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Branching processes in stationary random environment: The extinction problem revisited

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Abstract A classical result by Athreya and Karlin states that a supercritical Galton-Watson process in stationary ergodic environment $\mathbf{f} = (f_0, f_1, \dots)$ (these are the random generating functions of the successively picked offspring distributions) has a positive chance of survival $1 - q(\mathbf{f})$ for almost all realizations of $\mathbf{f}$ provided that $\mathbb{E}\log(1 - f_0(0)) > -\infty$. While in some cases like when $f_0, f_1, \dots$ are i.i.d., this last condition together with supercriticality, viz. $\mathbb{E}\log f_0'(1) > 0$, is actually equivalent to $q(\mathbf{f}) < 1$ a.s., there are others where it is not. This is demonstrated by giving a rather simple counterexample which in turn draws on the main result of this paper. The latter is intended to shed further light on the relation between $\mathbb{E}\log(1 - f_0(0)) > -\infty$ and the almost sure noncertain extinction property, the most interesting outcome being that, if $\mathbb{E}\log f_0'(1)$ is also finite, then $q(\mathbf{f}) < 1$ a.s. holds iff $\mathbb{E}\log\left(\frac{1-f_0\circ\ldots\circ f_T(0)}{1-f_1\circ\ldots\circ f_T(0)}\right) > -\infty$ for some random time T. The use of random times in connection with the stationary environment $\mathbf{f}$ will lead us quite naturally to the use of Palm-duality theory in some of our arguments.

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2.1 Introduction

The *Galton–Watson process in random environment (GWPRE)* constitutes one of the various generalizations of the classical Galton–Watson branching process and is

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characterized by a random variation of the offspring distribution over generations. When originating from one ancestor, it is given by an integer-valued stochastic sequence $(Z_n)_{n\ge 0}$ on a probability space $(\Omega,\mathfrak{A},\mathbb{P})$, recursively defined as

$$Z_0 \;=\; 1 \quad \text{and} \quad Z_n \;=\; \sum_{k=1}^{Z_{n-1}} X_{n,k} \quad \text{for } n\ge 1,$$

where Z_n denotes the size of the n^{th} generation of the considered population and the following assumptions hold true:

(A1) $(X_{n,k})_{n,k\ge 1}$ forms a double array of integer-valued random variables that are conditionally independent and rowwise identically distributed given a random sequence $(f_n)_{n\ge 0}$ (the random environment);

(A2) each f_n is a random generating function, i.e., a random element in

$$\Gamma \stackrel{\text{def}}{=} \left\{ g:[0,1]\to[0,1] : g(s)=\sum_{n\ge 0} p_n s^n \text{ with } p_0,p_1,\ldots\ge 0 \text{ and } g(1)=1 \right\}.$$

(A3) $\mathbb{E}\left(s^{X_{n,k}}|f_0,f_1,\ldots\right)=f_n(s)$ for all $n\ge 0$ and $k\ge 1$.

In this note we will consider the case where

(A4) $(f_n)_{n\ge 0}$ forms a stationary ergodic sequence satisfying $\mathbb{P}(f_0(0)+f_0'(0)=1)<1$ and $\min(\mathbb{E}\log^+ f_0'(1),\mathbb{E}\log^- f_0'(1))<\infty$.

This model was first studied by Athreya and Karlin in two seminal papers [3], [4] shortly after the work by Smith and Wilkinson [17] who dealt with the special case of i.i.d. $f_0,f_1,\ldots$

It is well known that many of the properties of $(Z_n)_{n\ge 0}$ are intimately related to the behavior of the random walk $(S_n)_{n\ge 0}$ with $S_0\stackrel{\text{def}}{=}0$ and stationary ergodic increments

$$X_n \;\stackrel{\text{def}}{=}\; \log f_{n-1}'(1), \quad n\ge 1.$$

Indeed, we have

$$\mathbb{E}(s^{Z_n}|Z_0,\ldots,Z_{n-1},f_0,f_1,\ldots) \;=\; f_{0:n-1}(s) \;\stackrel{\text{def}}{=}\; f_0\circ\ldots\circ f_{n-1}(s)$$

for each $n\ge 1$ and $s\in[0,1]$, and therefore

$$\mu_n \;\stackrel{\text{def}}{=}\; \mathbb{E}(Z_n|Z_0,f_0,f_1,\ldots) \;=\; Z_0 e^{S_n} \quad \text{a.s.} \tag{2.1}$$

We can extend $(f_n)_{n\ge 0}$ to a doubly infinite stationary sequence $(f_n)_{n\in\mathbb{Z}}$ which in turn leads to a doubly infinite extension of $(X_n)_{n\ge 1}$ as well. Define S_n recursively by $S_{n-1}\stackrel{\text{def}}{=}S_n-X_n$ for $n=-1,-2,\ldots$ The sequence $(f_{0:n})_{n\ge 0}$, constitutes the backward system associated with the forward (iterated function) system $f_{n:0}\stackrel{\text{def}}{=}f_n\circ\ldots\circ f_0$, $n\ge 0$, of Lipschitz maps on the unit interval. Defining the usual Lipschitz constant

$$l(g) \stackrel{\text{def}}{=} \sup_{r,s\in[0,1],r\neq s} \frac{|g(s)-g(r)|}{|s-r|}$$

for $g \in \Gamma$, the fact that g and its derivative are nondecreasing and convex implies

$$l(g) \;=\; g'(1)$$

which may be infinite. Consequently,

$$l(f_{0:n}) \;=\; f_{0:n}'(1) \;=\; \prod_{k=0}^{n} f_k'(1) \;=\; \prod_{k=0}^{n} l(f_k) \;\in\; [0,\infty] \quad \text{a.s.}$$

and the product is not a.s. equal to zero because $\mathbb{P}(f_0(0)=1)<1$ by (A4) which in turn ensures $\mathbb{P}(f_0'(1)>0)>0$. With the help of this observation and Birkhoff's ergodic theorem we infer that $(f_{n:0})_{n\ge 0}$ possesses the (by ergodicity necessarily constant) Lyapunov exponent $\chi=\mathbb{E}\log f_0'(1)\in[-\infty,\infty]$, for

$$\chi \;=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log f_k'(1) \;=\; \lim_{n\to\infty}\frac{S_n}{n} \;=\; \mathbb{E}\log f_0'(1).$$

The cases where χ is positive, negative or zero mark quite different types of behavior for the associated GWPRE and lead to the following definition.

Definition 2.1. A GWPRE $(Z_n)_{n\ge 0}$ satisfying (A1–4) and associated random walk $(S_n)_{n\ge 0}$ is called

- *subcritical*, if $\mathbb{E}\log f_0'(1)<0$ and thus $\lim_{n\to\infty}S_n=-\infty$ a.s.
- *supercritical*, if $\mathbb{E}\log f_0'(1)>0$ and thus $\lim_{n\to\infty}S_n=\infty$ a.s.
- *critical*, if $\mathbb{E}\log f_0'(1)=0$.
- *strongly critical*, if $\mathbb{E}\log f_0'(1)=0$, $\liminf_{n\to\infty}S_n=-\infty$ and $\limsup_{n\to\infty}S_n=\infty$ a.s.

