Series Expansions in Queues with Server Vacation

Fazia Rahmoune and Djamil Aïssani

Abstract This paper provides series expansions of the stationary distribution of finite Markov chains. The work presented is a part of research project on numerical algorithms based on series expansions of Markov chains with finite state-space S. We are interested in the performance of a stochastic system when some of its parameters or characteristics are changed. This leads to an efficient numerical algorithm for computing the stationary distribution. Numerical examples are given to illustrate the performance of the algorithm, while numerical bounds are provided for quantities from some models like manufacturing systems to optimize the requirement policy or reliability models to optimize the preventive maintenance policy after modelling by vacation queuing systems.

1 Introduction

Let **P** denote the transition kernel of a Markov chain defined on a finite state-space S having unique stationary distribution π_P. Let **Q** denote the Markov transition kernel of the Markov chain modeling the alternative system and assume that **Q** has unique stationary distribution π_Q. The question about the effect of switching from **P** to **Q** on the stationary behavior is expressed by $\pi_P - \pi_Q$, the difference between the stationary distributions (Heidergott and Hordijk, 2003). In this work, we show that the performance measure of some stochastic models, which are gouverned by a finite Markov chain, can be obtained from other performance of more simple models, via series expansion method. Let $\| \, . \, \|_{tv}$ denote the total variation norm, then the above problem can be phrased as follows: Can $\| \pi_P - \pi_Q \|_{tv}$ be approximated or bounded

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in terms of $\| \mathbf{P} - \mathbf{Q} \|_{tv}$? This is known as *perturbation analysis of Markov chains (PAMC)* in the literature.

This paper is considered as a continuity of the work (Rahmoune and Aïssani, 2008), where quantitative estimate of performance measure has been established via strong stability method for some vacation queueing models. In this work, we will show that $\pi_P - \pi_Q$ can be arbitrarily closely approximated by a polynomial in $(\mathbf{Q} - \mathbf{P})D_P$, where D_P denotes the deviation matrix associated with $\mathbf{P}$. A precise definitions and notations will be given later. Starting point is the representation of π_Q given by:

$$\pi_Q = \sum_{n=0}^{k} \pi_P((\mathbf{Q} - \mathbf{P})D_P)^n + \pi_Q((\mathbf{Q} - \mathbf{P})D_P)^{k+1}; \tag{1}$$

for any $k \geq 0$. This series expansion of π_Q provides the means of approximating π_Q by $\mathbf{Q}$ and entities given via the $\mathbf{P}$ Markov chain only. We obtain a bound for the remainder term working with the weighted supremum norm, denoted by $\| \cdot \|_v$, where v is some vector with positive non-zero elements, and for any $w \in \mathrm{R}^S$

$$\| w \|_v = \sup_{i \in S} \frac{w(i)}{v(i)}, \tag{2}$$

see, for example (Meyn and Tweedie, 1993). We will show that for our models

$$\pi_Q(s) - \left(\sum_{n=0}^{k} \pi_P((\mathbf{Q} - \mathbf{P})D_P)^n \right)(s) \leq d \, \| ((\mathbf{Q} - \mathbf{P})D_P) \|_v^{k+1}$$

for any $k \in \mathrm{N}$ and any $s \in \mathrm{S}$, where v can be any vector satisfying $v(s) \geq 1$ for $s \in \mathrm{S}$, and d is some finite computable constant. In particular, the above error bound can be computed without knowledge of π_Q.

The key idea of the approach is to solve for all k the optimization problem

$$\begin{cases} \min \| ((\mathbf{Q} - \mathbf{P})D_P)^k \|_v, & \text{subject to} \\ v(s) \geq 1, & \text{for } s \in \mathrm{S}. \end{cases} \tag{3}$$

The vector $v^\star$ thus yields the optimal measure of the rate of convergence of the series in (1). Moreover, the series in (1) tends to converge extremely fast which is due to the fact that in many examples $v^\star$ be found such that $\| ((\mathbf{Q} - \mathbf{P})D_P)^k \|_{v^\star} << 1$. The limit of the series (1) first appeared in (Cao, 1998), however, neither upper bounds for the remainder term nor numerical examples were given there. The derivation of this has been done in (Heidergott and Hordijk, 2003), which is a generalization of (Cao, 1998). The use of series expansion for computational purposes is not new. It has been used in the field of linear algebra (Cho and Meyer, 2001).
The work presented in this paper is part of research project on numerical algorithms based on series expansions of Markov chains as it was in Heidergott and Hordijk

(2003). The present paper establishes the main theoretical results. In particular, numerical examples are provided for vacation queueing systems.

