

Chapter 2

Basic Elements of System Reliability

It is difficult to get where you want to go if you don't know where that is.

Abstract This chapter presents the basic principles and functional relationships used for reliability assessment of systems with simple interconnections. Systems with simple interconnections are those that can be reduced to a single equivalent element or block through a sequence, however complex, of series and parallel reductions. The analysis of systems with complex interconnections is treated in Chapter 3. Although most multichannel systems have complex interconnections, many fundamental principles of redundant system reliability can be developed and understood using the principles discussed in the present chapter. The objective of this chapter is to first determine the overall reliability of a system that is composed of multiple subsystem elements or components, each with known reliability, and then to determine the reliability of the system that comprises these components.

2.1 The Reliability Function

Reliability is defined as the probability that an element (that is, a component, subsystem or full system) will accomplish its assigned task within a specified time, which is designated as the interval $t = [0, t_M]$. This book deals only with systems consisting of elements that can take on one of two states: either the element is operational (designated as the 1 state) or the element has failed (designated as the 0 state). Furthermore, the book considers only coherent systems, which have the following characteristics: (a) the reliability of the system increases if the reliability of its components increases, and (b) the system has no irrelevant components. The failure of any component or set of components in a coherent system cannot cause an increase in reliability, and every component has some effect, however small, on the overall reliability.

If the reliability of the i th component of a system is p_i and if this component has an unreliability q_i , then

$$p_i = 1 - q_i \quad (2.1)$$

and

$$q_i = 1 - p_i . \quad (2.2)$$

Also, since the component is always in one of the two possible states (operational or failed),

$$q_i + p_i = 1 . \quad (2.3)$$

To perform quantitative system reliability analysis, it is necessary to ascribe a probability that the individual components p_i either are operational or have failed. A *reliability function*, also called a *survivor function*, defines the probability that the component will perform its intended task (usually subject to some stated set of environmental conditions, such as vibration and temperature) for some specified performance period. The performance period may be a function of cycles, distance or time. Although the techniques presented here can employ a reliability function that depends on any of these three parameters, the focus is on determining the probability of system failure as a function of time. Additionally, although the estimation of system element reliabilities is outside the scope of this book, it is essential that the failure characteristics of the elements be determined using an approach that appropriately accounts for the environment in which they operate. It is also critical that the reliabilities are estimated in a legitimate and appropriately conservative fashion.

Several different functions have been used to characterize the probability distribution of failures as a function of time. Some of the more common reliability functions include the exponential, normal, log-normal and Weibull distributions. In this book, however, the exponential probability distribution is used almost exclusively.¹ The exponential distribution is appropriate for components with a failure rate that is time independent. Most electronic devices demonstrate such a constant failure rate during their useful lifetime, which is the time following a “burn in” that eliminates any weak or faulty components. The reliability function for a single-component system associated with the exponential distribution is

$$r(\lambda, t) = e^{-\lambda t} , \quad (2.4)$$

where $r(\lambda, t)$ is the probability that a component with failure rate λ will be operational at time t . The *Mathematica* function implementing Equation (2.4) is

$$r[\lambda, t_]:=e^{-\lambda t};$$

This book typically depicts system reliability graphically using log-log plots of the probability of failure as a function of time. Figure 2.1 shows the probability of failure for components with failure rates $\lambda = 100, 200$ and 400 failures per million hours (fpmh).

Figure 2.1 illustrates several noteworthy points. Obviously, the probability of component failure increases with time and with λ . Note that the curves, when shown on a log-log plot, are nearly straight lines in the time range of interest. Also, the

¹ The techniques illustrated in this book, however, are not limited to use of the exponential distribution; substitution of another distribution in lieu of the exponential is perfectly valid.

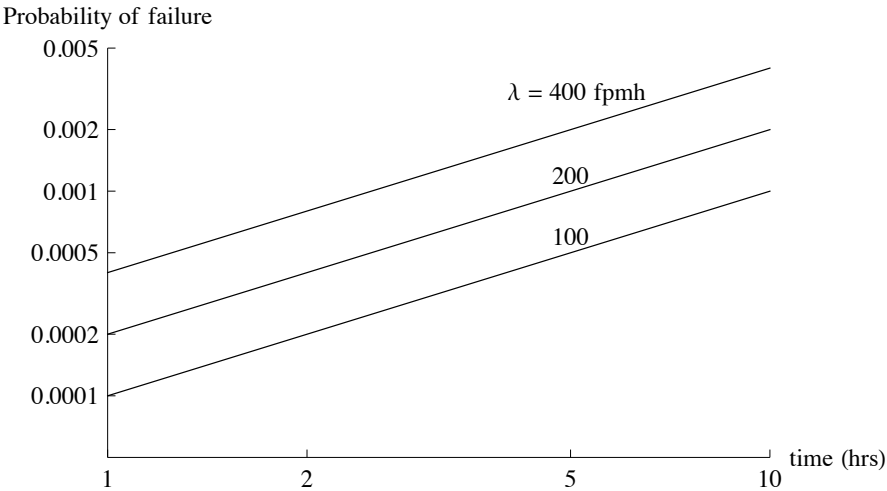


Fig. 2.1 Probability of component failure for various failure rates, λ

probability of failure curves increase uniformly by equal amounts for each doubling of the failure rate, λ . The slope of these curves is approximately one decade of unreliability per decade of time, which is a general feature of all simplex systems with a high degree of reliability over the time span of interest.² It is shown later that the slope of a system unreliability curve is related to the level of system redundancy, with the slope increasing as the level of redundancy increases.

