a_{ji}(t)$:

$$a_{ji}(t) = \lim_{\Delta t \rightarrow 0} \frac{\pi_{ji}(t, t) - \pi_{ji}(t, t + \Delta t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \frac{\pi_{ji}(t, t + \Delta t)}{\Delta t}. \quad (2.23)$$

The function $a_{ji}(t)$ is called the *transition intensity* from state i to state j at time t . For homogeneous Markov processes, the transition intensities do not depend on t and therefore are constant.

If the process is in state j at a given moment, in the next Δt time interval there is either a transition from j to some state i or the process remains at j . Therefore

$$\pi_{jj}(\Delta t) + \sum_{i \neq j} \pi_{ji}(\Delta t) = 1. \quad (2.24)$$

Designating $a_{jj} = -a_j$ and combining (2.24) with (2.22) one obtains

$$a_{jj} = -a_j = \lim_{\Delta t \rightarrow 0} -\frac{1}{\Delta t} \sum_{i \neq j} \pi_{ji}(\Delta t) = -\sum_{i \neq j} a_{ji}. \quad (2.25)$$

Let $p_i(t)$ be the state probabilities of $X(t)$ at time t :

$$p_i(t) = \Pr\{X(t) = i\}, \quad j = 1, \dots, K; \quad t \geq 0. \quad (2.26)$$

Expression (2.26) defines the probability mass function (pmf) of $X(t)$ at time t . Since at any given time the process must be in one of K states,

$$\sum_{i=1}^K p_i(t) = 1 \quad (2.27)$$

for any $t \geq 0$.

By using the theorem of total probability, for given $t > t_1$, we can express the pmf of $X(t)$ in terms of the transition probabilities $\pi_{ij}(t_1, t)$ and the pmf of $X(t_1)$:

$$\begin{aligned} p_j(t) &= \Pr(X(t) = j) = \sum_{i \in S} \Pr\{X(t) = j \mid X(t_1) = i\} \Pr\{X(t_1) = i\} \\ &= \sum_{i \in S} \pi_{ij}(t_1, t) p_i(t_1). \end{aligned} \quad (2.28)$$

If we let $t_1 = 0$ in (2.28), we obtain the following equation:

$$p_j(t) = \sum_{i \in S} \pi_{ij}(0, t) p_i(0). \quad (2.29)$$

This means that the probabilistic behavior of a continuous-time Markov chain in the future is completely determined by the transition probabilities $\pi_{ij}(0, t)$ and the initial probability vector $\mathbf{p}(0) = [p_1(0), \dots, p_K(0)]$.

The transition probabilities of a continuous-time Markov chain $\{X(t) \mid t \geq 0\}$ satisfy for all $i, j \in S$, the Chapman–Kolmogorov equation, which can be written for this case in the following form:

$$\pi_{ij}(t_1, t) = \sum_{k \in S} \pi_{ik}(t_1, t_2) \pi_{kj}(t_2, t), \quad 0 \leq t_1 < t_2 \leq t. \quad (2.30)$$

The proof of this equation is based on the theorem of total probability:

$$\begin{aligned} \Pr\{X(t) = j \mid X(t_1) = i\} \\ = \sum_{k \in S} \Pr\{X(t) = j \mid X(t_2) = k, X(t_1) = i\} \Pr\{X(t_2) = k \mid X(t_1) = i\}. \end{aligned} \quad (2.31)$$

The subsequent application of the Markov property (2.20) to expression (2.31) yields (2.30).

The state probabilities at instant $t + \Delta t$ can be expressed based on state probabilities at instant t by using the following equations:

$$p_j(t + \Delta t) = p_j(t) \left[1 - \sum_{i \neq j} a_{ji} \Delta t \right] + \sum_{i \neq j} p_i(t) a_{ij} \Delta t, \quad i, j = 1, \dots, K. \quad (2.32)$$

Equation 2.32 can be obtained by using the following considerations.

The process can achieve state j at instant $t + \Delta t$ in two ways.

1. The process may already be in state j at instant t and does not leave this state up to the instant $t + \Delta t$. These events have probabilities $p_j(t)$ and $1 - \sum_{i \neq j} a_{ji} \Delta t$, respectively.
2. At instant t the process may be in one of the states $i \neq j$ and during time Δt transits from state i to state j . These events have probabilities $p_i(t)$ and $a_{ij} \Delta t$, respectively. These probabilities should be multiplied and summarized for all $i \neq j$ because the process can achieve state j from any state i .

Now one can rewrite (2.32) by using (2.29) and obtain the following:

$$p_j(t + \Delta t) = p_j(t) [1 + a_{jj} \Delta t] + \sum_{i \neq j} p_i(t) a_{ij} \Delta t \quad (2.33)$$

or

$$\begin{aligned} p_j(t + \Delta t) - p_j(t) \\ = \sum_{i=1}^K p_i(t) a_{ij} \Delta t + p_j(t) a_{jj} \Delta t = \sum_{i=1}^K p_i(t) a_{ij} \Delta t - p_j(t) \sum_{i=1, i \neq j}^K a_{ji} \Delta t. \end{aligned} \quad (2.34)$$

After dividing both sides of Equation 2.34 by Δt and passing to limit $\Delta t \rightarrow 0$, we get

$$\frac{dp_j(t)}{dt} = \sum_{i=1, i \neq j}^K p_i(t) a_{ij} - p_j(t) \sum_{i=1, i \neq j}^K a_{ji}, \quad j = 1, 2, \dots, K. \quad (2.35)$$

The system of differential equations (2.35) is used for finding the state probabilities $p_j(t)$, $j = 1, \dots, K$ for the homogeneous Markov process when the initial conditions are given as

$$p_j(t) = \alpha_j, \quad j = 1, \dots, K. \quad (2.36)$$

More mathematical details about (2.35) may be found in Trivedi (2002) or in Ross (1995).

When a state-transition diagram for continuous-time Markov chain is built, Equation 2.35 can be written by using the following rule: the time derivative of $p_j(t)$ for any arbitrary state j equals the sum of the probabilities of the states that have transitions to state j multiplied by the corresponding transition intensities minus the probability of state j multiplied by the sum of the intensities of all transitions from state j .

Introducing the row-vector $\mathbf{p}(t) = [p_1(t), p_2(t), \dots, p_K(t)]$ and the transition intensity matrix \mathbf{a}

$$\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1K} \\ a_{21} & a_{22} & \dots & a_{2K} \\ \dots & \dots & \dots & \dots \\ a_{K1} & a_{K2} & \dots & a_{KK} \end{bmatrix}, \quad (2.37)$$

in which the diagonal elements are defined as $a_{jj} = -a_j$, we can rewrite system (2.35) in matrix notation:

$$\frac{d\mathbf{p}(t)}{dt} = \mathbf{p}(t)\mathbf{a}. \quad (2.38)$$

Note that the sum of the matrix elements in each row equals 0: $\sum_{j=1}^K a_{ij} = 0$ for each i ($1 \leq i \leq K$).

When the system state transitions are caused by failures and repairs of its elements, the corresponding transition intensities are expressed by the element's failure and repair rates.

An element's failure rate $\lambda(t)$ is the instantaneous conditional density of the probability of failure of an initially operational element at time t given that the element has not failed up to time t . Briefly, one can say that $\lambda(t)$ is the time-to-failure conditional probability density function (pdf). It expresses a hazard of fail-

ure in time instant t under a condition where there was no failure up to time t . The failure rate of an element at time t is defined as

$$\lambda(t) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[\frac{F(t + \Delta t) - F(t)}{R(t)} \right] = \frac{f(t)}{R(t)}, \quad (2.39)$$

where $R(t) = 1 - F(t)$ is the reliability function of the element, $F(t)$ is the CDF of the time to failure of the element, and $f(t)$ is pdf of the time to failure of the element.

For homogeneous Markov processes the failure rate does not depend on t and can be expressed as

$$\lambda = MTTF^{-1}, \quad (2.40)$$

where $MTTF$ is the mean time to failure. Similarly, the repair rate $\mu(t)$ is the time-to-repair conditional pdf. For homogeneous Markov processes a repair rate does not depend on t and can be expressed as

$$\mu = MTTR^{-1}, \quad (2.41)$$

where $MTTR$ is the mean time to repair.

A state i is said to be an *absorbing state*; if once entered, the process is destined to remain in that state. A state j is said to be *reachable* from state i if for some $t > 0$, $\pi_{ij}(t) > 0$. A continuous-time Markov chain is said to be *irreducible* if every state is reachable from every other state.

In many applications, the long-run (final) or steady-state probabilities $p_i = \lim_{t \rightarrow \infty} p_i(t)$ are of interest. For an irreducible continuous-time Markov chain these limits always exist for every state $i \in S$,

$$p_i = \lim_{t \rightarrow \infty} p_i(t) = \lim_{t \rightarrow \infty} \pi_{ij}(t) = \lim_{t \rightarrow \infty} \pi_i(t) \quad (2.42)$$

and they are independent of the initial state $j \in S$. If the steady-state probabilities exist, the process is called *ergodic*. For the steady-state state probabilities, the computations become simpler. The set of differential equations (2.35) is reduced to a set of K algebraic linear equations because for the constant probabilities all time-derivatives are equal to zero, so $\frac{dp_i(t)}{dt} = 0$, $i = 1, \dots, K$.

Let the steady-state probabilities $p_i = \lim_{t \rightarrow \infty} p_i(t)$ exist. For this case in steady state, all derivatives of state probabilities on the left-hand side of (2.35) will be ze-

roes. So, in order to find the long-run probabilities the following system of algebraic linear equations should be solved:

$$0 = \sum_{\substack{i=1 \\ i \neq j}}^k p_i(t) a_{ij} - p_j(t) \sum_{\substack{i=1 \\ i \neq j}}^K a_{ji}, \quad j = 1, 2, \dots, K. \quad (2.43)$$

The K equations in (2.43) are not linearly independent (the determinant of the system is zero). An additional independent equation can be provided by the simple fact that the sum of the state probabilities is equal to 1 at any time:

$$\sum_{i=1}^K p_i = 1. \quad (2.44)$$

Thus, steady-state probabilities of ergodic continuous-time Markov chains can be found using expressions (2.43) and (2.44).

Now we consider additional important parameters of the process in steady state: state frequency and mean time of staying in state. The frequency f_i of state i is defined as the expected number of arrivals into this state per unit time. Usually the concept of frequency is associated with the long-term (steady-state) behavior of the process. In order to relate the frequency, probability, and mean time of staying in state i , we consider the system evolution in the state space as consisting of two alternating periods – the stays in i and the stays outside i . Thus, the process is represented by two states. Designate the mean duration of the stays in state i as \bar{T}_i and that of the stays outside i , \bar{T}_{oi} . The mean cycle time, \bar{T}_{ci} , is then

$$\bar{T}_{ci} = \bar{T}_i + \bar{T}_{oi}. \quad (2.45)$$

From the definition of the state frequency it follows that, in the long run, f_i equals the reciprocal of the mean cycle time

$$f_i = \frac{1}{\bar{T}_{ci}}. \quad (2.46)$$

Multiplying by \bar{T}_i both sides of Equation 2.46 one gets

$$\bar{T}_i f_i = \frac{\bar{T}_i}{\bar{T}_{ci}} = p_i. \quad (2.47)$$

Therefore

$$f_i = \frac{p_i}{T_i}. \quad (2.48)$$

This is a fundamental equation, which provides the relation between the three state parameters in the steady state.

Unconditional random value T_i is minimal from all random values T_{ij} that characterize the conditional random time of staying in state i if the transition is performed from state i to any state $j \neq i$:

$$T_i = \min\{T_{i1}, \dots, T_{ij}\}. \quad (2.49)$$

All conditional times T_{ij} are distributed exponentially with the following cumulative distribution functions $F_{ij}(T_{ij} \leq t) = 1 - e^{-a_{ij}t}$. All transitions from state i are independent and, therefore, the cumulative distribution function of unconditional time T_i of staying in state i can be computed as follows:

$$\begin{aligned} F_i(T_i \leq t) &= 1 - \Pr\{T_i > t\} = 1 - \prod_{j \neq i} \Pr\{T_{ij} > t\} \\ &= 1 - \prod_{j \neq i} [1 - F_{ij}(T_{ij} \leq t)] = 1 - \prod_{j \neq i} e^{-a_{ij}t} = 1 - e^{-\sum_{j \neq i} a_{ij}t}. \end{aligned} \quad (2.50)$$

This means that unconditional time T_i is distributed exponentially with parameter $a_i = \sum_j a_{ij}$, and the mean time of staying in state i is as follows:

$$\bar{T}_i = \frac{1}{\sum_{j \neq i} a_{ij}}. \quad (2.51)$$

Substituting \bar{T}_i in expression (2.48) we finally get

$$f_i = p_i \sum_{j \neq i} a_{ij}. \quad (2.52)$$

Once state probabilities, p_i or $p_i(t)$, have been computed, reliability measures are usually obtained as corresponding functionals of these probabilities.

2.3.2 Markov Models for Evaluating the Reliability of Multi-state Elements

According to the generic MSS model (Chapter 1), any system element j can have k_j different states corresponding to the performance rates, represented by the set $\mathbf{g}_j = \{g_{j1}, \dots, g_{jk_j}\}$. The current state of the element j and, therefore, the current value of the element performance rate $G_j(t)$ at any instant t are random variables. $G_j(t)$ takes values from \mathbf{g}_j : $G_j(t) \in \mathbf{g}_j$. Therefore, for the time interval $[0, T]$, where T is the MSS operation period, the performance rate of element j is defined as a stochastic process. Note that we consider only the Markov process where the state probabilities at a future instant do not depend on the states occupied in the past.

In this subsection, when we deal with a single multi-state element, we can omit index j for the designation of a set of the element's performance rates. Thus, this set is denoted as $\mathbf{g} = \{g_1, \dots, g_k\}$. We also assume that this set is ordered so that $g_{i+1} \geq g_i$ for any i .

The elements can be divided into two groups. Those elements that are observed only until they fail belong to the first group. These elements either cannot be repaired, or the repair is uneconomical, or only the life history up to the first failure is of interest. Those elements that are repaired upon failure and whose life histories consist of operating and repair periods belong to the second group. In the following subsections, both groups are discussed.

2.3.2.1 Non-repairable Multi-state Element

As mentioned above, the lifetime of a non-repairable element lasts until its first entrance into the subset of unacceptable states. In general, the acceptability of an element's state depends on the relation between the element's performance and the desired level of this performance (demand). The demand $W(t)$ is also a random process that takes discrete values from the set $w = \{w_1, \dots, w_M\}$. The desired relation between the system performance and the demand can be expressed by the acceptability function $F(G(t), W(t))$.

First consider a multi-state element with only minor failures defined as failures that cause element transition from state i to the adjacent state $i-1$. In other words, a minor failure causes minimal degradation of element performance. The state-space diagram for such an element is presented in Figure 2.3.

The element evolution in the state space is the only performance degradation that is characterized by the stochastic process $\{G(t) \mid t \geq 0\}$. The transition intensity for any transition from state i to state $i-1$ is $\lambda_{i,i-1}$, $i = 2, \dots, k$.

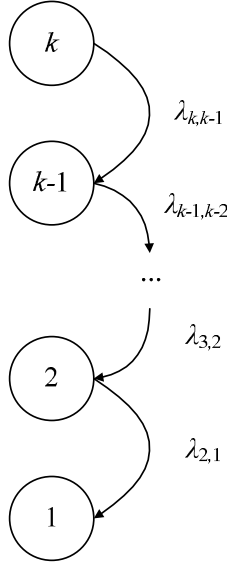


Fig. 2.3 State-transition diagram for non-repairable element with minor failures

When the sojourn time in any state i (or in other words, the time up to a minor failure in state i) is exponentially distributed with parameter $\lambda_{i,i-1}$, the process is a continuous-time Markov chain. Moreover, it is the widely known *pure death process* (Trivedi 2002). Let us define the auxiliary discrete-state continuous time stochastic process $\{X(t) | t \geq 0\}$, where $X(t) \in \{1, \dots, k\}$. This process is strictly associated with the stochastic process $\{G(t) | t \geq 0\}$. When $X(t) = i$, the corresponding performance rate of a multi-state element is $g_i : G(t) = g_i$. The process $X(t)$ is a discrete-state stochastic process decreasing by 1 at the points t_i , $i = 1, \dots, k$, when the corresponding transitions occur. The state probabilities of $X(t)$ are

$$p_i(t) = \Pr\{X(t) = i\}, \quad i = 1, \dots, k \quad \text{for } t \geq 0. \quad (2.53)$$

Note that

$$\sum_{i=1}^k p_i(t) = 1 \quad (2.54)$$

for any $t \geq 0$, since at any given time the process must be in some state.

According to the system (2.35), the following differential equations can be written in order to find state probabilities for the Markov process presented in Figure 2.3:

$$\begin{cases} \frac{dp_k(t)}{dt} = -\lambda_{k,k-1} p_k(t), \\ \frac{dp_i(t)}{dt} = \lambda_{i+1,i} p_{i+1}(t) - \lambda_{i,i-1} p_i(t), \quad i = 2, 3, \dots, k-1, \\ \frac{dp_1(t)}{dt} = \lambda_{2,1} p_2(t). \end{cases} \quad (2.55)$$

One can see that in state k there is only one transition from this state to the state $k-1$ with the intensity of $\lambda_{k,k-1}$ and there are no transitions to state k . In each state i , $i = 2, 3, \dots, k-1$, there is one transition to this state from the previous state $i+1$ with the intensity $\lambda_{i+1,i}$ and there is one transition from this state to state $i-1$ with the intensity $\lambda_{i,i-1}$. Observe that there are no transitions from state 1. This means that if the process enters this state, it is never left. State 1 for non-repairable multi-state elements is the absorbing state.

We assume that the process begins from the best state k with a maximal element performance rate of g_k . Hence, the initial conditions are

$$p_k(0) = 1, p_{k-1}(0) = p_{k-2}(0) = \dots = p_1(0) = 0. \quad (2.56)$$

Using widely available software tools, one can obtain the numerical solution of the system of differential equations (2.55) under initial conditions (2.56) even for large k . The system (2.55) can also be solved analytically using the Laplace–Stieltjes transform (Gnedenko and Ushakov 1995). Using this transform and taking into account the initial conditions (2.56) one can represent (2.55) in the form of linear algebraic equations:

$$\begin{cases} s\tilde{p}_k(s) - 1 = -\lambda_{k,k-1}\tilde{p}_k(s), \\ s\tilde{p}_i(s) = \lambda_{i+1,i}\tilde{p}_{i+1}(s) - \lambda_{i,i-1}\tilde{p}_i(s), \quad i = 2, 3, \dots, k-1, \\ s\tilde{p}_1(s) = \lambda_{2,1}\tilde{p}_2(s), \end{cases} \quad (2.57)$$

where $\tilde{p}_k(s) = L\{p_k(t)\} = \int_0^\infty e^{-st} p_k(t) dt$ is the Laplace–Stieltjes transform of a function $p_k(t)$ and $L\left\{\frac{dp_k(t)}{dt}\right\} = s\tilde{p}_k(s) - p_k(0)$ is the Laplace–Stieltjes transform of the derivative of a function $p_k(t)$.

The system (2.57) may be rewritten in the following form:

$$\begin{cases} \tilde{p}_k(s) = \frac{1}{s + \lambda_{k,k-1}}, \\ \tilde{p}_i(s) = \frac{\lambda_{i+1,i}}{s + \lambda_{i,k-1}} \tilde{p}_{i+1}(s), \quad i = 2, 3, \dots, k-1, \\ \tilde{p}_1(s) = \frac{\lambda_{2,1}}{s} \tilde{p}_2(s). \end{cases} \quad (2.58)$$

Starting to solve this system from the first equation and sequentially substituting the obtained results into the next equation, one obtains

$$\begin{cases} \tilde{p}_k(s) = \frac{1}{s + \lambda_{k,k-1}}, \\ \tilde{p}_i(s) = \frac{\lambda_{i+1,i}}{(s + \lambda_{i,i-1})} \frac{\lambda_{i+2,i+1}}{(s + \lambda_{i+1,i})} \dots \frac{\lambda_{k,k-1}}{(s + \lambda_{k-1,k-2})} \frac{1}{(s + \lambda_{k,k-1})}, \quad i = 2, 3, \dots, k-1, \\ \tilde{p}_1(s) = \frac{\lambda_{2,1}}{s} \frac{\lambda_{3,2}}{(s + \lambda_{2,1})} \frac{\lambda_{4,3}}{(s + \lambda_{3,2})} \dots \frac{\lambda_{k,k-1}}{(s + \lambda_{k-1,k-2})} \frac{1}{(s + \lambda_{k,k-1})}. \end{cases} \quad (2.59)$$

Now in order to find the functions $p_k(t)$, the inverse Laplace–Stieltjes transform $L^{-1}\{\tilde{p}_k(s)\} = p_k(t)$ should be applied (Korn and Korn 2000).

In the most common case when $F(g_i, w) = g_i - w$ (the element performance should not be less than the demand) for the constant demand level $g_{i+1} \geq w > g_i$ ($i = 1, \dots, k-1$) the acceptable states are the states $i+1, \dots, k$, where the element performance is above level g_i .