2 Preliminaries on Finite Markov chains

Let S denote a finite set $\{1,\cdots,S\}$, with $0 < S < \infty$ elements. We consider Markov kernels on state space S, where the Markov kernel $\mathbf{P}^n$ is simply obtained from taking the nth power of $\mathbf{P}$. Provided it exists, we denote the unique stationary distribution of $\mathbf{P}$ by π_P and its ergodic projector by Π_P. For simplicity, we identify π_P and π_Q with Π_P and Π_P, respectively. Throughout the paper, we assume that $\mathbf{P}$ is aperiodic and unichain, which means that there is one closed irreducible set of states and a set of transient states. Let $|A|(i;j)$ denote the $(i;j)$th element of the matrix of absolute values of $A \in \mathrm{R}^{S\times S}$, and additionally we use the notation $|A|$ for the matrix of absolute values of A.

The main tool for this analysis is the v-norm, as defined in (2). For a matrix $A \in \mathrm{R}^{S\times S}$ the v-norm is given by

$$\| A \|_v \overset{def}{=} \sup_{i,\|w\|_v\leq 1} \frac{\sum_{j=1}^{S} |A(i,j)w(j)|}{v(i)}$$

Next we introduce v-geometric ergodicity of $\mathbf{P}$, see Meyn and Tweedie (1993) for details.

Definition 0.1. A Markov chain $\mathbf{P}$ is v-geometric ergodic if $c < \infty, \beta < 1$ and $N < \infty$ exist such that

$$\| \mathbf{P}^n - \Pi_P \|_v \leq c\beta^n, \text{ for all } n \geq N.$$

The following lemma shows that any finite-state aperiodic Markov chain is v-geometric ergodic.

Lemma 0.1. *For finite-state and aperiodic* $\mathbf{P}$ *a finite number N exists such that*

$$\| \mathbf{P}^n - \Pi_P \|_v \leq c\beta^n, \textit{ for all } n \geq N;$$

where $c < \infty$ *and* $\beta < 1$.

Proof. Because of the finite state space and aperiodicity.

3 Series Expansions in Queues with Server Vacation

We are interested in the performance of a queuing system with single vacation of the server when some of its parameters or characteristics are changed. The system as given is modeled as a Markov chain with kernel **P**, the changed system with kernel **Q**. We assume that both Markov chains have a common finite state space S. We assume too, as indicated earlier, that both Markov kernels are aperiodic and unichain. The goal of this section is to obtain the stationary distribution of **Q**, denoted by π_Q, via a series expansion in **P**. In the next section, we comment on the speed of convergence of this series. We summarize our results in an algorithm, presented in an other section. Finally, we illustrate our approach with numerical examples.

3.1 Description of the models

Let us consider the $M/G/1//N$ queueing systems with multiples vacations of the server modelling the reliability system with multiple preventives maintenances. We suppose that there is on the whole N machines in the system. Our system consists of a source and a waiting system (queue + service). Each machine is either in source or in the waiting system at any time. A machine in source arrives at the waiting system precisely, with the durations of inter-failure exponentially distributed with parameter λ) for repairment (corrective maintenance). The distribution of the service time is general with a distribution function $B(.)$ and mean b. The repairmen take maintenance period at each time the system is empty. If the server returns from maintenance finding the queue impty, he takes an other maintenance period (mulitiple maintenance). In addition, let us consider the $M/G/1//N$ queuing system with unique vacation of the server modelling the reliability system with periodic prenventives maintenances, having the same distributions of the inter-arrivals and the repair time previously described. In this model, the server (repairman) will wait until the end of the next activity period during which at least a customer will be served, before beginning another maintenance period. In other words, there is exactly only one maintenance at the end of each activity period at each time when the queue becomes empty *(exhaustive service)*. If the server returns from maintenane finding the queue nonempty, then the maintenance period finishes for beginning another activity period. We also suppose that the maintenance times V of the server are independent and iid, with general distribution function noted $V(x)$.