2.2 Reliability Functional Block Diagrams

To support the analysis of system reliability, the analyst should first, after careful study of the entire system, depict the overall system design in the form of a reliability functional block diagram.³ The purpose of the block diagram is to describe the system at its simplest level, while still retaining all of the significant subsystem or component failure information, and to describe the effect that these failures have on the overall reliability of the system. Generally, this means that the functional block diagram represents the system as a collection of “black boxes,” each of which is subject to independent failure with respect to the other system elements. Note that

² Obviously, since $p = e^{-\lambda t}$ and $q = 1 - p$, $q \rightarrow 1$ as $t \rightarrow \infty$, and therefore, all of the curves in Figure 2.1 asymptotically approach unity after a sufficiently long period of time.

³ This book uses functional block diagrams to describe the functional relationships between system elements that are capable of independent failure. Functional block diagrams are similar to reliability block diagrams but do not strictly adhere to the same conventions. Also, functional block diagrams often require an accompanying explanation to unambiguously describe the characteristics of the system.

a given element may be inoperative owing to the failure of other elements on which it depends, but it still may be capable of failure independently of the other elements in the system. In the following discussion, each element of the overall system is represented as a block, with the appropriate inputs and outputs representing its relationship to the remainder of the system. Each block contains an element p_i that is assumed to have a failure rate λ_i .

Figure 2.2 represents a single component, p_1 , of a system that in turn could be part of another block diagram depicting a larger system. This component block has a single input and a single output, but in the more general case, the block might have multiple inputs and multiple outputs. By convention, inputs are generally assumed to enter either from the left side or from the top side of the box, and outputs exit either from the right side or from the bottom side.

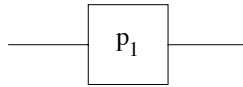


Fig. 2.2 Single-component block diagram

An overall system, of course, is made up of multiple blocks or elements. In general, these elements or groups of elements are arranged either in series or in parallel.

2.3 Elements in Series

The simplest system arrangement involves two elements, each of which has an independent failure mode, that are arranged so that the input of one block is dependent on the output of the previous block. As a result, this system (or portion of a larger system) requires that both blocks be operational. Such an arrangement is termed *elements in series*.

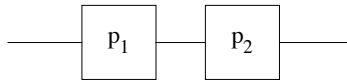


Fig. 2.3 Block diagram for elements in series

The reliability of the system made up of elements p_1 and p_2 , as shown in Figure 2.3, is the probability that both components are operational. If the symbols p_1 and p_2 also represent the respective reliability of elements p_1 and p_2 from Fig-

ure 2.3⁴ and if R_S represents the total reliability of the system that comprises components p_1 and p_2 , then

$$R_S = p_1 p_2 .$$

It should be clear that this relationship is easily generalized to a system of n elements arranged in series:

$$R_S = \prod_{i=1}^n p_i . \quad (2.5)$$

During analysis of complex systems, series elements that do not interact with other system elements should be collapsed into a single equivalent element with an assigned reliability equal to the product of the series reliabilities. If the elements have a constant failure rate, which implies an exponential distribution, then the component failure rates are simply added:

$$\lambda_{series} = \sum_{i=1}^n \lambda_i . \quad (2.6)$$

Thus, λ_{series} is the failure rate of the *new* single element.

A *Mathematica* function can be defined to represent the combination of elements in series. The success of a system made up of two elements arranged in series depends on both elements being operational:

rAND[p1_,p2_] := p1p2;

The function name **rAND** was chosen because it represents the probability that both **p1** and **p2** are operational. The continued usefulness of the Boolean function analogy is made apparent in later sections.

In a series system, the overall system reliability is a function of individual element reliabilities and the number of elements in series. This relationship is shown in Figure 2.4 for systems composed of 1 to 20 elements,⁵ each having an individual element reliability of 0.99, 0.98 or 0.95. Obviously, if the overall reliability of a system consisting of series elements is to be increased, either the individual element reliabilities must be increased or the number of elements in the system must be decreased.

⁴ This book uses a symbol, such as p_i , to represent both the given system element and also the reliability of that element. Although this definition leads to some ambiguity, it should not cause confusion in context.

⁵ Note that the curves shown in Figure 2.4 are actually defined only for integral numbers of elements in series.

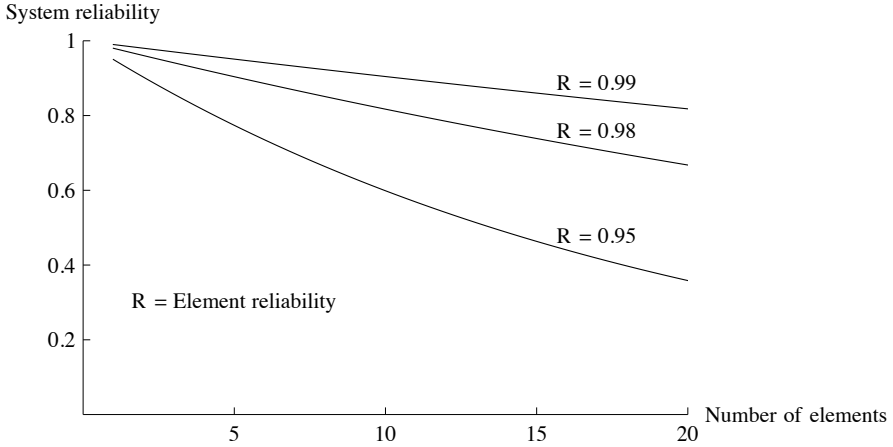


Fig. 2.4 Reliability of series systems

2.4 Elements in Parallel

The second basic arrangement of system components is shown in Figure 2.5. In this arrangement, the system is composed of elements p_1 and p_2 and is operational if either element or both elements are operational.