The probability of the state with the lowest performance $p_1(t)$ determines the unreliability function of the multi-state element for the constant demand level $g_2 \geq w > g_1$. Therefore, the reliability function defined as the probability that the element is not in its worst state (total failure) is

$$R_1(t) = 1 - p_1(t). \quad (2.60)$$

In general, if the constant demand is $g_{i+1} \geq w > g_i$, $i = 1, \dots, k-1$, the unreliability function for the multi-state element is a sum of the probabilities of the unacceptable states $1, 2, \dots, i$. Thus, the reliability function is

$$R_i(t) = 1 - \sum_{j=1}^i p_j(t). \quad (2.61)$$

The mean time up to multi-state element failure for this constant demand level can be interpreted as the mean time up to the process entering state i . It can be calculated as the sum of the time periods during which the process remains in each state $j > i$. Since the process begins from the best state k with the maximal element performance rate g_k [the initial conditions (2.56)], we have

$$MTTF_i = \sum_{j=i+1}^k \frac{1}{\lambda_{j,j-1}}, \quad i = 1, 2, \dots, k-1. \quad (2.62)$$

According to (1.23) one can obtain the element mean instantaneous performance at time t as

$$E_t = \sum_{i=1}^k g_i p_i(t). \quad (2.63)$$

The element mean instantaneous performance deficiency for the constant demand w according to (1.29) is

$$D_t = \sum_{i=1}^k p_i(t) \max(w - g_i, 0). \quad (2.64)$$

Example 2.2 We consider an electric generator installed in an airplane where its maintenance is impossible during flight. This generator assumed as a non-repairable multi-state element that can have only minor failures. The generator has 4 possible performance levels (in states 4, 3, 2, and 1 its capacities are $g_4 = 10$ KW, $g_3 = 8$ KW, $g_2 = 5$ KW and $g_1 = 0$, respectively) and the following failure rates: $\lambda_{4,3} = 2 \text{ year}^{-1}$, $\lambda_{3,2} = 1 \text{ year}^{-1}$, and $\lambda_{2,1} = 0.7 \text{ year}^{-1}$. The initial state is the best state 4.

Each flight duration is $T_{\text{flight}} = 10$ h. The airplane was designed for $N_{\text{flight}} = 50$ flights up to general maintenance on the ground. Thus, the service time up to the general maintenance is defined as $T_{\text{service}} = 500$ h. The failure is defined as decreasing of generating capacity down the demand level 6 KW.

Our objective is to find the expected energy not supplied to the airplane's consumers during the airplane service time, the probability that the failure occurs during the service time, and the mean time up to the failure.

Solution. In order to find state probabilities the following system of differential equations should be solved according to (2.55):

$$\begin{cases} \frac{dp_4(t)}{dt} = -\lambda_{4,3}p_4(t), \\ \frac{dp_3(t)}{dt} = \lambda_{4,3}p_4(t) - \lambda_{3,2}p_3(t), \\ \frac{dp_2(t)}{dt} = \lambda_{3,2}p_3(t) - \lambda_{2,1}p_2(t), \\ \frac{dp_1(t)}{dt} = \lambda_{2,1}p_2(t), \end{cases}$$

with the initial conditions $p_4(0) = 1$, $p_3(0) = p_2(0) = p_1(0) = 0$.

Using the Laplace–Stieltjes transform, we obtain

$$\begin{aligned} \tilde{p}_4(s) &= \frac{1}{s + \lambda_{4,3}}, \quad \tilde{p}_3(s) = \frac{\lambda_{4,3}}{(s + \lambda_{3,2})(s + \lambda_{4,3})}, \\ \tilde{p}_2(s) &= \frac{\lambda_{3,2}\lambda_{4,3}}{(s + \lambda_{2,1})(s + \lambda_{3,2})(s + \lambda_{4,3})}, \quad \tilde{p}_1(s) = \frac{\lambda_{2,1}\lambda_{3,2}\lambda_{4,3}}{s(s + \lambda_{2,1})(s + \lambda_{3,2})(s + \lambda_{4,3})}. \end{aligned}$$

Using the inverse Laplace–Stieltjes transform, we find the state probabilities as functions of time t :

$$\begin{aligned} p_4(t) &= e^{-\lambda_{4,3}t}, \\ p_3(t) &= \frac{\lambda_{4,3}}{\lambda_{4,3} - \lambda_{3,2}}(e^{-\lambda_{3,2}t} - e^{-\lambda_{4,3}t}), \\ p_2(t) &= \frac{\lambda_{3,2}\lambda_{4,3}[(\lambda_{4,3} - \lambda_{3,2})e^{-\lambda_{2,1}t} + (\lambda_{2,1} - \lambda_{4,3})e^{-\lambda_{3,2}t} + (\lambda_{3,2} - \lambda_{2,1})e^{-\lambda_{4,3}t}]}{(\lambda_{3,2} - \lambda_{2,1})(\lambda_{4,3} - \lambda_{3,2})(\lambda_{2,1} - \lambda_{4,3})}, \\ p_1(t) &= 1 - p_2(t) - p_3(t) - p_4(t). \end{aligned}$$

These probabilities are presented in Figure 2.4.

Now we can obtain the reliability measures for this multi-state element. The reliability functions for different demand levels are according to (2.61):

$$\begin{aligned} R_1(t) &= 1 - p_1(t), \text{ for } g_1 < w \leq g_2, \\ R_2(t) &= 1 - p_1(t) - p_2(t), \text{ for } g_2 < w \leq g_3, \\ R_3(t) &= 1 - p_1(t) - p_2(t) - p_3(t) = p_4(t), \text{ for } g_3 < w \leq g_4. \end{aligned}$$

These reliability functions are also presented in Figure 2.4.

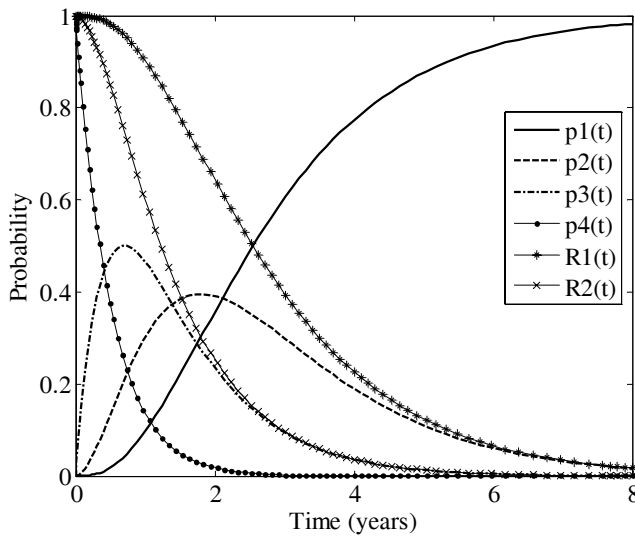


Fig. 2.4 State probabilities and reliability measures for non-repairable element with minor failures

According to (2.63) we obtain the element mean instantaneous performance at time t :

$$E_t = \sum_{i=1}^4 g_i p_i(t) = 10p_4(t) + 8p_3(t) + 5p_2(t) + 0p_1(t).$$

The demand is constant during the flight and $w = 6$ KW. Therefore, according to (2.64), the element mean instantaneous performance deficiency is

$$D_t = \sum_{i=1}^4 p_i(t) \max(w - g_i, 0) = 1p_2(t) + 6p_1(t).$$

Functions E_t and D_t are presented in the Figure 2.5.

Note that the expected energy not supplied ($EENS$) to the airplane consumers during the service time $T_{service} = 500$ h will be as follows:

$$EENS = \int_0^{T_{service}} D_t dt \approx 0.547 \text{ KWh}.$$

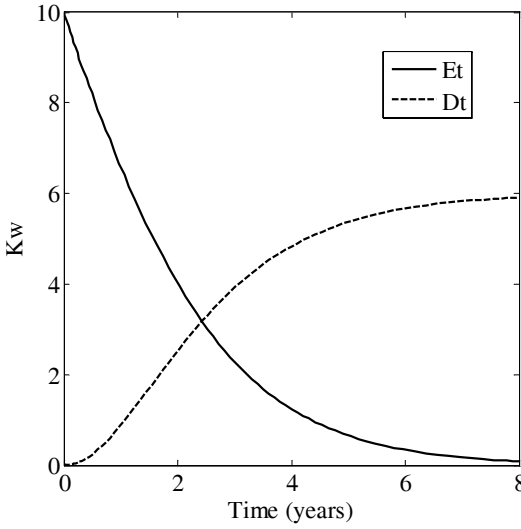


Fig. 2.5 Mean instantaneous performance and mean instantaneous performance deficiency for non-repairable element with minor failures

Now based on (2.62) we obtain the mean times to failure

$$MTTF_1 = \frac{1}{\lambda_{4,3}} + \frac{1}{\lambda_{3,2}} + \frac{1}{\lambda_{2,1}} = 2.93 \text{ year for } g_1 < w \leq g_2,$$

$$MTTF_2 = \frac{1}{\lambda_{4,3}} + \frac{1}{\lambda_{3,2}} = 1.5 \text{ year for } g_2 < w \leq g_3,$$

$$MTTF_3 = \frac{1}{\lambda_{4,3}} = 0.5 \text{ year for } g_3 < w \leq g_4.$$

For the constant demand $w = 6 \text{ KW}$, the mean time to failure is equal to $MTTF_2 = 1.5$ years. The probability that this failure (decreasing the generating capacity lower than a demand level of 6 KW) will not occur during the service time according to the graph in Figure 2.4 will be as follows:

$$R_2(t = T_{\text{service}}) = R_2(500 \text{ h}) = 0.997.$$

Now consider a non-repairable multi-state element that can have both minor and major failures (a major failure is a failure that causes the element transition from state i to state j : $j < i - 1$). The state-space diagram for such an element representing transitions corresponding to both minor and major failures is presented in Figure 2.6.

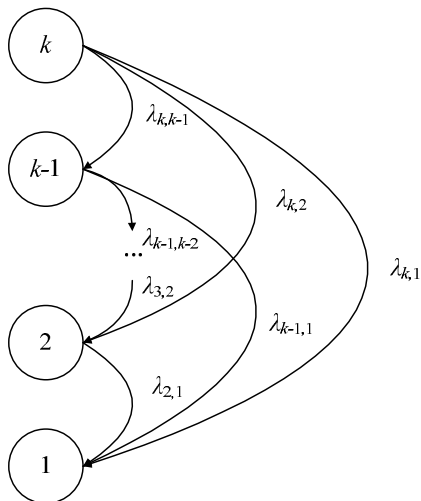


Fig. 2.6 State-transition diagram for non-repairable element with minor and major failures

For the continuous-time Markov chain that is represented by this state-space diagram, the following system of differential equations for state probabilities can be written according to Equations 2.35:

$$\begin{cases} \frac{dp_k(t)}{dt} = -p_k(t) \sum_{e=1}^{k-1} \lambda_{k,e}, \\ \frac{dp_i(t)}{dt} = \sum_{e=i+1}^k \lambda_{e,i} p_e(t) - p_i(t) \sum_{e=1}^{i-1} \lambda_{i,e}, \quad i = 2, 3, \dots, k-1, \\ \frac{dp_1(t)}{dt} = \sum_{e=2}^k \lambda_{e,1} p_e(t), \end{cases} \quad (2.65)$$

with the initial conditions (2.56).

After solving this system and obtaining the state probabilities $p_i(t)$, $i = 1, \dots, k$, the mean instantaneous performance and the mean instantaneous performance deficiency can be determined by using (2.63) and (2.64).

As in the case of the non-repairable multi-state element with minor failures, the unavailability of the element with both minor and major failures is equal to the sum of the probabilities of unacceptable states. Therefore, for the constant demand w ($g_i < w \leq g_{i+1}$) one can use expression (2.61) for determining the element reliability function.

The straightforward method for finding the mean time up to failure is not applicable for multi-state elements with minor and major failures. The general method for solving this problem is based on the Markov reward model and is presented in a later section.

2.3.2.2 Repairable Multi-state Elements

The more general model of a multi-state element is the model with repair. The repairs can also be both minor and major. A minor repair returns an element from state j to state $j+1$ while a major repair returns it from state j to state i , where $i > j+1$.

The special case of the repairable multi-state element is an element with only minor failures and minor repairs. The stochastic process corresponding to such an element is called the birth and death process. The state-space diagram of this process is presented in Figure 2.7 (a).

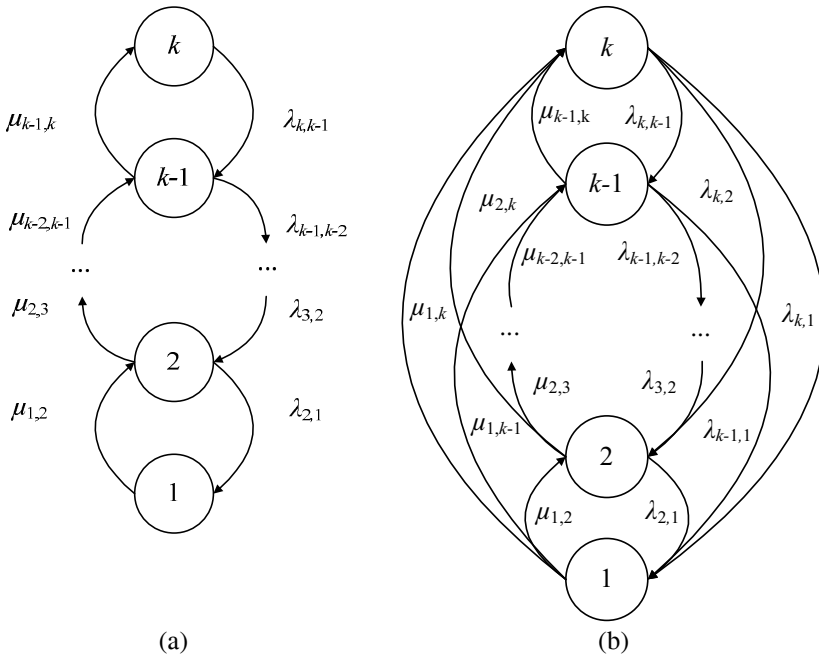


Fig. 2.7 State-transition diagrams for repairable element with minor failures and repairs (a) and for repairable element with minor and major failures and repairs (b)

The state-space diagram for the general case of the repairable multi-state element with minor and major failures and repairs is presented in Figure 2.7 (b). The following system of differential equations can be written for the state probabilities of such elements:

$$\begin{cases} \frac{dp_k(t)}{dt} = \sum_{e=1}^{k-1} \mu_{e,k} p_e(t) - p_k(t) \sum_{e=1}^{k-1} \lambda_{k,e}, \\ \frac{dp_i(t)}{dt} = \sum_{e=i+1}^k \lambda_{e,i} p_e(t) + \sum_{e=1}^{i-1} \mu_{e,i} p_e(t) - p_i(t) \left(\sum_{e=1}^{i-1} \lambda_{i,e} + \sum_{e=i+1}^k \mu_{i,e} \right), \\ \frac{dp_1(t)}{dt} = \sum_{e=2}^k \lambda_{e,1} p_e(t) - p_1(t) \sum_{e=2}^k \mu_{1,e}, \end{cases} \quad i = 2, 3, \dots, k-1 \quad (2.66)$$

with the initial conditions (2.56). Solving this system one obtains the state probabilities $p_i(t)$, $i = 1, \dots, k$.

When $F(g_i, w) = g_i - w$ for the constant demand level $g_i < w \leq g_{i+1}$, the acceptable states where the element performance is above level g_i are $i+1, \dots, k$. Thus, the instantaneous availability is

$$A_i(t) = \sum_{e=i+1}^k p_e(t). \quad (2.67)$$

The element mean instantaneous performance and mean instantaneous performance deficiency can be determined by using (2.63) and (2.64).

In many applications the steady-state probabilities $\lim_{t \rightarrow \infty} p_i(t)$ are of interest for the repairable element. As was said above, if the steady-state probabilities exist, the process is called *ergodic*. For the steady-state probabilities the computations become simpler. The set of differential equations (2.66) is reduced to a set of k algebraic linear equations because for the constant probabilities all time-derivatives are equal to zero, thus, $\frac{dp_i(t)}{dt} = 0$, $i = 1, \dots, k$.

Let the steady-state probabilities $p_i = \lim_{t \rightarrow \infty} p_i(t)$ exist. In order to find the probabilities the following system of algebraic linear equations should be solved

$$\begin{cases} 0 = \sum_{e=1}^{k-1} \mu_{e,k} p_e - p_k \sum_{e=1}^{k-1} \lambda_{k,e}, \\ 0 = \sum_{e=i+1}^k \lambda_{e,i} p_e + \sum_{e=1}^{i-1} \mu_{e,i} p_e - p_i \left(\sum_{e=1}^{i-1} \lambda_{i,e} + \sum_{e=i+1}^k \mu_{i,e} \right), \quad i = 2, 3, \dots, k-1, \\ 0 = \sum_{e=2}^k \lambda_{e,1} p_e - p_1 \sum_{e=2}^k \mu_{1,e}. \end{cases} \quad (2.68)$$

The k equations in (2.68) are not linearly independent (the determinant of the system is zero). An additional independent equation can be provided by the simple fact that the sum of the state probabilities is equal to 1 at any time:

$$\sum_{i=1}^k p_i = 1. \quad (2.69)$$

The determination of the reliability function for repairable multi-state elements is based on finding the probability of the event when the element enters the set of unacceptable states the first time. It does not matter which one of the unacceptable states is visited first. It also does not matter how the element behaves after entering the set of unacceptable states the first time.

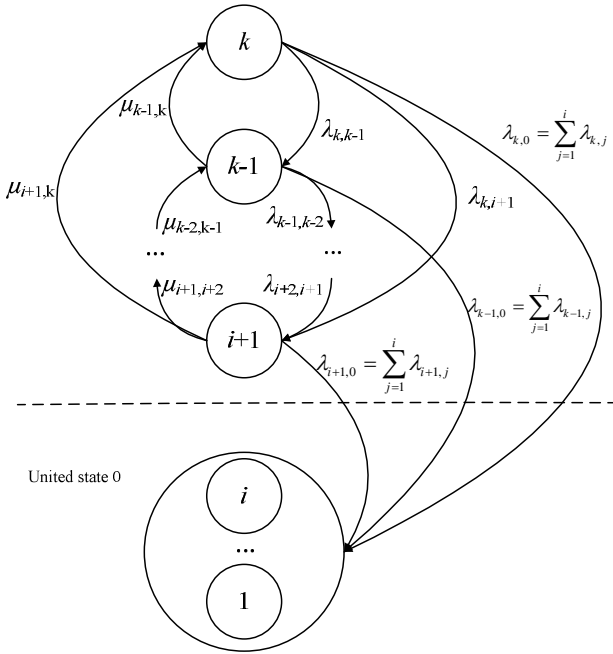


Fig. 2.8 State-transition diagram for determination of reliability function $R_i(t)$ for repairable element (for a constant demand rate w : $g_i < w \leq g_{i+1}$)

In order to find the element reliability function $R_i(t)$, for the constant demand w ($g_i < w \leq g_{i+1}$), an additional Markov model should be built. All states $1, 2, \dots, i$ of the element corresponding to the performance rates that are lower than the demand w should be united in one absorbing state. This absorbing state can be considered now as state 0 and all repairs that return the element from this state back to the set of acceptable states should be forbidden. This corresponds to zeroing all

the transition intensities $\mu_{0,m}$ for $m = i+1, \dots, k$. The transition rate $\lambda_{m,0}$ from any acceptable state m ($m > i$) to the united absorbing state 0 is equal to the sum of the transition rates from state m to all the unacceptable states (states $1, 2, \dots, i$):

$$\lambda_{m,0} = \sum_{j=1}^i \lambda_{m,j}, \quad m = k, k-1, \dots, i+1. \quad (2.70)$$

The state-transition diagram for computation of the reliability function is presented in Figure 2.8. For this diagram, the state probability $p_0(t)$ characterizes the reliability function of the element because after the first entrance into the absorbing state 0 the element never leaves it: $R_i(t) = 1 - p_0(t)$.

The system of differential equations for determining the reliability function of the element takes the following form:

$$\begin{cases} \frac{dp_k(t)}{dt} = \sum_{e=i+1}^{k-1} \mu_{e,k} p_e(t) - p_k(t) \left(\sum_{e=i+1}^{k-1} \lambda_{k,e} + \lambda_{k,0} \right), \\ \frac{dp_j(t)}{dt} = \sum_{e=j+1}^k \lambda_{e,j} p_e(t) + \sum_{e=1}^{j-1} \mu_{e,j} p_e(t) - p_j(t) \left(\sum_{e=i+1}^{j-1} \lambda_{j,e} + \lambda_{j,0} + \sum_{e=j+1}^k \mu_{j,e} \right), \\ \frac{dp_0(t)}{dt} = \sum_{e=i+1}^k \lambda_{e,0} p_e(t). \end{cases} \quad \text{for } i < j < k \quad (2.71)$$

Solving this system under initial conditions

$$p_k(0) = 1, p_{k-1}(0) = \dots = p_i(0) = p_0(0) = 0$$

one obtains the reliability function as

$$R_i(t) = 1 - p_0(t) = \sum_{j=i+1}^k p_j(t). \quad (2.72)$$

Obviously, the final state probabilities for system (2.71) are as follows:

$$p_k = p_{k-1} = \dots = p_{i+1} = 0, p_0 = 1,$$

because the element always enters the absorbing state 0 when $t \rightarrow \infty$.

Based on the computed reliability function $R_i(t) = \sum_{j=i+1}^k p_j(t)$ one can find the

mean time to first failure, when the element performance drops for the first time under demand level w , where $g_i < w \leq g_{i+1}$:

$$MTTF_i = \int_0^{\infty} R_i(t) dt. \quad (2.73)$$

Once state probabilities, p_i or $p_i(t)$, have been computed, reliability measures are usually obtained based on these probabilities.

Example 2.3 (Lisnianski and Levitin 2003). Consider a data processing unit that has $k = 4$ possible performance levels with corresponding task processing speeds: $g_4 = 100 \text{ s}^{-1}$, $g_3 = 80 \text{ s}^{-1}$, $g_2 = 50 \text{ s}^{-1}$, and $g_1 = 0 \text{ s}^{-1}$.

The unit has the following failure rates

$$\begin{aligned} \lambda_{4,3} &= 2 \text{ year}^{-1}, \lambda_{3,2} = 1 \text{ year}^{-1}, \lambda_{2,1} = 0.7 \text{ year}^{-1} \text{ (for minor failures),} \\ \lambda_{3,1} &= 0.4 \text{ year}^{-1}, \lambda_{4,2} = 0.3 \text{ year}^{-1}, \lambda_{4,1} = 0.1 \text{ year}^{-1} \text{ (for major failures)} \end{aligned}$$

and the following repair rates

$$\begin{aligned} \mu_{3,4} &= 100 \text{ year}^{-1}, \mu_{2,3} = 80 \text{ year}^{-1}, \mu_{1,2} = 50 \text{ year}^{-1} \text{ (for minor repairs),} \\ \mu_{1,4} &= 32 \text{ year}^{-1}, \mu_{1,3} = 40 \text{ year}^{-1}, \mu_{2,4} = 45 \text{ year}^{-1} \text{ (for major repairs).} \end{aligned}$$

The demand is constant $w = 60 \text{ s}^{-1}$.