3.2 Transition Kernels

Let X_n (resp. $\bar{X}_n$) the imbedded Markov chains at the end moments of repair t_n for the n^{th} machine associated with the $M/G/1//N$ system with multiple maintenance (resp.to the system with the unique maintenance). In the same way, we define the following probabilities:

$$f_k = P[k \text{ broken down machines at the end of the preventive maintenance period}]$$

$$= C_N^k \int_0^{\infty} (1-e^{-\lambda t})^k e^{-(N-k)\lambda t} dV(t),\ k = \overline{0,N}. \tag{4}$$

for witch

$$\bar{\alpha}_k = \begin{cases} f_0 + f_1, & \text{for } k = 1 \\ f_k, & \text{for } 2 \le k \le N. \end{cases}$$

and

$$\alpha_k = \frac{f_k}{1-f_0} \ \text{for} \quad k = \overline{1,N}.$$

The one stage transition probabilities of the imbedded Markov chains X_n and $\bar{X}_n$ allow us to describe the general expression of the transition kernels $\mathbf{P} = (\mathbf{P}_{ij})_{ij}$ and $\mathbf{Q} = (\mathbf{Q}_{ij})_{ij}$ summarized below respectively.

$$\mathbf{P}_{ij} = \begin{cases} \sum\limits_{k=1}^{j+1} P_{j-k+1}\alpha_k, & \text{if} \quad i = 0, j = \overline{0,N-1},\ k = \overline{1,N}, \\ P_{j-i+1} & \text{if} \quad 1 \ \le i \le j+1 \ \le \ N-1, \\ 0 & \text{else}. \end{cases}$$

$$\mathbf{Q}_{ij} = \begin{cases} \sum\limits_{k=1}^{j+1} P_{j-k+1}\bar{\alpha}_k, & \text{if} \quad i = 0, j = \overline{0,N-1},\ k = \overline{1,N}, \\ P_{j-i+1} & \text{if} \quad 1 \ \le i \le j+1 \ \le \ N-1, \\ 0 & \text{else}. \end{cases}$$

Clearly, the Markov chain $\{\bar{X}_n\}_{n\in \mathrm{N}}$ is irreducible, aperiodic with finite state space $\mathrm{S} = \{0,1,\cdots,N-1\}$. So, we can applied the main theorical results established in this paper to this model, in order to approach another Markov chain whose transition kernel is neighborhood of its transition kernel $\mathbf{Q}$.

3.3 Series Development for π_Q

We write D_P for the deviation matrix associated with $\mathbf{P}$; in symbols:

$$D_P = \sum_{m=0}^{\infty} (\mathbf{P}^m - \Pi_P) \tag{5}$$

Note that D_P is finite for any aperiodic finite-state Markov chain. Moreover, the deviation can be rewritten as

$$D_P = \sum_{m=0}^{\infty} (\mathbf{P} - \Pi_P)^m - \Pi_P,$$

where $\sum\limits_{m=0}^{\infty} (\mathbf{P} - \Pi_P)^m$ is often referred to as the group inverse, see for instance Cao (1998) or Coolen-Schrijner and van Doorn (2002). A general definition which is valid for periodic Markov chain, can be found in, e.g., Puterman (1994).

Let $\mathbf{P}$ be unichain. Using the definition of D_P, we obtain:

$$(I-\mathbf{P})D_P = I - \Pi_P.$$

This is the Poisson equation in matrix format.

Let the following equation:

$$\Pi_Q = \Pi_P \sum_{n=0}^{k} ((\mathbf{Q}-\mathbf{P})D_P)^n + \Pi_Q((\mathbf{Q}-\mathbf{P})D_P)^{k+1}. \tag{6}$$

for $k \geq 0$, where:

$$H(k) \stackrel{def}{=} \Pi_P \sum_{n=0}^{k} ((\mathbf{Q}-\mathbf{P})D_P)^n,$$

is called a *series approximation of degree k* for $\Pi_Q, T(k)$, with

$$T(k) \stackrel{def}{=} \Pi_P((\mathbf{Q}-\mathbf{P})D_P)^k, \tag{7}$$

denotes the kth element of H(k), and

$$R(k) \stackrel{def}{=} \Pi_Q((\mathbf{Q}-\mathbf{P})D_P)^{k+1}, \tag{8}$$

is called the *remainder term* (see Heidergott et al, 2007, for details). The quality of the approximation provided by $H(k)$ is given through the remainder term $R(k)$.

