2 Exploring the Fractal Universe

Arthur C. Clarke

In November 1989, when receiving the Association of Space Explorers' Special Achievement Award in Riyadh, Saudi Arabia, I had the privilege of addressing the largest gathering of astronauts and cosmonauts ever assembled at one place (more than fifty, including Apollo 11's Buzz Aldrin and Mike Collins, and the first 'space walker' Alexei Leonov … I decided to expand their horizons by introducing them to something really large, and, with astronaut Prince Sultan bin Salman bin Abdul Azïz in the chair, delivered a lavishly illustrated lecture.

Arthur C. Clarke is the world's most prophetic and prolific writer of science fiction, much of which has become science fact. The author of well over eighty best-selling works, he is perhaps best known for originating satellite communication in 1945 and for writing the book and screenplay of *2001: A Space Odyssey*, directed by Stanley Kubrick. Arthur C. Clarke has lived in Sri Lanka since the 1950s and was knighted in 2000.

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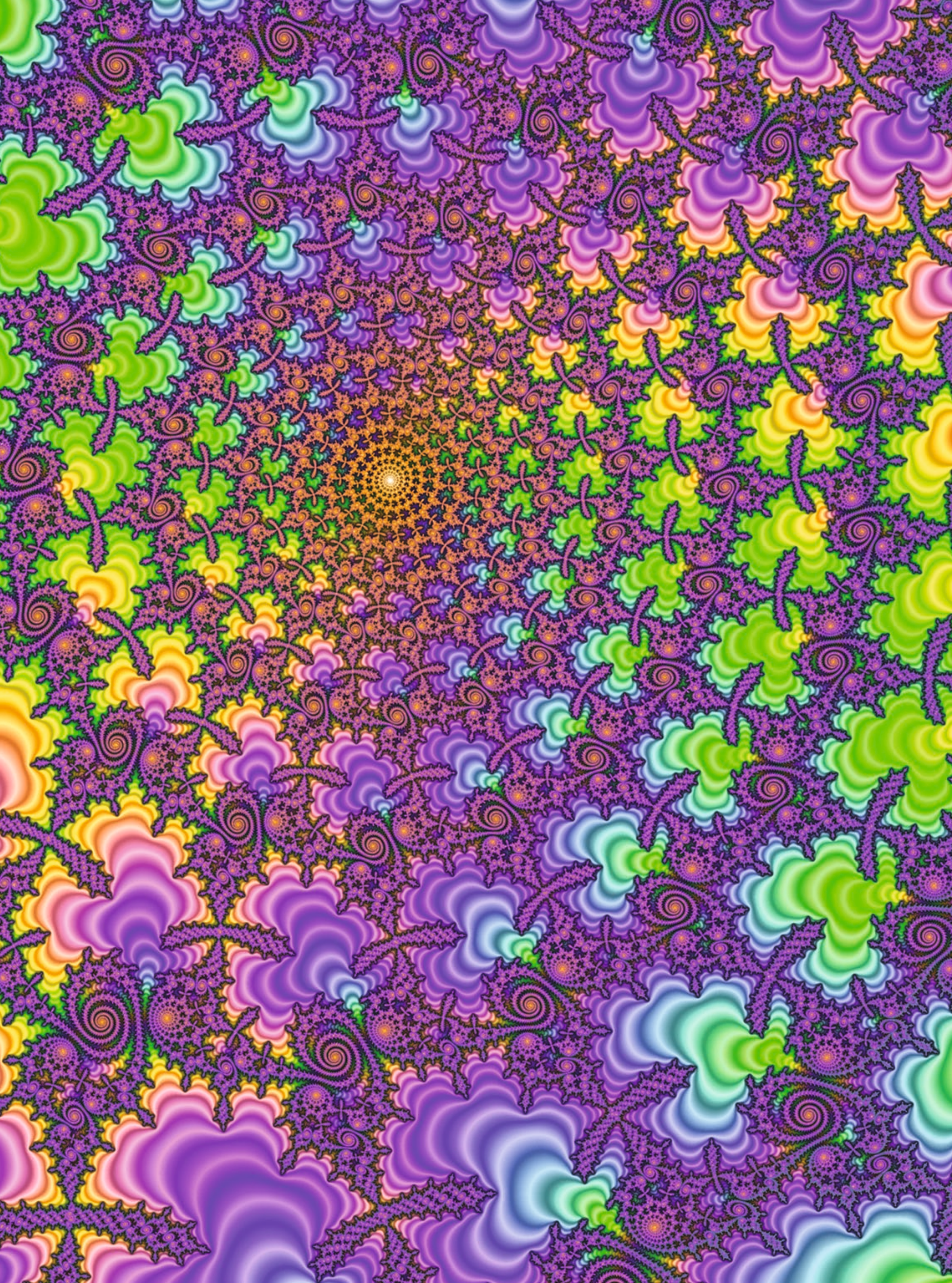