Let us note that strong criticality really constitutes a genuine subcase. In fact, Lalley [13, Prop. 6] has shown that $(S_n)_{n\ge 0}$ is L_1-bounded, i.e. $\sup_{n\ge 0}\mathbb{E}|S_n|<\infty$, iff $(\log f_n'(1))_{n\ge 0}$ is *null-homologous* which means that, for some measurable function $\xi:\mathbb{R}^{\mathbb{Z}}\to\mathbb{R}$,

$$\log f_0'(1) \;=\; \xi(\mathbf{f}_0)-\xi(\mathbf{f}_1) \quad \text{a.s.}$$

and thus

$$S_n \;=\; \xi(\mathbf{f}_0)-\xi(\mathbf{f}_n) \quad \text{a.s. for } n\ge 0,$$

where $\mathbf{f}=\mathbf{f}_0 \stackrel{\text{def}}{=} (\ldots,f_{-1},f_0,f_1,\ldots)$ and $\mathbf{f}_n \stackrel{\text{def}}{=} (\ldots,f_{n-1},f_n,f_{n+1},\ldots)$ denotes the n-shift of this sequence for $n\in\mathbb{Z}$. Hence, if ξ is bounded we see that strong criticality fails albeit $\mathbb{E}\log f_0'(1)=0$. Conversely, if $(\log f_n'(1))_{n\ge 0}$ is not null-homologous, then $\mathbb{E}\log f_0'(1)=0$ does indeed imply strong criticality, see [2, Remark (b) after Thm. 3].

2.2 Classical results revisited

We continue with a short survey of some classical results concerning the asymptotic behavior of $(Z_n)_{n\geq 0}$. Church [5] and Lindvall [14] proved that Z_n always converges a.s. to a random variable Z_∞ taking values in $\mathbb{N}_0\cup\{\infty\}$, while Tanny [19] added to this that

$$\mathbb{P}(Z_\infty=0 \text{ or } =\infty)=1 \quad\text{iff}\quad \mathbb{P}(f_0'(0)=1)<1.$$

In view of (A4) it is thus clear in the given setup that the considered population either explodes or dies out a.s. Turning to the extinction probability

$$q(\xi)\ \stackrel{\text{def}}{=}\ \mathbb{P}(Z_\infty=0|\mathbf{f}=\xi)$$

for a given environment $\xi\in\Gamma^{\mathbb{Z}}$, let us first note that

$$\mathbb{P}(Z_\infty=0|\mathbf{f},Z_0=k)\ =\ q(\mathbf{f})^k$$

for each $k\in\mathbb{N}_0$ whence it suffices to look at the case $Z_0=1$. Athreya and Karlin [3] proved that

$$q(\mathbf{f})\ =\ f_0(q(\mathbf{f}_1))\quad\text{a.s.}$$

and that $\{q(\mathbf{f})=1\}$ is a.s. shift-invariant, i.e. $\{q(\mathbf{f})=1\}=\{q(\mathbf{f}_1)=1\}$ a.s. Hence, by ergodicity,

$$\mathbb{P}(q(\mathbf{f})=1)\in\{0,1\}.$$

Putting

$$g_n(s)\ \stackrel{\text{def}}{=}\ \frac{1}{1-f_n(s)}-\frac{1}{f_n'(1)(1-s)},\quad s\in[0,1),$$

and $\eta_{k,n}\stackrel{\text{def}}{=}g_k\circ f_{k+1:n-1}(0)$ for $0\leq k<n$, Geiger and Kersting [10] derived the useful formula

$$\mathbb{P}(Z_n>0|\mathbf{f})\ =\ \left(e^{-S_n}+\sum_{k=0}^{n-1}\eta_{k,n}e^{-S_k}\right)^{-1} \tag{2.2}$$

for $n\geq 0$ which shows quite clearly the importance of $(S_n)_{n\geq 0}$ for the asymptotic behavior of the survival probability $\mathbb{P}(Z_n>0)$ as $n\to\infty$. The latter was then studied by them and later also in [11, 1] and [9] for critical and subcritical GWPRE in i.i.d. random environment; for earlier results of this type see Kozlov [12] and Dekking [7]. In the subcritical and strongly critical case, formula (2.2) also provides a quick proof of the following result first obtained by Athreya and Karlin [3, Cor. 1]:

Proposition 2.1. *Let $(Z_n)_{n\geq 0}$ be a GWPRE satisfying (A1–4). Then $q(\mathbf{f})=1$ a.s. if $(Z_n)_{n\geq 0}$ is subcritical or critical.*

Proof (in the subcritical and strongly critical case). If $(Z_n)_{n\geq 0}$ is subcritical or strongly critical then $\liminf_{n\to\infty}S_n=-\infty$ a.s. By using formula (2.2) and the monotonicity of $\mathbb{P}(Z_n>0|\mathbf{f})$, we hence infer

$$1-q(\mathbf{f}) \;=\; \lim_{n\to\infty}\mathbb{P}(Z_n>0|\mathbf{f}) \;\le\; \liminf_{n\to\infty} e^{S_n} \;=\; 0 \quad \text{a.s.},$$

that is $q(\mathbf{f})=1$ a.s.

In the subcritical case, there is yet another quick argument, for $\mathbb{E}\log f_0'(1)<1$ means nothing but the iterated function system $(f_{n:0})_{n\ge 0}$ to be strictly mean contractive with unique stationary distribution δ_1, the Dirac measure at 1. Hence the associated backward system $(f_{0:n}(s))_{n\ge 0}$ converges a.s. to 1 for any initial state s (see e.g. [8]), in particular $q(\mathbf{f})=\lim_{n\to\infty} f_{0:n}(0)=1$ a.s.

The proof is completed after Lemma 2.1. The next result, also obtained by Athreya and Karlin [3, Thm. 3], provides conditions ensuring $q(\mathbf{f})<1$ a.s.

Proposition 2.2. *Let $(Z_n)_{n\ge 0}$ be a GWPRE satisfying (A1–4). Then $q(\mathbf{f})<1$ a.s. if $(Z_n)_{n\ge 0}$ is supercritical and $\mathbb{E}\log(1-f_0(0))>-\infty$.*

The additionally occurring condition $\mathbb{E}\log(1-f_0(0))>-\infty$ in the supercritical case naturally raises the question of its necessity. Certain special cases for which the answer is positive are stated in the next result.

Proposition 2.3. *Let $(Z_n)_{n\ge 0}$ be a GWPRE satisfying (A1–4) and $\mathbb{E}|\log f_0'(1)|<\infty$. Then*

$$\left\{\begin{array}{c}\mathbb{E}\log(1-f_0(0))>-\infty\\ \mathbb{E}\log f_0'(1)>0\end{array}\right\} \quad\Longleftrightarrow\quad \mathbb{P}(q(\mathbf{f})<1)=1 \tag{2.3}$$

in any of the following two cases:

(C1) $f_0,f_1,\ldots$ are independent or m-dependent for some $m\in\mathbb{N}$.
(C2) $f_0,f_1,\ldots$ take values in a finite subset Γ_0 of Γ.