Find such element reliability measures as availability, mean performance, mean performance deficiency, reliability function, and mean time to first failure.

Solution. The state-space diagram for the unit is presented in Figure 2.9 (a). We assume that the initial state is the best state 4.

In order to find the state probabilities, the following system of differential equations should be solved:

$$\begin{cases} \frac{dp_4(t)}{dt} = -(\lambda_{4,3} + \lambda_{4,2} + \lambda_{4,1})p_4(t) + \mu_{3,4}p_3(t) + \mu_{2,4}p_2(t) + \mu_{1,4}p_1(t), \\ \frac{dp_3(t)}{dt} = \lambda_{4,3}p_4(t) - (\lambda_{3,2} + \lambda_{3,1} + \mu_{3,4})p_3(t) + \mu_{1,3}p_1(t) + \mu_{2,3}p_2(t), \\ \frac{dp_2(t)}{dt} = \lambda_{4,2}p_4(t) + \lambda_{3,2}p_3(t) - (\lambda_{2,1} + \mu_{2,3} + \mu_{2,4})p_2(t) + \mu_{1,2}p_1(t), \\ \frac{dp_1(t)}{dt} = \lambda_{4,1}p_4(t) + \lambda_{3,1}p_3(t) + \lambda_{2,1}p_2(t) - (\mu_{1,2} + \mu_{1,3} + \mu_{1,4})p_1(t), \end{cases}$$

with the initial conditions $p_4(0) = 1$, $p_3(0) = p_2(0) = p_1(0) = 0$.

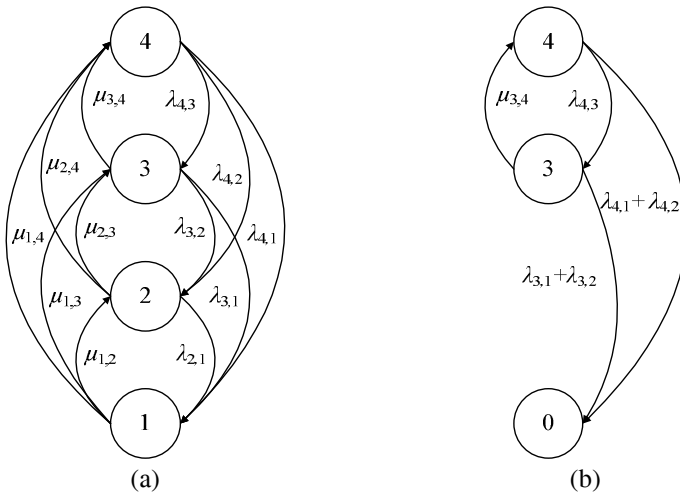


Fig. 2.9 State-transition diagrams for four-state element with minor and major failures and repairs

The element instantaneous availability can be obtained for different constant demand levels:

$$\begin{aligned} A_3(t) &= p_4(t), \text{ for } g_3 < w \leq g_4, \\ A_2(t) &= p_4(t) + p_3(t), \text{ for } g_2 < w \leq g_3, \\ A_1(t) &= p_4(t) + p_3(t) + p_2(t) = 1 - p_1(t), \text{ for } g_1 < w \leq g_2. \end{aligned}$$

These element instantaneous availabilities are presented in Figure 2.10.

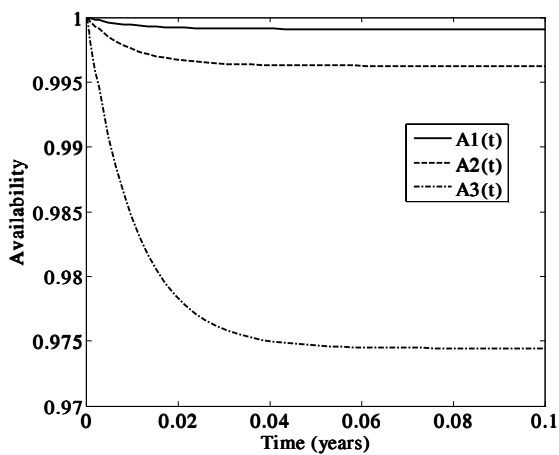


Fig 2.10 Instantaneous availability of four-state element

The element mean instantaneous performance at time t is

$$E_t = \sum_{k=1}^4 g_k p_k(t) = 100p_4(t) + 80p_3(t) + 50p_2(t) + 0p_1(t).$$

For demand $w = 60 \text{ s}^{-1}$ the element availability will be the following

$$A_w(t) = A_2(t).$$

The mean instantaneous performance deficiency (for constant demand $w = 60 \text{ s}^{-1}$) is

$$D_t = \sum_{k=1}^4 p_k(t) \max(w - g_k, 0) = 10p_2(t) + 60p_1(t).$$

The indices D_t and E_t , as functions of time, are presented in Figure 2.11.

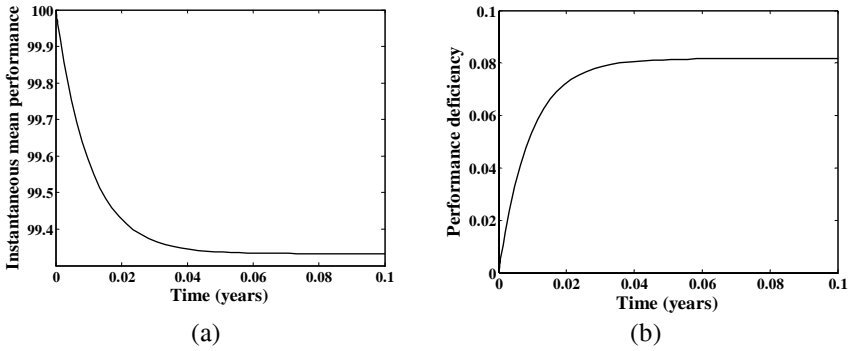


Fig. 2.11 Instantaneous mean performance (a) and performance deficiency (b) of the four-state element

If one wants to find only the final state probabilities he can do it without solving the system of differential equations. As was shown above, the final state probabilities can be found by solving the system of linear algebraic equations (2.68) in which one of the equations is replaced with Equation 2.69. In our example, the system of linear algebraic equations that should be solved takes the form

$$\begin{cases} (\lambda_{4,3} + \lambda_{4,2} + \lambda_{4,1})p_4 = \mu_{3,4}p_3 + \mu_{2,4}p_2 + \mu_{1,4}p_1, \\ (\lambda_{3,2} + \lambda_{3,1} + \mu_{3,4})p_3 = \lambda_{4,3}p_4 + \mu_{2,3}p_2 + \mu_{1,3}p_1, \\ (\lambda_{2,1} + \mu_{2,3} + \mu_{2,4})p_2 = \lambda_{4,2}p_4 + \lambda_{3,2}p_3 + \mu_{1,2}p_1, \\ p_1 + p_2 + p_3 + p_4 = 1. \end{cases}$$

Solving this system, we obtain the final state probabilities:

$$\begin{aligned} p_1 &= \frac{\mu_{1,4}(b_2c_3 - b_3c_2) + \mu_{1,2}(a_2b_3 - a_3b_2) + \mu_{1,3}(a_3c_2 - a_2c_3)}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \\ p_2 &= \frac{\mu_{2,3}(a_1c_3 - a_3c_1) + \mu_{2,4}(b_3c_1 - b_1c_3) + (\lambda_{2,1} + \mu_{2,3} + \mu_{2,4})(a_1b_3 - a_3b_1)}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \\ p_3 &= \frac{\lambda_{3,2}(a_1b_2 - a_2b_1) + (\lambda_{3,2} + \lambda_{3,1} + \mu_{3,4})(a_1c_2 - a_2c_1) + \mu_{3,4}(b_1c_2 - b_2c_1)}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \\ p_4 &= 1 - p_1 - p_2 - p_3, \end{aligned}$$

where

$$\begin{aligned} a_1 &= \mu_{1,4} - \mu_{2,4}, \quad a_2 = \mu_{1,4} - \mu_{3,4}, \quad a_3 = \mu_{1,4} + \lambda_{4,3} + \lambda_{4,2} + \lambda_{4,1}, \\ b_1 &= \mu_{1,3} - \mu_{2,3}, \quad b_2 = \mu_{1,3} + \lambda_{3,2} + \lambda_{3,1} + \mu_{3,4}, \quad b_3 = \mu_{1,3} - \lambda_{4,3}, \\ c_1 &= \mu_{1,2} + \lambda_{2,1} + \mu_{2,3} + \mu_{2,4}, \quad c_2 = \mu_{1,2} - \lambda_{3,2}, \quad c_3 = \mu_{1,2} - \lambda_{4,2}. \end{aligned}$$

The steady-state availability of the element for constant demand $w = 60 \text{ s}^{-1}$ is

$$A = p_4 + p_3,$$

the mean steady-state performance is

$$E_\infty = \sum_{k=1}^4 g_k p_k = 100p_4 + 80p_3 + 50p_2 + 0p_1,$$

and the mean steady-state performance deficiency is

$$D_\infty = \sum_{k=1}^4 p_k \max(w - g_k, 0) = 10p_2 + 60p_1.$$

As one can see in Figures 2.10 and 2.11 the steady-state values of the state probabilities are achieved during a short time period. After 0.07 years, the process becomes stationary. Due to this consideration, only the final solution is important

in many practical cases. This is especially so for elements with a relatively long lifetime. This is the case in our example if the element lifetime is at least several years. However, if one deals with highly responsible components and takes into account even small information losses at the beginning of the process, an analysis based on a system of differential equations should be performed.

In order to find the element reliability function $R_w(t)$, for the constant demand $w = 60 \text{ s}^{-1}$ ($g_2 < w \leq g_3$), an additional Markov model should be built. States 1 and 2 corresponding to performance rates that are lower than the demand w should be united in one absorbing state. This absorbing state can be considered now as state 0 and all repairs that return the element from this state back to the set of acceptable states should be forbidden. This corresponds to zeroing the transition intensities $\mu_{0,3}$ and $\mu_{0,4}$. The transition rates from the acceptable states 3 and 4 to the united absorbing state 0 are equal to the sum of the corresponding transition rates from these states to the unacceptable states 1 and 2. According to (2.70) we obtain $\lambda_{4,0} = \lambda_{4,1} + \lambda_{4,2}$, $\lambda_{3,0} = \lambda_{3,1} + \lambda_{3,2}$.

The state-space diagram for computation of the reliability function $R_w(t)$ is presented in Figure 2.9 (b). For this state-space diagram, the state probability $p_0(t)$ characterizes the reliability function of the element because after the first entrance into the absorbing state 0 the element never leaves it.

The system of differential equations for determining the reliability function of the element takes the form

$$\begin{cases} \frac{dp_4(t)}{dt} = -(\lambda_{4,3} + \lambda_{4,2} + \lambda_{4,1})p_4(t) + \mu_{3,4}p_3(t), \\ \frac{dp_3(t)}{dt} = \lambda_{4,3}p_4(t) - (\lambda_{3,2} + \lambda_{3,1} + \mu_{3,4})p_3(t), \\ \frac{dp_0(t)}{dt} = (\lambda_{4,1} + \lambda_{4,2})p_4(t) + (\lambda_{3,1} + \lambda_{3,2})p_3(t). \end{cases}$$

Solving this system under initial conditions $p_4(0) = 1$, $p_3(0) = p_0(0) = 0$ we obtain the reliability function as $R_w(t) = 1 - p_0(t)$. This function is presented in Figure 2.12.

When the reliability function is known, the mean time to first failure (element's capacity dropping below to demand $w = 60 \text{ s}^{-1}$) can be found by using (2.73):

$$MTTF_w = \int_0^{\infty} R_w(t) dt \approx 2.3 \text{ years}.$$

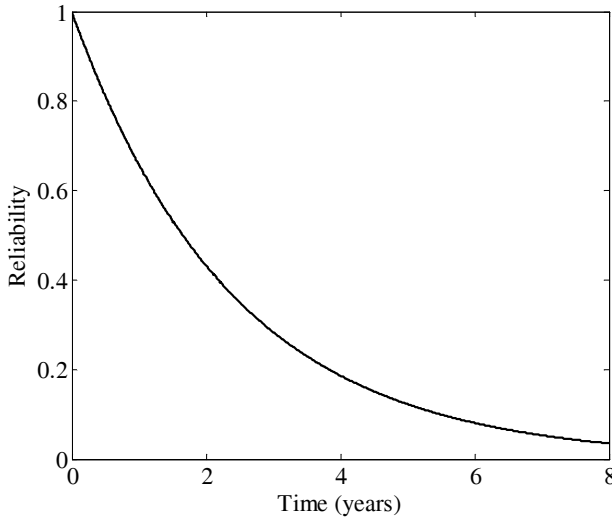


Fig. 2.12 Reliability $R_w(t)$ of a four-state element

2.3.3 Markov Models for Evaluating the Reliability of Multi-state Systems

Consider a system consisting of several multi-state elements. Each combination of the states of these elements constitutes a unique system state. Any system element j can have k_j different states corresponding to the performance rates, represented by the set $\mathbf{g}_j = \{g_{j1}, g_{j2}, \dots, g_{jk_j}\}$. The current state of element j and, therefore, the current value of the element performance rate $G_j(t)$ at any instant t are random variables. $G_j(t)$ takes values from \mathbf{g}_j : $G_j(t) \in \mathbf{g}_j$. The performance rate of any element j is defined as a continuous-state Markov process in the time interval $[0, T]$, where T is the MSS operation period. Such models for different types of MSS elements were studied in the previous section.

According to the generic MSS model, we assume that

$$L^n = \{g_{11}, \dots, g_{1k_1}\} \times \{g_{21}, \dots, g_{2k_2}\} \times \dots \times \{g_{n1}, \dots, g_{nk_n}\}$$

is a space of possible combinations of performance rates for all n system elements and $\mathbf{g} = \{g_1, g_2, \dots, g_K\}$ is a space of possible values of the performance rate for the entire system. The transform $\varphi(G_1(t), \dots, G_n(t)): L^n \rightarrow \mathbf{g}$, which maps the

space of the performance rates of the elements into the space of the system performance rates at any instant t , defines the system structure function. Therefore, by using the structure function, the entire MSS performance rate can be computed for any combination of performance rates of system elements. The current state of the entire MSS and, therefore, the current value of the system output performance rate $G(t)$ at any instant t are random variables. $G(t)$ is a continuous-time Markov chain that takes values from $\mathbf{g} : G(t) \in \mathbf{g} = \{g_1, g_2, \dots, g_K\}$.

We suppose that Markov processes for different elements are independent and that there are no simultaneous state transitions of any different elements. In other words, there may be only one failure or one repair in a system at any instant t .

The traditional application of the Markov technique to MSS reliability evaluation consists of two stages: development of the state-space diagram for the entire system and the evaluation of the system's reliability based on solving a system of differential equations corresponding to the diagram.

The proper design of the state-transition diagram is a critical task in Markov analysis, especially for the MSS. The explosion of the number of states when the modeled system is large enough is still a major problem. In such cases a state-space diagram representation in its pictorial form often becomes impossible. One of the possible solutions is to use a formalized description of the system. When such a description is used, the state-space diagram is not actually presented in pictorial form, but knowledge of the rules that govern the MSS evolution enable us to explore the state-space graph systematically by using a computer. In addition, it is important to understand that the state-space diagram plays only an auxiliary role. The main aim here is to determine the transition intensity matrix \mathbf{a} that defines the system of differential equations (2.38) and hence the corresponding Markov model. Therefore, in this context we speak about the formalized generation of the transition intensity matrix and, therefore, about the Markov model generation. Based on this idea, efficient algorithms are built for the reliability evaluation. One possible algorithm for Markov model generation for the MSS is as follows.

Algorithm for the generation of the Markov model

1. Arrangement of the failure and repair rate sets

For every element j of the MSS, the given element failure and repair rates should be arranged in the following ordered set of failure rates:

$$\left\{ \lambda_{k_j, k_j-1}^{(j)}, \lambda_{k_j, k_j-2}^{(j)}, \dots, \lambda_{k_j, 1}^{(j)}, \lambda_{k_j-1, k_j-2}^{(j)}, \lambda_{k_j-1, k_j-3}^{(j)}, \dots, \lambda_{k_j-1, 1}^{(j)}, \dots, \lambda_{3, 2}^{(j)}, \lambda_{3, 1}^{(j)}, \lambda_{2, 1}^{(j)} \right\}$$

and the ordered set of repair rates

$$\left\{ \mu_{1,2}^{(j)}, \dots, \mu_{1,k_j-1}^{(j)}, \mu_{1,k_j}^{(j)}, \mu_{2,3}^{(j)}, \dots, \mu_{2,k_j-1}^{(j)}, \mu_{2,k_j}^{(j)}, \dots, \mu_{k_j-2,k_j-1}^{(j)}, \mu_{k_j-2,k_j}^{(j)}, \mu_{k_j-1,k_j}^{(j)} \right\}.$$

If for element j there is no failure that causes a decrease in the element performance from level g_{jm} to level $g_{jm-\Delta m}$, the corresponding failure rate $\lambda_{m,m-\Delta m}^{(j)}$ is equal to zero in the failure rate set. In the same manner, if there is no repair that returns the performance of element j from level $g_{jm-\Delta m}$ to level g_{jm} , the corresponding repair rate $\mu_{m-\Delta m,n}^{(j)}$ is equal to zero in the repair rate set.

2. Generation of MSS states

All the $K=k_1k_2\dots k_n$ possible MSS states are generated as different combinations of all the possible performance levels of the system's elements. To each system state a set $\{g_{li}, \dots, g_{nl}\}$, $i \in [1, k_1], \dots, l \in [1, k_n]$, of corresponding states of the system elements should be assigned.

3. Enumeration of the system states and the computation of the MSS output performance

All system states should be enumerated. For computer-based algorithms the enumeration order is not important. What is really important is the correspondence among the number of states n_s ($n_s \in [1, K]$), the set of performance rates of elements in this state $\{g_{li}, \dots, g_{nl}\}$, and the MSS output performance rate g_{n_s} in this state that is determined by the MSS structure function

$$g_{n_s} = \varphi(g_{li}, \dots, g_{nl}), \quad n_s = 1, \dots, K.$$

4. State-transition analysis and generating the transition matrix

At this stage, the existence of connections between any system state n_s and other states must be determined. These connections are defined by failures and repairs of the system elements.

According to the assumption that there are no simultaneous transitions in any different elements, the transition from an arbitrary system state characterized by the set of element performances $\{g_{li}, \dots, g_{jm}, \dots, g_{nh}\}$ is possible only to one of the states in which just one of the elements changes its performance:

$$\{g_{li}, \dots, g_{jm}, \dots, g_{nh}\} \rightarrow \{g_{li}, \dots, g_{jf}, \dots, g_{nh}\}, \quad \text{where } m \neq f, \quad 1 \leq j \leq n.$$

The transition in which $f < m$ corresponds to the element failure (with transition intensity $\lambda_{m,f}^{(j)}$) and the transition in which $f > m$ corresponds to the element repair (with transition intensity $\mu_{m,f}^{(j)}$).

In order to determine all the transitions in the MSS state-space diagram, one has to choose all the pairs of system states that differ by the state of a single element. For each pair, the corresponding transition intensities (failure and repair rates) should be chosen from the corresponding ordered sets.

If the MSS transits from state n_1 to state n_2 because of a failure with the intensity $\lambda_{m,f}^{(j)}$ ($f < m$) of the arbitrary element j , then the element $a_{n_1 n_2}$ of transition matrix \mathbf{a} located in the intersection of row n_1 and column n_2 is

$$a_{n_1 n_2} = \lambda_{m,f}^{(j)}. \quad (2.74)$$

If the MSS transits from state n_1 to state n_2 because of repair with intensity $\mu_{m,f}^{(j)}$ ($f > m$) of an arbitrary element j , then the element $a_{n_1 n_2}$ of transition matrix \mathbf{a} located in the intersection of row n_1 and column n_2 is

$$a_{n_1 n_2} = \mu_{m,f}^{(j)}. \quad (2.75)$$

If the transition from state n_1 to state n_2 does not exist, then the element $a_{n_1 n_2}$ of transition matrix \mathbf{a} located in the intersection between row n_1 and column n_2 is zero:

$$a_{n_1 n_2} = 0. \quad (2.76)$$

5. Determination of diagonal elements in the transition intensity matrix

The last step in generating the transition intensity matrix \mathbf{a} is the determination of its diagonal elements. As is known (see previous section), the elements in each row of matrix \mathbf{a} add up to 0. Hence, diagonal elements of the transition intensity matrix should be defined as follows:

$$a_{ii} = - \sum_{\substack{n=1 \\ n \neq i}}^K a_{in}, \quad i = 1, \dots, K. \quad (2.77)$$

By applying the five-step algorithm, one obtains a transition intensity matrix for MSS. Based on the matrix, the system of differential equations (2.38) describing the system behavior can be directly derived.

The algorithm described above is general. It can build a Markov model for quite complex MSS and reduces the risk of errors and misrepresentations.

MSS reliability indices such as instantaneous availability, instantaneous expected performance, and instantaneous performance deficiency can be found in the same way as was demonstrated for multi-state element (the only difference is the greater order of the system of differential equations).

At first, the system of differential equations must be solved and probabilities $p_i(t)$ must be found for all system states $i=1, \dots, K$.