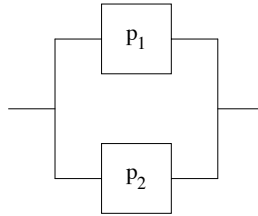


Fig. 2.5 Block diagram for elements in parallel

If p_1 and p_2 represent the reliability of elements p_1 and p_2 , respectively, and if R_P is the reliability of the system composed of these two elements, then the system reliability is

$$R_P = 1 - (1 - p_1)(1 - p_2) .$$

The general relationship for a system of n components arranged in parallel is

$$R_P = 1 - \prod_{i=1}^n (1 - p_i) . \quad (2.7)$$

Absent any reasons for showing the redundancy involved in the use of parallel elements, simple parallel arrangements (such as that shown in Figure 2.5) should be collapsed into a single equivalent element with an assigned reliability R_p . This same principle applies to series elements.

The following *Mathematica* function, which represents the combination of elements in parallel, is based on Equation (2.7). The success of a system made up of two elements arranged in parallel depends on either element or both elements being operational:

rOR[p1_,p2_] := 1 - (1 - p1)(1 - p2);

The Boolean function analogy is employed again by using the name **rOR** for the function.

Figure 2.6 shows the effect of an increasing number of parallel elements on system reliability.⁶ The redundant elements have reliabilities of 0.95, 0.9 or 0.8. The reliability of a system with components arranged in parallel increases rapidly with an increasing number of elements, or levels of redundancy, and asymptotically approaches unity with increasing n .

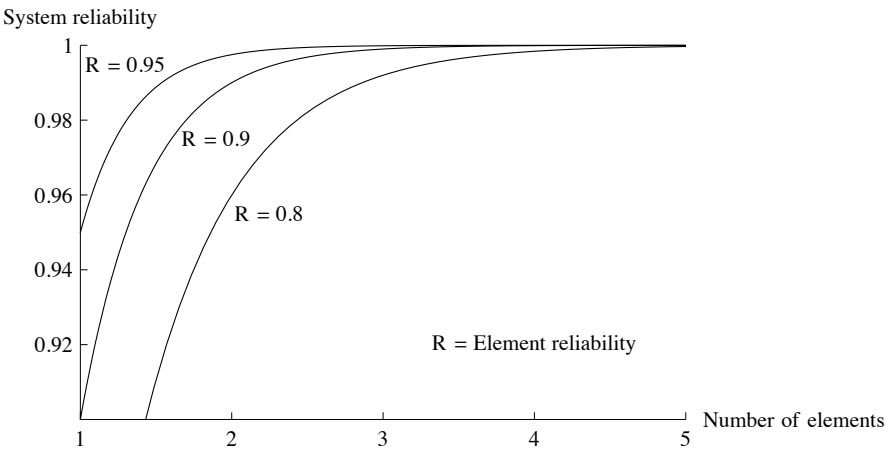


Fig. 2.6 Reliability of parallel systems

2.5 Combined Series/Parallel Systems

Figure 2.7 depicts a system comprising seven elements arranged in series and in parallel. If the elements p_i in Figure 2.7 also have individual element reliabilities

⁶ Again, these curves are defined only for integral values of n .

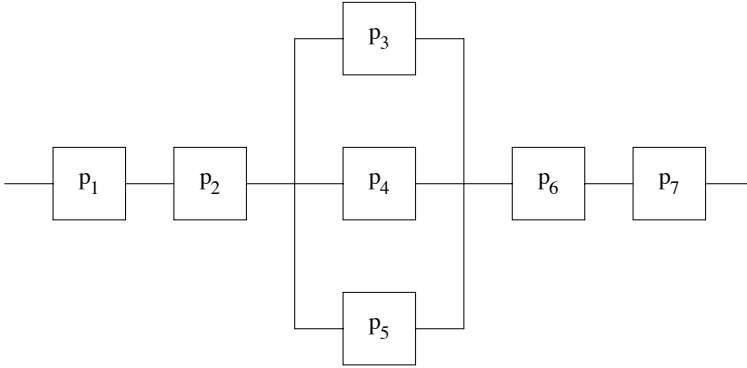


Fig. 2.7 System with elements in series and parallel

p_i , then the overall system reliability R_{sys} can be determined using the *Mathematica* functions defined above:

```
p12 = rAND[p1, p2]
p1p2
```

```
p345 = rOR[p3, rOR[p4, p5]]
1 - (1 - p3)(1 - p4)(1 - p5)
```

```
p67 = rAND[p6, p7]
p6p7
```

```
Rsys = rAND[p12, rAND[p345, p67]]
p1p2(1 - (1 - p3)(1 - p4)(1 - p5))p6p7
```

The same result is obtained with a single deeply nested expression:

```
Rsys = rAND[rAND[p1, p2], rAND[rOR[p3, rOR[p4, p5]], rAND[p6, p7]]]
p1p2(1 - (1 - p3)(1 - p4)(1 - p5))p6p7
```

This simplification technique can be used to reduce a system of multiple elements, arranged in series and parallel, to an equivalent single-block system as long as all of the elements are simply interconnected. Most real-world complex systems, however, are *not* simply interconnected.