For the subcases where $f_0,f_1,\ldots$ are i.i.d. or a finite irreducible Markov chain, this result is due to Smith [16] and once again to Athreya and Karlin [3, Thm. 2], respectively. For a proof of the result in the stated form we refer to Section 5.

2.3 Main result and a counterexample

We proceed with a statement of our main result after some necessary notation. It provides necessary and sufficient conditions in the supercritical case for $q(\mathbf{f})<1$ a.s. as well as for $\mathbb{E}\log(1-f_0(0))>-\infty$. Then we give a counterexample which demonstrates that even in the case where the environment $\mathbf{f}$ constitutes a positive recurrent discrete Markov chain we cannot generally conclude $\mathbb{E}\log(1-f_0(0))>-\infty$ from $q(\mathbf{f})<1$ a.s. Thus equivalence (2.3) in Proposition 2.3 for finite irreducible Markov chains does not extend to general positive recurrent Markov chains with countable state space; for a positive result in the latter situation see again [3, Thm. 4].

Given a doubly infinite environment $\mathbf{f}$, denote by $\mathbb{T}=\mathbb{T}_{\mathbf{f}}$ the set of all a.s. finite random times T of the form

$$T \;=\; \inf\{n\ge m:\mathbf{f}_n\in C\}$$

for $m \in \mathbb{N}$ and some measurable subset C of Γ (the definition of a suitable σ-field on Γ is standard and will not be spelled out here, see e.g. [8] in the context of iterated random Lipschitz functions). We note that with T any $T+k$, $k \in \mathbb{Z}$, is also an element of $\mathbb{T}$, for $T = \inf\{n \geq m+k : \mathbf{f}_n \in \Theta^k C\}$ where Θ denotes the usual forward shift mapping $(\ldots, f_{-1}, f_0, f_1, \ldots) \mapsto (\ldots, f_0, f_1, f_2, \ldots)$. The following truncation has been introduced in [6]: For any function $v : \Gamma^{\mathbb{Z}} \to \mathbb{N}$ and given $\mathbf{f} = (f_n)_{n \in \mathbb{Z}}$ with $f_n(s) = \sum_k p_{n,k} s^k$, let $\mathbf{f}^v \stackrel{\text{def}}{=} (f_{n,v})_{n \in \mathbb{Z}}$ be the sequence of generating functions defined by

$$f_{n,v}(s) \stackrel{\text{def}}{=} \sum_{k=0}^{v(\mathbf{f}_n)-1} p_{n,k} s^k \,+\, s^{v(\mathbf{f}_n)} \sum_{k \geq v(\mathbf{f}_n)} p_{n,k}.$$

In the case of a constant truncation function $v \equiv c$ we simply write $f_{n,c}$ for $f_{n,v}$. It is easily verified that $\mathbf{f}^v$ is again stationary and ergodic if this holds true for $\mathbf{f}$.

Theorem 2.1. *Let $(Z_n)_{n \geq 0}$ be a GWPRE satisfying (A1–4). and consider the following assumptions:*

(B1) $\mathbb{E}\log(1 - f_0(0)) > -\infty$;
(B2) There exists $c \in \mathbb{N}$ *such that* $0 < \mathbb{E}\log f'_{0,c}(1) < \infty$;
(B3) $\mathbb{P}(q(\mathbf{f}) < 1) = 1, \quad \mathbb{E}\left|\log\left(\dfrac{1-q(\mathbf{f}_0)}{1-q(\mathbf{f}_1)}\right)\right| < \infty$ *and* $\mathbb{E}\log\left(\dfrac{1-q(\mathbf{f}_0)}{1-q(\mathbf{f}_1)}\right) = 0$;
(B4) There exists $T \in \mathbb{T}$ *such that* $\mathbb{E}\log\left(\dfrac{1 - f_0 \circ \ldots \circ f_T(0)}{1 - f_1 \circ \ldots \circ f_T(0)}\right) > -\infty$;
(B5) There exists $v : \Gamma^{\mathbb{Z}} \to \mathbb{N}$ *such that* $\begin{Bmatrix} 0 < \mathbb{E}\log f'_{0,v}(1) < \infty \\ \lim_{n\to\infty} n^{-1} \log v(\mathbf{f}_n) = 0 \text{ a.s.} \\ \mathbb{P}(q(\mathbf{f}^v) < 1) = \mathbb{P}(q(\mathbf{f}) < 1) = 1 \end{Bmatrix}$;
(B6) $\lim_{n\to\infty} n^{-1} \log(1 - q(\mathbf{f}_n)) = 0$ *a.s.*;
(B7) $\lim_{n\to\infty} n^{-1} \log(1 - f_n(0)) = 0$ *a.s.*

(a) If $\mathbb{E}\log f'_0(1) > 0$, *then*

$$\begin{array}{ccc} (B1) & & (B4) \Rightarrow (B5) \Rightarrow (B6) \Rightarrow (B7) \\ \Updownarrow & & \Updownarrow \\ (B2) & \Rightarrow & (B3) \end{array}$$

(b) If $0 < \mathbb{E}\log f'_0(1) < \infty$, *then (B3–5) are equivalent.*

The proof of this result is rather long and postponed to Sect. 2.5. Let us point out that condition (B4) actually coincides with condition (B1) if $T \equiv 0$ and the term $f_1 \circ \ldots \circ f_T(0)$ is interpreted as 1 in this case. We will further see in Lemma 2.1 that, if $0 < \mathbb{E}\log f'_0(1) < \infty$, then $\mathbb{P}(q(\mathbf{f}) < 1) = 1$ alone already entails the other assertions in (B3). Consequently, in this situation we have equivalence of $\mathbb{P}(q(\mathbf{f}) < 1) = 1$ with condition (B4) for *some* random time T but not necessarily for $T \equiv 0$. This is exemplified by the subsequent already announced counterexample:

Example 2.1. Let $\Gamma_0 = \{g_0, g_1, \ldots\}$ be the countable subset of Γ, defined by

$$g_n(s) \stackrel{\text{def}}{=} 1 - e^{-n} + e^{-n} s^{m_n} \quad \text{for } n \geq 0,$$

where m_n is the smallest integer greater than e^{n+1}. Hence g_n is the generating function of the two-point distribution $Q_n \stackrel{\text{def}}{=} (1-e^{-n})\delta_0 + e^{-n}\delta_{m_n}$ and

$$e \;<\; g_n'(1) \;=\; m_n e^{-n} \;\leq\; e+1 \tag{2.4}$$

for each $n \geq 0$.