For the constant demand level w the MSS instantaneous availability can be obtained as the sum of probabilities of all acceptable states (the states where MSS output performance is greater than or equal to w). Therefore, MSS instantaneous availability can be defined as

$$A(t) = \sum_{i=1}^K p_i(t) \cdot 1(g_i \geq w), \quad (2.78)$$

MSS mean instantaneous performance can be defined as

$$E_t = \sum_{i=1}^K g_i p_i(t), \quad (2.79)$$

and MSS mean instantaneous performance deficiency can be defined as

$$D_t = \sum_{i=1}^K p_i(t) \max(w - g_i, 0). \quad (2.79)$$

In order to find the MSS reliability function $R_i(t)$ for the constant demand w , $g_i < w \leq g_{i+1}$, the Markov model should be changed. All the system states from the unacceptable area where the performance rate is lower than demand w , should be united in one absorbing state with the number 0. Transitions from state 0 to any acceptable state should be forbidden. The transition rate from any acceptable state j to the absorbing state should be determined as the sum of the transition rates from state j to all the unacceptable states. After performing these changes, we obtain the new transition intensity matrix. By solving the differential equation (2.38) with this matrix one obtains the probability of state 0 $p_0(t)$ and determines the system reliability function as $R(t) = 1 - p_0(t)$.

Example 2.4 (Lisnianski and Levitin 2003). Consider the flow transmission system from Example 1.2 (Chapter 1) that was presented in Figure 1.8 (a). It consists of three elements (pipes). The oil flow is transmitted from point C to point E. The performance of the pipes is measured by their transmission capacity (tons per minute). Elements 1 and 2 are repairable and each has two possible states. A state of

total failure for both elements corresponds to a transmission capacity of 0 and the operational state corresponds to capacities of 1.5 and 2 tons per minute, respectively, so that $G_1(t) \in \{g_{11}, g_{12}\} = \{0, 1.5\}$ and $G_2(t) \in \{g_{21}, g_{22}\} = \{0, 2\}$.

The failure rates and repair rates corresponding to these two elements are

$$\lambda_{2,1}^{(1)} = 7 \text{ year}^{-1}, \mu_{1,2}^{(1)} = 100 \text{ year}^{-1} \text{ for element 1,}$$

$$\lambda_{2,1}^{(2)} = 10 \text{ year}^{-1}, \mu_{1,2}^{(2)} = 80 \text{ year}^{-1} \text{ for element 2.}$$

Element 3 is a multi-state element with only minor failures and minor repairs. It can be in one of three states: a state of total failure corresponding to a capacity of 0, a state of partial failure corresponding to a capacity of 1.8 tons per minute, and a fully operational state with a capacity of 4 tons per minute. Therefore, $G_3(t) \in \{g_{31}, g_{32}, g_{33}\} = \{0, 1.8, 4\}$.

The failure rates and repair rates corresponding to element 3 are

$$\lambda_{3,2}^{(3)} = 10 \text{ year}^{-1}, \lambda_{2,1}^{(3)} = 7 \text{ year}^{-1},$$

$$\mu_{1,2}^{(3)} = 120 \text{ year}^{-1}, \mu_{2,3}^{(3)} = 110 \text{ year}^{-1}.$$

The system output performance rate is defined as the maximum flow that can be transmitted from C to E. As was shown in Example 1.2, the MSS structure function is

$$G_s(t) = f(G_1(t), G_2(t), G_3(t)) = \min\{G_1(t) + G_2(t), G_3(t)\}.$$

The demand is constant: $w = 1.0$ ton per minute.

The MSS structure is presented in Figure 2.13.

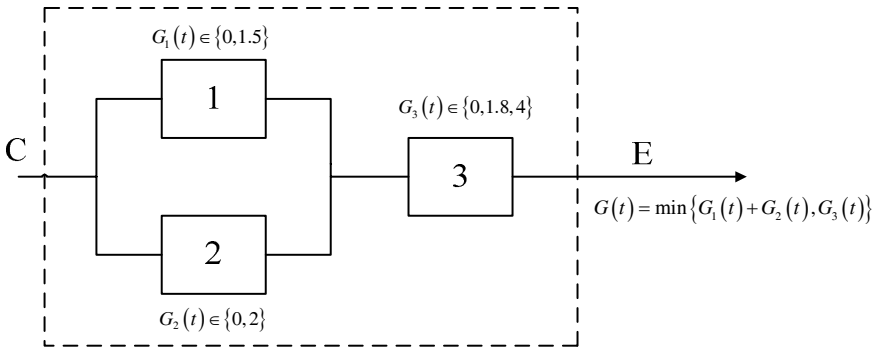


Fig. 2.13 MSS structure

The state-transition diagrams of system elements are presented in Figure 2.14.

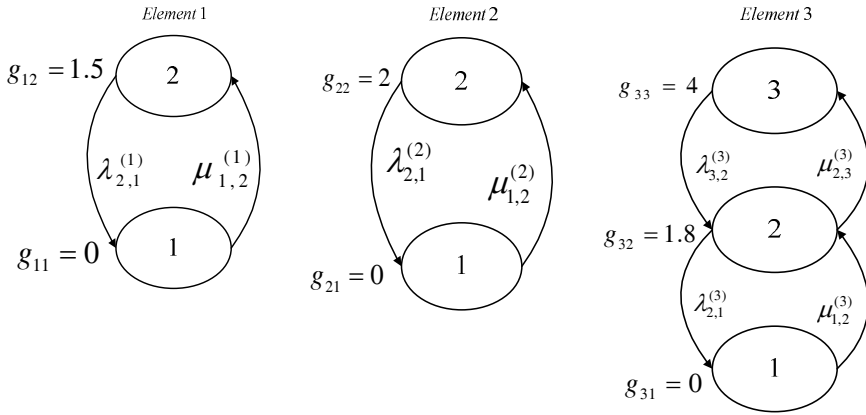


Fig. 2.14 State-transition diagrams of the system elements

In order to derive the system of differential equations for the MSS we apply the algorithm described above:

1. The failure and repair rate sets for the system elements are:

$$\begin{aligned} \text{element 1: } & \{\lambda_{2,1}^{(1)}\}, \{\mu_{1,2}^{(1)}\}; \text{ element 2: } \{\lambda_{2,1}^{(2)}\}, \{\mu_{1,2}^{(2)}\}; \\ \text{element 3: } & \{\lambda_{3,2}^{(3)}, \lambda_{3,1}^{(3)} = 0, \lambda_{2,1}^{(1)}\}, \{\mu_{1,2}^{(3)}, \mu_{1,3}^{(3)} = 0, \mu_{2,3}^{(3)}\}. \end{aligned}$$

2. All the system states are generated as combinations of all possible states of system elements (characterized by their performance levels). The total number of different system states is $K = k_1 k_2 k_3 = 2 \times 2 \times 3 = 12$.
3. A unique number is assigned to each system state. All the system states with their numbers n_s corresponding performance rates are presented in columns 1 to 5 of Table 2.1. For every state, the system output performance rate is computed based on the MSS structure function. For example, in state 1 we have $G_1(t) = g_{12} = 1.5$, $G_2(t) = g_{22} = 2.0$, $G_3(t) = g_{33} = 4.0$. Using the system structure function, we obtain the entire MSS output performance in state 1 as

$$G(t) = g_1 = f(g_{12}, g_{22}, g_{33}) = \min\{g_{12} + g_{22}, g_{33}\} = \min\{1.5 + 2.0, 4.0\} = 3.5.$$

4. The state transition analysis is performed for all pairs of system states. For example, for state number 2 where the states of the elements are $\{g_{11}, g_{22}, g_{33}\} = \{2, 4, 2\}$ the transitions to states 1, 5, and 6 exist with the intensities $\mu_{1,2}^{(1)}, \lambda_{2,1}^{(2)}, \lambda_{3,2}^{(3)}$, respectively. All the existing transitions and corresponding transition intensities are also presented in Table 2.1. Based on Table 2.1 one can easily find the non-diagonal elements of the transition intensity matrix that describes an evolution of the MSS in the state space (the elements of the matrix corresponding to the absence of transitions should be zeroed).

Table 2.1 Markov model generated for MSS of Example 2.4

Performance					$\mathbf{n_s}$											
$\mathbf{n_s}$	$\mathbf{G_1}$	$\mathbf{G_2}$	$\mathbf{G_3}$	\mathbf{G}	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{3}$	$\mathbf{4}$	$\mathbf{5}$	$\mathbf{6}$	$\mathbf{7}$	$\mathbf{8}$	$\mathbf{9}$	$\mathbf{10}$	$\mathbf{11}$	$\mathbf{12}$
1	1.5	2.0	4.0	3.5		$\lambda_{2,1}^{(1)}$	$\lambda_{2,1}^{(2)}$	$\lambda_{3,2}^{(3)}$								
2	0	2.0	4.0	2.0	$\mu_{1,2}^{(1)}$				$\lambda_{2,1}^{(2)}$	$\lambda_{3,2}^{(3)}$						
3	1.5	0	4.0	1.5	$\mu_{1,2}^{(2)}$				$\lambda_{2,1}^{(1)}$		$\lambda_{3,2}^{(3)}$					
4	1.5	2.0	1.8	1.8	$\mu_{2,3}^{(3)}$					$\lambda_{2,1}^{(1)}$	$\lambda_{2,1}^{(2)}$	$\lambda_{2,1}^{(3)}$				
5	0	0	4.0	0		$\mu_{1,2}^{(2)}$	$\mu_{1,2}^{(1)}$						$\lambda_{3,2}^{(3)}$			
6	0	2.0	1.8	1.8		$\mu_{2,3}^{(3)}$		$\mu_{1,2}^{(1)}$					$\lambda_{2,1}^{(2)}$	$\lambda_{2,1}^{(3)}$		
7	1.5	0	4.0	1.5			$\mu_{2,3}^{(3)}$	$\mu_{1,2}^{(2)}$					$\lambda_{2,1}^{(1)}$		$\lambda_{2,1}^{(3)}$	
8	1.5	2.0	0	0				$\mu_{1,2}^{(3)}$						$\lambda_{2,1}^{(1)}$	$\lambda_{2,1}^{(2)}$	
9	0	0	4	0					$\mu_{2,3}^{(3)}$	$\mu_{1,2}^{(2)}$	$\mu_{1,2}^{(1)}$					$\lambda_{2,1}^{(3)}$
10	0	2.0	0	0						$\mu_{1,2}^{(3)}$		$\mu_{1,2}^{(1)}$				$\lambda_{2,1}^{(2)}$
11	1.5	0	0	0							$\mu_{1,2}^{(3)}$	$\mu_{1,2}^{(2)}$				$\lambda_{2,1}^{(1)}$
12	0	0	0	0									$\mu_{1,2}^{(3)}$	$\mu_{1,2}^{(2)}$	$\mu_{1,2}^{(1)}$	

5. The diagonal elements of the transition intensity matrix are determined in such a way that the sum of elements of each row of the matrix equals zero. These diagonal elements are as follows:

$$\begin{aligned}
 a_{11} &= -(\lambda_{2,1}^{(1)} + \lambda_{2,1}^{(2)} + \lambda_{3,2}^{(3)}) & a_{77} &= -(\mu_{2,3}^{(3)} + \mu_{1,2}^{(1)} + \lambda_{2,1}^{(1)} + \lambda_{2,1}^{(3)}) \\
 a_{22} &= -(\mu_{1,2}^{(1)} + \lambda_{2,1}^{(2)} + \lambda_{3,2}^{(3)}) & a_{88} &= -(\mu_{1,2}^{(3)} + \lambda_{2,1}^{(1)} + \lambda_{2,1}^{(2)}) \\
 a_{33} &= -(\mu_{1,2}^{(1)} + \lambda_{2,1}^{(2)} + \lambda_{3,2}^{(3)}) & a_{99} &= -(\mu_{2,3}^{(3)} + \mu_{1,2}^{(2)} + \mu_{1,2}^{(1)} + \lambda_{2,1}^{(3)}) \\
 a_{44} &= -(\mu_{2,3}^{(3)} + \lambda_{2,1}^{(1)} + \lambda_{2,1}^{(2)} + \lambda_{2,1}^{(3)}) & a_{10,10} &= -(\mu_{1,2}^{(3)} + \mu_{1,2}^{(1)} + \lambda_{2,1}^{(2)}) \\
 a_{55} &= -(\mu_{1,2}^{(1)} + \mu_{1,2}^{(1)} + \lambda_{3,2}^{(3)}) & a_{11,11} &= -(\mu_{1,2}^{(3)} + \mu_{1,2}^{(2)} + \lambda_{2,1}^{(1)}) \\
 a_{66} &= -(\mu_{2,3}^{(3)} + \mu_{1,2}^{(1)} + \lambda_{2,1}^{(2)} + \lambda_{2,1}^{(3)}) & a_{12,12} &= -(\mu_{1,2}^{(3)} + \mu_{1,2}^{(2)} + \mu_{1,2}^{(1)})
 \end{aligned}$$

The state-space diagram of the system is presented in Figure 2.15 (in this diagram the corresponding system performance is presented in the lower parts of the circle).

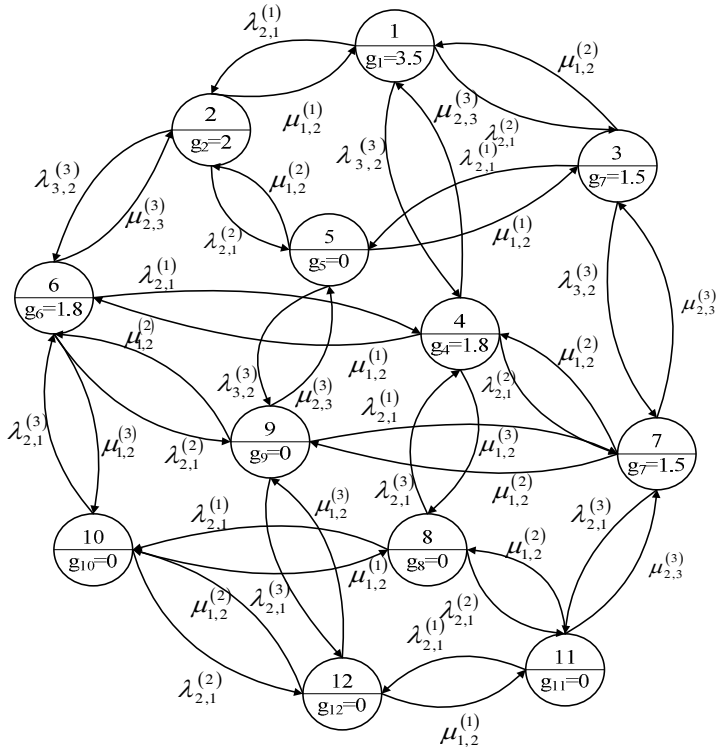


Fig. 2.15 State-transition diagram for MSS

According to (2.37) and (2.38) the corresponding system of differential equations for the state probabilities $p_i(t)$, $1 \leq i \leq 12$ takes the following form:

$$\begin{aligned}
 \frac{dp_1(t)}{dt} &= -\left(\lambda_{2,1}^{(1)} + \lambda_{2,1}^{(2)} + \lambda_{3,2}^{(3)}\right) p_1(t) + \mu_{1,2}^{(1)} p_2(t) + \mu_{1,2}^{(2)} p_3(t) + \mu_{2,3}^{(3)} p_4(t), \\
 \frac{dp_2(t)}{dt} &= \lambda_{2,1}^{(1)} p_1(t) - \left(\mu_{1,2}^{(1)} + \lambda_{2,1}^{(2)} + \lambda_{3,2}^{(3)}\right) p_2(t) + \mu_{1,2}^{(2)} p_5(t) + \mu_{2,3}^{(3)} p_6(t), \\
 \frac{dp_3(t)}{dt} &= \lambda_{2,1}^{(2)} p_1(t) - \left(\mu_{1,2}^{(2)} + \lambda_{2,1}^{(1)} + \lambda_{3,2}^{(3)}\right) p_3(t) + \mu_{1,2}^{(1)} p_5(t) + \mu_{2,3}^{(3)} p_7(t), \\
 \frac{dp_4(t)}{dt} &= \lambda_{3,2}^{(3)} p_1(t) - \left(\mu_{2,3}^{(3)} + \lambda_{2,1}^{(1)} + \lambda_{2,1}^{(2)} + \lambda_{2,1}^{(3)}\right) p_4(t) + \mu_{1,2}^{(1)} p_6(t) \\
 &\quad + \mu_{1,2}^{(2)} p_7(t) + \mu_{1,2}^{(3)} p_8(t), \\
 \frac{dp_5(t)}{dt} &= \lambda_{2,1}^{(2)} p_2(t) + \lambda_{2,1}^{(1)} p_3(t) - \left(\mu_{1,2}^{(2)} + \mu_{1,2}^{(1)} + \lambda_{3,2}^{(3)}\right) p_5(t) + \mu_{2,3}^{(3)} p_9(t),
 \end{aligned}$$

$$\begin{aligned}
\frac{dp_6(t)}{dt} &= \lambda_{3,2}^{(3)} p_2(t) + \lambda_{2,1}^{(1)} p_4(t) - \left(\mu_{2,3}^{(3)} + \mu_{1,2}^{(1)} + \lambda_{2,1}^{(2)} + \lambda_{2,1}^{(3)} \right) p_6(t) \\
&\quad + \mu_{1,2}^{(2)} p_9(t) + \mu_{1,2}^{(3)} p_{10}(t), \\
\frac{dp_7(t)}{dt} &= \lambda_{3,2}^{(3)} p_3(t) + \lambda_{2,1}^{(2)} p_4(t) - \left(\mu_{2,3}^{(3)} + \mu_{1,2}^{(2)} + \lambda_{2,1}^{(1)} + \lambda_{2,1}^{(3)} \right) p_7(t) \\
&\quad + \mu_{1,2}^{(1)} p_9(t) + \mu_{2,3}^{(3)} p_{11}(t), \\
\frac{dp_8(t)}{dt} &= \lambda_{2,1}^{(3)} p_4(t) - \left(\mu_{1,2}^{(3)} + \lambda_{2,1}^{(1)} + \lambda_{2,1}^{(2)} \right) p_8(t) + \mu_{1,2}^{(1)} p_{10}(t) + \mu_{1,2}^{(2)} p_{11}(t), \\
\frac{dp_9(t)}{dt} &= \lambda_{3,2}^{(3)} p_5(t) + \lambda_{2,1}^{(2)} p_6(t) + \lambda_{2,1}^{(1)} p_7(t) - \left(\mu_{2,3}^{(3)} + \mu_{1,2}^{(2)} + \mu_{1,2}^{(1)} + \lambda_{2,1}^{(3)} \right) p_9(t) \\
&\quad + \mu_{1,2}^{(3)} p_{12}(t), \\
\frac{dp_{10}(t)}{dt} &= \lambda_{2,1}^{(3)} p_6(t) + \lambda_{2,1}^{(1)} p_8(t) - \left(\mu_{1,2}^{(3)} + \mu_{1,2}^{(1)} + \lambda_{2,1}^{(2)} \right) p_{10}(t) + \mu_{1,2}^{(2)} p_{12}(t), \\
\frac{dp_{11}(t)}{dt} &= \lambda_{2,1}^{(3)} p_7(t) + \lambda_{2,1}^{(2)} p_8(t) - \left(\mu_{1,2}^{(3)} + \mu_{1,2}^{(2)} + \lambda_{2,1}^{(1)} \right) p_{11}(t) + \mu_{1,2}^{(1)} p_{12}(t), \\
\frac{dp_{12}(t)}{dt} &= \lambda_{2,1}^{(3)} p_9(t) + \lambda_{2,1}^{(2)} p_{10}(t) + \lambda_{2,1}^{(1)} p_{11}(t) - \left(\mu_{1,2}^{(3)} + \mu_{1,2}^{(2)} + \mu_{1,2}^{(1)} \right) p_{12}(t).
\end{aligned}$$

Solving this system with the initial conditions $p_1(0) = 1$, $p_i(0) = 0$ for $2 \leq i \leq 12$ one obtains the probability of each state at time t .

According to Table 2.1, in different states a MSS has the following performance rates: in state 1 $g_1 = 3.5$, in state 2 $g_2 = 2.0$, in states 4 and 6 $g_4 = g_6 = 1.8$, in states 3 and 7 $g_3 = g_7 = 1.5$, in states 5, 8, 9, 10, 11 and 12 $g_5 = g_8 = g_9 = g_{10} = g_{11} = g_{12} = 0$. Therefore,

$$\begin{aligned}
\Pr\{G = 3.5\} &= p_1(t), \\
\Pr\{G = 2.0\} &= p_2(t), \\
\Pr\{G = 1.5\} &= p_3(t) + p_7(t), \\
\Pr\{G = 1.8\} &= p_4(t) + p_6(t), \\
\Pr\{G = 0\} &= p_5(t) + p_8(t) + p_9(t) + p_{10}(t) + p_{11}(t) + p_{12}(t).
\end{aligned}$$

For the constant demand level $w = 1$ one obtains the MSS instantaneous availability as a sum of state probabilities where the MSS output performance is greater than or equal to 1. States 1, 2, 3, 4, 6, and 7 are acceptable. Hence

$$A(t) = p_1(t) + p_2(t) + p_3(t) + p_4(t) + p_6(t) + p_7(t).$$

Instantaneous availability and probabilities of different MSS performance levels are presented in Figure 2.16.

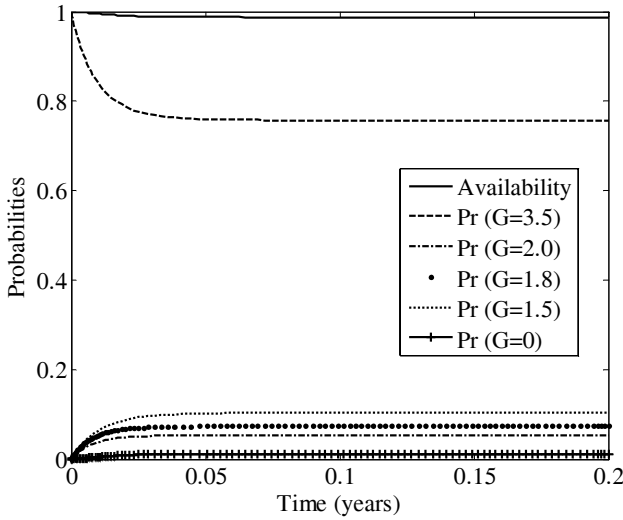


Fig. 2.16 Instantaneous availability and probabilities of different MSS performance levels

The MSS mean instantaneous output performance is

$$E_t = \sum_{i=1}^{12} p_i(t) g_i.$$

The MSS mean instantaneous performance deficiency is

$$D_t = \sum_{i=1}^{12} p_i(t) \max(w - g_i, 0).$$

These functions are presented in Figure 2.17.