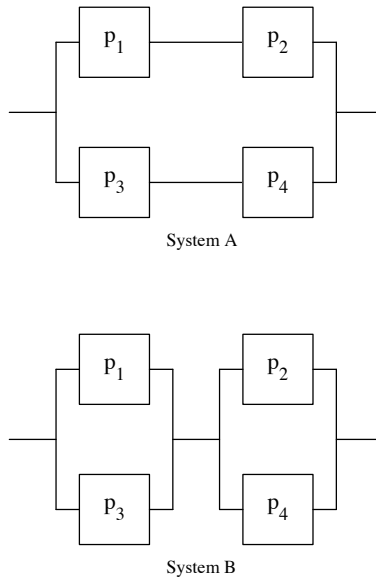


Fig. 2.8 Parallel system arrangements—high-level (System A) and low-level redundancy (System B)

2.6 Parallel System Arrangements

Consider the two alternative parallel system arrangements shown in Figure 2.8. Both of these systems are composed of identical components p_1, \dots, p_4 , but the system reliabilities are not equal. For System A to be operational, both p_1 and p_2 or both p_3 and p_4 must be operational. By contrast, System B is operational if at least elements p_1 and p_2 , p_1 and p_4 , p_3 and p_2 , or p_3 and p_4 are operational. System A demonstrates high-level redundancy; System B demonstrates low-level redundancy. The reliability of the two systems can be computed as shown below.

The reliability for System A (high-level redundancy) is

$$\begin{aligned} \mathbf{rSysA} &= \mathbf{rOR}[\mathbf{rAND}[p_1, p_2], \mathbf{rAND}[p_3, p_4]] \text{ // Expand} \\ p_1p_2 + p_3p_4 - p_1p_2p_3p_4 \end{aligned}$$

and for System B (low-level redundancy) is

$$\begin{aligned} \mathbf{rSysB} &= \mathbf{rAND}[\mathbf{rOR}[p_1, p_3], \mathbf{rOR}[p_2, p_4]] \text{ // Expand} \\ p_1p_2 + p_2p_3 - p_1p_2p_3 + p_1p_4 - p_1p_2p_4 + p_3p_4 - p_1p_3p_4 - p_2p_3p_4 + \\ &\quad p_1p_2p_3p_4 \end{aligned}$$

Clearly, the two systems do not have equivalent reliability. If numerical reliability values are assigned to each of the four elements, the difference between the total

system reliabilities can be calculated:

$$p1 = .95; p2 = .95; p3 = .9; p4 = .9;$$

$$\begin{aligned} &\mathbf{rSysA} \\ &0.981475 \end{aligned}$$

$$\begin{aligned} &\mathbf{rSysB} \\ &0.990025 \end{aligned}$$

Frequently, however, it is more instructive to look at the system unreliability (that is, the probability of failure) when comparing system alternatives.

$$\begin{aligned} &\mathbf{qSysA = 1 - rSysA} \\ &0.018525 \end{aligned}$$

$$\begin{aligned} &\mathbf{qSysB = 1 - rSysB} \\ &0.009975 \end{aligned}$$

$$\begin{aligned} &\mathbf{qSysA/qSysB} \\ &1.85714 \end{aligned}$$

Note that for these specific component reliabilities, System A is 1.86 times more likely to fail than is System B. In general, the low-level redundancy of System B has greater reliability than the high-level redundancy of System A.

Consider versions of System A and System B in which the components are identical and, consequently, each component has the same reliability. Figure 2.9 shows the ratio of the reliability of System B to that of System A as the element reliabilities are varied over the interval $[0 + \epsilon, 1]$. These are general results; systems with low-level redundancy outperform those with high-level redundancy by as much as a factor of two (in the case of low component reliability). Nevertheless, it should be noted that although the components that compose System A and System B may be identical, System B will have an additional level of complexity for most real-world systems. In most cases, to manage the system redundancy, the configuration of System B requires additional switching logic that would not be required for System A. A basic tenet of system design for reliability, however, is that low-level redundancy outperforms high-level redundancy.

2.7 Redundancy and System Reliability

As shown in the previous section, redundant systems are more reliable than single-strand or series systems. Each additional level of redundancy reduces the likelihood of system failure. Consider the following simplex, duplex and quadruplex systems,

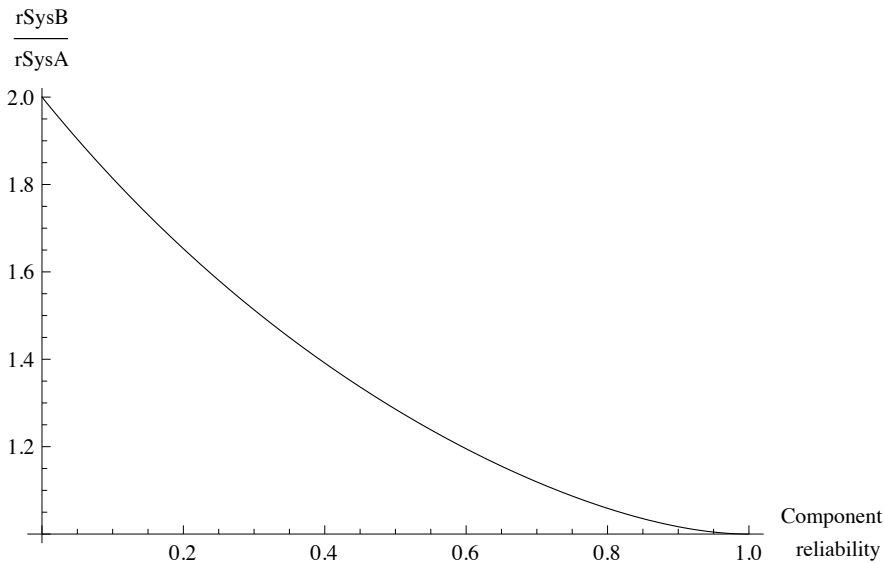


Fig. 2.9 Ratio of low-level to high-level redundant system reliabilities

i.e. systems with one, two, three and four levels of redundancy:

$$\begin{aligned} \mathbf{rSys1} &= \mathbf{p1} \\ p1 \end{aligned}$$

$$\begin{aligned} \mathbf{rSys2} &= \mathbf{rOR[p1, p2]} \\ 1 - (1 - p1)(1 - p2) \end{aligned}$$

$$\begin{aligned} \mathbf{rSys3} &= \mathbf{rOR[p1, rOR[p2, p3]]} \\ 1 - (1 - p1)(1 - p2)(1 - p3) \end{aligned}$$

$$\begin{aligned} \mathbf{rSys4} &= \mathbf{rOR[p1, rOR[p2, rOR[p3, p4]]]} \\ 1 - (1 - p1)(1 - p2)(1 - p3)(1 - p4) \end{aligned}$$