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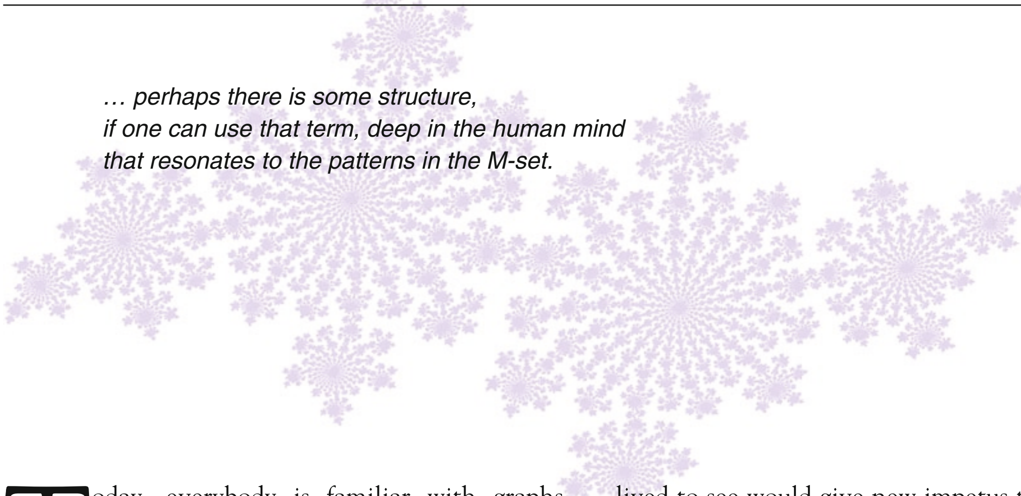

Today, everybody is familiar with graphs, especially the one with Time along the horizontal axis, and the Cost of Living climbing steadily up the vertical one. The idea that any point on a plane can be expressed by two numbers, usually written x and y, now appears so obvious that it seems quite surprising that the world of mathematics had to wait until 1637 for Descartes to invent it.

We are still discovering the consequences of that apparently simple idea, and the most amazing is now just a few years old. It's called the Mandelbrot Set (from now on, the 'M-set') and you're soon going to meet it everywhere – in the design of fabrics, wallpaper, jewellery and linoleum. And, I'm afraid, it will be popping out of your TV screen in every other commercial.

The stunning beauty of the images the M-set generates has an appeal that is both emotional and universal: I have seen people almost hypnotized by the computer-produced films that explore its – literally infinite – ramifications.

Resonating to the M-set

The psychological reasons for this appeal are still a mystery, and may always remain so; perhaps there is some structure, if one can use that term, deep in the human mind that resonates to the patterns in the M-set. Carl Jung would have been surprised – and delighted – to know that thirty years after his death, the computer revolution whose beginnings he just lived to see would give new impetus to his theory of archetypes, and his belief in the existence of a 'collective unconscious'.

Many patterns in the M-set are strongly reminiscent of the abstract, curvilinear motifs of Islamic decorative art; the comma-shaped Paisley design is one example. Others resemble organic structures – tentacles, compound insect eyes, armies of sea-horses, elephant trunks … then, abruptly, they become transformed into angular shapes like the crystals and snowflakes of the world before any life existed.

Yet perhaps the most astonishing feature of the M-set is its basic simplicity. Unlike almost everything else in modern mathematics, any schoolchild can understand how it is produced. Its generation involves nothing more advanced than addition and multiplication; there's no need for such complexities as subtraction and – heaven forbid! – division, let alone any of the more exotic beasts from the mathematical menagerie.

Another of those equations

There can be few people in the civilized world who have not encountered Einstein's famous $E = mc^2$, or who would consider it too hopelessly complicated to understand. Well, the equation that defines the M-set contains the same number of terms, indeed looks very similar. Here it is:

$$Z \rightleftharpoons z^2 + c.$$

Not very terrifying, is it? Yet the lifetime of the Universe would not be long enough to explore all its ramifications.

The zs and the c in Mandelbrot's equation are all numbers, not (as in Einstein's) physical quantities like mass and energy. They are coordinates, which specify the position of a point, and the equation controls the way in which it moves, to trace a pattern.

There's a very simple analogue familiar to everyone – those children's books with blank pages sprinkled with numbers, which when joined up in the right order reveal hidden – and often surprising – pictures. The image on a TV screen is produced by a sophisticated application of the same principle.

In theory, anyone who can add and multiply could plot out the M-set with pen or pencil on a sheet of squared paper. However, as we'll see later, there are certain practical difficulties – notably the fact that a human lifespan is seldom more than a hundred years. So the M-set is invariably computer-generated, and usually shown on a computer screen.

Take any point in space

Now, there are two ways of locating a point in space. The more common employs some kind of grid reference – West–East, North–South, or, on squared graph paper, a horizontal X-axis and a vertical Y-axis.

But there's also the system used in radar, now familiar to most people thanks to countless movies. Here the position of an object is given by (1) its distance from the origin, and (2) its direction, or compass bearing. Incidentally, this is the natural system – the one you use automatically and unconsciously when you play any ball game. Then you're concerned with distances and angles, with yourself as the origin.

