Dynamics of Rational Surface Automorphisms

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Abstract This is a 2-part introduction to the dynamics of rational surface automorphisms. Such maps can be written in coordinates as rational functions or polynomials. The first part concerns polynomial automorphisms of complex 2-space and includes the complex Henon family.

The second part concerns compact (complex) rational surfaces. The basic properties of automorphisms of positive entropy are given, as well as the construction of invariant currents and measures. This is illustrated by a number of examples.

1 Polynomial Automorphisms of $\mathbf{C}^2$

1.1 Hénon Maps

A surface is said to be rational if it is birationally equivalent to the plane. The purpose of these notes is to give an entry into the dynamics of the automorphisms of rational surfaces. The first part is devoted to the complex Hénon family of maps, which has been the most heavily studied family of invertible holomorphic maps. Up to this point, the investigations of the Hénon maps have been guided by the study of polynomial maps of one variable. The dynamics of polynomials in the one-dimensional case has developed into a very rich topic, and these maps are understood in considerable detail. Although the Hénon family is only partially understood, its methods and results should provide motivation and guidance for the understanding of other automorphisms in dimension 2. We have selected for discussion only a part of what is known on the subject, and the reader is recommended to consult the expository treatments in [MNTU] and [S], as well as the original works [HOV1,2, FS1] and the series of papers [BS1,2,...], [BLS].

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G. Gentili et al. (eds.), *Holomorphic Dynamical Systems*,
Lecture Notes in Mathematics 1998, DOI 10.1007/978-3-642-13171-4_2,

Only certain compact complex manifolds carry automorphisms of positive entropy (see [C1]), and the majority of these are rational surfaces. The second part of these notes considers the geometry of compact rational surfaces, and then presents some examples of nontrivial automorphisms. In contrast with the case of the polynomial automorphisms of $\mathbf{C}^2$, not much is known about the set of all rational surface automorphisms — neither a description of all the automorphisms are nor a dynamical classification of them. As will be seen from the second part, the rational surface automorphisms have a close connection with certain birational maps of the plane. The reader is referred to [CD] for further discussion of the Cremona group of all birational transformations of the plane.

We say that a map $f = (f_1, f_2)$ of $\mathbf{C}^2$ to itself is *polynomial* if both coordinates are given by polynomials. We let $PolyAut(\mathbf{C}^2)$ denote the set of polynomial automorphisms of $\mathbf{C}^2$. That is, these are invertible maps of $\mathbf{C}^2$; it is a Theorem that if a polynomial map is one-to-one and onto, then its inverse must be polynomial. Our goal here is to study dynamical properties of such automorphisms, by which we mean the behavior of the iterates $f^n := f \circ \cdots \circ f$ as $n \to \infty$. If $f, g \in PolyAut(\mathbf{C}^2)$, then we may consider the conjugate $g^{-1} f g$ of f. However, we note that $(g^{-1} f g)^n = g^{-1} f^n g$, and so the dynamics of f and its conjugate are essentially the same. Thus in the study of $PolyAut(\mathbf{C}^2)$, we need to understand representatives of the various conjugacy classes. Friedland and Milnor [FM] have done this and have shown that, in order to understand the whole family $PolyAut(\mathbf{C}^2)$, it suffices to consider the family of *generalized Hénon mappings*, which are the class

$$\mathscr{H} = \{f(x,y) = (y, p(y) - \delta x), p(y) = y^d + a_{d-2} y^{d-2} + \cdots + a_0, \ \ d \geq 2, \delta \in \mathbf{C}, \delta \neq 0\}$$

In fact, much of what we do will remain valid if we replace f^n by certain more general compositions $f_n \circ \cdots \circ f_1$, where $f_j \in \mathscr{H}$. While these mappings may appear to be somewhat special, in fact they are the key to understanding invertible polynomial dynamics in $\mathbf{C}^2$.

Exercises:

1. Each map $f \in \mathscr{H}$ is invertible, and f^{-1} is polynomial.
2. The Jacobian of f is δ, and in particular it is constant.
3. $\deg(f^n) = (d^{n-1}, d^n)$.
4. Consider the dilation by t: $D(x,y) = (tx, ty)$, and the translation by (s,s): $T(x,y) = (x+s, y+s)$. Show that if $f \in \mathscr{H}$, then the conjugate $T \circ D \circ f \circ D^{-1} \circ T^{-1}$ has the form of an element of $\mathscr{H}$, except that the coefficient of y^d is an arbitrary nonzero number, and the coefficient of y^{d-1} is arbitrary.
5. Define the involution $\tau(x,y) = (y,x) = \tau^{-1}$. Then $f^{-1} \notin \mathscr{H}$, but $\tau \circ f^{-1} \circ \tau$ has the form of an element of $\mathscr{H}$, except that p is not monic.

We will use the notation $(x_n, y_n) = f^n(x,y)$ for $n \in \mathbf{Z}$. Thus we have $x_n = y_{n-1}$, and the whole orbit $\{(x_n, y_n) : n \in \mathbf{Z}\}$ is essentially given by the bi-infinite scalar sequence

$$\ldots, y_{-2}, y_{-1}, y_0, y_1, y_2, \ldots.$$

One advantage of this reduction is that the action of f may be replaced by the shift. That is, if the f-orbit of the point $(x', y') = f(x, y)$ corresponds to the sequence

$$\ldots, y'_{-2}, y'_{-1}, y'_0, y'_1, y'_2, \ldots,$$

then $y'_n = y_{n+1}$.