Figure 2.10 shows the probability of failure for each of the systems, given that the reliability of the elements is 0.9 ($p1 = p2 = p3 = p4 = 0.9$). For each additional level of redundancy, the overall reliability of the system increases by an order of magnitude. In this example, the simplex system is one thousand times more likely to fail than is the quadruplex system. The combination of redundancy and high component reliability can yield very low probabilities of system failure.

Figure 2.11 illustrates the relationship between component reliability and the level of redundancy. Note that even with low-reliability elements ($p_i = 0.5$), a comparatively high system reliability (approximately 0.94) can be achieved with four

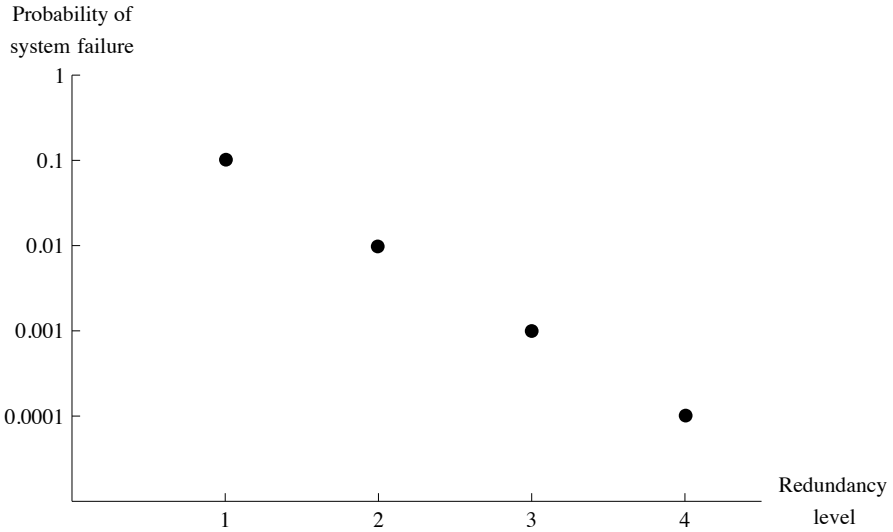


Fig. 2.10 Probability of system failure as a function of redundancy level

elements in parallel. Also note that the benefit of additional parallel elements diminishes rapidly as the redundancy level increases beyond three or four.

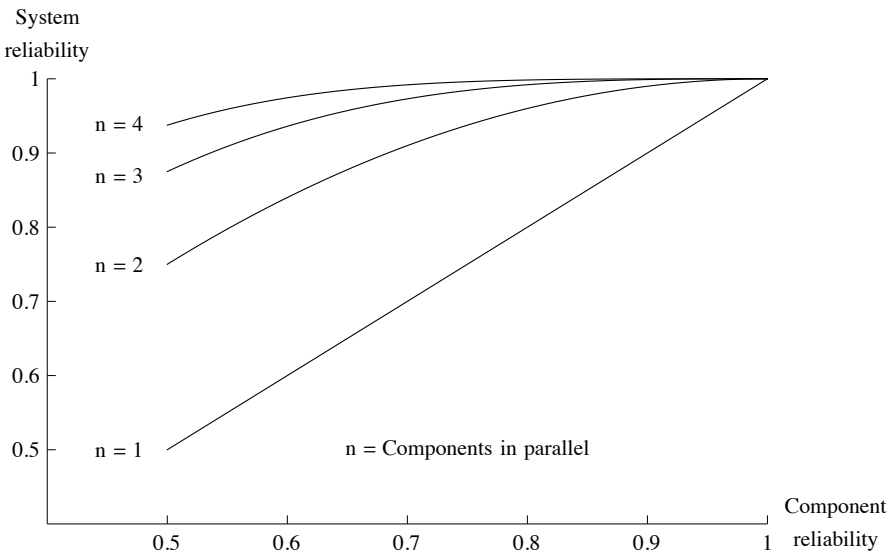


Fig. 2.11 Redundant system reliability as a function of component reliability

The results shown in Figures 2.10 and 2.11 are for redundant components with constant reliabilities; for real systems, understanding the system reliability as a function of time may be more useful. As previously discussed, components with a constant failure rate λ have a probability of failure function $p_i = e^{-\lambda_i t}$. Figure 2.12 shows the probability of failure for simplex, duplex and quadruplex systems with identical components for which $\lambda = 1000$ fpmh.