So think of a computer's screen as a radar screen with a single blip on it, whose movements are going to trace out the M-set. However, before we switch on our radar, I want to make the equation even simpler, to: $Z = z^2$. I've thrown c away, for the moment, and left only the zs. Now let me define them more precisely. Small z is the initial range of the blip – the distance at which it starts. Big Z is its final distance from the origin. Thus if a point was initially 2 units away, by obeying this equation it would promptly hop to a distance of 4.

The iteration loop

Nothing to get very excited about, but now comes the modification that makes all the difference:

$$Z \rightleftharpoons z^2$$

That double arrow is a two-way traffic sign, indicating that the numbers flow in both directions. This time, we don't stop at Z = 4; we make that equal to a new z – which promptly give us a second Z of 16, and so on.

In no time we've generated the series 256,65536,4294967296 … and the spot that started only 2 units from the centre is heading towards infinity in giant steps of ever-increasing magnitude.

This process of going round and round a loop is called 'iteration'. It's like a dog chasing its own tail, except that a dog doesn't get anywhere. But mathematical iteration can take us to some very strange places indeed – as we shall soon discover.

Now we're ready to turn on our radar. Most displays have range circles at 10, 20 … 100 kilometres from the centre. We will require only a single circle, at a range of 1. There's no need to specify any units, as we're dealing with pure numbers. Make them centimetres or light years, as you please.

Let's suppose that the initial position of our blip is anywhere on this circle – the bearing doesn't matter. So z is 1.

As 1 squared is still 1, so is Z. And it remains at that value because no matter how many times you square 1, it always remains exactly 1. The blip may hop round and round the circle, but it always stays on it.

Shooting for infinity

Now consider the case where the initial *z* is greater than 1. We've already seen how rapidly the blip shoots to infinity if *z* equals 2, but the same thing happens even if it's only a microscopic shade more than 1: say 1.000000000000000000001. Watch:

At the first squaring, Z becomes

1.000000000000000000002
then
1.000000000000000000004
1.000000000000000000008
1.000000000000000000016
1.000000000000000000032
1.000000000000000000064

and so on for pages of printout. For all practical purposes, the value is still exactly 1. The blip hasn't moved visibly outwards or inwards; it's still on the circle at range 1.

But those zeros are slowly being whittled away, as the digits march inexorably across from the right. Quite suddenly, something appears in the third, second, first decimal place – and the numbers explode after a very few additional terms, as this example shows, reading left to right:

1.001	1.002	1.004	1.008
1.016	1.032	1.066	1.136
1.292	1.668	2.783	7.745
59.987	3598.467	12,948,970	
167,675,700,000,000			
28,115,140,000,000,000,000,000,000,000			

There could be a million – a billion – zeros on the right hand side, and the result would still be the same. Eventually the digit would creep up to the decimal point – and then Z would take to infinity.

The other side of infinity

Now let's look at the other case. Suppose *z* is a microscopic amount less than 1 – say something like 0.999999999999999999999.

As before, nothing much happens for a long time as we round the loop, except that the numbers on the far right get steadily smaller. But after a few thousand or million iterations – catastrophe! Z suddenly shrinks to nothing, dissolving in an endless string of zeros …

Check it out on your computer. It can only handle twelve digits? Well, no matter how many you had to play with, you'd get the same answer. Trust me …

The results of this 'program' can be summarized in the laws that may seem too trivial to be worth formulating. But mathematical truth is trivial, and in a few more steps these laws will take us into a universe of mind-boggling wonder and beauty.

The laws of squaring

Here are the three laws of the squaring program:

1. if the input *z* is exactly equal to 1, the output Z always remains 1;
2. if the input is more than 1, the output eventually becomes infinite; and
3. if the input is less than 1, the output eventually becomes zero.

That circle of radius 1 is therefore a kind of map, dividing the plane into two distinct territories. Outside it, numbers that obey the squaring law have the freedom of infinity; numbers inside it are prisoners, trapped and doomed to ultimate extinction.

At this point, someone may say: 'You've only talked about ranges – distances from the origin. To fix the blip's position, you have to give its bearing as well. What about that?'

Very true. Fortunately, in this selection process – this division of the *z*s into two distinct classes – bearings are

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irrelevant; the same thing happens in every direction. For this simple example – let's call it the S-set – we can ignore them.

When we come on to the more complicated case of the M-set, where the bearing is important, there's a very neat mathematical trick to take care of it. Many of you will have guessed that it uses complex or imaginary numbers (which really aren't at all complex, still less imaginary). But we don't need them for this discussion, and I promise not to mention them again.

Inside the map

The S-set lies inside a map, and its frontier is the circle enclosing it. That circle is simply a continuous line with no thickness. If you could examine it with a microscope of infinite power, it would always look exactly the same.

You could expand the S-set to the size of the universe; its boundary would still be a line of zero thickness. Yet there are no holes in it; it's an absolutely impenetrable barrier, forever separating the zs less than one from those greater than one.

Now, at last, we're ready to tackle the M-set, where these commonsense ideas are turned upside down. Fasten your seat belts.

