Chapter 2
Mass and Radius of the Star

In this chapter, we will discuss the most basic properties of a compact star, its mass and radius. We have already given typical values for these quantities above. Below we shall connect them with microscopic properties of nuclear and quark matter. This connection is made by the equation of state from which, in particular, an estimate for the maximum mass of the star can be obtained. Let us begin with a simple estimate of mass and radius from general relativity. For the stability of the star we need $R > R_s$ where R is the radius of the star, and $R_s = 2MG$ the Schwarzschild radius, with the mass of the star M and the gravitational constant $G = 6.672 \cdot 10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}} = 6.707 \cdot 10^{-39}\,\mathrm{GeV}^{-2}$. (We shall mostly use units common in nuclear and particle physics, $\hbar = c = k_B = 1$, although astrophysicists often use different units.) For $R < R_s$ the star becomes unstable with respect to the collapse into a black hole. Let us build a simple star made out of a number of nucleons A with mass $m \simeq 939\,\mathrm{MeV}$ and a distance $r_0 \simeq 0.5 \cdot 10^{-13}\,\mathrm{cm}$ (that's where the nucleon interaction becomes repulsive). We thus cover a volume $\sim r_0^3\,A$ and thus have a radius

$$R \sim r_0\,A^{1/3}\,, \tag{2.1}$$

(for our rough estimate we are not interested in factors of π), and a mass

$$M \sim m\,A\,. \tag{2.2}$$

Now from the limit $R = 2MG$ we obtain

$$A \sim \left(\frac{r_0}{2mG}\right)^{3/2} \sim 2.6 \cdot 10^{57}\,. \tag{2.3}$$

This is the number of nucleons up to which we can fill our star before it gets unstable. In other words, the Schwarzschild radius is proportional to the mass of the star and thus increases linearly in the number of nucleons A, while the radius increases with $A^{1/3}$; therefore, for A smaller than the limit $A \sim 2.6 \cdot 10^{57}$ the star

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is stable, while it collapses into a black hole for nucleon numbers larger than this limit. We can plug the limit value for A back into the radius and mass of the star to obtain

$$R \sim 7\,\text{km}\,, \qquad M \sim 2.3\, M_\odot\,. \tag{2.4}$$

Adding more nucleons would make the star too heavy in relation to its radius. We see that these values are not too far from the observed ones given in Sect. 1.2.

Besides giving an estimate for the baryon number in the star, we see from this simple exercise that general relativistic effects will be important because the Schwarzschild radius will be a significant fraction of the radius of the star. We can also estimate the gravitational energy of the star. To this end, we need the differential mass of the star at a given radius (i.e., the mass of a thin spherical layer)

$$dm = \rho(r)\, dV\,, \tag{2.5}$$

where $dV = 4\pi r^2 dr$ is the volume of the thin spherical layer at radius r. For a rough estimate let us (unrealistically) assume a constant density $\rho(r) = \rho$ such that the mass $m(r)$ of the star up to a radius $r \leq R$, is given by $m(r) = \frac{4\pi}{3} r^3 \rho$. Then we obtain

$$E_{\text{grav}} \simeq \int_0^R \frac{Gm(r)\, dm(r)}{r} \simeq \frac{3}{5}\frac{GM^2}{R} \simeq 0.12\, M\,, \tag{2.6}$$

where we have used the above realistic values $M \simeq 1.4\, M_\odot$ and $R \simeq 10\,\text{km}$. We thus see that the gravitational energy E_{grav} is more than 10% of the mass of the star. This suggests that for the mass-radius relation we need an equation that incorporates effects from general relativity. For simplicity, let us first derive the equation that relates mass and radius without general relativistic effects and include them afterwards. We are looking for an equation that describes equilibrium between the gravitational force, seeking to compress the star, and the opposing force coming from the pressure of the matter inside the star. In the case of a compact star, this pressure is the Fermi pressure plus the pressure coming from the strong interactions of the nuclear or quark matter inside. The differential pressure dP at a given radius r is related to the gravitational force dF via

$$dP = \frac{dF}{4\pi r^2}\,, \tag{2.7}$$

with

$$dF = -\frac{Gm(r)\, dm}{r^2}\,. \tag{2.8}$$

The equation for the differential mass (2.5), together with Eq. (2.7) [into which we insert Eqs. (2.5) and (2.8)], yields the two coupled differential equations,

$$\frac{dm}{dr} = 4\pi r^2 \varepsilon(r)\,, \tag{2.9a}$$

$$\frac{dP}{dr} = -\frac{G\varepsilon(r)m(r)}{r^2}\,. \tag{2.9b}$$

where we have expressed the mass density through the energy density $\varepsilon(r) = \rho(r)$ (in units where $c = 1$). The second equation, which is easy to understand from elementary Newtonian physics, receives corrections from general relativity. It is beyond the scope of these lectures to derive these corrections. We simply quote the resulting equation,

$$\frac{dP}{dr} = -\frac{G\varepsilon(r)m(r)}{r^2}\left[1+\frac{P(r)}{\varepsilon(r)}\right]\left[1+\frac{4\pi r^3 P(r)}{m(r)}\right]\left[1-\frac{2Gm(r)}{r}\right]^{-1}\,. \tag{2.10}$$

This equation is called Tolman–Oppenheimer–Volkov (TOV) equation and the derivation can be found for instance in Ref. [1]. In order to solve it, one first needs the energy density for a given pressure. Only then do we have a closed system of equations. This input is given from the microscopic physics which yields an equation of state in the form $P(\varepsilon)$. We have thus found a first example how the microscopic physics can potentially be "observed" from astrophysical data, namely from mass and radius of the star. We shall encounter many more of these examples. The equations of state for noninteracting nuclear and quark matter will be discussed in the subsequent sections.