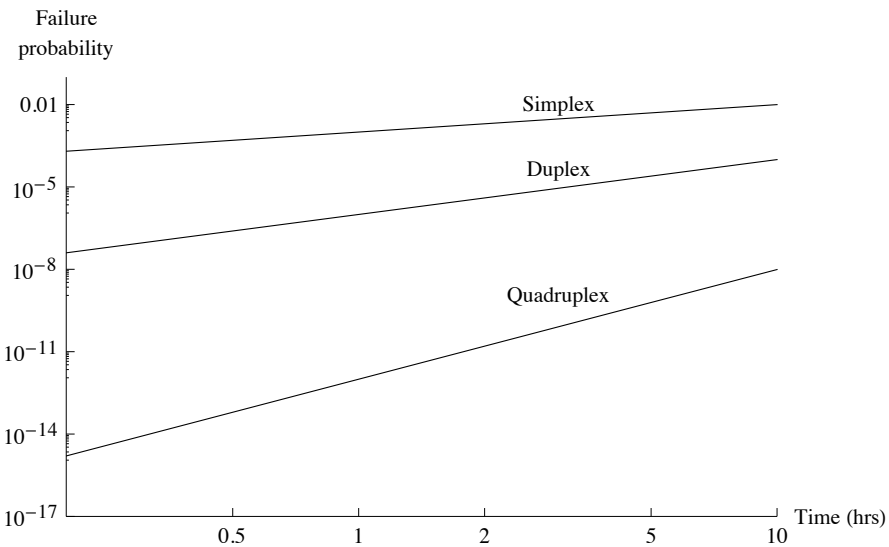


Fig. 2.12 Probability of failure for redundant systems ($\lambda = 1000$ fpmh)

When shown in log-log scale, each of the system failure probability curves is a straight line with a slope, measured in decades per decade (dpd), that is nearly equal to the redundancy of the system. For redundant systems with components having an exponential failure distribution, the slope of the probability of failure curve plotted on log-log axes is a measure of the system's redundancy level. In this book, this slope is referred to as the *equivalent redundancy level* (ERL) of the system.

Over the interval $[0, 10]$, the system failure probability curves shown in Figure 2.12 are very nearly straight lines. For large mission times, the system failure probability curves asymptotically approach unity over the interval $[0, 10^4]$, as shown in Figure 2.13. Once the curvature starts to appear, however, it can be argued that the system is past its useful lifetime.

Figure 2.14 illustrates the effect of increasing the component failure rates from 1000 fpmh (solid curves) to 2000 fpmh (dashed curves). The slope of the failure probability curve indicates the effective level of redundancy, and the vertical displacement is a function of the redundant component failure rate. Note that the vertical displacement is four times greater for the quadruplex system than for the simplex system. These relationships are general, and as a result, depicting the probability of

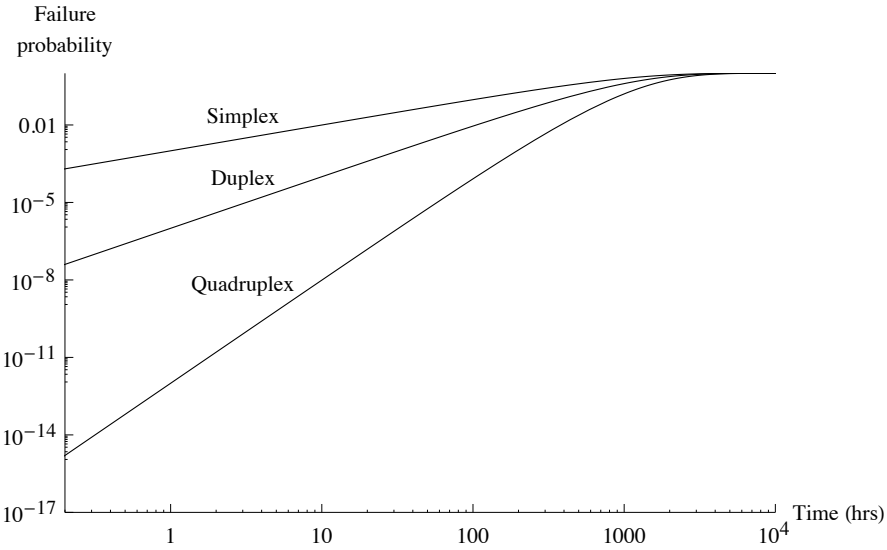


Fig. 2.13 Probability of failure for redundant systems for large mission times ($\lambda = 1000$ fpmh)

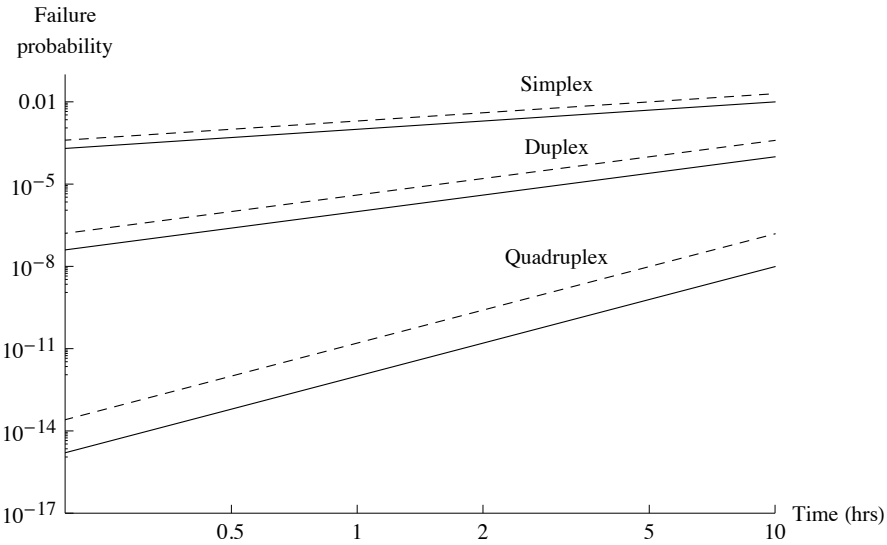


Fig. 2.14 Probability of failure for redundant systems, $\lambda = 1000$ fpmh (solid lines) and 2000 fpmh (dashed lines)

system failure on a log-log scale is helpful. Furthermore, the relationships will generally hold even if the redundant elements are composed of multiple components that constitute a full channel in a multichannel system.