In order to find the reliability function $R(t)$ for $w = 1$ one must unite all the unacceptable states into one absorbing state, forbid repairs that return the MSS from this state to the acceptable states, and replace the failure rates from each acceptable state j to absorbing state 0 by the sum of the failure rates from state j to all the unacceptable states.

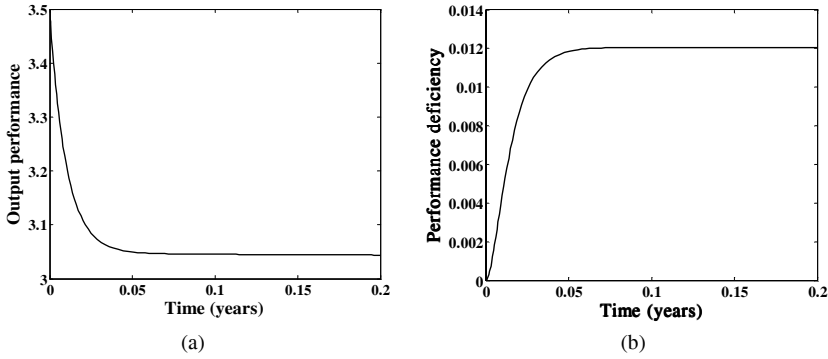


Fig. 2.17 Reliability indices of the flow transmission MSS

The corresponding state-space diagram is presented in Figure 2.18.

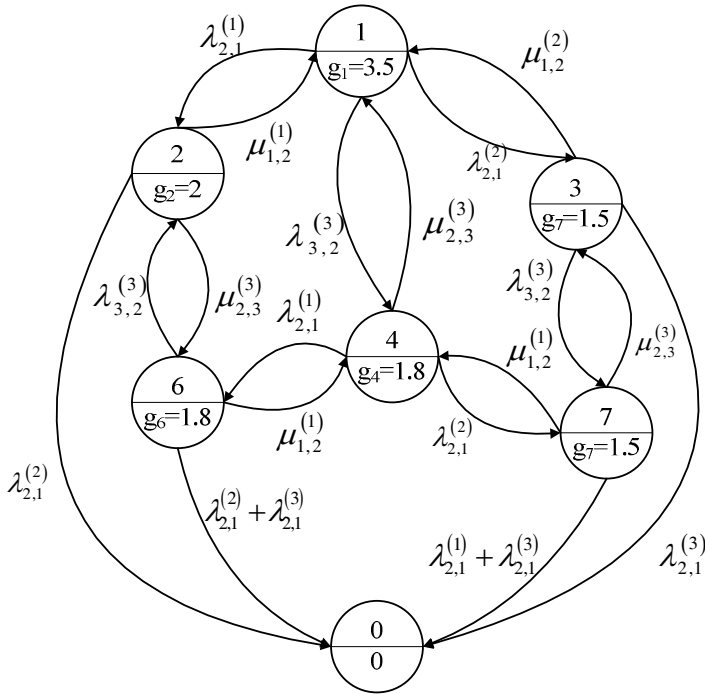


Fig. 2.18 State-transition diagram for evaluating MSS reliability function

In this state-space diagram all unacceptable states (with MSS output performance lower than $w = 1$) are united into one absorbing state 0. As was described above, the system unreliability is treated as the probability that the MSS enters the unacceptable area the first time at time instant t .

The corresponding system of differential equations is as follows:

$$\begin{aligned}
\frac{dp_1(t)}{dt} &= -\left(\lambda_{2,1}^{(1)} + \lambda_{3,2}^{(3)} + \lambda_{2,1}^{(2)}\right)p_1(t) + \mu_{1,2}^{(1)}p_2(t) + \mu_{1,2}^{(2)}p_3(t) + \mu_{2,3}^{(3)}p_4(t), \\
\frac{dp_2(t)}{dt} &= \lambda_{2,1}^{(1)}p_1(t) - \left(\lambda_{2,1}^{(2)} + \mu_{1,2}^{(1)} + \lambda_{3,2}^{(3)}\right)p_2(t) + \mu_{2,3}^{(3)}p_6(t), \\
\frac{dp_3(t)}{dt} &= \lambda_{2,1}^{(2)}p_1(t) - \left(\lambda_{2,1}^{(1)} + \mu_{1,2}^{(2)} + \lambda_{3,2}^{(3)}\right)p_3(t) + \mu_{2,3}^{(3)}p_7(t), \\
\frac{dp_4(t)}{dt} &= \lambda_{3,2}^{(3)}p_1(t) - \left(\lambda_{2,1}^{(1)} + \mu_{2,3}^{(3)} + \lambda_{2,1}^{(2)} + \lambda_{2,1}^{(3)}\right)p_4(t) + \mu_{1,2}^{(1)}p_6(t) + \mu_{1,2}^{(2)}p_7(t), \\
\frac{dp_6(t)}{dt} &= \lambda_{3,2}^{(3)}p_2(t) + \lambda_{2,1}^{(1)}p_4(t) - \left(\mu_{2,3}^{(3)} + \lambda_{2,1}^{(2)} + \lambda_{2,1}^{(3)} + \mu_{1,2}^{(1)}\right)p_6(t), \\
\frac{dp_7(t)}{dt} &= \lambda_{3,2}^{(3)}p_3(t) + \lambda_{2,1}^{(2)}p_4(t) - \left(\mu_{2,3}^{(3)} + \lambda_{2,1}^{(3)} + \lambda_{2,1}^{(1)} + \mu_{1,2}^{(2)}\right)p_7(t), \\
\frac{dp_0(t)}{dt} &= \lambda_{2,1}^{(2)}p_2(t) + \lambda_{2,1}^{(1)}p_3(t) + \lambda_{2,1}^{(3)}p_4(t) + \left(\lambda_{2,1}^{(2)} + \lambda_{2,1}^{(3)}\right)p_6(t) + \left(\lambda_{2,1}^{(3)} + \lambda_{2,1}^{(1)}\right)p_7(t).
\end{aligned}$$

Solving this system under the initial conditions $p_1(0) = 1$, $p_0(0) = p_2(0) = p_3(0) = p_4(0) = p_6(0) = p_7(0) = 0$, we find the probability $p_0(t)$ of absorbing state 0. This probability characterizes the reliability function $R(t) = 1 - p_0(t)$ for $w = 1$. The reliability function is presented in Figure 2.19.

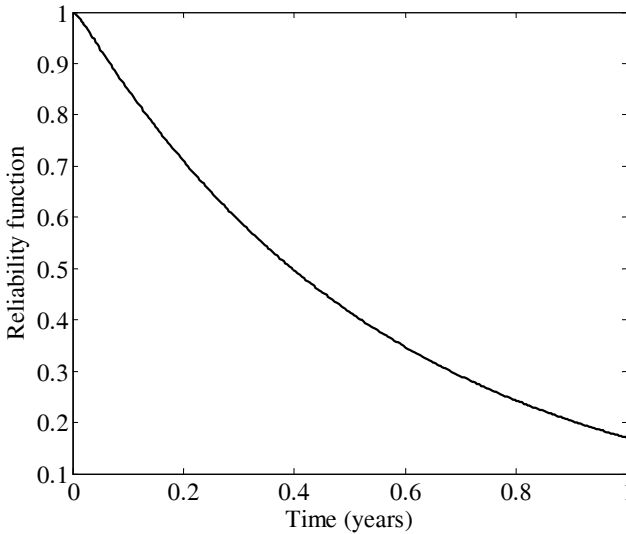


Fig. 2.19 Reliability function of flow transmission MSS

2.4 Markov Reward Models

2.4.1 Basic Definition and Model Description

In the preceding subsections, it was shown how some important MSSs' reliability indices can be found by using the Markov technique. Here we consider additional indices such as states frequencies and the mean number of system failures during an operating period. It is also very important that the Markov reward models considered here are very useful for MSS life cycle cost analysis and reliability-associated cost computation. Here we describe the common computational method, which is based on the general Markov reward model that was primarily introduced by Howard (1960) and then was essentially extended for different applications in Mine and Osaki (1970) and many other research works. The corresponding overview can be found in Reibman *et al.* (1989).

This model considers the continuous-time Markov chain with a set of states $\{1, \dots, K\}$ and transition intensity matrix $\mathbf{a} = [a_{ij}]$, $i, j = 1, \dots, K$. It is suggested that if the process stays in any state i during the time unit, a certain amount of money r_{ii} should be paid. It is also suggested that each time that the process transits from state i to state j a certain amount of money r_{ij} should be paid. These amounts of money r_{ii} and r_{ij} are called *rewards* (the reward may also be negative when it characterizes losses or penalties). Rewards may also be considered in other senses, not only as money. It may be, for example, the energy of a power generating system, information quantity of a communications system, the productivity of a production line, etc. Markov process with rewards associated with its states and transitions is called a Markov process with rewards. For these processes, an additional matrix $\mathbf{r} = [r_{ij}]$, $i, j = 1, \dots, K$ of rewards is determined. If all rewards are zeroes, the process reduces to the ordinary continuous-time discrete-state Markov process.

Note that the rewards r_{ii} and r_{ij} have different dimensions. For example, if r_{ij} is measured in cost units, reward r_{ii} is measured in cost units per time unit. The value that is of interest is the total expected reward accumulated up to time instant t under specified initial conditions.

Let $V_i(t)$ be the total expected reward accumulated up to time t , given the initial state of the process at time instant $t = 0$ is state i . According to Howard (1960), the following system of differential equations must be solved under specified initial conditions in order to find the total expected rewards:

$$\frac{dV_i(t)}{dt} = r_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^K a_{ij} r_{ij} + \sum_{j=1}^K a_{ij} V_j(t), \quad i = 1, \dots, K. \quad (2.80)$$

System (2.80) can be obtained in the following manner. Assume that at time instant $t = 0$ the process is in state i . During the time increment Δt , the process can remain in this state or transit to some other state j . If it remains in state i during time Δt , the expected reward accumulated during this time is $r_{ii}\Delta t$. Since at the beginning of the time interval $[\Delta t, \Delta t + t]$ the process is still in state i , the expected reward during this interval is $V_i(t)$ and the expected reward during the entire interval $[0, \Delta t + t]$ is $V_i(\Delta t + t) = r_{ii}\Delta t + V_i(t)$. The probability that the process will remain in state i during the time interval Δt equals 1 minus the probability that it will transit to any other state $j \neq i$ during this interval:

$$\pi_{ii}(0, \Delta t) = 1 - \sum_{\substack{j=1 \\ j \neq i}}^K a_{ij} \Delta t = 1 + a_{ii} \Delta t. \quad (2.81)$$

On the other hand, during time Δt the process can transit to some other state $j \neq i$ with the probability $\pi_{ij}(0, \Delta t) = a_{ij}\Delta t$. In this case the expected reward accumulated during the time interval $[0, \Delta t]$ is r_{ij} . At the beginning of the time interval $[\Delta t, \Delta t + t]$ the process is in state j . Therefore, the expected reward during this interval is $V_j(t)$ and the expected reward during the interval $[0, \Delta t + t]$ is $V_i(\Delta t + t) = r_{ij} + V_j(t)$.

In order to obtain the total expected reward one must summarize the products of rewards and corresponding probabilities for all of the states. Thus, for small Δt one has

$$V_i(\Delta t + t) \approx (1 + a_{ii}\Delta t)[r_{ii}\Delta t + V_i(t)] + \sum_{\substack{j=1 \\ j \neq i}}^K a_{ij}\Delta t[r_{ij} + V_j(t)], \quad i = 1, \dots, K. \quad (2.82)$$

Neglecting the terms with an order greater than Δt one can rewrite the last expression as follows:

$$\frac{V_i(\Delta t + t) - V_i(t)}{\Delta t} = r_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^K a_{ij} r_{ij} + \sum_{j=1}^K a_{ij} V_j(t), \quad i = 1, \dots, K. \quad (2.83)$$

Passing to the limit in this equation gives (2.80).

Defining the vector column of total expected rewards $\mathbf{V}(t)$ with components $V_1(t), \dots, V_K(t)$ and vector column \mathbf{u} with components

$$u_i = r_{ii} + \sum_{\substack{j \neq i \\ j=1}}^K a_{ij} r_{ij}, \quad i = 1, \dots, K, \quad (2.84)$$

one obtains Equation 2.80 in matrix notation:

$$\frac{d}{dt} \mathbf{V}(t) = \mathbf{u} + \mathbf{a} \mathbf{V}(t). \quad (2.85)$$

Usually system (2.80) should be solved under initial conditions $V_i(0) = 0$, $i = 1, \dots, K$.

In order to find the long-run (steady-state) solution of (2.80), the following system of algebraic equations must be solved

$$\mathbf{0} = \mathbf{u} + \mathbf{a} \mathbf{V}(t), \quad (2.86)$$

where $\mathbf{0}$ is a vector column with zero components.

Example 2.5 As an example of the method application we consider a line for ice-cream production. The nominal productivity (performance) of the line is $N_{ic} = 280$ ice creams per hour. The profit from the selling (delivery according contract) of this product is $r_{prf} = \text{US } \$15/\text{h}$. The line only has complete failures with a failure rate of $\lambda = 0.1 \text{ year}^{-1}$. If the line fails, the owner by contract is forced to pay US \$3 penalty for one unsupplied ice cream per hour, so $c_p = \text{US } \$3$ per ice-cream per hour. After the line failure, a repair is performed with a repair rate of $\mu = 200 \text{ year}^{-1}$. The mean cost of repair is $c_r = \text{US } \$14,000$.

The problem is to evaluate a total expected reward R_T associated with the production line operating during the time interval $[0, T]$.

Solution. The state-space diagram for the ice-cream production line is presented in Figure 2.20. It has only two states: perfect functioning with a nominal productivity (state 2) and complete failure where the unit generating capacity is zero (state 1). The profit from ice cream selling defines the reward associated with state 2, so $r_{22} = r_{prf}$. The transitions from state 2 to state 1 are associated with failures and have intensity λ . If the line is in state 1, the penalty cost $c_p N_{ic}$ should be paid for each time unit (hour). Hence, the reward r_{11} associated with state 1 is $r_{11} = c_p N_{ic}$.

The transitions from state 1 to state 2 are associated with repairs and have an intensity of μ . The repair cost is c_r , therefore the reward associated with the tran-

sition from state 1 to state 2 is $r_{12} = c_r$. There is no reward associated with the transition from state 2 to state 1, so $r_{21} = 0$.

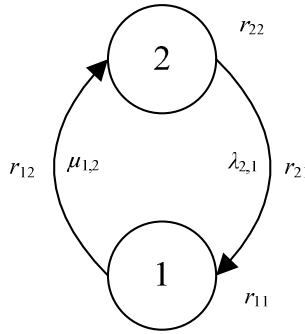


Fig. 2.20 Markov reward model for ice-cream production line

The reward matrix takes the form

$$\mathbf{r} = [r_{ij}] = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} c_p N_{ic} & c_r \\ 0 & r_{prf} \end{bmatrix}$$

and the transition intensity matrix will be as follows:

$$\mathbf{a} = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}.$$

Using (2.85) the following system of differential equations can be written in order to find the expected total rewards $V_1(t)$ and $V_2(t)$:

$$\begin{cases} \frac{dV_1(t)}{dt} = c_p N_{ic} + \mu c_r - \mu V_1(t) + \mu V_2(t), \\ \frac{dV_2(t)}{dt} = r_{prf} + \lambda V_1(t) - \lambda V_2(t). \end{cases}$$

A total expected reward R_T associated with the production line operating during the time interval $[0, T]$ is equal to the expected reward $V_2(t)$ accumulated up to time t , given the initial state of the process at time instant $t = 0$ is state 2.

Using the Laplace–Stieltjes transform under the initial conditions $V_1(0) = V_2(0) = 0$, we transform the system of differential equations into the following system of linear algebraic equations:

$$\begin{cases} sv_1(s) = \frac{c_p L + \mu c_r}{s} - \mu v_1(s) + \mu v_2(s), \\ sv_2(s) = \lambda v_1(s) - \lambda v_2(s), \end{cases}$$

where $v_k(s)$ is the Laplace–Stieltjes transform of a function $V_k(t)$.

The solution of this system is

$$v_2(s) = \frac{\lambda c_p L + \lambda \mu c_r}{s^2(s + \lambda + \mu)}.$$

After applying the inverse Laplace–Stieltjes transform we obtain

$$V_2(t) = L^{-1}\{v_2(s)\} = \frac{\lambda c_p L + \lambda \mu c_r}{(\mu + \lambda)^2} \left[e^{-(\lambda + \mu)t} + (\mu + \lambda)t - 1 \right].$$

The total expected cost C_T during the operation time T is

$$C_T = V_2(T) = \frac{\lambda c_p L + \lambda \mu c_r}{(\mu + \lambda)^2} \left[e^{-(\lambda + \mu)T} + (\mu + \lambda)T - 1 \right].$$

For relatively large T the term $e^{-(\lambda + \mu)T}$ can be neglected and the following approximation can be used:

$$C_T \approx \frac{\lambda(c_p L + \mu c_r)}{\mu + \lambda} T.$$

Therefore, for large T , the total expected reward is a linear function of time and the coefficient

$$c_{un} = \frac{\lambda(c_p L + \mu c_r)}{\mu + \lambda}$$

defines the annual expected cost associated with production line unreliability. For the data given in the example, $c_{un} = \$13.14 \cdot 10^6 \text{ year}^{-1}$.

2.4.2 Computation of Multi-state System Reliability Measures Using Markov Reward Models

In its general form the Markov reward model was intended to provide economic and financial calculations. From the preceding subsection and Example 2.5 it is clear that the Markov reward model is a very useful tool for life cycle cost analysis, and corresponding case studies will be presented in Chapters 6 and 7. However, it was shown by Lubkov and Stepanyans (1978) and Volik *et al.* (1988) that this tool may also be very suitable for reliability analysis and important reliability measures could be easily found by the corresponding determination of the rewards in matrix \mathbf{r} . In these works it was suggested that demand w is constant. The method was extended by Lisnianski (2007) to MSS with variable demand, where demand is assumed to be a continuous-time Markov chain with m different possible states (levels) w_1, \dots, w_m and corresponding constant transition intensities with a given matrix $\mathbf{b} = [b_{ij}]$, $i, j = 1, 2, \dots, m$. Here we apply this method for MSS reliability analysis.

2.4.2.1 Multi-state System with Variable Demand

In the previous section, MSS was considered to have constant demand. In practice, this is often not so. A MSS can fall into a set of unacceptable states in two ways: either through a performance decrease because of failures or through an increase in demand.

For example, consider the demand variation that is typical for power systems. Usually demand can be represented by a daily demand curve. This curve is cyclic in nature with a maximum level (peak) during the day and a minimum level at night (Endrenyi 1979); (Billinton and Allan 1996). Another example is a number of telephone calls arriving during a time unit to a telephone station. In the simplest and most frequently used model, the cyclic demand variation can be approximated by a two-level demand curve as shown in Figure 2.21 (a).

In this model, the demand is represented as a continuous-time Markov chain with two states: $w = \{w_1, w_2\}$ [Figure 2.21 (b)], where w_2 is a peak level of demand and w_1 is a low level. When the cycle time T_c and the mean duration of the peak t_p are known (usually $T_c = 24$ h), the transition intensities of the model can be obtained as

$$\lambda_p = \frac{1}{T_c - t_p}, \quad \lambda_l = \frac{1}{t_p}, \quad (2.87)$$

where λ_p is the transition intensity from a low demand level to a peak level and λ_l is the transition intensity from a peak demand level to a low level.

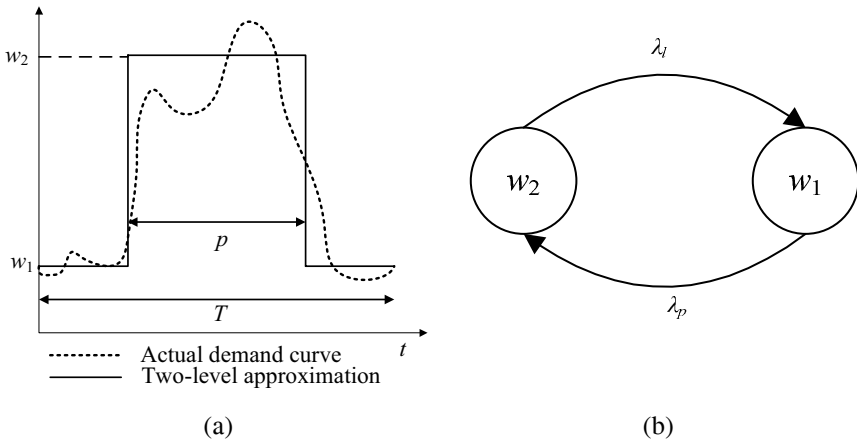


Fig. 2.21 Two-level demand model: (a) approximation of actual demand curve, and (b) state-transition diagram

In the further extension of the variable demand model the demand process can be approximated by defining a set of discrete values $\{w_1, w_2, \dots, w_m\}$ representing different possible demand levels and determining the transition intensities between each pair of demand levels (usually derived from the demand statistics). The realization of the stochastic process of the demand for a specified period and the corresponding state-space diagram are shown in Figure 2.22. b_{ij} is the transition intensity from demand level w_i to demand level w_j .