For a given equation of state one needs two boundary conditions for the TOV equation. The first is obviously $m(r = 0) = 0$, the second is a boundary value for the pressure in the center of the star, $P(r = 0) = P_0$. Then, the solution of the equations will produce a mass and pressure profile $m(r)$, $P(r)$ with the pressure going to zero at some point $r = R$, giving the radius of the star. The mass of the star is then read off at this point, $M = m(R)$. Doing this for varying initial pressures P_0 yields a curve $M(R)$ in the mass-radius plane, parametrized by P_0. This curve depends strongly on the chosen equation of state.

2.1 Noninteracting Nuclear Matter

We start with a very simple system where we neglect all interactions. In this case, all we need is basic statistical physics and thermodynamics. The thermodynamic potential for the grand-canonical ensemble is given by

$$\Omega = E - \mu N - TS\,, \tag{2.11}$$

with the energy E, chemical potential μ, particle number N, temperature T and entropy S. The pressure is then

$$P = -\frac{\Omega}{V} = -\varepsilon + \mu n + Ts \,, \tag{2.12}$$

where V is the volume of the system. Number density $n = N/V$, energy density $\varepsilon = E/V$, and entropy density $s = S/V$ are, for a fermionic system, given by

$$n = 2\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, f_k \,, \tag{2.13a}$$

$$\varepsilon = 2\int \frac{d^3\mathbf{k}}{(2\pi)^3}\, E_k\, f_k \,, \tag{2.13b}$$

$$s = -2\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[(1-f_k)\ln(1-f_k) + f_k \ln f_k\right] . \tag{2.13c}$$

We shall first be interested in a system of neutrons (n), protons (p), and electrons (e), each giving a contribution to the pressure according to Eqs. (2.12) and (2.13). Since they are spin-$\frac{1}{2}$ fermions, we have included a factor 2 for the two degenerate spin states. The Fermi distribution function is denoted by f_k,

$$f_k \equiv \frac{1}{e^{(E_k-\mu)/T}+1} \,, \tag{2.14}$$

and the single-particle energy is

$$E_k = \sqrt{k^2+m^2} \,. \tag{2.15}$$

Inserting number density, energy density, and entropy density into the pressure (2.12) yields

$$P = 2T\int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left[1+e^{-(E_k-\mu)/T}\right] . \tag{2.16}$$

This corresponds to the result obtained from field-theoretical methods in Appendix A.2, see Eq. (A.71). There also the antiparticle contribution is included, which can here, due to the large positive chemical potential, safely be neglected. One can easily check that number density and entropy are obtained from the pressure (2.16) via the usual thermodynamic relations, i.e., by taking the derivatives with respect to μ and T. For the following we now take the limit $T = 0$. This is a good approximation since the temperature of a compact star is typically in the keV range and thus much smaller than the chemical potentials and masses of the nucleons.

For $T = 0$ the Fermi distribution is a step function, $f_k = \Theta(k_F - k)$, and thus the k integrals will be cut off at the Fermi momentum k_F, i.e.,

$$n = \frac{1}{\pi^2}\int_0^{k_F} dk\, k^2 = \frac{k_F^3}{3\pi^2}\,, \tag{2.17a}$$

$$\begin{aligned}\varepsilon &= \frac{1}{\pi^2}\int_0^{k_F} dk\, k^2\sqrt{k^2+m^2} \\ &= \frac{1}{8\pi^2}\left[\left(2k_F^3 + m^2 k_F\right)\sqrt{k_F^2+m^2} - m^4 \ln\frac{k_F+\sqrt{k_F^2+m^2}}{m}\right].\end{aligned} \tag{2.17b}$$

Then, with $\mu = \sqrt{k_F^2+m^2}$, the pressure is

$$\begin{aligned}P &= \frac{1}{\pi^2}\int_0^{k_F} dk\, k^2\left(\mu - \sqrt{k^2+m^2}\right) \\ &= \frac{1}{24\pi^2}\left[\left(2k_F^3 - 3m^2 k_F\right)\sqrt{k_F^2+m^2} + 3m^4 \ln\frac{k_F+\sqrt{k_F^2+m^2}}{m}\right].\end{aligned} \tag{2.18}$$

This can either be obtained by inserting Eqs. (2.17a) and (2.17b) into Eq. (2.12) or, equivalently, by taking the $T = 0$ limit in Eq. (2.16). For the latter one makes use of $\lim_{T\to 0} T \ln(1 + e^{x/T}) = x\,\Theta(x)$.

For n, p, e matter, the total pressure is

$$P = \frac{1}{\pi^2}\sum_{i=n,p,e}\int_0^{k_{F,i}} dk\, k^2\left(\mu_i - \sqrt{k^2+m_i^2}\right). \tag{2.19}$$

The Fermi momenta can be thought of as variational parameters which have to be determined from maximizing the pressure, i.e., from the conditions

$$\frac{\partial P}{\partial k_{F,i}} = 0\,, \qquad i = n, p, e\,, \tag{2.20}$$

which implies

$$\mu_i = \sqrt{k_{F,i}^2 + m_i^2}\,. \tag{2.21}$$

We have additional constraints on the Fermi momenta from the following two conditions. Firstly, we have to require the star to be electrically neutral,[1], i.e., the densities of protons and electrons has to be equal,

$$n_e = n_p \,. \tag{2.25}$$

With Eq. (2.17a) this means

$$k_{F,e} = k_{F,p} \,. \tag{2.26}$$

Secondly, we require chemical equilibrium with respect to the weak processes

$$n \to p + e + \bar{\nu}_e \,, \tag{2.27a}$$

$$p + e \to n + \nu_e \,. \tag{2.27b}$$

The first of these processes is the usual *β-decay*, the second is sometimes called *inverse β-decay* or *electron capture*. We shall assume that the neutrino chemical potential vanishes, $\mu_{\nu_e} = 0$. This is equivalent to assuming that neutrinos and antineutrinos, once created by the above processes, simply leave the system without further interaction. This assumption is justified for compact stars since the neutrino mean free path is of the order of the size of the star or larger (except for the very