For a system with perfect fault coverage, the slope of the probability of failure curve, when plotted on a log-log scale, approximates the level of system redundancy; that is, a quadruplex system has a slope of approximately 4 dpd, a triplex system has a slope of approximately 3 dpd and so on. Again, for this reason, the slope of a system's probability of failure curve is referred to as its *equivalent redundancy level*, or ERL.

These results illustrate two general effects: the redundancy level determines the slope of a redundant system's probability of failure curve, and the redundant component reliability shifts the curve vertically.⁷

2.8 k -out-of- n :G Systems

The redundant systems discussed in Sections 2.4 and 2.7 are examples of at least 1-out-of- n :G systems, where the system is operational ($G \rightarrow$ good) if at least one of the n redundant components is operational. The series systems discussed in Section 2.3 are n -out-of- n :G systems, where all n of the components must be operational for the system to be operational. Both parallel and series systems are examples of the more general k -out-of- n :G system structure in which at least k of the n system components must be operational for the system to function.

In addition to determining *at least* k -out-of- n :G system reliability, there are circumstances for which determining the reliability of an *exactly* k -out-of- n :G system is also useful, and both are presented in the following sections.

2.8.1 At Least k -out-of- n :G Systems

If a system consists of identical components, then the components are independent and also have identical failure distributions; the components are said to be *independent and identically distributed* (i.i.d.). For an i.i.d. k -out-of- n :G system with perfect fault coverage, the system reliability can be readily computed using the following equation:

⁷ These general results are applicable to redundant systems that are *not* subject to imperfect fault coverage. In subsequent chapters, the effect of imperfect fault coverage is examined in some detail. The probability of failure curves plot on a log-log scale as straight lines as long as λt is within an effective range. Obviously, as $\lambda t \rightarrow \infty$, the probability of failure approaches unity. For highly reliable redundant elements, however, the probability of failure curve continues to be a straight line for reasonable mission times.

$$\begin{aligned}
 R(k, n) &= \sum_{i=k}^n \binom{n}{i} p^i (1-p)^{n-i} \\
 &= \sum_{i=k}^n \binom{n}{i} p^i q^{n-i} .
 \end{aligned} \tag{2.8}$$

In Equation (2.8), $q = 1 - p$. This equation uses a widely known function (see, for instance, page 21 in Barlow and Proschan [1]).

For the general case with non-identical components, computing the system reliability is somewhat more complex. Let $\mathbf{p} = \{p_1, \dots, p_n\}$ be a vector of component reliabilities (which are not necessarily i.i.d.), and likewise, let $\mathbf{q} = \{1-p_1, \dots, 1-p_n\}$ be a vector of component unreliabilities. Then define the following functions:

- pT**(i, \mathbf{p}) Set of products of the k -subsets of \mathbf{p} with exactly i elements
- qT**(i, \mathbf{q}) Set of products of the k -subsets of \mathbf{q} with exactly $(n - i)$ elements

A k -subset is a subset with exactly k elements.⁸ Note that the k -subsets must be generated in lexicographic order. The reliability of a non-i.i.d. k -out-of- n :G system is given by

$$R(k, n, \mathbf{p}) = \sum_{i=k}^n \sum_{j=1}^{\binom{n}{i}} \mathbf{qT}(i, \mathbf{q})_j \mathbf{pT}(i, \mathbf{p})_{\binom{n}{i}-j+1} . \tag{2.9}$$

For example, given $\mathbf{p} = \{p_1, p_2, p_3\}$ so that $n = |\mathbf{p}| = 3$, the required **pT** and **qT** sets are

$$\begin{aligned}
 \mathbf{pT}(1, \mathbf{p}) &= \{p_1, p_2, p_3\} \\
 \mathbf{pT}(2, \mathbf{p}) &= \{p_1 p_2, p_1 p_3, p_2 p_3\} \\
 \mathbf{pT}(3, \mathbf{p}) &= \{p_1 p_2 p_3\} \\
 \mathbf{qT}(1, \mathbf{p}) &= \{(1-p_1)(1-p_2), (1-p_1)(1-p_3), (1-p_2)(1-p_3)\} \\
 \mathbf{qT}(2, \mathbf{p}) &= \{(1-p_1), (1-p_2), (1-p_3)\} \\
 \mathbf{qT}(3, \mathbf{p}) &= 1 ,
 \end{aligned}$$

leading to the following:

$$\begin{aligned}
 R(1, 3, \mathbf{p}) &= p_1(1-p_2)(1-p_3) + (1-p_1)p_2(1-p_3) \\
 &\quad + p_1 p_2(1-p_3) + (1-p_1)(1-p_2)p_3 \\
 &\quad + p_1(1-p_2)p_3 + (1-p_1)p_2 p_3 + p_1 p_2 p_3 \\
 &= p_1 + p_2 - p_1 p_2 + p_3 - p_1 p_3 - p_2 p_3 \\
 &\quad + p_1 p_2 p_3 .
 \end{aligned} \tag{2.10}$$

⁸ Including the set itself and the empty set, a set of n elements has 2^n subsets. A k -subset is a subset with exactly k elements [4].

Additional detail on the derivation and use of Equation (2.9) is given in [3]. An algorithm and computer code for the generation of *k*-subsets in lexicographic order is given in [2]. In later chapters, Equation (2.9) is modified to include the calculation of general *k*-out-of-*n*:G systems subject to imperfect fault coverage.

2.8.2 Exactly *k*-out-of-*n*:G Systems

The probability that an *at least k*-out-of-*n*:G system is operational is simply the sum from *k* to *n* of the probability that *exactly k* out of *n* components are operational. This leads to straightforward functions for determining exactly *k*-out-of-*n*:G system reliability.