During the 1970s, the French mathematician Benoît Mandelbrot, working at Harvard and IBM, started to investigate the equation that has made him famous, and which I will now write in dynamic form:

$$Z \rightleftharpoons z^2 + c.$$

The only difference between this and the equation we have used to describe the S-set is the term c. This – not z – is now the starting point of our mapping operation. The first time round the loop, z is put equal to zero.

The M-set and the unimaginable universe

It seems a trifling change, and no-one could have imagined the universe it would reveal. Mandelbrot himself did not obtain the first crude glimpses until the spring of 1980, when vague patterns started to emerge on computer printouts. He had begun to peer through Keats's

> Charmed magic casements, opening on the foam
> Of perilous seas, in faery lands forlorn.

As we shall learn later; that word 'foam' is surprisingly appropriate.

The new equation asks and answers the same question as the earlier one: What shape is the 'territory' mapped out when we put numbers into it? For the S-set it was a circle of radius 1. Let's see what happens when we start with this value in the M-equation.

You should be able to do it in your head – for the first few steps. But after a few dozen even a supercomputer may blow a gasket.

For starters, $z = 0$, $c = 1$. So $Z = 1$.
First loop: $Z = 1^2 + 1 = 2$.
Second loop: $Z = 2^2 + 1 = 5$.
Third loop: $Z = 5^2 + 1 = 26$.
Fourth loop: $Z = 26^2 + 1$ … and so on.

I once set my computer to work out the higher terms (about the limit of my programming ability) and it produced only two more values before it had to start approximating. Starting from the beginning we get:

1
2
5
26
677
458,330
21,006,640,000
4,412,789,000,000,000,000,000.

At that point my computer gave up, because it doesn't believe there are any numbers with more than 38 digits.

However, even the first two or three terms are quite enough to show that the M-set must have a different shape from the perfectly circular S-set. A point at distance 1 is in the S-set; indeed, it defines its boundary. A point at that same distance may be outside the boundary of the M-set.

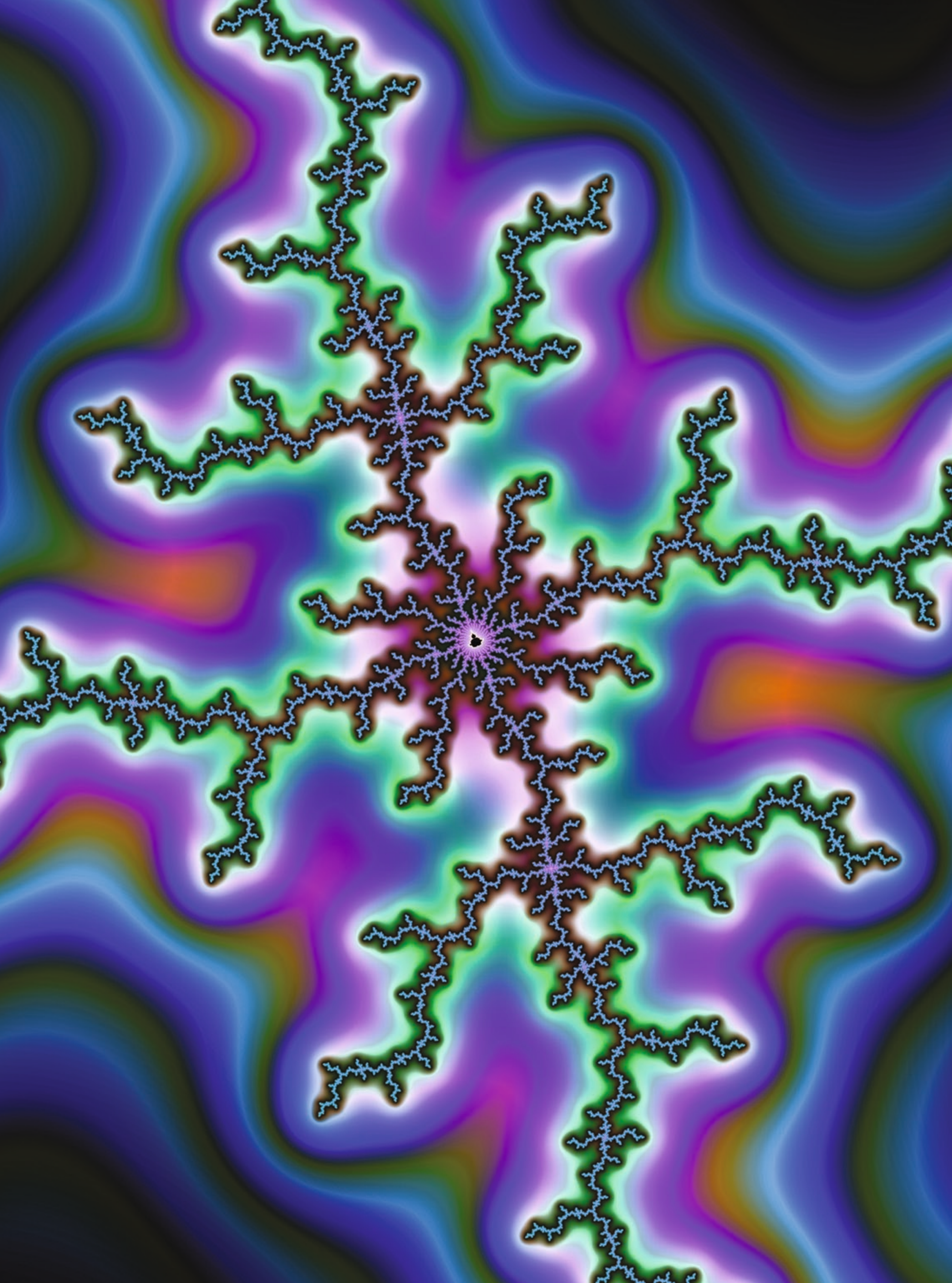
Note that I say 'may' not 'must'. It all depends on the initial direction, or bearing, of the starting point, which we have been able to ignore hitherto because it did not affect our discussion of the (perfectly symmetrical) S-set. As it turns out, the M-set is only symmetrical about the X, or horizontal, axis.