Suppose that $(f_n)_{n\geq 0}$ is a Markov chain on Γ_0 with transition matrix

$$\mathbf{P} \;=\; \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & \ldots \\ 1 & 0 & 0 & 0 & 0 & \ldots \\ 0 & 1 & 0 & 0 & 0 & \ldots \\ 0 & 0 & 1 & 0 & 0 & \ldots \\ \vdots & & & \ddots & \ddots & 0 \end{pmatrix}$$

for suitable $\alpha_n > 0$ satisfying $\sum_n \alpha_n = 1$, $\kappa \stackrel{\text{def}}{=} \sum_n n\alpha_n < \infty$ and $\sum_n n^2 \alpha_n = \infty$. Plainly, the (i,j)-component of $\mathbf{P}$ denotes the probability of $f_n = g_j$ given $f_{n-1} = g_i$ for any $n \geq 0$. As one can further readily check, $(f_n)_{n\geq 0}$ is positive recurrent with stationary distribution π, given by

$$\pi_n \stackrel{\text{def}}{=} \pi(g_n) \;=\; \frac{1}{\kappa+1} \sum_{k\geq n} \alpha_k, \quad n \geq 0.$$

and satisfying $\sum_n n\pi_n = \infty$, for $\sum_n n^2\alpha_n = \infty$. In the following, let $(f_n)_{n\geq 0}$ be in stationary regime under $\mathbb{P}$ (so $\mathbb{P} = \mathbb{P}_\pi$) and note that

$$\mathbb{E}\log f_0'(1) \;=\; \sum_{n\geq 0} \pi_n \log g_n'(1) \;\in\; (1, \log(e+1)] \quad \text{[by (2.4)]}$$
$$\text{and} \quad \mathbb{E}\log(1-f_0(0)) \;=\; \sum_{n\geq 0} \pi_n \log(1-g_n(0)) \;=\; -\sum_{n\geq 0} n\pi_n \;=\; -\infty.$$

So we are in the supercritical case, but with condition (B1) being violated. On the other hand, condition (B4) will now be shown to hold true with $T \equiv 1$. Indeed, using

$$\lim_{n\to\infty} n^{-1}\log\left(e^n(1-e^{-n})^3\right) = 1 \quad \text{and} \quad \lim_{n\to\infty}(1-e^{-n+1})^{m_n} = \exp(-e^2),$$

we infer

$$\begin{aligned} 0 \;<\; & \mathbb{E}\left(\frac{1-f_0\circ f_1(0)}{1-f_1(0)}\right) \\ = \;& \pi_0 \sum_{n\geq 1} \alpha_n \log\left(\frac{1-f_0\circ f_n(0)}{1-f_n(0)}\right) \;+\; \sum_{n\geq 1} \pi_n \log\left(\frac{1-f_n\circ f_{n-1}(0)}{1-f_{n-1}(0)}\right) \end{aligned}$$

$$\begin{aligned}
&= \pi_0 \sum_{n\geq 1} \alpha_n \log\left(\frac{f_n(0)^{m_0}}{1-f_n(0)}\right) + \sum_{n\geq 1} \pi_n \log\left(\frac{e^{-n}(1-f_{n-1}(0)^{m_n})}{1-f_{n-1}(0)}\right) \\
&= \pi_0 \sum_{n\geq 1} \alpha_n \log(e^n(1-e^{-n})^3) + \sum_{n\geq 1} \pi_n \log\left(e^{-1}(1-(1-e^{-n+1})^{m_n})\right) \\
&\leq C\Big(\pi_0 \sum_{n\geq 1} n\alpha_n + 1-\pi_0\Big) < \infty
\end{aligned}$$

for a suitable constant $C \in (0,\infty)$. Having verified condition (B4), we infer from Theorem 2.1 that $q(\mathbf{f}) < 1$ a.s.

The flavor of this example is that the offspring distributions Q_n are picked in such a way that, as $n \to \infty$, individuals have no offspring with a chance $1-e^{-n}$ exponentially approaching 1, while with probability e^{-n} they produce an exponentially growing number of descendants. This is combined with a transition mechanism for picking the Q_n resulting in long runs of the form $Q_0 \to Q_1 \to \ldots \to Q_k$ where the population grows very rapidly on the event of survival, before it starts anew with a pick of Q_0. It is this transition mechanism that makes for an almost certain positive chance of survival. Indeed, if the f_n are i.i.d. with distribution π then $\mathbb{E}\log(1-f_0(0)) = -\infty$ and $0 < \mathbb{E}\log f_0'(1) < \infty$ clearly persist to hold while Proposition 2.3 now tells us that $q(\mathbf{f}) = 1$ a.s.

Let us finally note that Tanny [19, Ex. 2] has produced a counterexample of a similar kind but less explicit insofar as it requires to review a rather long and technical argument given in [18] in order to conclude $q(\mathbf{f}) < 1$ a.s.

2.4 Some useful facts from Palm-duality theory

Consider a doubly infinite sequence $\mathbf{X} = (X_n)_{n\in\mathbb{Z}}$ of random variables defined on a probability space $(\Omega,\mathfrak{A},\mathbb{P})$ and taking values in a Borel space $(\Lambda,\mathfrak{C})$. Put $\mathbf{X}_n \overset{\text{def}}{=} \Theta^n\mathbf{X} = (\ldots,X_{n-1},X_n,X_{n+1},\ldots)$ for $n \in \mathbb{Z}$. Let further $\mathbf{T} = (T_n)_{n\in\mathbb{Z}}$ be an increasing doubly infinite sequence of a.s. finite random epochs such that $\mathbf{T} = \tau(\mathbf{X})$ for a measurable function τ and $T_0 \leq 0 < T_1$ holds true. The forward shift Θ is supposed to act on $(\mathbf{X},\mathbf{T})$ in the canonical manner, namely

$$\Theta \circ (\mathbf{X},\mathbf{T}) \overset{\text{def}}{=} (\Theta\mathbf{X}, \tau(\Theta\mathbf{X}))\,.$$

Hence $(\mathbf{X},\mathbf{T})$ is stationary if this holds true for $\mathbf{X}$. The T_n divide $\mathbf{X}$ into the cycles (or segments)

$$Y_n \overset{\text{def}}{=} (X_{T_n},\ldots,X_{T_{n+1}-1}), \quad n \in \mathbb{Z}$$

which are generally not stationary under $\mathbb{P}$. The Palm-duality theory for stationary point processes tells us how the latter can be achieved (leading to so-called cycle stationarity) under an appropriate change of measure which, roughly speaking, means to condition the stationary $(\mathbf{X},\mathbf{T})$ to have a point at 0 (here points are the epochs T_n). For a detailed exposition of this topic we refer to the monographs by Thorisson

[20, Chap. 8] or Sigman [15]. Here we confine ourselves with a statement of some basic facts. Defining

$$\widehat{\mathbb{P}}(d\omega) \stackrel{\text{def}}{=} \frac{1}{c(T_1(\omega)-T_0(\omega))}\mathbb{P}(d\omega), \quad c \stackrel{\text{def}}{=} \mathbb{E}\left(\frac{1}{T_1-T_0}\right), \tag{2.5}$$

the main result may be stated as follows:

Proposition 2.4. *Given a stationary sequence* $(\mathbf{X},\mathbf{T})$ *on a probability space* $(\Omega,\mathfrak{A},\mathbb{P})$ *as introduced before, the following assertions hold true:*

(a) $(Y_n)_{n\in\mathbb{Z}}$ *is stationary under* $\widehat{\mathbb{P}}$.
(b) *Under* $\widehat{\mathbb{P}}$, $\Theta^{T_n}\circ(\mathbf{X},\mathbf{T}) \stackrel{d}{=} \Theta^{T_0}\circ(\mathbf{X},\mathbf{T})$ *for all* $n\in\mathbb{Z}$.
(c) $\widehat{\mathbb{E}}(T_n-T_{n-1}) = \frac{1}{c}$ *for all* $n\in\mathbb{Z}$.