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[1] In fact, a compact star has to be electrically neutral to a very high accuracy, as one can see from the following simple estimate. Suppose the star has an overall charge of Z times the elementary charge, Ze, and we consider the Coulomb repulsion of a test particle, say a proton, with mass m and charge e (e having the same sign as the hypothetical overall charge of the star Ze). The Coulomb force, seeking to expel the test particle, has to be smaller than the gravitational force, seeking to keep the test particle within the star. This gives the condition

$$\frac{(Ze)e}{R^2} \leq \frac{GMm}{R^2} \,, \tag{2.22}$$

with the mass M and radius R of the star. Even if we are generous with the limit on the right-hand side by assigning the upper limit $M < Am$ to the mass of the star (if the star contains A nucleons, its total mass will be less than Am due to the gravitational binding energy), we will get a very restrictive limit on the overall charge. Namely, we find

$$\frac{(Ze)e}{R^2} < \frac{GAm^2}{R^2} \qquad \Rightarrow \qquad Z < G\frac{m^2}{e^2}A \,. \tag{2.23}$$

With the proton mass $m \sim 10^3$ MeV, the elementary charge $e^2 \sim 10^{-1}$ (remember $\alpha = e^2/(4\pi) \simeq 1/137$), and the gravitational constant $G \sim 7 \cdot 10^{-39}\,\mathrm{GeV}^{-2}$, we estimate

$$Z < 10^{-37}A \,, \tag{2.24}$$

i.e., the average charge per nucleon has to be extremely small in order to ensure the stability of the star. Since we have found such an extremely small number, it is irrelevant for the argument whether we use a proton or an electron as a test particle. The essence of this argument is the weakness of gravitation compared to the electromagnetic interactions: a tiny electric charge per unit volume, distributed over the star, is sufficient to overcome the stability from gravity.

early stages in the life of the star). Consequently, β-equilibrium, i.e., equilibrium with respect to the processes (2.27), translates into the following constraint for the chemical potentials,

$$\mu_n = \mu_p + \mu_e \,. \tag{2.28}$$

Inserting Eq. (2.21) into this constraint yields

$$\sqrt{k_{F,n}^2 + m_n^2} = \sqrt{k_{F,p}^2 + m_p^2} + \sqrt{k_{F,e}^2 + m_e^2} \,. \tag{2.29}$$

We can eliminate the electron Fermi momentum with the help of Eq. (2.26) and solve this equation to obtain the proton Fermi momentum as a function of the neutron Fermi momentum,

$$k_{F,p}^2 = \frac{(k_{F,n}^2 + m_n^2 - m_e^2)^2 - 2(k_{F,n}^2 + m_n^2 + m_e^2)m_p^2 + m_p^4}{4(k_{F,n}^2 + m_n^2)} \,. \tag{2.30}$$

To illustrate the physical meaning of this relation, let us consider some limit cases. First assume a vanishing proton contribution, $k_{F,p} = 0$. Then the equation gives [which is most easily seen from Eq. (2.29)]

$$k_{F,n}^2 = (m_p + m_e)^2 - m_n^2 < 0 \,. \tag{2.31}$$

This expression is negative because the neutron is slightly heavier than the electron and the proton together, $m_p \simeq 938.3\,\mathrm{MeV}$, $m_n \simeq 939.6\,\mathrm{MeV}$, $m_e \simeq 0.511\,\mathrm{MeV}$. Therefore, $k_{F,p} = 0$ is impossible and there always has to be at least a small fraction of protons. Now let's assume $k_{F,n} = 0$, which leads to

$$k_{F,p}^2 = \left(\frac{m_n^2 + m_e^2 - m_p^2}{2m_n}\right)^2 - m_e^2 \simeq 1.4\,\mathrm{MeV}^2 \,. \tag{2.32}$$

This is the threshold below which there are no neutrons and the charge neutral system in β-equilibrium contains only protons and electrons of equal number density.

In general, we may consider a given baryon density $n_B = n_n + n_p$ to express the neutron Fermi momentum as

$$k_{F,n} = (3\pi^2 n_B - k_{F,p}^3)^{1/3} \,. \tag{2.33}$$

Inserting this into Eq. (2.30) yields an equation for $k_{F,p}$ as a function of the baryon density. In the ultrarelativistic limit, i.e., neglecting all masses, Eq. (2.30) obviously yields $k_{F,p} = k_{F,n}/2$ and thus $n_p = n_n/8$ or

$$n_p = \frac{n_B}{9} \,. \tag{2.34}$$

By solving Eq. (2.30) numerically one can check that this limit is approached from below for large baryon densities, i.e., in a compact star containing nuclear matter we deal with neutron-rich matter, which justifies the term neutron star.

As a crude approximation we may thus consider the simple case of pure neutron matter. We also consider the nonrelativistic limit, $m_n \gg k_{F,n}$. In this case, the energy density (2.17b) and the pressure (2.18) become

$$\varepsilon \simeq \frac{m_n^4}{3\pi^2}\left[\frac{k_{F,n}^3}{m_n^3} + \mathcal{O}\left(\frac{k_{F,n}^5}{m_n^5}\right)\right], \tag{2.35a}$$

$$P \simeq \frac{m_n^4}{15\pi^2}\left[\frac{k_{F,n}^5}{m_n^5} + \mathcal{O}\left(\frac{k_{F,n}^7}{m_n^7}\right)\right]. \tag{2.35b}$$

(To see this, note that the ln term cancels the term linear in k_F in the case of ε, and the linear and cubic terms in k_F in the case of P.) Consequently, keeping the terms to lowest order in $k_{F,n}/m_n$,

$$P(\varepsilon) \simeq \left(\frac{3\pi^2}{m_n}\right)^{5/3} \frac{\varepsilon^{5/3}}{15 m_n \pi^2}. \tag{2.36}$$

We have thus found a particularly simple equation of state, where the pressure is given by a power of the energy density. The general (numerical) discussion of the equation of state, including protons and electrons, is left to the reader, see Problem 2.1.