3.4 Series Convergence

In this section we investigate the limiting behavior of $H(k)$ as k tends to ∞. We first establish sufficient conditions for the existence of the series.

Lemma 0.2. *(Heidergott and Hordijk, 2003) The following assertions are equivalent:*

(i) The series $\sum_{k=0}^{\infty} ((\mathbf{Q}-\mathbf{P})D_P)^k$ *is convergent.*
(ii) There are N *and* $\delta_N \in (0,1)$ *such that* $\| ((\mathbf{Q}-\mathbf{P})D_P)^N \|_v < \delta_N$.
(iii) There are κ *and* $\delta < 1$ *such that* $\| ((\mathbf{Q}-\mathbf{P})D_P)^k \|_v < \kappa\delta^k$ *for any k.*
(iv) There are N *and* $\delta \in (0;1)$ *such that* $\| ((\mathbf{Q}-\mathbf{P})D_P)^k \|_v < \delta^k$ *for any* $k \geq N$.

Proof. See Heidergott and Hordijk (2003).

The fact that the maximal eigenvalue of $|(\mathbf{Q}-\mathbf{P})D_P|$ is smaller than 1 is necessary for the convergence of the series $\sum_{k=0}^{\infty} ((\mathbf{Q}-\mathbf{P})D_P)^k$.

Remark 0.1. Existence of the limit of $H(k)$, see (i) in Lemma 0.3, is equivalent to an exponential decay in the v-norm of the elements of the series, see (iv) in Lemma 0.2. For practical purposes, one needs to identify the decay rate δ and the threshold value N after which the exponential decay occurs. The numerical experiments have shown that the condition (ii) in Lemma 0.2 is the most convenient to work with. More specifically, we work with the following condition (C) as in Heidergott and Hordijk (2003), which is similar to the geometric series convergence criterion.

The Condition(C)**:** There exists a finite number N such that we can find $\delta_N \in (0;1)$ which satisfies:

$$\| ((\mathbf{Q}-\mathbf{P})D_P)^N \|_v < \delta^N;$$

and we set

$$c^v_{\delta_N} \stackrel{def}{=} \frac{1}{1-\delta_N} \| \sum_{k=0}^{N-1} ((\mathbf{Q}-\mathbf{P})D_P)^k \|_v$$

As shown in the following lemma, the factor $c^v_{\delta_N}$ in condition (C) allows to establish an upper bound for the remainder term that is independent of Π_Q.

Lemma 0.3. *Under (C) it holds that:*

(i) $\| R(k-1) \|_v \leq c^v_{\delta_N} \| T(k) \|_v$ *for all* k,

(ii) $\lim_{k\to\infty} H(k) = \Pi_P \sum_{n=0}^{\infty} ((\mathbf{Q}-\mathbf{P})D_P)^n = \Pi_Q$

Proof. To proof the lemma it is sufficient to use the definition of the norm $\| \, . \, \|_v$ and the remainder term $R(k-1)$, using the condition (iv) of Lemma 0.2.

Remark 0.2. An example where the series $H(k)$ fails to converge is illustrated in Heidergott and Hordijk (2003).

Remark 0.3. The series expansion for Π_Q put forward in the assertion (ii) in Lemma 1 is well known; see Cao (1998) and Kirkland (2003) for the case of finite Markov chains and Heidergott and Hordijk (2003) for the general case. It is however worth noting that in the aforementioned papers, the series was obtained via a differentiation approach, whereas the representation is derived in this paper from the elementary equation 6.