For a general exactly *k*-out-of-*n*:G system with components that are not necessarily i.i.d.,

$$R_e(k, n, \mathbf{p}) = \sum_{j=1}^{\binom{n}{k}} \mathbf{qT}(k, \mathbf{p})_j \mathbf{pT}(k, \mathbf{p})_{\binom{n}{k}-j+1} . \quad (2.11)$$

For an exactly *k*-out-of-*n*:G system with i.i.d. components, Equation (2.11) can be simplified to the following well-known expression [1]:

$$\begin{aligned} R_e(k, n) &= \binom{n}{k} p^k (1-p)^{n-k} \\ &= \binom{n}{k} p^k q^{n-k} . \end{aligned} \quad (2.12)$$

2.8.3 Mathematica *k*-out-of-*n*:G Reliability

2.8.3.1 Mathematica i.i.d. *k*-out-of-*n*:G Reliability

The combinatorial functions presented in this section can be easily formulated as *Mathematica* functions. For Equation (2.8) in the case of i.i.d. components and at least *k*-out-of-*n*:G reliability,

```
Ral[k_,n_,p_]:=Module[{i,q=(1-p)},
  Sum[Binomial[n,i]piqn-i//Expand];
```

For Equation (2.12) in the case of i.i.d. components and exactly *k*-out-of-*n*:G reliability,

```
Rex[k_,n_,p_]:=Module[{q = (1 - p)},
Binomial[n,k]pkqn-k//Expand];
```

The **Ral** function returns a fully expanded polynomial representing the i.i.d. at least k -out-of- n :G system reliability, and the **Rex** function returns a fully expanded polynomial representing the i.i.d. exactly k -out-of- n :G system reliability. For example, the reliabilities for i.i.d. at least 3-out-of-7:G and exactly 3-out-of-7:G systems are

```
Ral[3,7,p]
35p3 - 105p4 + 126p5 - 70p6 + 15p7
```

and

```
Rex[3,7,p]
35p3 - 140p4 + 210p5 - 140p6 + 35p7
```

2.8.3.2 Mathematica Non i.i.d. k -out-of- n :G Reliability

The *Mathematica* functions for non i.i.d. k -out-of- n :G reliability are somewhat more complicated but are actually straightforward implementations of Equations (2.9) and (2.11).

Define the required **pT** and **qT** functions (the *Mathematica* function **KSubsets** requires the **Combinatorica** package):

```
Needs["Combinatorica`"];

pT[i_Integer,p_List]:=Module[{}],
Apply[Times,KSubsets[p,i],{1}]];

qT[i_Integer,p_List]:=
Module[{j,n = Length[p],q},q = Table[1 - p[[j]],{j,n]];
Apply[Times,KSubsets[q,n - i],{1}]];
```

Define the following functions:

```
Ral[k_,p_List]:=Module[{i,j,n = Length[p]},
 $\sum_{i=k}^n \sum_{j=1}^{\text{Binomial}[n,i]}$  qT[i,p][[j]]pT[i,p][[Binomial[n,i] - j + 1]]//Expand];
```

and

```
Rex[k,p.List]:=Module[{j,n=Length[p]},
  SumBinomial[n,k]qT[k,p][[j]]pT[k,p][[Binomial[n,k]-j+1]]//Expand];
```

Consider, for example, at least 1-out-of-4:G reliability and exactly 1-out-of-4:G reliability, both with non i.i.d. components:

```
Ral[1,{p1,p2,p3,p4}]
p1 + p2 - p1p2 + p3 - p1p3 - p2p3 + p1p2p3 + p4 -
p1p4 - p2p4 + p1p2p4 - p3p4 + p1p3p4 + p2p3p4 - p1p2p3p4
```

and

```
Rex[1,{p1,p2,p3,p4}]
p1 + p2 - 2p1p2 + p3 - 2p1p3 - 2p2p3 + 3p1p2p3 + p4 - 2p1p4 -
2p2p4 + 3p1p2p4 - 2p3p4 + 3p1p3p4 + 3p2p3p4 - 4p1p2p3p4
```

The *Mathematica* substitution **pn** \rightarrow **p** permits the computation of the same i.i.d. 3-out-of-7:G reliability as the one calculated above:

```
ToP = {p1  $\rightarrow$  p, p2  $\rightarrow$  p, p3  $\rightarrow$  p, p4  $\rightarrow$  p, p5  $\rightarrow$  p, p6  $\rightarrow$  p, p7  $\rightarrow$  p};
```

```
Ral[3,{p1,p2,p3,p4,p5,p6,p7}]/.ToP
35p3 - 105p4 + 126p5 - 70p6 + 15p7
```

and

```
Rex[3,{p1,p2,p3,p4,p5,p6,p7}]/.ToP
35p3 - 140p4 + 210p5 - 140p6 + 35p7
```

These results are identical to those obtained above using the i.i.d. variations of the **Ral** and **Rex** functions defined in Section 2.8.3.1. Note that even though the non i.i.d. **Ral** and **Rex** functions defined here have the same names as the previously defined functions, *Mathematica* is able to distinguish between them because they have different arguments. Consequently, the implementations do not conflict.

This section has presented combinatorial functions for the calculation of k -out-of- n :G system reliability along with implementation as *Mathematica* functions. These combinatorial expressions have a computational complexity of $O(\sum_{i=k}^n \binom{n}{i})$, which may be unacceptable for large n values ($n \gtrsim 10$). Other algorithms (table-based codes), which are presented in later chapters, yield identical results but have a complexity of $O(n \cdot (n - k))$. This order of complexity permits efficient calculation, even for large values of n .

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