The next step to obtain the mass-radius relation of the star is to insert the equation of state into the TOV equation. The simplest case is a power-law behavior as in Eq. (2.36). The general form of such a so-called "polytropic" equation of state is

$$P(\varepsilon) = K\,\varepsilon^\gamma. \tag{2.37}$$

Using the Newtonian form of the mass-radius equations, Eqs. (2.9), this yields

$$\frac{dm}{dr} = \frac{4\pi}{K^{1/\gamma}} r^2 P^{1/\gamma}(r), \tag{2.38a}$$

$$\frac{dP}{dr} = -\frac{G}{K^{1/\gamma}} \frac{P^{1/\gamma}(r) m(r)}{r^2}. \tag{2.38b}$$

Even in this simplest example, we need to solve the equations numerically, see Problem 2.2. The results of this problem show that the maximum mass reached within this model is about $M < 0.7 M_\odot$ which is well below observed neutron star masses. (See also Refs. [2–4] for a pedagogical introduction into the equation of state and mass-radius relation from solving the TOV equation.) This small maximum mass is a consequence of the assumption of noninteracting nucleons. Taking into account interactions will increase the maximum mass significantly.

2.2 Noninteracting Quark Matter

Whenever we talk about quark matter in these lectures we ignore the charm (c), bottom (b), and top (t) quarks. The quark chemical potential inside the star is at most of the order of 500 MeV and thus much too small to create a population of these states. Therefore, we only consider at most three quark flavors, namely up (u), down (d), and strange (s). We shall mostly neglect the masses of the u and d quarks; their *current masses* are $m_u \simeq m_d \simeq 5\,\text{MeV} \ll \mu \simeq (300 - 500)\,\text{MeV}$. The mass of the strange quark, however, is not negligible. The current strange quark mass is $m_s \simeq 90\,\text{MeV}$, and the density-dependent *constituent mass* can be significantly larger, making it non-negligible compared to the quark chemical potential.

If we consider free quarks, the energy density ε, the number density n, and the pressure P, are obtained in the same way as demonstrated for nucleons in the previous subsection. We only have to remember that there are three colors for each quark flavor, $N_c = 3$, i.e., the degeneracy factor for a single quark flavor is $2N_c = 6$, where the factor 2 counts the spin degrees of freedom. Consequently, for each quark flavor $f = u, d, s$ we have at zero temperature [cf. Eqs. (2.17) and (2.18)],

$$n_f = \frac{k_{F,f}^3}{\pi^2} \,, \tag{2.39a}$$

$$\varepsilon_f = \frac{3}{\pi^2} \int_0^{k_{F,f}} dk\, k^2 \sqrt{k^2 + m_f^2} \,, \tag{2.39b}$$

$$P_f = \frac{3}{\pi^2} \int_0^{k_{F,f}} dk\, k^2 \left(\mu_f - \sqrt{k^2 + m_f^2}\right) . \tag{2.39c}$$

Again, we need to impose equilibrium conditions with respect to the weak interactions. In the case of three-flavor quark matter, the relevant processes are the leptonic processes (including a neutrino or an antineutrino)

$$d \to u + e + \bar{\nu}_e \,, \qquad s \to u + e + \bar{\nu}_e \,, \tag{2.40a}$$

$$u + e \to d + \nu_e \,, \qquad u + e \to s + \nu_e \,, \tag{2.40b}$$

and the non-leptonic process

$$s + u \leftrightarrow d + u \,. \tag{2.41}$$

These processes yield the following conditions for the quark and electron chemical potentials,

$$\mu_d = \mu_e + \mu_u \, , \qquad \mu_s = \mu_e + \mu_u \, . \tag{2.42}$$

(This automatically implies $\mu_d = \mu_s$.) The charge neutrality condition can be written in a general way as

$$\sum_{f=u,d,s} q_f n_f - n_e = 0 \, , \tag{2.43}$$

with the electric quark charges

$$q_u = \frac{2}{3} \, , \qquad q_d = q_s = -\frac{1}{3} \, , \tag{2.44}$$

and the electron density n_e.

2.2.1 Strange Quark Matter Hypothesis

Before computing the equation of state, we discuss the *strange quark matter hypothesis* within the so-called *bag model*. The bag model is a very crude phenomenological way to incorporate confinement into the description of quark matter. The effect of confinement is needed in particular if we compare quark matter with nuclear matter (which is ultimately what we want to do in this section). Put another way, although we speak of noninteracting quarks, we need to account for a specific – in general very complicated – aspect of the interaction, namely confinement.

To understand how the bag constant accounts for confinement, we compare the pressure of a noninteracting gas of massless pions with the pressure of a noninteracting gas of quarks and gluons at finite temperature and zero chemical potential. The pressure of a single bosonic degree of freedom at $\mu = 0$ and at large temperatures compared to the mass of the boson is

$$P_{\rm boson} \simeq -T \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left(1 - e^{-k/T}\right) = \frac{\pi^2 T^4}{90} \, . \tag{2.45}$$

This is derived in Appendix A.1 within thermal field theory, see Eq. (A.37). Analogously, a single fermionic degree of freedom gives [see Eq. (A.72) of Appendix A.2]

$$P_{\rm fermion} \simeq T \int \frac{d^3\mathbf{k}}{(2\pi)^3} \ln\left(1 + e^{-k/T}\right) = \frac{7}{8}\frac{\pi^2 T^4}{90} \, . \tag{2.46}$$