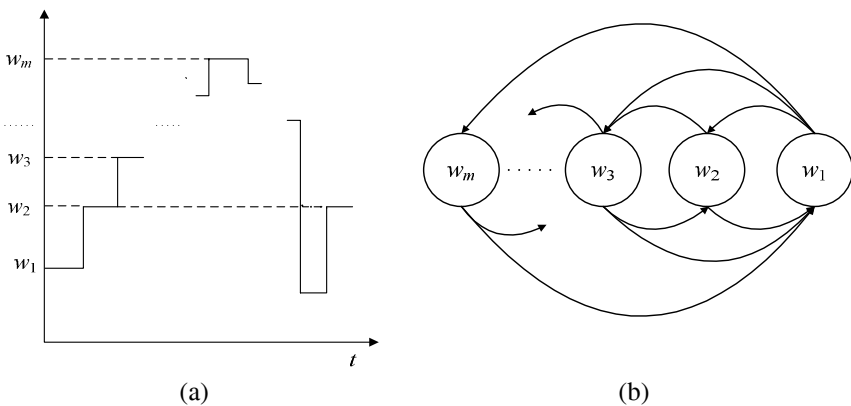


Fig. 2.22 Discrete variable demand: (a) realization of general Markov demand process, and (b) state-transition diagram for general Markov demand process

So, for a general case we assume that demand $W(t)$ is also a random process that can take on discrete values from the set $w = \{w_1, \dots, w_m\}$. The desired relation between the MSS output performance and the demand at any time instant t can be expressed by the acceptability function $\Phi(G(t), W(t))$. The acceptable system states correspond to $\Phi(G(t), W(t)) \geq 0$ and the unacceptable states correspond to $\Phi(G(t), W(t)) < 0$. The last inequality defines the system failure criterion. Usually in power systems, the system generating capacity should be equal to or exceed the demand. Therefore, in such cases the acceptability function takes on the following form:

$$\Phi(G(t), W(t)) = G(t) - W(t) \quad (2.88)$$

and the criterion of state acceptability can be expressed as

$$\Phi(G(t), W(t)) = G(t) - W(t) \geq 0. \quad (2.89)$$

Below we present a general method that proved to be very useful for the computation of system reliability measures when MSS output performance and demand are independent discrete-state continuous-time Markov processes.

2.4.2.2 Combined Performance-demand Model

Consider a MSS where its output performance is represented by a stochastic process $G(t)$ that is described as a continuous-time Markov chain Ch_1 with K different possible states g_1, \dots, g_K and corresponding transition intensities matrix $\mathbf{a} = [a_{ij}]$, $i, j = 1, 2, \dots, K$. Therefore, Ch_1 is a mathematical model for stochastic process $G(t)$ that represents MSS output performance. This process is graphically presented in Figure 2.23, where system output performances for each state are represented inside the ellipses and the state number is presented near the corresponding ellipse. Transition intensities are presented near the arcs connecting the corresponding states. The state with the largest performance g_K is the best state and all the states are ordered according to their capacity, so that $g_K > g_{K-1} > \dots > g_1$.

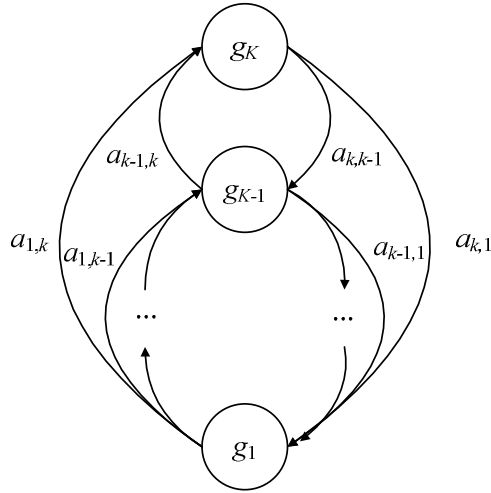


Fig. 2.23 Markov model for MSS output performance

The demand process $W(t)$ is also modeled as a continuous-time Markov chain Ch_2 with m different possible states w_1, \dots, w_m and corresponding constant transition intensities with the matrix $\mathbf{b} = [b_{ij}]$, $i, j = 1, 2, \dots, m$. Ch_2 is a mathematical model for the demand stochastic process $W(t)$ and is graphically represented in Figure 2.24. The demand levels for each state are presented inside the ellipses. As in the previous case, the state number is presented inside the corresponding circle and transition intensities are presented near the corresponding arcs (connecting corresponding states). State m is the state with the largest demand, and all states are ordered according to their demand levels, so that $w_m > w_{m-1} > \dots > w_1$.

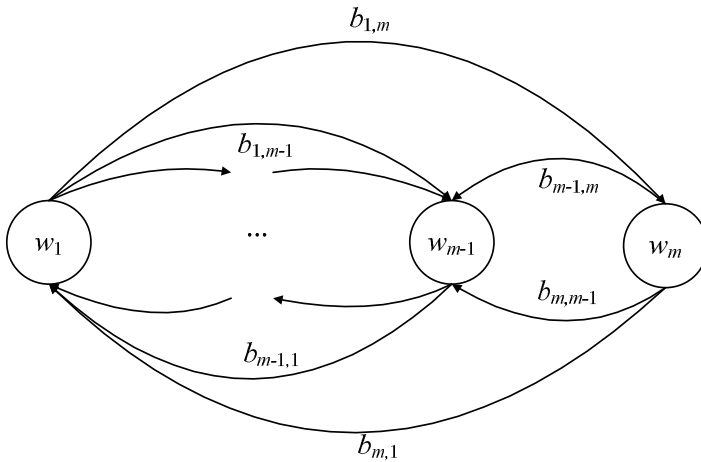


Fig. 2.24 Markov model for MSS demand

The performance and demand models can be combined based on the independence of events in these two models. The probabilities of transitions in each model are not affected by the events that occur in another one. The state-space diagram for the combined m -state demand model and K -state output capacity model is shown in Figure 2.25. Each state in the diagram is labeled by two indices indicating the demand level $w \in \{w_1, \dots, w_m\}$ and the element performance rate $g \in \{g_1, g_2, \dots, g_K\}$.

These indices for each state are presented in the lower part of the corresponding circle. The combined model is considered to have mK states. Each state corresponds to a unique combination of demand levels w_i and element performance g_j and is numbered according to the following rule:

$$z = (i - 1)K + j, \quad (2.90)$$

where z is a state number in the combined performance-demand model,

$$z = 1, \dots, mK;$$

$$i \text{ is the demand level number, } i = 1, \dots, m;$$

$$j \text{ is the MSS output performance level number, } j = 1, \dots, K.$$

In order to designate that state z in a combined performance-demand model corresponds to demand level w_i and performance g_j , we use the form

$$z \sim \{w_i, g_j\}. \quad (2.91)$$

In Figure 2.25 the number of each state is shown in the upper part of the corresponding circle.

In addition to transitions between states with different performance levels, there are transitions between states with the same performance levels but with different demand levels. All intensities of horizontal transitions are defined by transition intensities $b_{i,j}$, $i, j = 1, \dots, m$ of the Markov demand model Ch₂, and all intensities of vertical transitions are defined by transition intensities $a_{i,j}$, $i, j = 1, \dots, K$ of the performance model Ch₁. All other (diagonal) transitions are forbidden. We designate the transition intensity matrix for the combined performance-demand model as $\mathbf{c} = [c_{ij}]$, where $i, j = 1, 2, \dots, mK$.

Thus, the algorithm of the combined performance-demand model building based on separate performance and demand models Ch₁ and Ch₂ can be presented by the following steps.

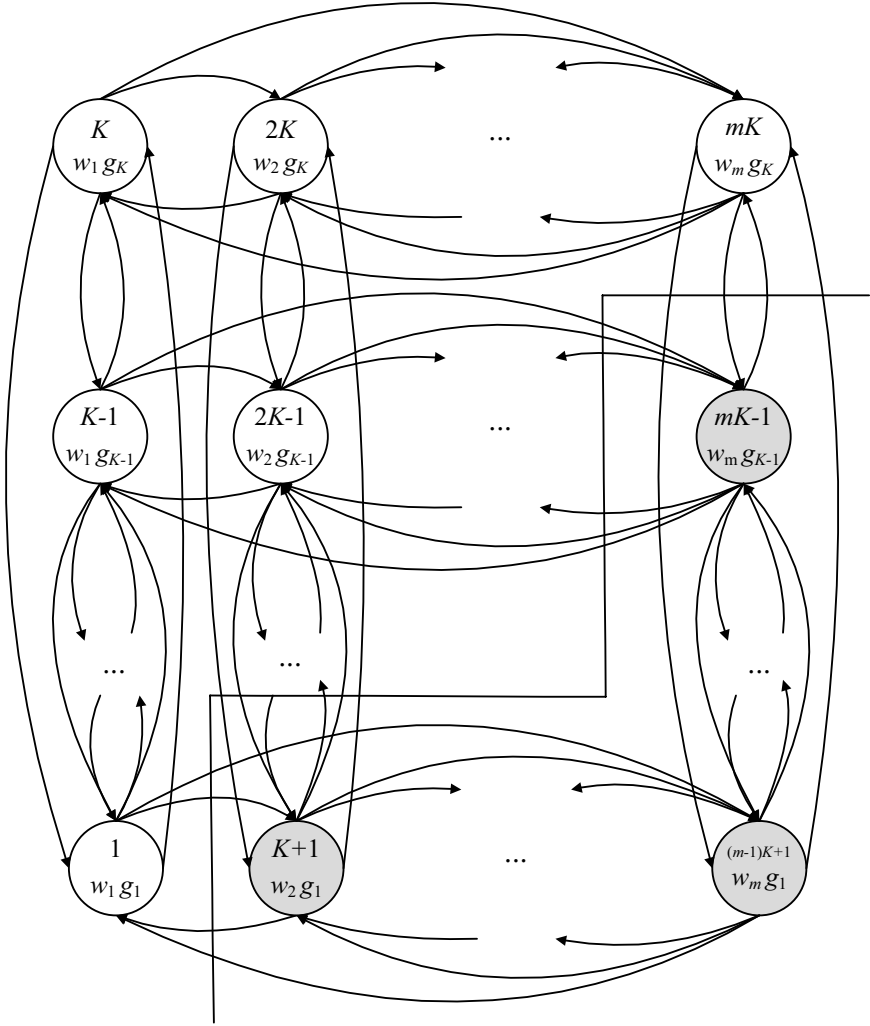


Fig. 2.25 Combined performance–demand model [Unacceptable states are grey]

Algorithm

1. The state-space diagram of a combined performance-demand model is shown in Figure 2.25, where the nodes represent system states and the arcs represent corresponding transitions.
2. The graph consists of mK nodes that should be ordered in K rows and m columns.
3. Each state (node) should be numbered according to rule (2.90).

4. All intensities c_{z_1, z_2} of horizontal transitions from state z_1 (corresponding to demand w_i and performance g_j) to state z_2 (corresponding to demand w_s and the same performance g_j according to rule (2.90)) are defined by demand transition intensities matrix \mathbf{b} ,

$$c_{z_1, z_2} = b_{i, s}, \quad (2.92)$$

where $z_1 \sim \{w_i, g_j\}, z_2 \sim \{w_s, g_j\}, i, s = 1, \dots, m, j = 1, \dots, K$.

5. All intensities of vertical transitions from state z_1 (corresponding to demand w_i and performance g_j) to state z_3 (corresponding to the same demand w_i and performance g_t according to rule (2.90)) are defined by the performance transition intensities matrix \mathbf{a} ,

$$c_{z_1, z_3} = a_{j, t}, \quad (2.93)$$

where

$$z_1 \sim \{w_i, g_j\}, z_3 \sim \{w_i, g_t\}, i = 1, \dots, m, j, t = 1, \dots, K. \quad (2.94)$$

6. All diagonal transitions are forbidden so that the corresponding transitions' intensities in matrix \mathbf{c} are zeroed.

2.4.2.3 Reward Determination for Computation of Multi-state System Reliability Indices

In the previous subsection we built the combined performance-demand model and, therefore, defined its transition intensity matrix \mathbf{c} based on matrices \mathbf{a} and \mathbf{b} for performance and demand processes.

When the combined performance-demand model is built we can consider it as a continuous-time Markov chain with a set of states $\{1, \dots, mK\}$ and a transition intensity matrix $\mathbf{c} = [c_{ij}]$, $i, j = 1, \dots, mK$. In general it is assumed that a certain reward r_{ii} is associated with the process of staying in any state i during a time unit. It is also assumed that each time the process transits from state i to state j a reward r_{ij} is associated with this transition.

Let $V_i(t)$ be the expected total reward accumulated up to time t , given the initial state of the process at time instant $t = 0$ is in state i . According to (2.80), the following system of differential equations must be solved under specified initial

conditions in order to find the total expected rewards for the combined performance-demand model:

$$\frac{dV_i(t)}{dt} = r_{ii} + \sum_{\substack{j=1 \\ j \neq i}}^{mK} c_{ij} r_{ij} + \sum_{j=1}^{mK} c_{ij} V_j(t), \quad i = 1, \dots, mK. \quad (2.95)$$

In the most common case, the MSS begins to accumulate rewards after time instant $t = 0$, therefore, the initial conditions are

$$V_i(0) = 0, \quad i = 1, \dots, mK. \quad (2.96)$$

If, for example, the state number K (Figure 2.25) with the highest performance level and the lowest demand level is defined as the initial state, the value $V_K(t)$ should be found as a solution of system (2.95).

In order to find reliability measures for a MSS the specific reward matrix \mathbf{r} should be defined for each measure. Based on the combined performance-demand model, the theory of the Markov reward processes can be applied for computation of reliability measures for Markov MSS. As was said above, we assume that demand $W(t)$ and MSS output performance $G(t)$ are mutually independent continuous-time Markov chains.

MSS average availability $\bar{A}(T)$ is defined as a mean fraction of time when the system resides in the set of acceptable states during the time interval $[0, T]$,

$$\bar{A}(T) = \frac{1}{T} \int_0^T A(t) dt, \quad (2.97)$$

where $A(t)$ is the instantaneous (point) availability – the probability that the MSS at instant $t > 0$ is in one of the acceptable states:

$$A(t) = \Pr\{\Phi(G(t), W(t)) \geq 0\}. \quad (2.98)$$

As was shown in the previous section, $A(t)$ can be found by solving differential equations (2.35) and summarizing the probabilities corresponding to all acceptable states. But based on the Markov reward model MSS average availability $\bar{A}(T)$ may be found more easily without using expression (2.97). For this purpose the rewards in matrix \mathbf{r} for the combined performance-demand model should be determined in the following manner:

- The rewards associated with all acceptable states should be defined as 1.
- The rewards associated with all unacceptable states should be zeroed as well as all the rewards associated with the transitions.

The mean reward $V_i(T)$ accumulated during the interval $[0, T]$ defines how long the power system will be in the set of acceptable states in the case where state i is the initial state. This reward should be found as a solution of system (2.95) under initial conditions (2.96). After solving (2.95) and finding $V_i(t)$, the MSS average availability can be obtained for each different initial state $i = 1, 2, \dots, mK$:

$$\bar{A}_i(T) = \frac{V_i(T)}{T}. \quad (2.99)$$

Usually the state K with the greatest performance level and minimum demand level is determined as an initial state.

The mean number $N_{fi}(T)$ of MSS failures during the time interval $[0, T]$, if state i is the initial state, can be treated as a mean number of MSS entrances into the set of unacceptable states during the time interval $[0, T]$. For its computation, the rewards associated with each transition from the set of acceptable states to the set of unacceptable states should be defined as 1. All other rewards should be zeroed.

In this case the mean accumulated reward $V_i(T)$, obtained by solving (2.95) provides the mean number of entrances into the unacceptable area during the time interval $[0, T]$:

$$N_{fi}(T) = V_i(T). \quad (2.100)$$

When the mean number of system failures is computed, the corresponding *frequency of failures or frequency of entrances into the set of unacceptable states* can be found:

$$f_{fi}(T) = \frac{1}{N_{fi}(T)}. \quad (2.101)$$

Expected accumulated performance deficiency (EAPD) can be defined as mean performance deficiency accumulated within the interval $[0, T]$. The rewards for any state number $z = (i-1)K + j$, in a combined model, where $w_j - g_i > 0$, should be defined as $r_{zz} = w_j - g_i$. All other rewards should be zeroed. Therefore, the mean reward $V_i(T)$ accumulated during the time interval $[0, T]$, if state i is in the initial state, defines the mean accumulated performance deficiency:

$$EAPD_i = V_i(T) = E \left\{ \int_0^T (W(t) - G(t)) dt \right\}. \quad (2.102)$$

Mean time to failure (MTTF) is the mean time up to the instant when the system enters the subset of unacceptable states for the first time. For its computation the

combined performance-demand model should be transformed – all transitions that return the MSS from an unacceptable states should be forbidden, as in this case all unacceptable states should be treated as absorbing states.

In order to assess MTTF for a MSS, the rewards in matrix \mathbf{r} for the transformed performance-demand model should be determined as follows:

- The rewards associated with all acceptable states should be defined as 1.
- The reward associated with unacceptable (absorbing) states should be zeroed, as should all rewards associated with transitions.

In this case, the mean accumulated reward $V_i(t)$ defines the mean time accumulated up to the first entrance into the subset of unacceptable states (MTTF), if state i is the initial state.

Probability of system failure during the time interval $[0, T]$ The combined performance-demand model should be transformed as in the previous section for calculating the MSS reliability function – all unacceptable states should be treated as absorbing states and, therefore, all transitions that return the system from unacceptable states should be forbidden.

- Rewards associated with all transitions to the absorbing state should be defined as 1.
- All other rewards should be zeroed.

The mean accumulated reward $V_i(T)$ in this case defines the probability of system failure during the time interval $[0, T]$ if state i is the initial state. Therefore, the MSS reliability function can be obtained as

$$R_i(T) = 1 - V_i(T), \quad i = 1, \dots, K. \quad (2.103)$$

Example 2.6 Consider reliability evaluation for a power system, whose output generating capacity is represented by a continuous-time Markov chain with three states. The corresponding capacity levels for states 1, 2, and 3 are $g_1 = 0$, $g_2 = 70$ MW, $g_3 = 100$ MW, respectively, and the transition intensity matrix is as the follows:

$$\mathbf{a} = [a_{ij}] = \begin{bmatrix} -500 & 0 & 500 \\ 0 & -1000 & 1000 \\ 1 & 10 & -11 \end{bmatrix}.$$

All intensities a_{ij} are represented in such units as 1/year.

The corresponding capacity model Ch_1 is graphically shown in Figure 2.27 (a).

The demand for the power system is also represented by a continuous-time Markov chain with three possible levels $w_1 = 0$, $w_2 = 60$, $w_3 = 90$. This demand is shown graphically in Figure 2.26.

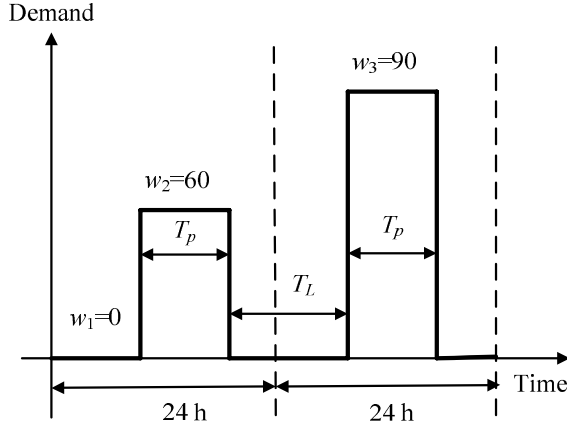


Fig. 2.26 Daily demand for the power system

Daily peaks w_2 and w_3 occur twice a week and five times a week, respectively, and the mean duration of the daily peak is $T_p = 8$ h. The mean duration of low demand level $w_1 = 0$ is defined as $T_L = 24 - 8 = 16$ h.

According to the approach presented in Endrenyi (1979) that is justified for a power system, peak duration and low level duration are assumed to be exponentially distributed random values.

The acceptability function is given: $\Phi(G(t), W(t)) = G(t) - W(t)$. Therefore, a failure is treated as an entrance into the state where the acceptability function is negative or $G(t) < W(t)$.

Find the mean number of generator entrances into the set of unacceptable states during the time interval $[0, T]$.

Solution. Markov performance model Ch₁ corresponding to the given capacity levels $g_1 = 0$, $g_2 = 70$, $g_3 = 100$ and transition intensity matrix **a** is graphically shown in Figure 2.27 (a).

Markov demand model Ch₂ is shown in Figure 2.27 (b). States 1, 2, and 3 represent the corresponding demand levels w_1 , w_2 , and w_3 . Transition intensities are such as follows:

$$b_{21} = b_{31} = \frac{1}{T_p} = \frac{1}{8} \text{ h}^{-1} = 1110 \text{ years}^{-1},$$

$$b_{12} = \frac{2}{7} \cdot \frac{1}{T_L} = \frac{2}{7} \cdot \frac{1}{16} = 0.0179 \text{ h}^{-1} = 156 \text{ years}^{-1},$$

$$b_{13} = \frac{5}{7} \cdot \frac{1}{T_L} = \frac{5}{7} \cdot \frac{1}{16} = 0.0446 \text{ h}^{-1} = 391 \text{ years}^{-1}.$$

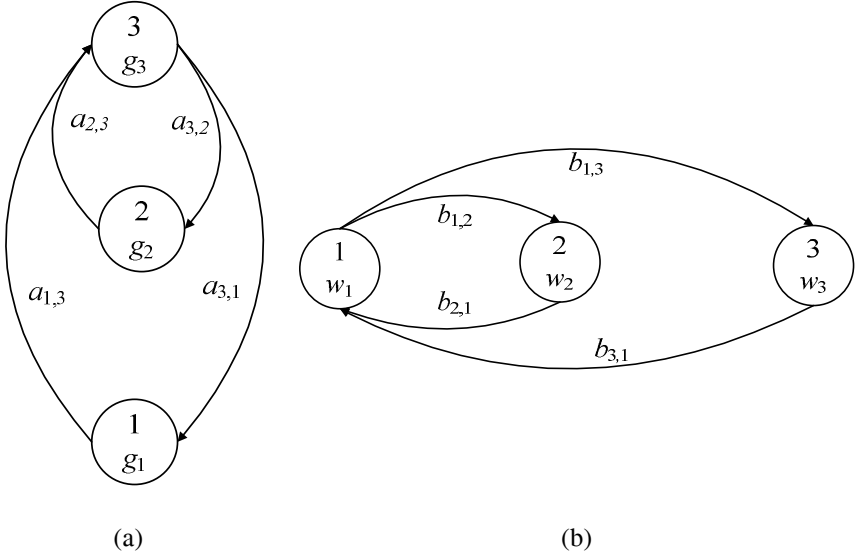


Fig. 2.27 Output performance model (a) and demand model (b)

There are no transitions between states 2 and 3, therefore $b_{23} = b_{32} = 0$.