Remark 0.4. Provided that $\det(I-(\mathbf{Q}-\mathbf{P})D_P) \neq 0$, one can obtain π_Q from

$$\pi_Q = \Pi_P(I-(\mathbf{Q}-\mathbf{P})D_P)^{-1} \tag{9}$$

Moreover, provided that the limit

$$\lim_{k\to\infty} H(k) = \lim_{k\to\infty} \pi_P \sum_{n=0}^{\infty} ((\mathbf{Q}-\mathbf{P})D_P)^n$$

exists (see Lemma 0.3 for sufficient conditions), it yields π_Q as $\pi_P \sum_{n=0}^{\infty}((\mathbf{Q}-\mathbf{P})D_P)^n$.

Remark 0.5. Note that a sufficient condition for (C) is

$$\| (\mathbf{Q}-\mathbf{P})D_P \|_v < \delta, \quad \delta < 1. \tag{10}$$

In Altman et al (2004); Cho and Meyer (2001) it is even assumed that

$$\| (\mathbf{Q}-\mathbf{P})D_P \|_v < g_1, \tag{11}$$

with $g_1 > 0$ a finite constant, and

$$\| D_P \|_v < \frac{c}{1-\beta}, \tag{12}$$

with $c > 0$ and $0 < \beta < 1$ finite constants. If

$$\frac{g_1 c}{1-\beta} < 1, \tag{13}$$

then (10) and hence (C) is clearly fulfilled. Hence, for numerical purposes these conditions are too strong.

3.5 The remainder term Bounds

The quality of approximation by $H(k-1)$ is given by the remainder term $R(k-1)$ and in applications v should be chosen such that it minimizes $c^v_{\delta_N} \parallel T(k) \parallel_v$, thus minimizing our upper bound for the remainder term. For finding an optimal upper bound, since $c^v_{\delta_N}$ is independent of k, we focus on $T(k)$. Specifically, we have to find a bounding vector v that minimizes $\parallel T(k) \parallel_v$ uniformly w.r.t. k. As the following theorem shows, the unit vector, denoted by 1, with all components equal to one, yields the minimal value for $\parallel T(k) \parallel_v$ for any k.

Theorem 0.1. *(Heidergott and Hordijk, 2003) The unit vector* 1 *minimizes* $\parallel T(k) \parallel_v$ *uniformly over k, i.e.,*

$$\forall k \geq 1 : inf_v \parallel T(k) \parallel_v = \parallel T(k) \parallel_1 \tag{14}$$

Remark 0.6. It can be shown as for the results in Altman et al (2004) and Cho and Meyer (2001), that the smallest $\frac{cg_1}{1-\beta}$ is precisely the maximal eigenvalue of $|D_P|$. Again we note that often the product of these maximal eigenvalues is not smaller than 1. If this is the case, then according to Altman et al (2004) and Cho and Meyer (2001) we cannot decide whether the series $H(k)$ converges to Π_Q. Hence, their condition is too restrictive for numerical purposes.

3.6 Algorithm

In this section we describe a numerical approach to computing our upper bound for the remainder term $R(k)$. We search for N such that $1 > \delta_N \stackrel{def}{=} \| ((\mathbf{Q}-\mathbf{P})D_P)^N \|_1$, which implies that for N and δ_N, the condition (C) holds. Then the upper bound for $R(k)$ is obtained from $c^1_{\delta_N} \| ((\mathbf{Q}-\mathbf{P})D_P)^{k+1} \|_1$. Based on the above, the algorithm that yields an approximation for π_Q with ε precision can be described, with two main parts. First $c^1_{\delta_N}$ is computed. Then, the series can be computed in an iterative way until a predefined level of precision is reached.

The Algorithm

Chose precision $\varepsilon > 0$. Set $k = 1, T(1) = \Pi_P(\mathbf{Q}-\mathbf{P})D_P$ and $H(0) = \Pi_P$.

Step 1: Find N such that $\| ((\mathbf{Q}-\mathbf{P})D_P)^N \|_1 < 1$. Set $\delta_N = \| ((\mathbf{Q}-\mathbf{P})D_P)^N \|_1$ and compute

$$c^1_{\delta_N} = \frac{1}{1-\delta_N} \| \sum_{k=0}^{N-1} ((\mathbf{Q}-\mathbf{P})D_P)^k \|_1 .$$

Step 2: If

$$c_\delta \| T(k) \|_1 < \varepsilon,$$

the algorithm terminates and $H(k-1)$ yields the desired approximation. Otherwise, go to step 3.