Therefore, since there are three types of pions, their pressure is

$$P_\pi = 3\,\frac{\pi^2 T^4}{90}\,. \tag{2.47}$$

This is a simple approximation for the pressure of the confined phase. In the deconfined phase, the degrees of freedom are gluons (8×2) and quarks ($4N_c N_f = 24$). Thus with $2 \times 8 + 7/8 \times 24 = 37$ we have

$$P_{q,g} = 37\,\frac{\pi^2 T^4}{90} - B\,, \tag{2.48}$$

where the *bag constant* B has been subtracted for the following reason. If B were zero, the deconfined phase would have the larger pressure and thus would be preferred for all temperatures. We know however, that at sufficiently small temperatures, the confined phase (that's the world we live in) must be preferred. This is phenomenologically accounted for by the bag constant B which acts like an energy penalty for the deconfined phase. Without this penalty, at least in this very simply model description, the deconfined phase would be "too favorable" compared to what we observe. As a consequence, by including the bag constant there is certain critical temperature T_c below which the confined phase is preferred, $P_\pi > P_{q,g}$, and above which the deconfined phase is preferred, $P_{q,g} > P_\pi$. This is indeed what one expects from QCD, where the deconfinement transition temperature is expected to be $T_c \simeq 170\,\text{MeV}$. (As can be seen in the QCD phase diagram in Fig. 1.1, this deconfinement transition is rather a crossover than a phase transition in the strict sense.)

In the context of compact stars we are not interested in such large temperatures. In this case, the chemical potential is large and the temperature practically zero. Nevertheless we compare nuclear (confined) with quark (deconfined) matter and thus have to include the bag constant in the pressure and the free energy of quark matter,

$$P + B = \sum_f P_f\,, \tag{2.49a}$$

$$\varepsilon = \sum_f \varepsilon_f + B\,. \tag{2.49b}$$

This phenomenological model of confinement is called the *bag model* [5, 6] because the quarks are imagined to be confined in a bag. One can view the microscopic pressure $\sum_f P_f$ of the quarks to be counterbalanced by the pressure of the bag B and an external pressure P.

Equipped with the bag model, we can now explain the strange quark matter hypothesis. For simplicity we consider massless quarks. A nonzero strange quark mass will slightly change the results but is not important for the qualitative argument. We will also ignore electrons. They are not present in three-flavor massless quark matter at zero temperature. They are however required in two-flavor quark matter to achieve electric neutrality. But also in this case their population is small enough to render their effect unimportant for the following argument.

With $m_f = 0$ we simply have

$$n_f = \frac{\mu_f^3}{\pi^2} \, , \qquad \varepsilon_f = \frac{3\mu_f^4}{4\pi^2} \, , \qquad P_f = \frac{\mu_f^4}{4\pi^2} \, , \tag{2.50}$$

which in particular implies

$$P_f = \frac{\varepsilon_f}{3} \, . \tag{2.51}$$

For the strange quark matter hypothesis we consider the energy E per nucleon number A,

$$\frac{E}{A} = \frac{\varepsilon}{n_B} \, , \tag{2.52}$$

where n_B is the baryon number density, given in terms of the quark number densities as

$$n_B = \frac{1}{3} \sum_f n_f \, , \tag{2.53}$$

because a baryon contains $N_c = 3$ quarks. At zero pressure, $P = 0$, Eqs. (2.49) and (2.51) imply $\varepsilon = 4B$ and thus

$$\frac{E}{A} = \frac{4B}{n_B} \, . \tag{2.54}$$

We now apply this formula first to three-flavor quark matter ("strange quark matter"), then to two-flavor quark matter of only up and down quarks. For strange quark matter, the neutrality constraint (2.43) becomes

$$2n_u - n_d - n_s = 0 \, . \tag{2.55}$$

Together with the conditions from chemical equilibrium (2.42) this implies

$$\mu_u = \mu_d = \mu_s \equiv \mu \, . \tag{2.56}$$

We see that strange quark matter is particularly symmetric. The reason is that the electric charges of an up, down, and strange quark happen to add up to zero. Now with $n_B = \mu^3/\pi^2$ and

$$B = \sum_f P_f = \frac{3\mu^4}{4\pi^2} \tag{2.57}$$

(still everything at $P = 0$) we have

$$\left.\frac{E}{A}\right|_{N_f=3} = (4\pi^2)^{1/4}\, 3^{3/4}\, B^{1/4} \simeq 5.714\, B^{1/4} \simeq 829\,\text{MeV}\, B_{145}^{1/4} \,. \tag{2.58}$$

We have expressed $B^{1/4}$ in units of 145 MeV, $B_{145}^{1/4} \equiv B^{1/4}/(145\,\text{MeV})$.

For two-flavor quark matter (neglecting the contribution of electrons), the charge neutrality condition is

$$n_d = 2n_u \,. \tag{2.59}$$

Hence,

$$\mu_d = 2^{1/3}\mu_u \,. \tag{2.60}$$

Then, with $n_B = \mu_u^3/\pi^2$ and

$$B = \sum_f P_f = \frac{(1+2^{4/3})\mu_u^4}{4\pi^2} \,, \tag{2.61}$$

we find

$$\left.\frac{E}{A}\right|_{N_f=2} = (4\pi^2)^{1/4}\, (1+2^{4/3})^{3/4}\, B^{1/4} \simeq 6.441\, B^{1/4} \simeq 934\,\text{MeV}\, B_{145}^{1/4} \,. \tag{2.62}$$

By comparing this to Eq. (2.58) we see that two-flavor quark matter has a larger energy per baryon number than three-flavor quark matter. This is a direct consequence of the Pauli principle: adding one particle species (and keeping the total number of particles fixed) means opening a set of new available low-energy states that can be filled, thus lowering the total energy of the system.