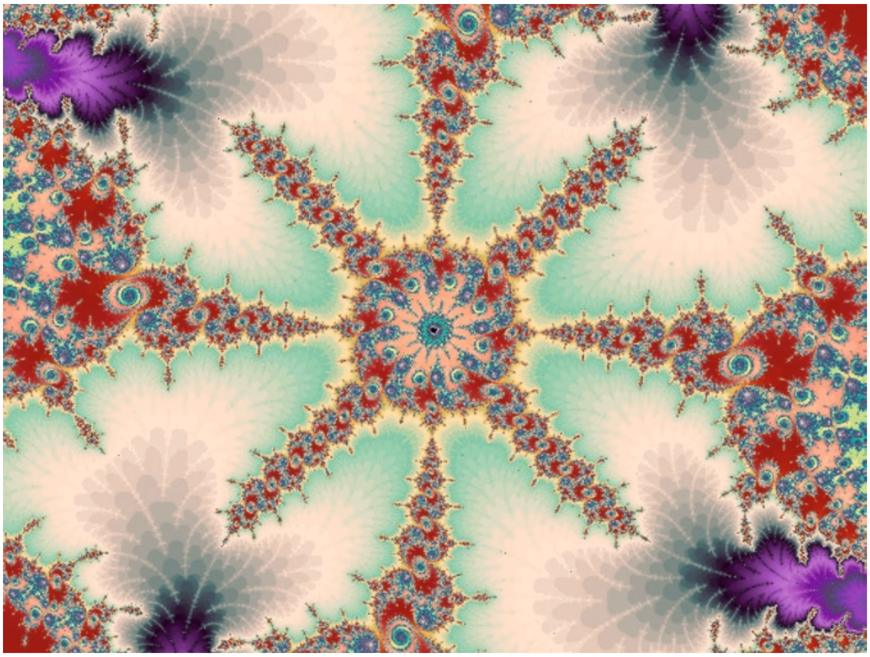
One might have guessed that, from the nature of the equation. But no-one could possibly have intuited its real appearance: if the question had been put to me in my virginal pre-Mandelbrot days, I would probably have hazarded: 'Something like an ellipse, squashed along the Y-axis.' I might even (though I doubt it) have correctly guessed that it would be shifted towards the left, or minus, direction.

The indescribable M-set

At this point, I would like to try a thought experiment on you. The M-set being literally indescribable, here's my best attempt describe it: imagine you're looking straight down on a rather plump turtle swimming westwards. It's been crossed with a swordfish, so has a narrow spike pointing ahead of it. Its entire perimeter is festooned with bizarre marine growths – and with baby turtles of assorted sizes, which have smaller weeds growing on them …

I defy you to find a description like that in any maths textbook. And if you think you can do better when you've met the real beast, you're welcome to try.





Computers can easily make snapshots of the M-set at any magnification, and even in black and white they are fascinating. However, by a simple trick they can be coloured, and transformed into objects of amazing, even surreal, beauty.

(I suspect that the insect world might provide better analogies; there may even be a Mandelbeetle lurking in the Brazilian rain forests. Too bad, we'll never know.)

Here is the first crude approximation, shorn of details; if you like to fill its blank spaces with the medieval cartographers' favourite 'here be dragons' you will hardly be exaggerating.

First of all, note that – as I've already remarked – it's shifted to the left (or West, if you prefer) of the S-set, which of course extends from +1 to −1 along the X-axis. The M-set only gets to 0.25 on the right of the horizontal axis line itself, though above and below the axis line it bulges out to just beyond 0.4.

The 'Utter West'

On the left-hand side, the map stretches to about −1.4, and then it sprouts a peculiar spike – or antenna – which reaches out to exactly −2.0. As far as the M-set is concerned, there is nothing beyond this point; it is the edge of the Universe.