Taking into account the sum of elements in each row of the matrix to be zero, we can find the diagonal elements in the matrix.

Therefore, a transition intensity matrix \mathbf{b} for the demand takes the form:

$$\mathbf{b} = [b_{ij}] = \begin{bmatrix} -547 & 156 & 391 \\ 1110 & -1110 & 0 \\ 1110 & 0 & -1110 \end{bmatrix}.$$

All intensities b_{ij} are also represented in 1/year.

By using the suggested method we find the *mean number $N_f(T)$ of system failures during the time interval $[0, T]$* if the state with maximal generating capacity and minimal demand level is given as the initial state.

First, the combined performance-demand model should be built according to the algorithm presented above. The model consists of $mK = 3 \times 3 = 9$ states (nodes) that should be ordered in $K = 3$ rows and $m = 3$ columns. Each state should be numbered according to rule (2.41). All intensities of horizontal transitions from state $z_1 \sim \{w_i, g_j\}$ to state $z_2 \sim \{w_s, g_j\}$, $i, s = \overline{1, 3}$, $j = \overline{1, 3}$ are defined by demand transition intensity matrix \mathbf{b}

$$c_{z_1 z_2} = b_{i, s}.$$

All intensities of vertical transitions from state $z_1 \sim \{w_i, g_j\}$ to state $z_3 \sim \{w_i, g_t\}$, $i = \overline{1,3}$, $j, t = \overline{1,3}$, are defined by the capacity transition intensity matrix **a**

$$c_{z_1 z_3} = a_{j,t}.$$

All diagonal transitions are forbidden; therefore, the corresponding transition intensities in matrix **c** are zeroed.

The state-space diagram for the combined performance-demand Markov model for this example is shown in Figure 2.28.

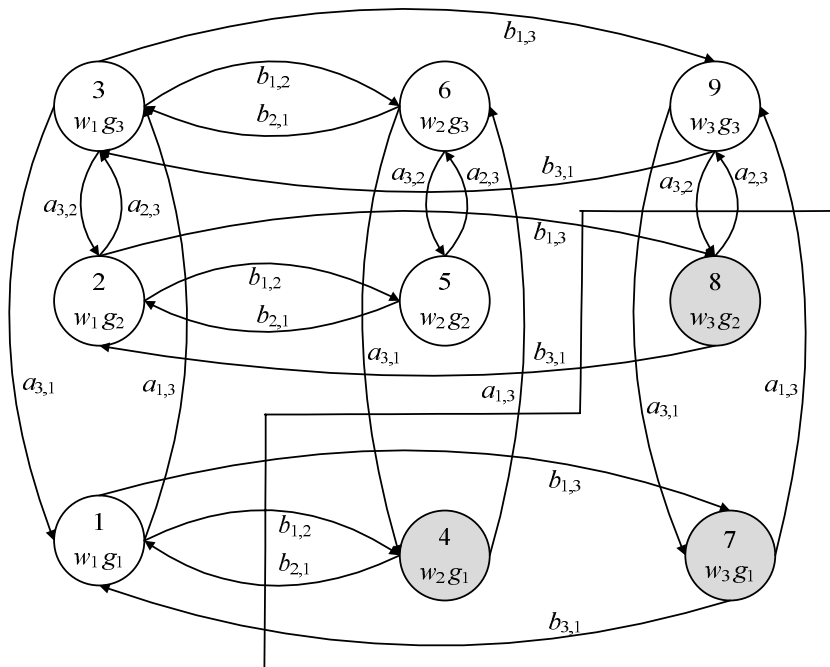


Fig. 2.28 Combined performance-demand model [Unacceptable states are grey]

Corresponding transition intensity matrix **c** for the combined performance-demand model can be written as follows:

$$\mathbf{c} = [c_{ij}] = \begin{bmatrix} x_1 & 0 & a_{1,3} & 0 & 0 & 0 & b_{1,3} & 0 & 0 \\ 0 & x_2 & a_{2,3} & 0 & 0 & 0 & 0 & b_{1,3} & 0 \\ a_{1,3} & a_{3,2} & x_3 & 0 & 0 & 0 & 0 & 0 & b_{1,3} \\ 0 & 0 & 0 & x_4 & 0 & a_{1,3} & b_{2,3} & 0 & 0 \\ 0 & 0 & 0 & 0 & x_5 & a_{2,3} & 0 & b_{2,3} & 0 \\ 0 & 0 & 0 & a_{3,1} & a_{3,2} & x_6 & 0 & 0 & b_{2,3} \\ b_{3,1} & 0 & 0 & b_{3,2} & 0 & 0 & x_7 & 0 & a_{1,3} \\ 0 & b_{3,1} & 0 & 0 & b_{3,2} & 0 & 0 & x_8 & a_{2,3} \\ 0 & 0 & b_{3,1} & 0 & 0 & b_{3,2} & a_{3,1} & a_{3,2} & x_9 \end{bmatrix}$$

where

$$\begin{aligned} x_1 &= -a_{1,3} - b_{1,3}, & x_2 &= -a_{2,3} - b_{3,1}, & x_3 &= -a_{1,3} - a_{3,2} - b_{1,3}, \\ x_4 &= -a_{1,3} - b_{2,3}, & x_5 &= -a_{2,3} - b_{2,3}, & x_6 &= -a_{3,1} - a_{3,2} - b_{2,3}, \\ x_7 &= -a_{1,3} - b_{3,1} - b_{3,2}, & x_8 &= -a_{2,3} - b_{3,1} - b_{3,2}, & x_9 &= -a_{3,1} - a_{3,2} - b_{1,3} - b_{3,2}. \end{aligned}$$

The state with the maximum performance $g_3 = 100$ MW and the minimum demand $w_1 = 0$ (state 3) is given as the initial state. In states 2, 5, and 8 the MSS performance is 70 MW, in states 3, 6, and 9 it is 100 MW, and in states 1, 4, and 7 it is 0. In states 4, 7, and 8 the MSS performance is lower than the demand. These states are unacceptable and have a performance deficiency: $D_4 = w_2 - g_1 = -60$ MW, $D_7 = w_3 - g_1 = -90$ MW, and $D_8 = w_3 - g_2 = -70$ MW. States 1, 2, 3, 5, 6, and 9 constitute the set of acceptable states.

In order to find the *mean number of failures* the reward matrix should be defined according to the suggested method. Each reward associated with transition from the set of acceptable states to the set of unacceptable states should be defined as 1. All other rewards should be zeroed. Therefore, in a reward matrix $r_{14} = r_{17} = r_{28} = r_{98} = r_{97} = 1$ and all other rewards are zeroes. So, reward matrix \mathbf{r} is obtained:

$$\mathbf{r} = [r_{ij}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

The corresponding system of differential equations is as follows:

$$\begin{aligned}
\frac{dV_1(t)}{dt} &= b_{1,3} - (a_{1,3} + b_{1,3})V_1(t) + a_{1,3}V_3(t) + b_{1,3}V_7(t), \\
\frac{dV_2(t)}{dt} &= b_{1,3} - (a_{2,3} + b_{1,3})V_2(t) + a_{2,3}V_3(t) + b_{1,3}V_8(t), \\
\frac{dV_3(t)}{dt} &= a_{1,3}V_1(t) + a_{3,2}V_2(t) - (a_{1,3} + b_{1,3} + a_{3,2})V_3(t) + b_{1,3}V_9(t), \\
\frac{dV_4(t)}{dt} &= - (a_{1,3} + b_{2,3})V_4(t) + a_{1,3}V_6(t) + b_{2,3}V_7(t), \\
\frac{dV_5(t)}{dt} &= b_{2,3} - (a_{2,3} + b_{2,3})V_5(t) + a_{2,3}V_6(t) + b_{2,3}V_8(t), \\
\frac{dV_6(t)}{dt} &= a_{3,1} + a_{3,1}V_4(t) + a_{3,2}V_5(t) - (a_{3,1} + a_{3,2} + b_{2,3})V_6(t) + b_{2,3}V_9(t), \\
\frac{dV_7(t)}{dt} &= b_{3,1}V_1(t) + b_{3,2}V_4(t) - (a_{1,3} + b_{3,1} + b_{3,2})V_7(t) + a_{1,3}V_9(t), \\
\frac{dV_8(t)}{dt} &= b_{3,1}V_2(t) + b_{3,2}V_5(t) - (a_{2,3} + b_{3,1} + b_{3,2})V_8(t) + a_{2,3}V_9(t), \\
\frac{dV_9(t)}{dt} &= a_{3,1} + a_{3,2} + b_{3,1}V_3(t) + b_{3,2}V_6(t) + a_{3,1}V_7(t) + a_{3,2}V_8(t) \\
&\quad - (a_{3,1} + a_{3,2} + b_{3,1} + b_{3,2})V_9(t).
\end{aligned}$$

By solving the system of these differential equations under the initial conditions $V_i(t) = 0$, $i = 1, \dots, 9$ all expected rewards $V_i(t)$, $i = 1, \dots, 9$ can be found as functions of time t .

The state $K=3$, in which the system has a maximum capacity level and a minimum demand, is given as the initial state. Then, according to expression (2.100) the value $V_3(T)$ is treated as the mean numbers of system entrances into the area of unacceptable states or the mean number of power system failures during the time interval $[0, T]$. The function $N_{f3}(t) = V_3(t)$ is graphically presented in Figure 2.29, where $N_{f3}(t)$ is the mean number of system failures in the case, when state 3 is an initial state.

The function $N_{f1}(t) = V_1(t)$ characterizes the mean number of system failures in the case where state 1 is given as the initial state. It is also presented in this figure. As shown, $N_{f3}(t) < N_{f1}(t)$ because state 1 is “closer” to the set of unacceptable states – it has the direct transition to the set in the unacceptable area and state 3 does not. Therefore, at the beginning of the process the system’s entrance into the set of unacceptable states is more likely from state 1 than from state 3. Figure 2.29 (a) graphically represents a number of power system failures for a short period – only 8 d. However, after this short period the function $N_{f3}(t)$ will

be a linear function. The reliability evaluation is usually performed over an extended period (years). See Figure 2.29 (b). For example, for 1 year we obtain $N_{f3}(T = 1 \text{ year}) \approx 132$.

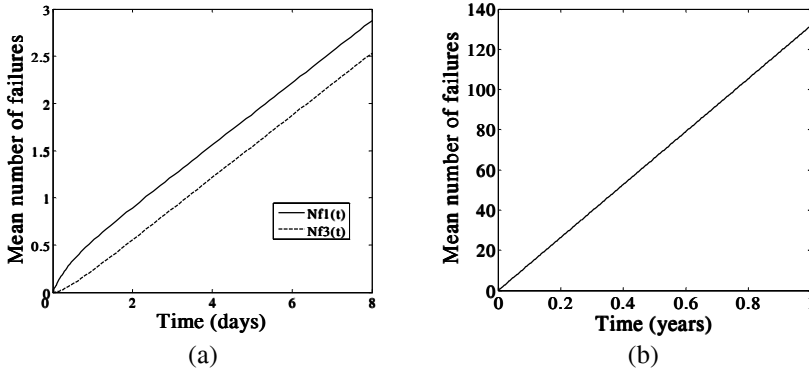


Fig. 2.29 Mean number of generator entrances to the set of unacceptable states: (a) – short time period, and (b) – 1 year time period

According to (2.101) the frequency of the power system failures can be obtained:

$$f_{f3} = \frac{1}{N_{f3}} = 0.0076 \text{ year}^{-1}.$$

2.5 Semi-Markov Models

As was mentioned above, a discrete-state, continuous-time stochastic process can only be represented as a continuous-time Markov chain when the transition time between any states is distributed exponentially. This fact seriously restricts the application of the Markov chain model to real-world problems. One of the ways to investigate processes with arbitrarily distributed sojourn times is to use a semi-Markov process model. The main advantage of a semi-Markov model is that it allows non-exponential distributions for transitions between states and generalizes several kinds of stochastic processes. Since in many real cases the lifetime and repair times are not exponential, this is very important.

The semi-Markov processes were introduced almost simultaneously by Levy (1954) and Smith (1955). At the same time, Takacs (1954) introduced essentially the same type of processes and applied them to some problems in counter theory. The foundations of the theory of semi-Markov processes can be found in Cinlar (1975), Gihman and Skorohod (2004), Korolyuk and Swishchuk (1995), and

Silverstov (1980). For readers interested in the field of semi-Markov processes applications to the reliability theory and performability analysis, the following books may be especially recommended: Limnios and Oprisan (2000), Kovalenko *et al.* (1997), and Sahner *et al.* (1996). Some interesting examples one can find in Grabski and Kolowrocki (1999). Using of Petri nets in semi-Markov process evaluation is also proved to be effective (Ulmeanu and Ionescu 1999).

The general theory of semi-Markov processes is quite complex. Here we study some aspects of reliability evaluation based on using semi-Markov processes that do not involve very complex computations. In many real-world problems, relatively simple computation procedures allow engineers to assess the reliability of MSSs with arbitrary transition times without Monte-Carlo simulation. This especially relates to MSS steady-state behavior.

2.5.1 Embedded Markov Chain and Definition of Semi-Markov Process

In order to define a semi-Markov process, consider a system that at any time instant $t \geq 0$ can be in one of various possible states g_1, g_2, \dots, g_K . The system behavior is defined by the discrete-state continuous-time stochastic performance process $G(t) \in \{g_1, g_2, \dots, g_K\}$. We assume that the initial state i of the system and one-step transition probabilities are given as follows:

$$\begin{aligned} G(0) &= g_i, \quad i \in \{1, \dots, K\}, \\ \pi_{jk} &= P\{G(t_m) = g_k \mid G(t_{m-1}) = g_j\}, \quad j, k \in \{1, \dots, K\}. \end{aligned} \quad (2.104)$$

Here π_{jk} is the probability that the system will transit from state j with performance rate g_j to state k with performance rate g_k . Probabilities π_{jk} , $j, k \in \{1, \dots, K\}$ define the one-step transition probability matrix $\pi = [\pi_{jk}]$ for the discrete-time chain $G(t_m)$, where transitions from one state to another may happen only at discrete time moments $t_1, t_2, \dots, t_{m-1}, t_m, \dots$. Such a Markov chain $G(t_m)$ is called Markov chain embedded in stochastic process $G(t)$, or *embedded Markov chain* for short.

To each $\pi_{jk} \neq 0$ a random variable corresponds T_{jk}^* with the cumulative distribution function

$$F_{jk}^*(t) = P\{T_{jk}^* \leq t\} \quad (2.105)$$

and probability density function $f_{jk}^*(t)$. This random variable is called a conditional sojourn time in state j and characterizes the system sojourn time in the j under condition that the system transits from state j to state k .

The graphical interpretation of possible realization of the considered process is shown in Figure 2.30. At the initial time instant $G(0) = g_i$. The process transits to state j (with performance rate g_j) from the initial state i with probability π_{ij} . Therefore, if the next state is state j , the process remains in state i during random time T_{ij}^* with cdf $F_{ij}^*(t)$. When the process transits to state j , the probability of the transition from this state to any state k is π_{jk} . If the system transits from state j to state k , it remains in state j during random time T_{jk}^* with cdf $F_{jk}^*(t)$ up to the transition to state k .

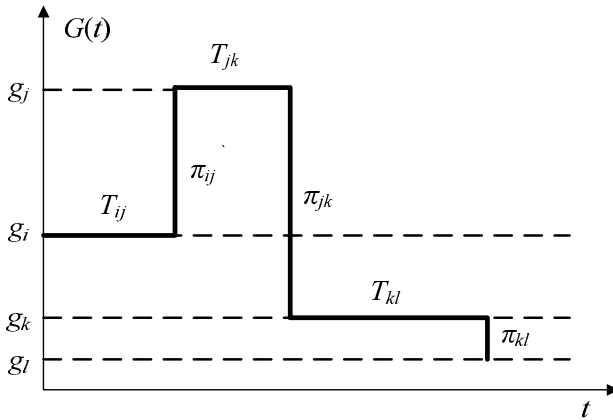


Fig. 2.30 Semi-Markov stochastic process

This process can be continued over an arbitrary period T .

The described stochastic process $G(t)$ is called a *semi-Markov process* if each time the next state and the corresponding sojourn time in the current state must be chosen independently of the previous history of the process.

The chain $G(t_m)$ in this case will be a Markov chain with one-step transition probabilities π_{jk} , $j, k \in \{1, \dots, K\}$ and be called an *embedded Markov chain*.

So, in order to define the semi-Markov process one has to define the initial state of the process and the matrices $\pi = [\pi_{jk}]$ and $\mathbf{F}^*(t) = [F_{ij}^*(t)]$ for $i, j \in \{1, \dots, K\}$.

Note that the process in which the arbitrarily distributed times between transitions are ignored and only time instants of transitions are of interest is a homogeneous discrete-time Markov chain. However, in a general case, if one takes into

account the sojourn times in different states, the process does not have Markov properties. (It remains a Markov process only if all the sojourn times are distributed exponentially.) Therefore, the process can be considered a Markov process only at time instants of transitions. This explains why the process was named semi-Markov.

The most general definition of the semi-Markov process is based on kernel matrix $\mathbf{Q}(t)$. Each element $Q_{ij}(t)$ of this matrix determines the probability that a *one-step transition* from state i to state j occurs during the time interval $[0, T]$. Using a kernel matrix, *one-step transition probabilities for embedded Markov chain* can be obtained as

$$\pi_{ij} = \lim_{t \rightarrow \infty} Q_{ij}(t) \quad (2.106)$$

and the CDF $F_{ij}^*(t)$ of the conditional sojourn time in state i can be obtained as

$$F_{ij}^*(t) = \frac{1}{\pi_{ij}} Q_{ij}(t). \quad (2.107)$$

Based on the kernel matrix, the CDF $F_i(t)$ of unconditional sojourn time T_i in any state i can be defined as

$$F_i(t) = \sum_{j=1}^K Q_{ij}(t) = \sum_{j=1}^K \pi_{ij} F_{ij}^*(t). \quad (2.108)$$

Hence, for pdf of the unconditional sojourn time in state i with performance rate g_i , we can write

$$f_i(t) = \frac{d}{dt} F_i(t) = \sum_{j=1}^K \pi_{ij} f_{ij}^*(t). \quad (2.109)$$

Based on (2.109), the mean unconditional sojourn time in state i can be obtained as

$$\bar{T}_i = \int_0^{\infty} t f_i(t) dt = \sum_{j=1}^K \pi_{ij} \bar{T}_{ij}^*, \quad (2.110)$$

where \bar{T}_{ij}^* is the mean conditional sojourn time in state i given that the system transits from state i to state j .

Kernel matrix $\mathbf{Q}(t)$ and the initial state completely define the stochastic behavior of a semi-Markov process.

In practice, when MSS reliability is studied, in order to find the kernel matrix for a semi-Markov process, one can use the following considerations (Lisnianski and Yeager 2000). Transitions between different states are usually executed as consequences of such events as failures, repairs, inspections, etc. For every type of event, the cdf of time between them is known. The transition is realized according to the event that occurs first in a competition among the events.

In Figure 2.31, one can see a state-transition diagram for the simplest semi-Markov process with three possible transitions from initial state 0. The process will transit from state 0 to states 1, 2, and 3 when events of some different types 1, 2, and 3, respectively, occurs. The time between events of type 1 is random variable $T_{0,1}$ distributed according to CDF $F_{0,1}(t)$. If an event of type 1 occurs first, the process transits from state 0 to state 1. The random variable $T_{0,2}$ that defines the time between events of type 2 is distributed according to cdf $F_{0,2}(t)$. If an event of type 2 occurs earlier than other events, the process transits from state 0 to state 2.

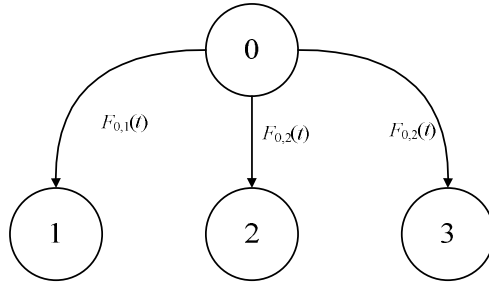


Fig. 2.31 State-transition diagram of simplest semi-Markov process

The time between events of type 3 is random variable $T_{0,3}$ distributed according to cdf $F_{0,3}(t)$. If an event of type 3 occurs first, the process transits from state 0 to state 3.

The probability $Q_{01}(t)$ that the process will transit from state 0 to state 1 up to time t (the initial time $t = 0$) may be determined as the probability that under condition $T_{0,1} \leq t$, the random variable $T_{0,1}$ is less than variables $T_{0,2}$ and $T_{0,3}$. Hence, we have

$$\begin{aligned}
 Q_{01}(t) &= \Pr\{(T_{0,1} \leq t) \& (T_{0,2} > t) \& (T_{0,3} > t)\} \\
 &= \int_0^t dF_{0,1}(u) \int_t^\infty dF_{0,2}(u) \int_t^\infty dF_{0,3}(u) \\
 &= \int_0^t [1 - F_{0,2}(u)] [1 - F_{0,3}(u)] dF_{0,1}(u).
 \end{aligned} \tag{2.111}$$

In the same way we obtain

$$Q_{02}(t) = \int_0^t [1 - F_{0,1}(u)] [1 - F_{0,3}(u)] dF_{0,2}(u), \quad (2.112)$$

$$Q_{03}(t) = \int_0^t [1 - F_{0,1}(u)] [1 - F_{0,2}(u)] dF_{0,3}(u). \quad (2.113)$$

For a semi-Markov process with the state-transition diagram presented in Figure 2.31, we have the following kernel matrix:

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & Q_{01}(t) & Q_{02}(t) & Q_{03}(t) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \quad (2.114)$$

Expressions (2.112) – (2.114) can be easily generalized to the arbitrary number of possible transitions from initial state 0.