We can now compare the results (2.58) and (2.62) with the energy per nucleon in nuclear matter. For pure neutron matter, it is simply given by the neutron mass,

$$\left.\frac{E}{A}\right|_{\text{neutrons}} = m_n = 939.6\,\text{MeV} \,. \tag{2.63}$$

For iron, ^{56}Fe, it is

$$\left.\frac{E}{A}\right|_{^{56}\text{Fe}} = \frac{56\, m_N - 56 \cdot 8.8\,\text{MeV}}{56} = 930\,\text{MeV} \,, \tag{2.64}$$

with the nucleon mass $m_N = 938.9\,\text{MeV}$ and the binding energy per nucleon in iron of 8.8 MeV. Since we observe iron rather than deconfined quark matter, we know that

$$\left.\frac{E}{A}\right|_{^{56}\text{Fe}} < \left.\frac{E}{A}\right|_{N_f=2} \qquad \Rightarrow \qquad B^{1/4} > 144.4\,\text{MeV}\,. \tag{2.65}$$

We have thus found a lower limit for the bag constant from the stability of iron with respect to two-flavor quark matter. Now what if the bag constant were only slightly larger than this lower limit? What if it were small enough for three-flavor quark matter to have lower energy than iron? The condition for this would be

$$\left.\frac{E}{A}\right|_{N_f=3} < \left.\frac{E}{A}\right|_{^{56}\text{Fe}} \qquad \Rightarrow \qquad B^{1/4} < 162.8\,\text{MeV}\,. \tag{2.66}$$

This would imply that strange quark matter is absolutely stable (stable at $P = 0$), while nuclear matter is metastable. This possibility, which would be realized by a bag constant in the window $145\,\text{MeV} < B^{1/4} < 162\,\text{MeV}$, is called *strange quark matter hypothesis*, suggested by Bodmer [7] and Witten [8], see also [9].

Note that the existence of ordinary nuclei does *not* rule out the strange quark matter hypothesis. The conversion of an ordinary nucleus into strange quark matter requires the simultaneous conversion of many (roughly speaking A many) u and d quarks into s quarks. Since this has to happen via the weak interaction, it is practically impossible. In other words, there is a huge energy barrier between the metastable (if the hypothesis is true) state of nuclear matter and absolutely stable strange quark matter. This means that strange quark matter has to be created in another way ("going around" the barrier), by directly forming a quark-gluon plasma. This can for instance happen in a heavy-ion collision. Or, more importantly in our context, it may happen in the universe, giving rise to stars made entirely out of quark matter, so-called *strange stars*.

Small "nuggets" of strange quark matter are called *strangelets* (a strange star would then in some sense simply be a huge strangelet). If a strangelet is injected into an ordinary compact star (a neutron star), it would, assuming the strange quark matter hypothesis to be true, be able to "eat up" the nuclear matter, converting the neutron star into a strange star. Note the difference between this transition and the above described impossible transition from ordinary nuclear matter to strange quark matter: once there is a sufficiently large absolutely stable strangelet, *successive* conversion of up and down quarks into strange quarks increase the size of the strangelet; the energy barrier originating from the *simultaneous* creation of a large number of strange quarks now cannot cause the system to relax back into its original nuclear (metastable) state. This argument has important consequences. If there exist enough sizable strangelets in the universe to hit neutron stars, *every* neutron star would be converted into a strange star. In other words, the observation of a single ordinary neutron star would rule out the strange quark matter hypothesis. Therefore, it is important to understand whether there are enough strangelets around. It has been discussed recently in the literature that there may not be enough strangelets [10], in contrast to what was assumed before.

2.2.2 Equation of State

Next we derive the equation of state for strange quark matter. We include the effect of the strange quark mass to lowest order and also include electrons. It is convenient to express the quark chemical potentials in terms of an average quark chemical potential $\mu = (\mu_u + \mu_d + \mu_s)/3$ and the electron chemical potential μ_e,

$$\mu_u = \mu - \frac{2}{3}\mu_e \,, \tag{2.67a}$$

$$\mu_d = \mu + \frac{1}{3}\mu_e \,, \tag{2.67b}$$

$$\mu_s = \mu + \frac{1}{3}\mu_e \,. \tag{2.67c}$$

Written in this form, the conditions from β-equilibrium (2.42) are automatically fulfilled. Taking into account the strange quark mass, the Fermi momenta for the approximately massless up and down quark and the massive strange quark are given by

$$k_{F,u} = \mu_u \,, \tag{2.68a}$$

$$k_{F,d} = \mu_d \,, \tag{2.68b}$$

$$k_{F,s} = \sqrt{\mu_s^2 - m_s^2} \,. \tag{2.68c}$$

The energy density and the pressure are

$$\sum_{i=u,d,s,e} \varepsilon_i = \frac{3\mu_u^4}{4\pi^2} + \frac{3\mu_d^4}{4\pi^2} + \frac{3}{\pi^2}\int_0^{k_{F,s}} dk\, k^2\sqrt{k^2+m_s^2} + \frac{\mu_e^4}{4\pi^2} \,, \tag{2.69a}$$

$$\sum_{i=u,d,s,e} P_i = \frac{\mu_u^4}{4\pi^2} + \frac{\mu_d^4}{4\pi^2} + \frac{3}{\pi^2}\int_0^{k_{F,s}} dk\, k^2\left(\mu_s - \sqrt{k^2+m_s^2}\right) + \frac{\mu_e^4}{12\pi^2} \,, \tag{2.69b}$$

where we have neglected the electron mass. The neutrality condition can now be written as

$$0 = \frac{\partial}{\partial\mu_e}\sum_{i=u,d,s,e} P_i = -\frac{2}{3}n_u + \frac{1}{3}n_d + \frac{1}{3}n_s + n_e \,. \tag{2.70}$$