Some Mandelbrot fans call it 'the Utter West', and you might like to see what happens when you make c equal to −2. Z doesn't converge to zero – but it doesn't escape to infinity either, so the point belongs to the Set – just. But if you make c equal to −2.0000001, before you know you're passing Pluto and heading for Quasar West.

Now we come to the most important distinction between the two sets. The S-set has a nice, clean line for its boundary. The frontier of the M-set is, to say the least, fuzzy. Just how fuzzy you will begin to understand when we start to zoom into it; only then will we see the incredible flora and fauna that flourish in that disputed territory.

The boundary – if one can call it that – of the M-set is not a simple line; it is something that Euclid never imagined, and for which there is no word in ordinary language. Mandelbrot, whose command of English (and American) is awesome, has ransacked the dictionary for suggestive nouns. A few examples: foams, sponges, dusts, webs, nets, curds. He himself coined the technical name fractal, and is now putting up a spirited rearward action to stop anyone defining it too precisely.

The colours of infinity

Computers can easily make snapshots of the M-set at any magnification, and even in black and white they are fascinating. However, by a simple trick they can be coloured, and transformed into objects of amazing, even surreal, beauty.

The original equation, of course, is no more concerned with colour than is Euclid's Elements of Geometry. But if we instruct the computer to colour any given region in accordance with the number of times z goes round the loop before it decides whether or not it belongs to the M-set the results are gorgeous.

Thus the colours, though arbitrary, are not meaningless. An exact analogy is found in cartography. Think of the contour lines on a relief map, which show elevations above sea level. The spaces between them are often coloured so that the eye can more easily grasp the information conveyed. Ditto with bathymetric charts; the deeper the ocean, the darker the blue. The map-maker can make the colours anything he likes, and is guided by aesthetics as much as geography.

It's just the same here – except that these contour lines are set automatically by the speed of the calculation – I won't go into details. I have not discovered what genius first had this idea – perhaps Monsieur M. himself, but it turns them into fantastic works of art. And you should see them when they're animated …

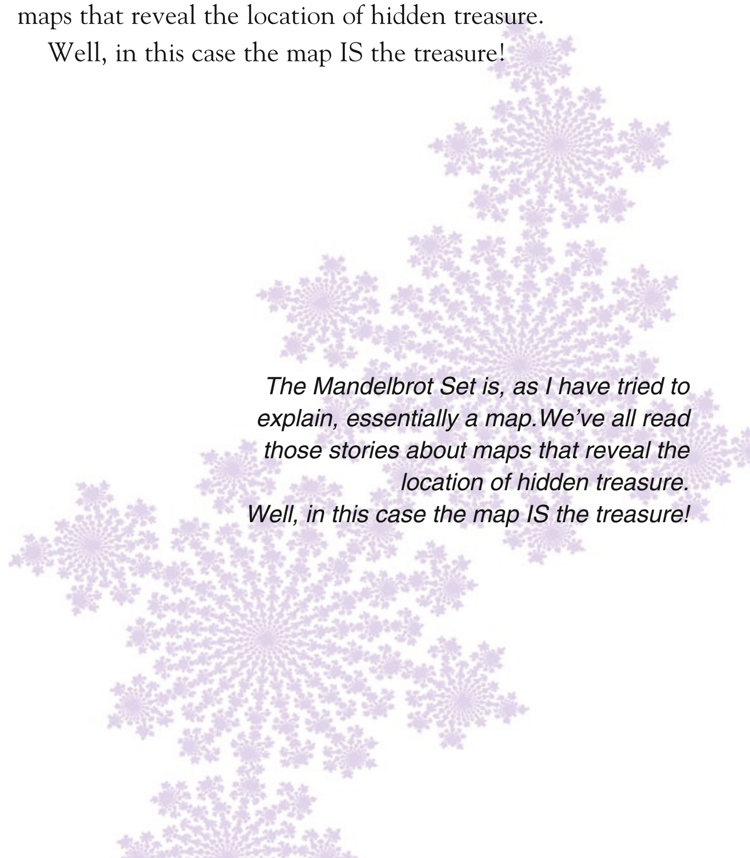
Only in the computer age

One of the many strange thoughts that the M-set generates is this. In principle, it could have been discovered as soon as the human race learned to count. In practice, since even a low magnification image may involve billions of calculations, there was no way in which it could even be glimpsed before computers were invented! And such movies as those on the DVD with this book would have required the entire present world population to calculate night and day for years – without making a single mistake in multiplying together trillions of hundred-digit numbers.

I began by saying that the Mandelbrot Set is the most extraordinary discovery in the history of mathematics. For who could have possibly imagined that so absurdly simple an equation could have generated such literally infinite complexity, and such unearthly beauty?

The Mandelbrot Set is, as I have tried to explain, essentially a map. We've all read those stories about maps that reveal the location of hidden treasure.

Well, in this case the map IS the treasure!