In order to demonstrate the technique of kernel matrix computation we consider the following example.

Example 2.7 We consider the simplest system with a state-transition diagram as shown in Figure 2.31. Two random variables $T_{0,1}$ and $T_{0,2}$ are exponentially distributed with CDFs $F_{0,1}(t) = 1 - e^{-\lambda_{0,1}t}$ and $F_{0,2}(t) = 1 - e^{-\lambda_{0,2}t}$, respectively, and the third random variable $T_{0,3}$ has the following CDF:

$$F_{0,3}(t) = \begin{cases} 0, & \text{if } t < T_c, \\ 1, & \text{if } t \geq T_c \end{cases}$$

(such a CDF corresponds to the arrival of events with constant period T_c).

Find:

1. one-step transition probabilities $Q_{01}(t)$, $Q_{02}(t)$, $Q_{03}(t)$ for the kernel matrix;
2. cumulative distribution function for unconditional sojourn time T_0 in state 0;
3. one-step transition probabilities for the embedded Markov chain.

Solution. Using (2.111) – (2.113) we obtain one-step probabilities for the kernel matrix:

$$\begin{aligned}
Q_{01}(t) &= \begin{cases} \frac{\lambda_{0,1}}{\lambda_{0,1} + \lambda_{0,2}} [1 - e^{-(\lambda_{0,1} + \lambda_{0,2})t}], & \text{if } t < T_c, \\ \frac{\lambda_{0,1}}{\lambda_{0,1} + \lambda_{0,2}} [1 - e^{-(\lambda_{0,1} + \lambda_{0,2})T_c}], & \text{if } t \geq T_c, \end{cases} \\
Q_{02}(t) &= \begin{cases} \frac{\lambda_{0,2}}{\lambda_{0,1} + \lambda_{0,2}} [1 - e^{-(\lambda_{0,1} + \lambda_{0,2})t}], & \text{if } t < T_c, \\ \frac{\lambda_{0,2}}{\lambda_{0,1} + \lambda_{0,2}} [1 - e^{-(\lambda_{0,1} + \lambda_{0,2})T_c}], & \text{if } t \geq T_c, \end{cases} \\
Q_{03}(t) &= \begin{cases} 0, & \text{if } t < T_c, \\ e^{-(\lambda_{0,1} + \lambda_{0,2})T_c}, & \text{if } t \geq T_c. \end{cases}
\end{aligned}$$

According to (2.108), unconditional sojourn time T_0 in state 0 is distributed as follows:

$$F_0(t) = \sum_{j=1}^3 Q_{0j}(t) = \begin{cases} 1 - e^{-(\lambda_{0,1} + \lambda_{0,2})t}, & \text{if } t < T_c, \\ 1, & \text{if } t \geq T_c. \end{cases}$$

One-step transition probabilities for embedded Markov chain are defined according to (2.106):

$$\begin{aligned}
\pi_{01} &= \frac{\lambda_{0,1}}{\lambda_{0,1} + \lambda_{0,2}} [1 - e^{-(\lambda_{0,1} + \lambda_{0,2})T_c}], \\
\pi_{02} &= \frac{\lambda_{0,2}}{\lambda_{0,1} + \lambda_{0,2}} [1 - e^{-(\lambda_{0,1} + \lambda_{0,2})T_c}], \\
\pi_{03} &= e^{-(\lambda_{0,1} + \lambda_{0,2})T_c}.
\end{aligned}$$

2.5.2 Evaluation of Reliability Indices Based on Semi-Markov Processes

In order to find the MSS reliability indices, the system state-space diagram should be built as was done in previous sections for Markov processes. The only difference is that, in the case of the semi-Markov model, the transition times may be distributed arbitrarily. Based on transition time distributions $F_{i,j}(t)$, the kernel matrix $\mathbf{Q}(t)$ should be defined according to the method presented in the previous section.

The main problem of semi-Markov process analysis is to find the state probabilities. Let $\theta_{ij}(t)$ be the probability that the process that starts in initial state i at instant $t = 0$ will be in state j at instant t . It was shown that probabilities $\theta_{ij}(t)$, $i, j \in \{1, \dots, K\}$, can be found from the solution of the following system of integral equations:

$$\theta_{ij}(t) = \delta_{ij}[1 - F_i(t)] + \sum_{k=1}^K \int_0^t q_{ik}(\tau) \theta_{kj}(t - \tau) d\tau, \quad (2.115)$$

where

$$q_{ik}(\tau) = \frac{dQ_{ik}(\tau)}{d\tau}, \quad (2.116)$$

$$F_i(t) = \sum_{j=1}^K Q_{ij}(t), \quad (2.117)$$

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases} \quad (2.118)$$

The system of linear integral equations (2.115) is the main system in the theory of semi-Markov processes. By solving this system, one can find all the probabilities $\theta_{ij}(t)$, $i, j \in \{1, \dots, K\}$, for a semi-Markov process with a given kernel matrix $[Q_{ij}(t)]$ and given initial state.

Based on the probabilities $\theta_{ij}(t)$, $i, j \in \{1, \dots, K\}$, important reliability indices can easily be found. Suppose that system states are ordered according to their performance rates $g_K \geq g_{K-1} \geq \dots \geq g_2 \geq g_1$ and demand $g_m \geq w > g_{m-1}$ is constant. State K with performance rate g_K is the initial state. In this case system instantaneous availability is treated as the probability that a system starting at instant $t = 0$ from state K will be at instant $t \geq 0$ in any state g_K, \dots, g_m . Hence, we obtain

$$A(t, w) = \sum_{j=m}^K \theta_{Ki}(t). \quad (2.119)$$

The mean system instantaneous output performance and the mean instantaneous performance deficiency can be obtained, respectively, as

$$E_i = \sum_{i=1}^K g_i \theta_{Ki}(t) \quad (2.120)$$

and

$$D_i(w) = \sum_{i=1}^{m-1} (w - g_i) \theta_{Ki}(t) \mathbf{1}(w > g_i). \quad (2.121)$$

In the general case, the system of integral Equations 2.115 can be solved only by numerical methods. For some of the simplest cases the method of the Laplace–Stieltjes transform can be applied in order to derive an analytical solution of the system. As was done for Markov models, we designate a Laplace–Stieltjes transform of function $f(x)$ as

$$\tilde{f}(s) = L\{f(x)\} = \int_0^t e^{-sx} f(x) dx. \quad (2.122)$$

Applying the Laplace–Stieltjes transform to both sides of (2.115) we obtain

$$\tilde{\theta}_{ij}(s) = \delta_{ij} \tilde{\Psi}_i(s) + \sum_{k=1}^K \pi_{ik} \tilde{f}_{ik}(s) \tilde{\theta}_{kj}(s), \quad 1 \leq i, j \leq K, \quad (2.123)$$

where $\tilde{\Psi}_i(s)$ is the Laplace–Stieltjes transform of the function

$$\Psi_i(t) = 1 - F_i(t) = \int_t^\infty f_i(t) dt = \Pr\{T_i > t\} \quad (2.124)$$

and, therefore,

$$\tilde{\Psi}_i(s) = \frac{1}{s} [1 - \tilde{f}_i(s)]. \quad (2.125)$$

The system of algebraic equations (2.123) defines Laplace–Stieltjes transform of probabilities $\theta_{ij}(t)$, $i, j \in \{1, \dots, K\}$, as a function of the main parameters of a semi-Markov process.

By solving this system, one can also find steady-state probabilities. A detailed investigation is beyond the scope of this book and we only give here the resulting formulae for computation of steady-state probabilities. Steady-state probabilities $\theta_{ij} = \lim_{t \rightarrow \infty} \theta_{ij}(t)$ (if they exist) do not depend on the initial state of process I , and for their designation, one can use only one index: θ_j . It is proven that

$$\theta_j = \frac{p_j \bar{T}_j}{\sum_{j=1}^K p_j \bar{T}_j}, \quad (2.126)$$

where $p_j, j=1, \dots, K$ are steady-state probabilities of the embedded Markov chain. These probabilities are the solutions of the following system of algebraic equations:

$$\begin{cases} p_j = \sum_{i=1}^K p_i \pi_{ij}, & j=1, \dots, K, \\ \sum_{i=1}^K p_i = 1. \end{cases} \quad (2.127)$$

Note that the first K equations in (2.127) are linearly dependant and we cannot solve the system without the last equation $\sum_{i=1}^K p_i = 1$.

In order to find the reliability function, an additional semi-Markov model should be built in analogy with the corresponding Markov models: all states corresponding to performance rates lower than constant demand w should be united in one absorbing state with the number 0. All transitions that return the system from this absorbing state should be forbidden. The reliability function is obtained from this new model as $R(w, t) = \theta_{K0}(t)$.

Example 2.8 (Lisnianski and Levitin 2003). Consider an electric generator that has four possible performance (generating capacity) levels $g_4 = 100$ MW, $g_3 = 70$ MW, $g_2 = 50$ MW, and $g_1 = 0$. The constant demand is $w = 60$ MW. The best state with performance rate $g_4 = 100$ MW is the initial state. Only minor failures and minor repairs are possible. Times to failures are distributed exponentially with following parameters: $\lambda_{3,2} = 5 \times 10^{-4} \text{ h}^{-1}$, $\lambda_{2,1} = 2 \times 10^{-4} \text{ h}^{-1}$. Hence, times to failures $T_{4,3}$, $T_{3,2}$, $T_{2,1}$ are random variables distributed according to the corresponding CDF:

$$F_{4,3}(t) = 1 - e^{-\lambda_{4,3}t}, \quad F_{3,2}(t) = 1 - e^{-\lambda_{3,2}t}, \quad F_{2,1}(t) = 1 - e^{-\lambda_{2,1}t}.$$

Repair times are normally distributed. $T_{3,4}$ has a mean time to repair of $\bar{T}_{3,4} = 240$ h and a standard deviation of $\sigma_{3,4} = 16$ h, $T_{2,3}$ has a mean time to repair of $\bar{T}_{2,3} = 480$ h and standard deviation $\sigma_{2,3} = 48$ h, $T_{1,2}$ has a mean time to repair $\bar{T}_{1,2} = 720$ h and standard deviation $\sigma_{1,2} = 120$ h. Hence, the CDF of random variables $T_{3,4}$, $T_{2,3}$, and $T_{1,2}$ are, respectively:

$$F_{3,4}(t) = \frac{1}{\sqrt{2\pi\sigma_{3,4}^2}} \int_0^t \exp\left[-\frac{(u-\bar{T}_{3,4})}{2\sigma_{3,4}}\right] du,$$

$$F_{2,3}(t) = \frac{1}{\sqrt{2\pi\sigma_{2,3}^2}} \int_0^t \exp\left[-\frac{(u-\bar{T}_{2,3})}{2\sigma_{2,3}}\right] du,$$

$$F_{1,2}(t) = \frac{1}{\sqrt{2\pi\sigma_{1,2}^2}} \int_0^t \exp\left[-\frac{(u-\bar{T}_{1,2})}{2\sigma_{1,2}}\right] du.$$

Find the generator steady-state availability, mean steady-state performance (generating capacity), and mean steady-state performance deficiency and the generator reliability function.

Solution. The state-transition diagram of the generator is shown in Figure 2.32 (a).

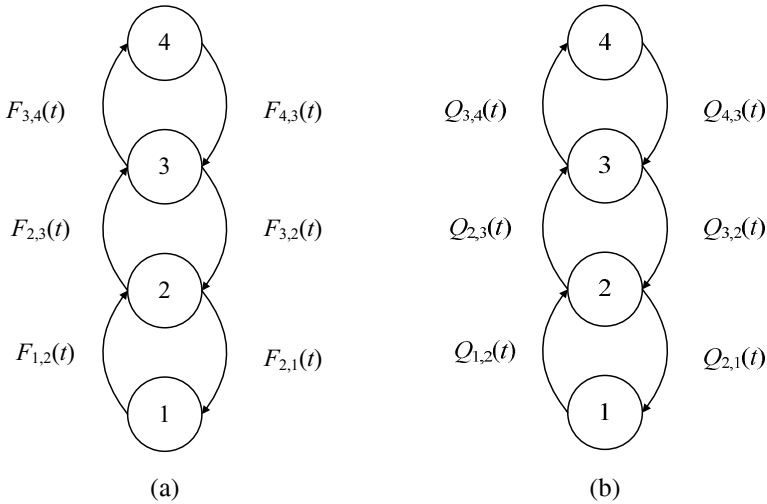


Fig. 2.32 Generator representation by stochastic process: (a) generator evolution in the state space, and (b) semi-Markov model

State 4 is an initial state with generating capacity g_4 . After the failure, which occurs according to distribution $F_{4,3}(t)$, the generator transits from state 4 to state 3 with reduced generating capacity g_3 .

If a random repair time in state 3, which is distributed according to CDF $F_{3,4}(t)$, is lower than the time up to the failure in state 3, which is distributed according to $F_{3,2}(t)$, the generator will come back to state 4. If the repair time is greater than the time up to the failure in state 3, the generator will fall down to state 2 with generating capacity g_2 .

If the random repair time in state 2, which is distributed according to CDF $F_{2,3}(t)$, is lower than the time up to the failure in state 2, which is distributed ac-

cording to $F_{2,1}(t)$, the generator will come back to state 3. If the repair time is greater than the time up to the failure in state 2, the generator will fall down to state 1 with generating capacity g_1 .

In state 1 after repair time, which is distributed according to $F_{1,2}(t)$, the generator will come back to state 2.

Based on (2.111)–(2.113), we obtain the following kernel matrix $\mathbf{Q}(t) = [Q_{ij}(t)]$, $i, j = 1, 2, 3, 4$:

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & Q_{12}(t) & 0 & 0 \\ Q_{21}(t) & 0 & Q_{23}(t) & 0 \\ 0 & Q_{32}(t) & 0 & Q_{34}(t) \\ 0 & 0 & Q_{43}(t) & 0 \end{bmatrix}$$

in which

$$\begin{aligned} Q_{12}(t) &= F_{1,2}(t), & Q_{21}(t) &= \int_0^t [1 - F_{2,3}(t)] dF_{2,1}(t), \\ Q_{23}(t) &= \int_0^t [1 - F_{2,1}(t)] dF_{2,3}(t), & Q_{32}(t) &= \int_0^t [1 - F_{3,4}(t)] dF_{3,2}(t), \\ Q_{34}(t) &= \int_0^t [1 - F_{3,2}(t)] dF_{3,4}(t), & Q_{43}(t) &= F_{4,3}(t). \end{aligned}$$

The corresponding semi-Markov process is presented in Figure 2.32 (b).

Based on the kernel matrix, the cdf of unconditional sojourn times in states 1, 2, 3, and 4 can be written according to (2.108) as

$$\begin{aligned} F_1(t) &= Q_{12}(t) & F_2(t) &= Q_{12}(t) + Q_{23}(t) \\ F_3(t) &= Q_{32}(t) + Q_{34}(t) & F_4(t) &= Q_{43}(t) \end{aligned}$$

According to (2.109) and (2.110) we have the following mean unconditional sojourn times: $\bar{T}_1 = 720$ h, $\bar{T}_2 = 457$ h, $\bar{T}_3 = 226$ h, $\bar{T}_4 = 1000$ h.

Using (2.106) we obtain one-step probabilities for the embedded Markov chain:

$$\begin{aligned} \pi_{12} &= F_{1,2}(\infty) = 1, & \pi_{21} &= \int_0^\infty [1 - F_{2,3}(t)] dF_{2,1}(t), & \pi_{23} &= \int_0^\infty [1 - F_{2,1}(t)] dF_{2,3}(t), \\ \pi_{32} &= \int_0^\infty [1 - F_{3,4}(t)] dF_{3,2}(t), & \pi_{34} &= \int_0^\infty [1 - F_{3,2}(t)] dF_{3,4}(t), & \pi_{43} &= F_{4,3}(\infty) = 1. \end{aligned}$$

Calculating the integrals numerically, we obtain the following one-step probability matrix for the embedded Markov chain:

$$\pi = \lim_{t \rightarrow \infty} \mathbf{Q}(t) = \begin{bmatrix} 0 & \pi_{12} & 0 & 0 \\ \pi_{21} & 0 & \pi_{23} & 0 \\ 0 & \pi_{32} & 0 & \pi_{34} \\ 0 & 0 & \pi_{43} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.0910 & 0 & 0.9090 & 0 \\ 0 & 0.1131 & 0 & 0.8869 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

In order to find steady-state probabilities p_j , $j=1,2,3,4$, for the embedded Markov chain, we have to solve the system of algebraic equations (2.127) that takes the form

$$\begin{cases} p_1 = \pi_{21}p_2, \\ p_2 = \pi_{12}p_1 + \pi_{32}p_3, \\ p_3 = \pi_{23}p_2 + \pi_{43}p_4, \\ p_4 = \pi_{34}p_3, \\ p_1 + p_2 + p_3 + p_4 = 1. \end{cases}$$

By solving this system we obtain: $p_1 = 0.0056$, $p_2 = 0.0615$, $p_3 = 0.4944$, $p_4 = 0.4385$.

Now using (2.126) we obtain the steady state probabilities

$$\begin{aligned} \theta_1 &= \frac{p_1 \bar{T}_1}{\sum_{j=1}^4 p_j \bar{T}_j} = 0.0069, & \theta_2 &= \frac{p_2 \bar{T}_2}{\sum_{j=1}^4 p_j \bar{T}_j} = 0.0484, \\ \theta_3 &= \frac{p_3 \bar{T}_3}{\sum_{j=1}^4 p_j \bar{T}_j} = 0.1919, & \theta_4 &= \frac{p_4 \bar{T}_4}{\sum_{j=1}^4 p_j \bar{T}_j} = 0.7528. \end{aligned}$$

The steady-state availability of the generator for the given constant demand is

$$A(w) = \theta_3 + \theta_4 = 0.9447.$$

According to (2.120), we obtain the mean steady-state performance

$$E_\infty = \sum_{k=1}^4 g_k \theta_k = 91.13 \text{ MW},$$

and according to (2.121), we obtain the mean steady-state performance deficiency

$$D_{\infty} = (w - g_2)\theta_2 + (w - g_1)\theta_1 = 0.50 \text{ MW}.$$

In order to find the reliability function for the given constant demand $w = 60 \text{ MW}$, we unite states 1 and 2 into one absorbing state 0. The modified graphical representation of the system evolution in the state space for this case is shown in Figure 2.33 (a). Figures 2.33 (b) shows the state-space diagram for the corresponding semi-Markov process.

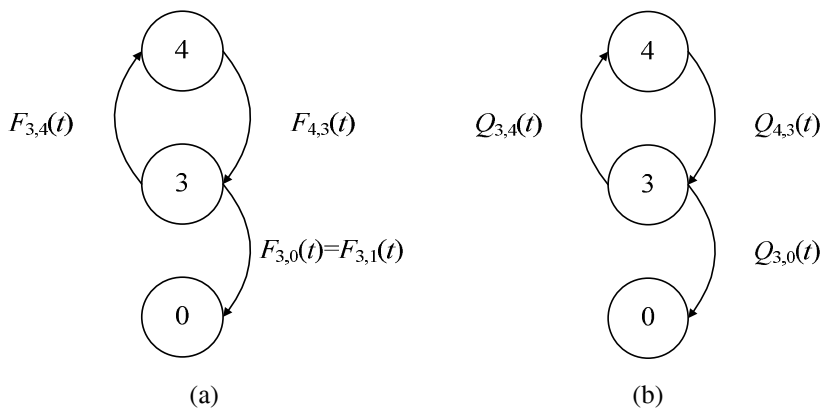


Fig. 2.33 State-transition diagrams for evaluating reliability function of generator: (a) evolution in modified state space, and (b) semi-Markov model

As in the previous case, we define the kernel matrix for the corresponding semi-Markov process based on expressions (2.111) – (2.113):

$$\mathbf{Q}(t) = \begin{bmatrix} 0 & 0 & 0 \\ Q_{30}(t) & 0 & Q_{34}(t) \\ 0 & Q_{43}(t) & 0 \end{bmatrix},$$

where

$$Q_{30}(t) = \int_0^t [1 - F_{3,4}(t)] dF_{3,1}(t), \quad Q_{34}(t) = \int_0^t [1 - F_{3,1}(t)] dF_{3,4}(t), \quad Q_{43}(t) = F_{4,3}(t).$$

The reliability function for constant demand $w = 60 \text{ MW}$ is defined as

$$R(w, t) = \theta_{40}(t).$$

According to (2.115), the following system of integral equations can be written in order to find the probability $\theta_{40}(t)$:

$$\begin{cases} \theta_{40}(t) = \int_0^t q_{43}(\tau) \theta_{30}(t-\tau) d\tau, \\ \theta_{30}(t) = \int_0^t q_{34}(\tau) \theta_{40}(t-\tau) d\tau + \int_0^t q_{30}(\tau) \theta_{00}(t-\tau) d\tau, \\ \theta_{00}(t) = 1. \end{cases}$$

The reliability function obtained by solving this system numerically is presented in Fig. 2.34.

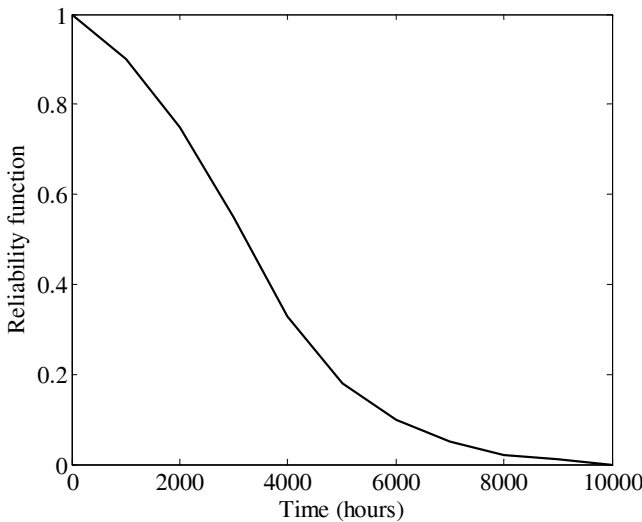


Fig. 2.34 Reliability function of generator

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