(Note that μ_e is defined as the chemical potential for *negative* electric charge.) Solving this equation to lowest order in the strange quark mass yields

$$\mu_e \simeq \frac{m_s^2}{4\mu} \, . \tag{2.71}$$

Consequently, the quark Fermi momenta become

$$k_{F,u} \simeq \mu - \frac{m_s^2}{6\mu} \, , \tag{2.72a}$$

$$k_{F,d} \simeq \mu + \frac{m_s^2}{12\mu} \, , \tag{2.72b}$$

$$k_{F,s} \simeq \mu - \frac{5m_s^2}{12\mu} \, . \tag{2.72c}$$

We see that the Fermi momenta are split by an equal distance of $m_s^2/(4\mu)$, and $k_{F,s} < k_{F,u} < k_{F,d}$, see Fig. 2.1. The splitting and the order of the Fermi momenta can be understood from the following physical picture: start from the symmetric situation $m_s = \mu_e = 0$. In this case, all quark flavors fill their Fermi spheres to a common Fermi momentum given by μ, and the system is neutral. Now switch on the strange quark mass. This lowers the Fermi momentum of the strange quark according to Eq. (2.68c). Consequently, there are fewer strange quarks in the system and thus there is a lack of negative charge. To counterbalance this missing negative charge, the system responds by switching on a chemical potential μ_e. Because of β-equilibrium, the Fermi momenta of all quark flavors are rigidly coupled to this change. Electric neutrality is regained by lowering the up quark Fermi momentum and raising the down and strange quark Fermi momenta (which makes for the catchy phrase "the Fermi momentum of the *down* goes *up*"). Since the strange quark Fermi momentum was already lowered by the finite mass, it is clear that the resulting order

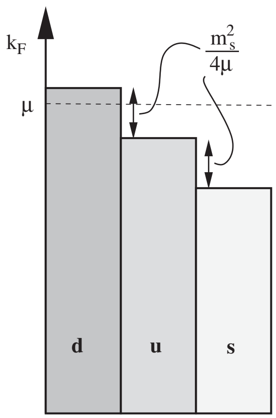

Fig. 2.1 Illustration of the Fermi momenta for neutral, unpaired quark matter in β-equilibrium with quark chemical potential μ. The splitting of the Fermi momenta is due to the strange quark mass m_s which is assumed to be small compared to μ

is $k_{F,s} < k_{F,u} < k_{F,d}$. The electron contribution to the negative charge density is negligibly low, $n_e \propto \mu_e^3 \propto m_s^6/\mu^3$, while the contribution of the quarks due to the strange quark mass is proportional to μm_s^2. The splitting of the Fermi momenta due to the effects of the strange quark mass, β-equilibrium, and electric neutrality is very important in the context of *color superconductivity*. Since color superconductivity is usually based on Cooper pairing of quarks of different flavor, a mismatch in Fermi surfaces tends to disfavor this pairing. We shall discuss superconductivity in quark and nuclear matter in Chap. 4 and give a brief qualitative discussion of the consequences of Fermi surface splitting for color superconductivity at the end of that chapter.

Here we continue with unpaired quark matter and insert the result for μ_e (2.71) back into the energy density and the pressure. Again keeping only terms to lowest order in the strange quark mass yields

$$\sum_i \varepsilon_i \simeq \frac{9\mu^4}{4\pi^2} - \frac{3\mu^2 m_s^2}{4\pi^2}\,, \tag{2.73a}$$

$$\sum_i P_i \simeq \frac{3\mu^4}{4\pi^2} - \frac{3\mu^2 m_s^2}{4\pi^2}\,. \tag{2.73b}$$

Consequently,

$$\sum_i \varepsilon_i \simeq 3\sum_i P_i + \frac{3\mu^2 m_s^2}{2\pi^2}\,. \tag{2.74}$$

With Eq. (2.49a) the pressure, including the bag constant, becomes

$$P \simeq \frac{3\mu^4}{4\pi^2} - \frac{3\mu^2 m_s^2}{4\pi^2} - B\,, \tag{2.75}$$

and, expressing P in terms of the energy density, we obtain with the help of Eq. (2.49b)

$$P \simeq \frac{\varepsilon - 4B}{3} - \frac{\mu^2 m_s^2}{2\pi^2}\,. \tag{2.76}$$

This is the equation of state of noninteracting, unpaired strange quark matter within the bag model with strange quark mass corrections to lowest order.

2.3 Mass-Radius Relation Including Interactions

Let us briefly discuss the results for the mass-radius relation of a compact star for given equations of state for nuclear and quark matter. Since the underlying calculations in general are complicated and have to be done on a computer, we only quote some results to illustrate the physical conclusions. So far we have only discussed the simplest cases of noninteracting matter. Interactions have a significant effect on both the equation of state and the mass-radius relation. We now discuss these effects briefly, only in the subsequent chapters shall we study the nature and details of these interactions (and discuss their relevance to other observables than the mass and the radius of the star).

The maximum mass of a star for noninteracting nuclear matter is $\sim 0.7M_\odot$ (see for instance Ref. [4] or solve Problem 2.2); including interactions increases the mass to values well above $2M_\odot$. The significance of the equation of state and interactions for the maximum mass is easy to understand: if the pressure $P(\varepsilon)$ for a given energy density ε is large, the system is able to sustain a large gravitational force that seeks to compress it. Comparing two equations of state over a given energy density range, the one with the larger pressure (for all energy densities in the given range) is thus termed stiff, the one with the smaller pressure is termed soft. Soft equations of state can sustain less gravitational force and thus lead to stars with lower maximum masses. In the case of noninteracting nuclear matter, it is only the Fermi pressure from the Pauli exclusion principle that prevents the star from the collapse. Interactions increase this pressure because the dominant effect in the case of nuclear matter at the relevant densities is the short-range repulsion between the nucleons. Therefore, the maximum mass is significantly larger in this case.

In Figs. 2.2 and 2.3 several models for the nuclear equation of state are applied to obtain maximum masses up to $2.4\,M_\odot$. For the case of quark matter, we can understand some of the corrections through interactions in the following simple way. A generalization of the pressure (2.75) is

$$P = \frac{3\mu^4}{4\pi^2}(1-c) - \frac{3\mu^2}{4\pi^2}(m_s^2 - 4\Delta^2) - B\,. \tag{2.77}$$

This equation contains two corrections compared to Eq. (2.75). One is included in the coefficient c and originates from the (leading order) correction of the Fermi momentum due to the QCD coupling α_s,

$$k_F = \mu\left(1 - \frac{2\alpha_s}{3\pi}\right), \tag{2.78}$$

resulting in a correction of the μ^4 term in the pressure with $c = 2\alpha_s/\pi$. (This modification of the Fermi momentum will also become important in the context of neutrino emissivity in Chap. 5.) Higher order calculations suggest $c \gtrsim 0.3$ at densities relevant for compact stars. However, the exact value of c is unknown because perturbative calculations are not valid in the relevant density regime, cf. discussion in Sect. 1.1. Therefore, c can only be treated as a parameter with values around 0.3, as done for example in Fig. 2.2. To get an idea about perturbative calculations beyond leading order in α_s, you may consult the recent Ref. [11].