Step 3: Set $H(k) = H(k-1) + T(k)$. Set $k := k+1$ and $T(k) = T(k-1)(\mathbf{Q}-\mathbf{P})D_P$. Go to step 2.

Remark 0.7. Algorithm 1 terminates in a finite number of steps, since $\sum_{k=0}^{\infty} ((\mathbf{Q}-\mathbf{P})D_P)^k$ is finite, .

3.7 Numerical Application

The present paper established the main theoretical results, and the analysis provided applies to the case of optimization of preventive maintenance in reparable reliability models. The application of this algorithm step by step gives us the following results.

This part of the paper is reserved for theoretical and numerical results obtained via series expansion method to obtain the development of the stationary distribution of the $M/G/1//N$ queueing models with single server vacation, witch modeless reliability system with preventive maintenance.

Let S the state space of the imbedded Markov chains X_n and $\bar{X}_n$ of the both considered queueing systems. Note that the both chains are irreducible and aperiodic, with finite state space S, so they are v-geometric ergodic. We note by D_P the deviation matrix associated to $\bar{X}_n$ chain, and by π_P its stationary distribution, with stationary projector Π_P. In the same time, π_Q is the stationary distribution of X_n, with the projector Π_Q.

We want to express π_Q in terms of puissance series on $(P-Q)D_P$ and π_P as follows:

$$\pi_Q = \sum_{n=0}^{\infty} \pi_P((Q-P)D_P)^n; \tag{15}$$

We show that this series is convergent. In fact, since the state space of the both chains is finite, so we can give the first following elementary result:

Lemma 0.4. *Let X_n and $\bar{X}_n$ the imbedded Markov chains of the $M/G/1//N$ queueing system with server vacation and the classical $M/G/1//N$system respectively. Then, the finite number N exist and verified the following:*

$$\| P^n - \Pi_P \|_v \leq c\beta^n, \ \textit{for all} \ \ n \geq N; \tag{16}$$

where $c < \infty$, $\beta < 1$.

For the same precedent raisons we give the most important result about the deviation matrix D_P associated to the imbedded Markov chain $\bar{X}_n$.

Lemma 0.5. *Let $\bar{X}_n$ the imbedded Markov of the classical $M/G/1//N$ queueing system and D_P its deviation matrix. Then, D_P is finite.*

Using Lemma 0.2, we obtain the following result about the required series expansion:

Lemma 0.6. *Let π_P (resp. π_Q) the stationary distribution of the $M/G/1//N$ classical system, (resp. $M/G/1//N$ system with unique vacation), and D_P the associated deviation matrix. Then, the series*

$$\sum_{n=0}^{\infty} \pi_P((Q-P)D_P)^n; \tag{17}$$

converge normally then uniformly.

This result is equivalent to say that the reminder term $R(k)$ is uniformly convergent to zero.

From the condition (C) and the Lemma 0.3, the sum function of the series 15 is the stationary vector π_Q.

Lemma 0.7. *Let π_P (resp. π_Q) the stationary distribution of the $M/G/1//N$ classical system, (resp. $M/G/1//N$ with vacation of the server), and D_P the associated deviation matrix. Then, the series*

$$\pi_Q = \sum_{n=0}^{\infty} \pi_P((Q-P)D_P)^n; \tag{18}$$

converge uniformly to the stationary vector π_Q.

From the work of Heidergot, we describe in this section a numerical approach to compute the supremum borne of the reminder term $R(k)$. We ask about the number N as:

$$\delta_N = \| ((Q-P)D_P)^N \|_1 < 1,$$

witch implies that the condition (C) is verified for N and δ_n. Then the limit of $R(k)$ is obtained from:

$$\| ((Q-P)D_P)^{k+1} \|_1 < c^1_{\delta_N}.$$

The performance measure for witch we are interesting is the mean number of costumers at the stationary state in the system.

The considered entries parameters are: $\bar{N} = 5$, $\lambda = 2$, service rate $\rightarrow Exp(\mu_s = 5)$, vacation rate $\rightarrow Exp(\mu_v = 300)$.

Our goal is to compute approximatively the quantities $\pi^{\star} w$.

The error to predict the stationary queue length via the quantities $H(n)$ is then given and illustrated in the Figure1.