A consequence of this result is the subsequent useful formula reminiscent of Dynkin's formula in the theory of Markov processes. We will use it in several places in the proof of Theorem 2.1.

Corollary 2.1. *Suppose that* $(\mathbf{X},\mathbf{T})$ *is stationary under* $\mathbb{P}$. *Let* $h:\Lambda^\infty\to\mathbb{R}$ *be any measurable function such that* $\mathbb{E}h^-(\mathbf{X})$ *or* $\mathbb{E}h^+(\mathbf{X})$ *is finite. Then*

$$\widehat{\mathbb{E}}\left(\sum_{k=T_0}^{T_1-1} h(\mathbf{X}_k)\right) = \mathbb{E}h(\mathbf{X})\,\widehat{\mathbb{E}}(T_1-T_0) = \frac{\mathbb{E}h(\mathbf{X})}{c}.$$

Proof. It suffices to consider the case of bounded h, for then the general assertion follows in the usual manner by monotone approximation of h^+ and h^- through non-negative bounded functions. Observe that, by part (a) of the previous proposition, $\left(\sum_{k=T_{n-1}}^{T_n-1} h(\mathbf{X}_k)\right)_{n\in\mathbb{Z}}$ is stationary under $\widehat{\mathbb{P}}$. For bounded h, we now infer

$$\widehat{\mathbb{E}}\left(\sum_{k=T_0}^{T_1-1} h(\mathbf{X}_k)\right) = \frac{1}{n}\widehat{\mathbb{E}}\left(\sum_{k=T_0}^{T_n-1} h(\mathbf{X}_k)\right) = \frac{1}{n}\sum_{j=1}^{n}\widehat{\mathbb{E}}\left(\frac{T_j-T_{j-1}}{T_n-T_0}\sum_{k=T_0}^{T_n-1} h(\mathbf{X}_k)\right)$$

and each expectation in the last sum converges to $\mathbb{E}h(\mathbf{X})\,\widehat{\mathbb{E}}(T_1-T_0)$ as $n\to\infty$ by an appeal to Birkhoff's ergodic theorem (giving $(T_n-T_0)^{-1}\sum_{k=T_0}^{T_n-1} h(\mathbf{X}_k)\to\mathbb{E}h(\mathbf{X})$ $\mathbb{P}$-a.s. and thus $\widehat{\mathbb{P}}$-a.s.), the dominated convergence theorem and Proposition 2.4(b). This clearly yields the asserted result.

2.5 Proofs

Let us begin by recalling the following result by Athreya and Karlin [3, Thm. 1].

Lemma 2.1. *Let* $(Z_n)_{n\geq 0}$ *be a GWPRE satisfying (A1–4) and* $\mathbb{E}(\log f_0'(1))^+ < \infty$. *Then* $\mathbb{P}(q(\mathbf{f})<1) = 1$ *implies*

$$\mathbb{E}|\log f_0'(1)| < \infty \quad \textit{and} \quad \mathbb{E}\log f_0'(1) > 0$$

as well as

$$\mathbb{E}\left|\log\left(\frac{1-q(\mathbf{f}_0)}{1-q(\mathbf{f}_n)}\right)\right| < \infty \quad \textit{and} \quad \mathbb{E}\log\left(\frac{1-q(\mathbf{f}_0)}{1-q(\mathbf{f}_n)}\right) = 0 \tag{2.6}$$

for all $n \in \mathbb{N}$.

Proof (of Proposition 2.1 (Completion)). In view of the previous lemma we see that criticality (i.e., $\mathbb{E}\log f_0'(1) = 0$ and particularly $\mathbb{E}(\log f_0'(1))^+ < \infty$) always entails $q(\mathbf{f}) = 1$ a.s.

Proof (of Proposition 2.3). Suppose $\mathbb{E}|\log f_0'(1)| < \infty$. In view of Proposition 2.2 it remains to show that $q(\mathbf{f}) < 1$ a.s. implies $\mathbb{E}\log(1 - f_0(0)) > -\infty$. That $\mathbb{E}\log f_0'(1) > 0$ holds true as well follows by an appeal to Proposition 2.1.

For all $n \geq 1$ and $\varepsilon \in (0,1)$, we infer with the help of Lemma 2.1 that

$$\begin{aligned}
\infty &> \mathbb{E}\left|\log\left(\frac{1-q(\mathbf{f}_0)}{1-q(\mathbf{f}_n)}\right)\right| \\
&= \mathbb{E}\left|\log\left(\frac{1-f_0\circ\ldots\circ f_{n-1}(q(\mathbf{f}_n))}{1-q(\mathbf{f}_n)}\right)\right| \\
&\geq \mathbb{E}\left|\log\left(\frac{1-f_0\circ\ldots\circ f_{n-1}(q(\mathbf{f}_n))}{1-q(\mathbf{f}_n)}\right)\right| \mathbb{1}_{[0,1-\varepsilon]}(q(\mathbf{f}_n)) \\
&\geq -\mathbb{E}\log(1-f_0\circ\ldots\circ f_{n-1}(q(\mathbf{f}_n)))\mathbb{1}_{[0,1-\varepsilon]}(q(\mathbf{f}_n)) + \log\varepsilon \\
&\geq -\mathbb{E}\log(1-f_0(0))\mathbb{1}_{[0,1-\varepsilon]}(q(\mathbf{f}_n)) + \log\varepsilon \\
&\geq -\mathbb{E}\Big(\log(1-f_0(0))\mathbb{P}(q(\mathbf{f}_n) \leq 1-\varepsilon | f_0)\Big) + \log\varepsilon.
\end{aligned} \tag{2.7}$$

Now we have

$$\mathbb{P}(q(\mathbf{f}_n) \leq 1-\varepsilon | f_0) \;=\; \mathbb{P}(q(\mathbf{f}) \leq 1-\varepsilon) \;>\; 0 \quad \text{a.s.}$$

for all $n > m$ and ε sufficiently small, if the f_n are m-dependent (notice that $q(\mathbf{f}_n)$ does actually only depend on $f_n, f_{n+1}, \ldots$), while in the case (C2)

$$\mathbb{P}(q(\mathbf{f}_n) \leq 1-\varepsilon | f_0) \;\geq\; \min_{g\in\Gamma_0}\mathbb{P}(q(\mathbf{f}_n) \leq 1-\varepsilon | f_0 = g) \;>\; 0 \quad \text{a.s.}$$

for ε sufficiently small. Consequently, by choosing n and ε in a suitable manner in (2.7), we infer $\mathbb{E}\log(1 - f_0(0)) > -\infty$.