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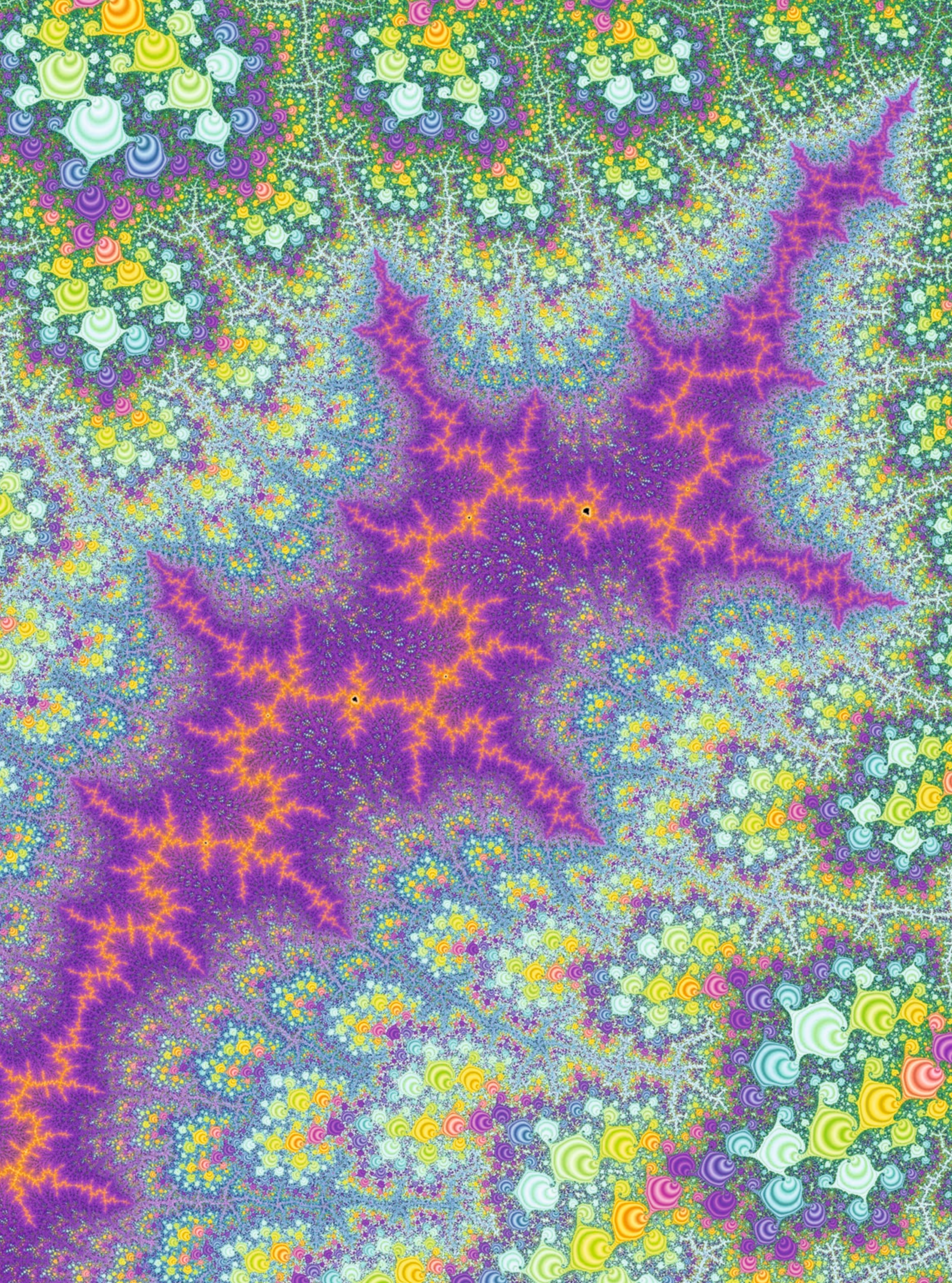

Mathematical Appendix

One way of appreciating where the curiously-shaped country of the M-set is located on the map of all possible (complex) numbers is to pin down its Eastern and Western frontiers, ignoring everything to the North and South.

The Western, or negative, limit is easily identified; for once, the calculation can be done mentally, without the aid of a computer! If we take the basic equation: $Z = z^2 + c$ and set the initial value of c equal to −2, the first time round the loop gives Z = −2. The second value is $Z = (-2)^2 - 2 = 2$. The third value is $Z = 2^2 - 2 = 2$.

And so on for ever: Z is stuck at 2! It does not shrink to zero, but neither does it go racing off to infinity. Thus the point at −2 on the X-axis, or 2 units to the left of the origin, definitely belongs to the M-set. It masks the Utter West – the very tip of the strangely ornamented spike that extends in that direction.