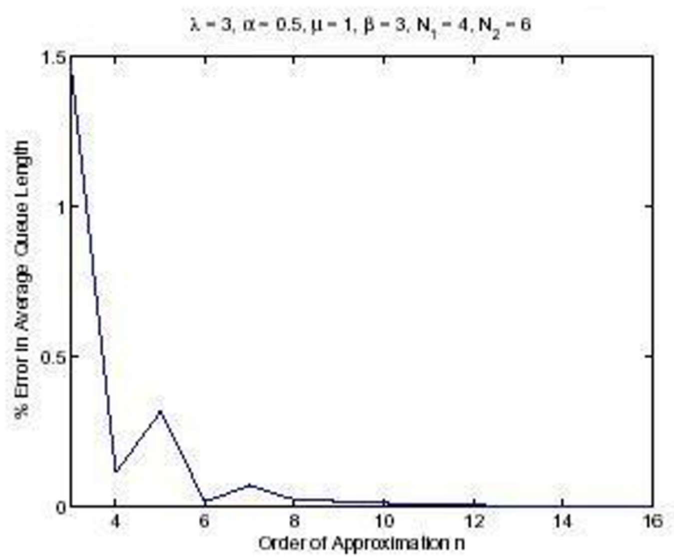

Fig. 1 Error in Average queue length

The figure show that

$$|\frac{\pi^{\star}w - H(n)w}{\pi^{\star}w}| \tag{19}$$

is a graph on n. The numerical value of πw is 2.4956. For this example, we have obtained $N = 14, \delta_N = 0.9197$ and $c^1_{\delta_N} = 201.2313$.

The algorithm terminates when the upper bound for $\| R \|_1$ given by $c^1_{\delta_N} \| R \|_1$ is under the value ε. By taking $\varepsilon = 10^{-2}$, the algorithm compute $\pi^{\star}w$ juste to the precision $10^{-2} \| w \|_1$.

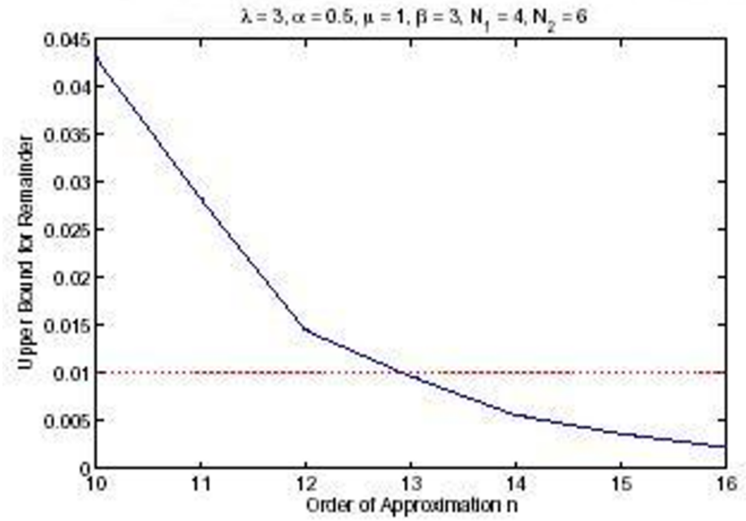

Fig. 2 Relative error of the upper bound of the remainder term

From this figure we conclude that $\pi \sum_{k=0}^{13} ((Q^{\star} - Q)D)^k w$ approximates $\pi^{\star}w$ with a maximal absolute error $\varepsilon \| w \|_1 = 3 * 10^{-2}$.

4 Conclusion

In this work, we have presented a part of research project on numerical algorithms based on series expansions of finite Markov chains. We are interested in the performance of a stochastic system when some of its parameters or characteristics are perturbed. This leads to an efficient numerical algorithm for computing the stationary distribution. We have shown theoretically and numerically that introducing a small disturbance on the structure of maintenance policy in $M/G/1//N$ system with multiples maintenances after modelling by queues with server vacation, we obtain the $M/G/1//N$ system with single maintenance policy (periodic maintenance). Then characteristics of this system can be approximated by those of the $M/G/1//N$ system with periodic maintenance, with a precision which depends on the disturbance, in other words on the maintenance parameter value.

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