Proof (of Theorem 2.1). Put $h(\xi) \stackrel{\text{def}}{=} -\log\left(\frac{1-q(\xi)}{1-q(\Theta\xi)}\right)$ for any $\xi \in \Gamma^{\mathbb{Z}}$.

(a) Suppose $\mathbb{E}\log f_0'(1) > 0$.

"(B1)⇒(B2)" As, by convexity, $f_{0,c}'(1) \geq 1 - f_{0,c}(0) = 1 - f_0(0)$ for all $c \in \mathbb{N}$, we see that (B1) implies $\mathbb{E}\log f_{0,c}'(1) > -\infty$ for all $c \in \mathbb{N}$ and thereupon, by the monotone convergence theorem, that $\lim_{c\to\infty}\mathbb{E}\log f_{0,c}'(1)=\mathbb{E}\log f_0'(1)>0$. This shows (B2).

"(B2)⇒(B1)" Using $f_{0,c}''(1)<\infty$ and the inequality $\frac{1-f_{0,c}(s)}{1-s}\ge f_{0,c}'(1)-\frac{1}{2}f_{0,c}''(1)(1-s)$ for $s\in[0,1)$, we infer

$$\mathbb{E}\log\left(\frac{1-f_{0,c}(s)}{1-s}\right)\ \ge\ \frac{1}{2}\mathbb{E}\log f_{0,c}'(1)\ >\ 0$$

by making $1-s$ sufficiently small. Picking such an s, (B1) now follows from

$$\begin{aligned}\mathbb{E}\log(1-f_0(0))\ &=\ \mathbb{E}\log(1-f_{0,c}(0))\ \ge\ \mathbb{E}\log(1-f_0(s))\\ &=\ \mathbb{E}\log\left(\frac{1-f_{0,c}(s)}{1-s}\right)\ +\ \log(1-s)\ \ge\ \log(1-s).\end{aligned}$$

"(B3)⇒(B4)" As $\mathbb{P}(q(\mathbf{f})<1)=1$, Birkhoff's ergodic theorem implies the existence of $\overline{q}\in(0,1)$ such that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}_{[0,\overline{q}]}\big(q(\mathbf{f}_k)\big)\ =\ \mathbb{P}(q(\mathbf{f})\le\overline{q})\ >\ 0\quad\text{a.s.}$$

which in turn yields

$$\mathbb{P}\big(q(\mathbf{f}_n)\le\overline{q}\text{ infinitely often}\big)=1.$$

Denote by $(T_n)_{n\in\mathbb{Z}}$ the sequence of successive random epochs with $q(\mathbf{f}_{T_n})\le\overline{q}$, where $T_0\le 0<T_1$ a.s. Defining $\widehat{\mathbb{P}}$ as in (2.5) for this sequence, Proposition 2.4 ensures that

$$\widehat{\mathbf{f}}\ =\ (\widehat{f}_n)_{n\in\mathbb{Z}},\quad \widehat{f}_n\ \stackrel{\text{def}}{=}\ f_{T_n}\circ\ldots\circ f_{T_{n+1}-1},$$

is stationary and ergodic under $\widehat{\mathbb{P}}$. The latter holds true in particular for $(T_n-T_{n-1})_{n\in\mathbb{Z}}$ and $\widehat{\mathbb{E}}(T_1-T_0)=\big[\mathbb{E}\big(\frac{1}{T_1-T_0}\big)\big]^{-1}<\infty$ (see Proposition 2.4(c)). Now consider a stochastic sequence $(\widehat{Z}_n)_{n\in\mathbb{Z}}$ satisfying

$$\widehat{\mathbb{P}}((\widehat{Z}_n)_{n\ge0}\in\cdot|\widehat{\mathbf{f}}=\xi)=\mathbb{P}((\widehat{Z}_n)_{n\ge0}\in\cdot|\widehat{\mathbf{f}}=\xi)=\mathbb{P}((Z_n)_{n\ge0}\in\cdot|\mathbf{f}=\xi)$$

for all $\xi\in\Gamma^{\mathbb{Z}}$. It follows that $(\widehat{Z}_n)_{n\in\mathbb{Z}}$ forms a GWPRE under $\mathbb{P}$ as well as $\widehat{\mathbb{P}}$ with $\widehat{Z}_0=1$, (under $\widehat{\mathbb{P}}$ stationary ergodic) environment $(\widehat{f}_n)_{n\ge0}$ and

$$\widehat{\mathbb{P}}(\widehat{Z}_\infty=0|\widehat{\mathbf{f}}=\xi)=\mathbb{P}(\widehat{Z}_\infty=0|\widehat{\mathbf{f}}=\xi)=q(\xi).$$

By the choice of the T_n, we have

$$q(\widehat{\mathbf{f}})\ =\ q(\mathbf{f}_{T_0})\ \le\ \overline{q}\quad\mathbb{P}\text{-a.s.}$$

and therefore (since $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are equivalent measures and $\widehat{f}_0(0)\le q(\widehat{\mathbf{f}})$)

$$\widehat{\mathbb{E}}\log(1-\widehat{f}_0(0)) \;\ge\; \log(1-\overline{q}) \;>\; -\infty. \tag{2.8}$$

Next use $\mathbb{E}(\log f_0'(1))^- < \infty$ (by assumption) and Corollary 2.1 to obtain

$$\begin{aligned}
\widehat{\mathbb{E}}(\log \widehat{f}_0'(1))^- \;&=\; \widehat{\mathbb{E}}\big(\log(f_{T_0}\circ\ldots\circ f_{T_1-1})'(1)\big)^- \\
&=\; \widehat{\mathbb{E}}\left(\sum_{k=T_0}^{T_1-1}\log f_k'(1)\right)^- \\
&=\; \widehat{\mathbb{E}}\left(\sum_{k=T_0}^{T_1-1}(\log f_k'(1))^-\right) \\
&=\; \mathbb{E}(\log f_0'(1))^-\,\widehat{\mathbb{E}}(T_1-T_0) \;<\; \infty
\end{aligned}$$

and then once again Corollary 2.1 to conclude

$$\widehat{\mathbb{E}}\log \widehat{f}_0'(1) \;=\; \mathbb{E}\log f_0'(1)\,\widehat{\mathbb{E}}(T_1-T_0) \;>\; 0.$$