The second correction in Eq. (2.77) is the quantity Δ. This is the energy gap arising from color superconductivity whose microscopic origin we discuss in Chap. 4. It

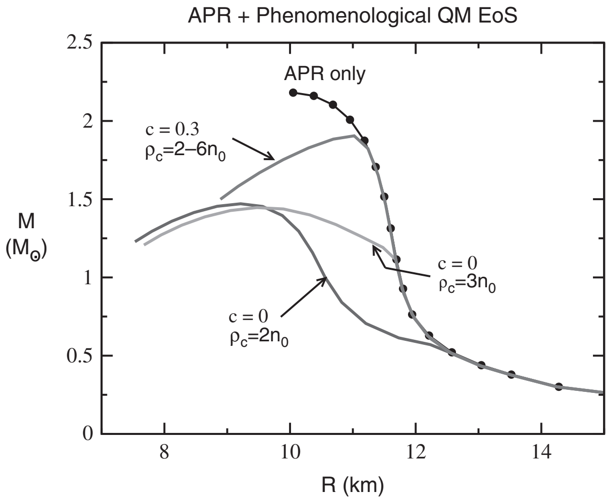

Fig. 2.2 Mass-radius plot from Ref. [12] which shows the dependence of the mass-radius curve on the (uncertain) parameters of the quark matter equation of state in a hybrid star. We see that reasonable choices of the parameters lead to similar curves as for nuclear matter (here with the APR equation of state). In this plot, the transition density ρ_c (in units of the nuclear ground state density n_0) between quark matter and nuclear matter has been used as a parameter, rather than the bag constant. From our discussion it is clear that one can be translated into the other. The coefficient c describes QCD corrections to the quark Fermi momentum and thus to the μ^4 term in the pressure, see Eq. (2.77)

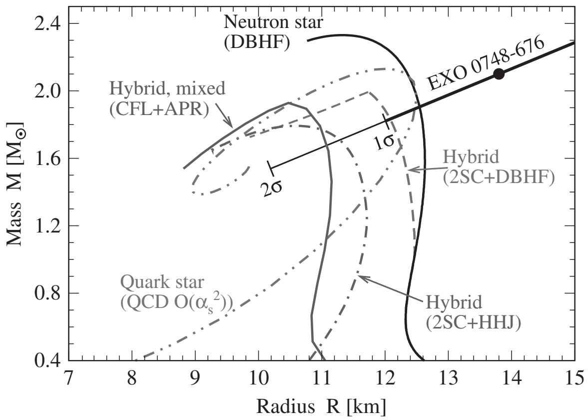

Fig. 2.3 Mass-radius plot from Ref. [13]. A comparison of a neutron star, different hybrid stars, and a quark star is shown, using several nuclear equations of state (DBHF, APR, HHJ) and several quark phases (CFL, 2SC). For more details and explanations of the various abbreviations, see Ref. [13]

gives a correction to the μ^2 term in the pressure. One might think that this correction is negligible compared to the μ^4 term and the bag constant. However, it turns out that for reasonable values of the bag constant these two terms largely cancel each other and the μ^2 term becomes important. However, the effect of superconductivity is still hard to determine. Firstly, it would require a precise knowledge of the strange quark mass. Secondly, it turns out that the maximum mass of a hybrid star is not very sensitive to the value of $m_s^2 - 4\Delta^2$ [12].

As a result of this discussion and the results in Figs. 2.2 and 2.3, two points are important for the further contents of these lectures. Firstly, we should now be motivated to learn more about the nature and the consequences of interactions in nuclear and quark matter. Secondly, we have learned that, given our ignorance of the precise quantitative effects of the strong interaction and the uncertainty in astrophysical observations, the mass and the radius of the star are not sufficient to distinguish between a neutron star, a hybrid star, and possibly a quark star. Therefore, we also have to take into account other observables which are linked to the microscopic physics. While the equation of state is a bulk property, i.e., it is determined by the whole Fermi sea, there are other phenomena which are only sensitive to the low-energy excitations at the Fermi surface. One class of such phenomena is given by transport properties. They can possibly be related to observables which are more restrictive than mass and radius for the question of the matter composition of the star. We shall discuss such observables in Chap. 5 where we relate the cooling of the star to neutrino emissivity, and in Chap. 6 where we qualitatively discuss other such observables.

Problems

2.1 Equation of state for noninteracting nuclear matter

Find the full equation of state for noninteracting n, p, e matter at $T = 0$ numerically by plotting P versus ε. You should see the onset of neutrons and identify a region where the equation of state is well approximated by the power-law behavior of pure neutron matter in the nonrelativistic limit, Eq. (2.36).

2.2 Mass-radius relation

(a) Solve Eqs. (2.38) numerically (for nonrelativistic pure neutron matter, i.e., $\gamma = 5/3$) and plot $m(r)$, $P(r)$ for a given value of the pressure $P_0 = P(r = 0)$.
(b) Use P_0 as a parameter to find the mass-radius relation $M(R)$. To this end, you need to do (a) for several values of P_0 and find for each P_0 the radius R at which $P(R) = 0$ and the corresponding mass $M(R)$.
(c) You may incorporate general relativistic effects from the TOV equation (2.10) and/or the full (numerical) equation of state for noninteracting nuclear matter from Problem 2.1.

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