It's interesting to see what happens for values of c on either side of −2, and for that we certainly do need a computer. Take c = −1.99999. Table 2.1 shows what happens to Z as it goes round and round the loop:

Table 2.1 c = −1.99999 (reading the numbers left to right)

1.999970	1.999890	1.999570	1.998290	1.993174
1.972752	1.891762	1.578773	0.492534	−1.757400
1.088466	−0.815231	−1.335388	−0.216729	−1.953019
1.814292	1.291665	−0.331592	−1.890037	1.572248
0.471975	−1.777230	1.158556	−0.657737	−1.567371
0.456663	−1.791449	1.209298	−0.537588	−1.710989
0.927494	−1.139744	−0.700973	−1.508626	0.275963
−1.923834	1.701 149	0.893918	−1.200901	−0.557826
−1.688820	0.852124	−1.273875	−0.377232	−1.857686
1.451007	0.105431	−1.988874	1.955631	1.824503
1.328822	−0.234222	−1.945130	1.783540	1.181027
−0.605166	−1.633764	0.669195	−1.552168	0.409236
−1.832516	1.358123	−0.155492	−1.975812	1.903845
1.624634	0.639446	−1.591099	0.531605	−1.717386
0.949426	−1.098581			

The value then goes on oscillating, presumably forever (my computer has been round the loop only about ten thousand times) between the limits of plus and minus 2. Perhaps after ten million iterations Z might change its mind and suddenly shoot off to infinity, but it seems reasonable to assume that this value of c is definitely inside the M-set.

Table 2.2 c = −2.00001

2.00003	2.00011	2.00043	2.00171	2.00683
2.02737	2.11023	2.45306	4.01748	14.14011
197.942	39,179.2	1.5E+9		
2.4E+18	5.5E+36	3.1E+73		

The fate of the point only 0.00002 units further 'west', on the other hand, is very quickly decided, as we see in Table 2.2:

As far as an Apple Mac is concerned, the numbers in that last line are infinite, and I doubt if even a Super Cray would disagree. So −2.00001 is definitely outside the M-set.

On the Eastern, or positive side of the Set, the limit is not so easily defined.

Obviously, it is closer to the origin (0,0) than the point + 1, which gives a value shooting off to infinity after only a few times round the loop. A few minutes' work with pencil and paper shows that it is even closer than 0.5, for putting c = $^1/_2$ also gives a rapidly soaring Z. It is, in fact, at 0.25 – though this is by no means easy to prove.

When I set c = 0.25 in the program I have painfully written, the screen is flooded with a torrent of numbers, which after hundreds of iterations finally settle down to the odd value 0.4998505. I assume that this should be exactly 0.5, with the difference due to rounding-off errors. In any event, Z doesn't shoot off to infinity, so the Eastern limit of the M-set is definitely at 0.25. (On the centre line, that is; above and below, it bulges considerably further eastwards.)

It's interesting to check what happens when bracketing this value and setting c equal to 0.24999 and 0.25001. Table 2.3 gives the result of the first:

Table 2.3 c = .24999

.3124850	.3476369	.3708414	.3875133	.4181846
.4248683	.4305031	.4431468	.4463691	.4492353
.4562108	.4581183	.4598624	.4643020	.4655664
.4667420	.4698221	.4707228	.4715699	.4738342
.4745088	.4751486	.4768840	.4774084	.4779088
.4792815	.4797008	.4801028	.4812158	.4815586
.4818887	.4828091	.4830946	.4833704	.4001566
.4101153	.4353229	.4394960	.4518024	.4541154
.4614634	.4629385	.4678381	.4688625	.4723682
.4731217	.4757562	.4763340	.4783868	.4788439
.4804887	.4808594	.4822067	.4825133	.4833704

and then, after 12 more screens-full of figures …

.4968333	.4968333	.4968334	.4968334	.4968334
.4968334	.4968335	.4968335	.4968335	.4968335
.4968336	.4968336	.4968336	.4968337	.4968337
.4968337	.4968337	.4968338	.4968338	.4968338
.4968338	.4968339	.4968339	.4968339	.4968339
.4968339	.4968340	.4968340	.4968340	.4968340
.4968341	.4968341	.4968341	.4968341	.4968342

and so on forever.

c = .24999 is therefore definitely inside the M-set. If we increase its value very slightly, to .25001, Table 2.4 reveals a quite different result, though it takes almost as long to arrive at it.

Table 2.4 c = .25001

.3125150	.3476756	.3708883	.3875682	.4002191
.4305951	.4354222	.4493631	.4519372	.4600251
.4616331	.4669395	.4680425	.4718018	.4726070
.4754148	.4760292	.4101853	.4182620	.4249531
.4396025	.4432603	.4464897	.4542572	.4563596
.4582741	.4631151	.4644856	.4657569	.4690738
.4700402	.4709478	.4733673	.4740866	.4747681

and then, after some eight screens-full of figures…

.5611078	.5648520	.5690677	.5738481	.5793116
.5856120	.5929514	.6016013	.61 19342	.6244734
.6399771	.6595806	.6850566	.7193126	.7674206
.8389443	.9538376	1.159816	1.595183	2.794620
8.059914	6.5E+1	4.3E+3	1.8E+7	3.3E+14
1.07E+29	1.1E+58	1.3E+116	1.7E+232	

c = 0.25001 is therefore outside the M-set.

Although all these calculations involve only the X-coordinate, and ignore complex numbers by setting Y = 0, they can be very time-consuming. Tables 2.3 and 2.4 demonstrate how impossible it would have been to discover – let alone map in detail! – the Mandelbrot Set before the advent of modern computers.

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