Write next

$$\begin{aligned}
\widehat{\mathbb{E}}\log(1-\widehat{f}_0(0)) \;&=\; \widehat{\mathbb{E}}\left(\sum_{k=T_0}^{T_1-1}\log\left(\frac{1-f_k\circ\ldots\circ f_{T_1-1}(0)}{1-f_{k+1}\circ\ldots\circ f_{T_1-1}(0)}\right)\right) \\
&=\; \widehat{\mathbb{E}}\left(\sum_{k=T_0}^{T_1-1} g(\mathbf{f}_k)\right),
\end{aligned} \tag{2.9}$$

where, for any $\xi = (\xi_n)_{n\in\mathbb{Z}} \in \varGamma^{\mathbb{Z}}$,

$$g(\xi) \;\stackrel{\text{def}}{=}\; \log\left(\frac{1-\xi_0\circ\ldots\circ\xi_{\tau(\xi)-1}(0)}{1-\xi_1\circ\ldots\circ\xi_{\tau(\xi)-1}(0)}\right), \qquad \tau(\xi) \;\stackrel{\text{def}}{=}\; \inf\{n\ge 1 : q(\Theta^n\xi)\le\overline{q}\}.$$

We show now that $\mathbb{E}g(\mathbf{f}) > -\infty$, which proves (B4) with $T = T_1 - 1 \in \mathbb{T}$. It follows with the help of the monotonicity of $\frac{1-f_0(s)}{1-s}$ that

$$\begin{aligned}
g^+(\mathbf{f}_n) \;&=\; \left(\log\left(\frac{1-f_n(f_{n+1}\circ\ldots\circ f_{T_1-1}(0))}{1-f_{n+1}\circ\ldots\circ f_{T_1-1}(0)}\right)\right)^+ \\
&\le\; \left(\log\left(\frac{1-f_n(q(\mathbf{f}_{n+1}))}{1-q(\mathbf{f}_{n+1})}\right)\right)^+ \\
&=\; \left(\log\left(\frac{1-q(\mathbf{f}_n)}{1-q(\mathbf{f}_{n+1})}\right)\right)^+ \;=\; h^-(\mathbf{f}_n)
\end{aligned}$$

for $T_0 \le n < T_1$. Since $\mathbb{E}|h(\mathbf{f})| < \infty$ by assumption (B3), we obtain by another appeal to Corollary 2.1

$$\widehat{\mathbb{E}}\left(\sum_{k=T_0}^{T_1-1} g^+(\mathbf{f}_k)\right) \le \widehat{\mathbb{E}}\left(\sum_{k=T_0}^{T_1-1} h^-(\mathbf{f}_k)\right) = \mathbb{E}h^-(\mathbf{f})\,\widehat{\mathbb{E}}(T_1-T_0) < \infty$$

which in combination with (2.8) and (2.9) gives $\widehat{\mathbb{E}}(\sum_{k=T_0}^{T_1-1} g^-(\mathbf{f}_k)) < \infty$ as well and thus

$$\mathbb{E}|g(\mathbf{f})| = \frac{1}{\widehat{\mathbb{E}}(T_1-T_0)}\,\widehat{\mathbb{E}}\left(\sum_{k=T_0}^{T_1-1} |g(\mathbf{f}_k)|\right) < \infty.$$

"(B4)⇒(B3)" Assuming (B4) with $T = \inf\{n \ge m : \mathbf{f}_n \in C\}$ for some measurable $C \subset \Gamma^{\mathbb{Z}}$ and $m \ge 0$, let T_n', $n \in \mathbb{Z}$, be the increasing sequence of successive random epochs where $\mathbf{f}_{T_n'} \in C$ and $T_0' \le 0 < T_1'$. Putting $T_n \stackrel{\text{def}}{=} T_{(m+1)n}$ for $n \in \mathbb{Z}$, we clearly have $T \le T_1 - 1$ and therefore

$$\begin{aligned} -\infty &< \mathbb{E}\log\left(\frac{1-f_0(f_1\circ\ldots\circ f_T(0))}{1-f_1\circ\ldots\circ f_T(0)}\right) \\ &\le \mathbb{E}\log\left(\frac{1-f_0(f_1\circ\ldots\circ f_{T_1-1}(0))}{1-f_1\circ\ldots\circ f_{T_1-1}(0)}\right). \end{aligned} \tag{2.10}$$

Now let $\widehat{\mathbb{P}}$ and $\widehat{\mathbf{f}}$ be defined as in the previous part but for the T_n just defined. Then Corollary 2.1 (applicable because of (2.10)) and (2.9) (with g adapted to the present T_n) implies

$$\widehat{\mathbb{E}}\log(1-\widehat{f}_0(0)) = \widehat{\mathbb{E}}(T_1-T_0)\,\mathbb{E}\log\left(\frac{1-f_0(f_1\circ\ldots\circ f_{T_1-1}(0))}{1-f_1\circ\ldots\circ f_{T_1-1}(0)}\right) > -\infty. \tag{2.11}$$

We further infer $\widehat{\mathbb{E}}\log\widehat{f}_0'(1) > 0$ as in the previous part which in combination with (2.11) gives

$$1 = \widehat{\mathbb{P}}(q(\widehat{\mathbf{f}}) < 1) = \mathbb{P}(q(\widehat{\mathbf{f}}) < 1) = \mathbb{P}(q(\mathbf{f}_{T_0}) < 1)$$

by an appeal to Proposition 2.2 and thus also $q(\mathbf{f}) < 1$ a.s.

Left with the proof of $\mathbb{E}|h(\mathbf{f})| < \infty$, we first note that $\mathbb{E}h^-(\mathbf{f}) < \infty$, for

$$\log\left(\frac{1-f_0(f_1\circ\ldots\circ f_T(0))}{1-f_1\circ\ldots\circ f_T(0)}\right) \le \log\left(\frac{1-f_0(q(\mathbf{f}_1))}{1-q(\mathbf{f}_1)}\right) = -h(\mathbf{f}) \quad \text{a.s.}$$

Then use $\sum_{k=0}^{n-1} h(\mathbf{f}_k) = -\log(1-q(\mathbf{f}_0)) + \log(1-q(\mathbf{f}_n))$ to infer

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} h(\mathbf{f}_k) \le \lim_{n\to\infty}\frac{-\log(1-q(\mathbf{f}))}{n} = 0 \quad \text{a.s.}$$

which in combination with $n^{-1}\sum_{k=0}^{n-1} h^-(\mathbf{f}_k) = \mathbb{E}h^-(\mathbf{f}) < \infty$ a.s. by Birkhoff's ergodic theorem leads us to the conclusion

$$\mathbb{E}h^+(\mathbf{f}) \;=\; \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}h^+(\mathbf{f}_k) \;\le\; \mathbb{E}h^-(\mathbf{f}) \;<\; \infty.$$

and thereupon to $\mathbb{E}h(\mathbf{f}) = \lim_{n\to\infty} n^{-1}\sum_{k=0}^{n-1} h(\mathbf{f}_k) = 0$.

"(B3)⇒(B5)" We first point out that for any truncation function v the inequality

$$f_{0,v}'(1) \;\ge\; \frac{1-q(\mathbf{f}_0)-q(\mathbf{f}_1)^{v(\mathbf{f})}}{1-q(\mathbf{f}_1)} \quad \text{a.s.}$$

holds true as was shown by Coffey and Tanny [6, Lemma 2]. Choose $\tilde{v}(\xi) \stackrel{\text{def}}{=} \inf\{n \ge 1 : q(\Theta\xi)^n \le (1-q(\xi))/2\}$ which is a.s. finite as $q(\mathbf{f}) < 1$ a.s. It follows that

$$\mathbb{E}\log f_{0,\tilde{v}}'(1) \;\ge\; \mathbb{E}\left(\frac{1-q(\mathbf{f}_0)}{2(1-q(\mathbf{f}_1))}\right) \;>\; -\infty.$$

Hence, by the monotone convergence theorem,

$$\mathbb{E}\log f_{0,\tilde{v}+n}'(1) \;=\; \mathbb{E}\log f_0'(1) \;>\; 0.$$

Nox fix N large enough such that, for $v \stackrel{\text{def}}{=} \tilde{v}+N$, we have $\mathbb{E}\log f_{0,v}'(1) > 0$. It is no loss of generality to assume $\mathbb{E}\log f_{0,v}'(1) < \infty$ as well, for otherwise we can choose m so large that

$$\mathbb{E}\log f_{0,v}'(1)\mathbb{1}_{\{\log f_{0,v}'(1)\le m\}} \;>\; 0$$

and replace $v(\mathbf{f})$ on the event $\{\log f_{0,v}'(1) > m\}$ by $v(\mathbf{f}) - w(\mathbf{f})$, where for $\xi \in \Gamma^{\mathbb{Z}}$ (recall $f_0(s) = \sum_k p_{0,k}s^k$)

$$w(\xi) \;\stackrel{\text{def}}{=}\; \inf\left\{n \ge 1 : \sum_{k=0}^{v(\xi)-n-1} kp_{0,k} + (v(\xi)-n)\sum_{k\ge v(\xi)-n} p_{0,k} \le m\right\}.$$

If $v(\mathbf{f}_n)$ satisfies the second condition of (B5) still to be verified, then this is obviously also true for the just defined modification. Turning to the verification of

$$\lim_{n\to\infty} n^{-1}\log v(\mathbf{f}_n) \;=\; 0, \quad \text{i.e.} \quad \lim_{n\to\infty} v(\mathbf{f}_n)^{1/n} \;=\; 1 \quad \text{a.s.}$$

note that it suffices to do so for $\tilde{v}$ instead of v. Since, by assumption and Birkhoff's ergodic theorem,

$$0 \;=\; \mathbb{E}h(\mathbf{f}) \;=\; -\lim_{n\to\infty}\sum_{k=0}^{n-1}h(\mathbf{f}_k) \;=\; \lim_{n\to\infty}\frac{1}{n}\log\left(\frac{1-q(\mathbf{f}_0)}{1-q(\mathbf{f}_n)}\right) \quad \text{a.s.} \tag{2.12}$$

we infer $n^{-1}\log(1-q(\mathbf{f}_n)) \to 0$ a.s. and thereby

$$\sum_{n\ge 0}\mathbb{1}_{[1-\exp(-\varepsilon n),1]}(q(\mathbf{f}_n)) \;<\; \infty \quad \text{a.s.} \tag{2.13}$$

for all $\varepsilon > 0$. Consequently,

$$\begin{aligned}\limsup_{n\to\infty} q(\mathbf{f}_n)^{\exp(2\varepsilon n)} &\le \lim_{n\to\infty}(1-e^{-\varepsilon})^{\exp(2\varepsilon n)} \quad \text{a.s.}\\ &= \lim_{n\to\infty}\exp\left(e^{2\varepsilon n}\log(1-e^{-\varepsilon n})\right) = \lim_{n\to\infty}\exp(-e^{\varepsilon n}) = 0\end{aligned}$$

which in turn implies

$$\lim_{n\to\infty} e^{-2\varepsilon n}\tilde{v}(\mathbf{f}_{n+1}) = 0 \quad \text{a.s.}$$

for all $\varepsilon > 0$ by going back to the definition of $\tilde{v}$ and using (2.13). Namely, for any sufficiently large n (depending on the fixed ε),

$$\begin{aligned}q(\mathbf{f}_{n+1})^{\exp(2\varepsilon n)} &\le (1-e^{-\varepsilon(n+1)})^{\exp(2\varepsilon n)} = \exp\left(e^{2\varepsilon n}\log(1-e^{-\varepsilon(n+1)})\right)\\ &\le 2\exp\left(-e^{\varepsilon(n-1)}\right) \le \frac{e^{-\varepsilon n}}{2} \le \frac{1-q(\mathbf{f}_n)}{2}.\end{aligned}$$

Since $\tilde{v} \ge 1$, we have thus particularly shown that

$$1 \le \limsup_{n\to\infty}\tilde{v}(\mathbf{f}_n)^{1/n} \le e^{2\varepsilon} \quad \text{a.s.}$$

for all $\varepsilon > 0$, and this clearly gives the desired assertion.

Since truncation always increases the chance of extinction, we have $q(\mathbf{f}^v) \ge q(\mathbf{f})$ a.s. and must therefore still verify $\mathbb{P}(q(\mathbf{f}^v) < 1) = 1$. Let $(Z_{n,v})_{n\ge 0}$ be a GWPRE with one ancestor and stationary environment $(f_{n,v})_{n\ge 0}$. Put $\mu_{n,v} \stackrel{\text{def}}{=} \mathbb{E}(Z_n|f_{0,v}, f_{1,v}, \ldots)$ (compare (2.1)) and notice that $W_{n,v} \stackrel{\text{def}}{=} Z_{n,v}/\mu_{n,v}$, $n \ge 0$, constitutes a mean one nonnegative martingale. Check that $f''_{n,v}(1) \le v(\mathbf{f}_n)^2$ a.s. for all $n \ge 0$ whence

$$\lim_{n\to\infty}\frac{1}{n}\log^+ f''_{n,v}(1) \le \lim_{n\to\infty}\frac{2}{n}v(\mathbf{f}_n) = 0 \quad \text{a.s.}$$

But a combination of this fact with $0 < \mathbb{E}\log f'_{0,v}(1) < \infty$ and $\mathbb{E}\log f''_{0,v}(1) < \infty$ is now easily seen to imply the L_2-boundedness of W_n which in turn entails $\mathbb{P}(q(\mathbf{f}^v) < 1) = 1$ as claimed.

"(B5)⇒(B6)" As $\mathbb{P}(q(\mathbf{f}^v) < 1) = 1$ and $0 < \mathbb{E}\log f'_{0,v}(1) < \infty$, we infer from Lemma 2.1 that $\mathbb{E}\log\left(\frac{1-q(\mathbf{f}_0)}{1-q(\mathbf{f}_n)}\right) = 0$ for all $n \ge 1$. This yields the assertion by recalling (2.12) from above.

"(B6)⇒(B7)" This is immediate from $\log(1-q(\mathbf{f}_n)) \le \log(1-f_n(0)) \le 0$.

(b) Here it remains to show **"(B5)⇒(B3)"**. Since part (a) already gives $\mathbb{P}(q(\mathbf{f}) < 1) = 1$ and we now additionally assume $\mathbb{E}|\log f'_0(1)| < \infty$, the remaining assertions in (B3) are once again following from Lemma 2.1.

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