1.2 Filtration

We will find it useful to look at Hénon maps in terms of their escape to infinity. Let us define

$$V = \{|x|, |y| \le R\}, \;\; V^+ = \{|y| \ge \max\{|x|, R\}\}, \;\; V^- = \{|x| \ge \max\{|y|, R\}\}.$$

Thus V is a bi-disk, and the sets $V^\pm$ are (topologically) the product of a disk and an annulus. Let us choose R to be sufficiently large that

$$|p(t)| - |\delta t| \ge \max\left\{\frac{1}{2}|t^d|, 2|t|\right\}, \text{ for all } |t| \ge R. \tag{1}$$

It follows that:

$$(x, y) \in V^+ \;\; \Rightarrow \;\; |y_1| \ge 2|y_0| = 2|x_1| \;\; \Rightarrow \;\; (x_1, y_1) \in V^+$$

and thus

$$(x, y) \in V^+ \;\; \Rightarrow \;\; |y_n| \ge 2^n |y| \ge 2^n R \tag{2}$$

for $n = 1, 2, 3 \ldots$ In fact, it is possible to show that for any $\varepsilon > 0$ we may choose R sufficiently large that for $(x_0, y_0) \in V^+$ we have

$$(1 - \varepsilon)|y_0|^{d^n} \le |y_n| \le (1 + \varepsilon)|y_0|^{d^n}. \tag{3}$$

Theorem 1.1. *If $(x, y) \in V^-$, then there is a number N such that $f^N(x, y) \in V \cup V^+$.*

Proof. Suppose not. Then we have $(x_n, y_n) \in V^-$ for all $n \ge 0$. Thus $|y_n| \le |x_n|$. Since $x_n = y_{n-1}$, we have $|x_1| \ge |x_2| \ge \cdots \ge R$, and $|y_1| \ge |y_2| \ge \cdots$, and the both approach the same limit M. Thus for N sufficiently large, we will have $|x_N| \sim |y_N| \sim M \ge R$. Thus by (*), we will have $|y_{N+1}| \ge 2M$, which is a contradiction. □

We summarize this discussion in the following Theorem:

Theorem 1.2. *1. $fV^+ \subset V^+$, and for each $(x, y) \in V^+$, the forward orbit escapes to infinity.*
2. $f(V \cup V^+) \subset V \cup V^+$.

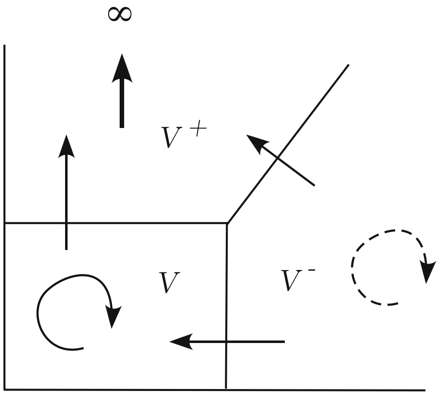

Fig. 1 Filtration behavior of Hénon maps

3. *If $(x,y) \in V^-$, then the orbit $f^n(x,y)$, $n \geq 0$, can stay in V^- for only finite time. After $f^n(x,y)$ leaves V^-, it cannot reenter, which means that it stays in $V \cup V^+$. In particular, if an orbit is bounded in forward time, then it must enter V and remain there.*

This Theorem is illustrated in Figure 1. The arrows give the possibilities where a point might map; the dashed circular arrow indicates that an orbit can remain in V^- for only finite time.

$$\begin{aligned} K^{\pm} &= \{(x,y) \in C^2 : \{f^{\pm n}(x,y), n \geq 0\} \textit{ is bounded}\} \\ &= \{(x,y) \in C^2 : \textit{with bounded forward/backward orbit}\} \end{aligned}$$

$$K := K^+ \cap K^-, \quad J^{\pm} = \partial K^{\pm}, \quad J = J^+ \cap J^-, \quad U^+ = \mathbf{C}^2 - K^+$$

Proposition 1.3. *The iterates $\{f^n, n \geq 0\}$ are a normal family on the interior of K^+, and for $(x_0, y_0) \in J^+$, there is no neighborhood $U \ni (x_0, y_0)$ on which this family is normal.*

Proof. If $p \in int(K^+)$, then the forward orbit cannot enter V^+. It can remain in V^- for only finite time, so a neighborhood of p must ultimately be in V. Thus the forward iterates of a neighborhood of p are ultimately bounded by R in a neighborhood of p, so the iterates are a normal family in a neighborhood of p. If U is an open set that intersects J^+, then $U \cap K^+ \neq \emptyset$ and $U \cap (\mathbf{C}^2 - K^+) \neq \emptyset$. Thus U contains points where the forward iterates are bounded and points whose orbits tend to ∞. Thus f^n cannot be normal on U. □

Exercise: Show that if $p \in int(K^+)$ is fixed, then the eigenvalues of Df_p have modulus ≤ 1.

Proposition 1.4. *We have $K \subset V$ and $K^+ \subset V \cup V^-$. Further, $U^+ = \bigcup_{n \geq 0} f^{-n} V^+$, where the union $V^+ \subset f^{-1} V^+ \subset \cdots$ is increasing.*

Proof. This is a consequence of the previous Theorem. □

A point p is *periodic* if $f^N p = p$ for some N. The minimal such $N > 0$ is called the *period* of p.

Proposition 1.5. *For each N there are only finitely many periodic points of period N.*

Proof. The set $P_N := \{(x,y) \in \mathbf{C}^2 : f^N(x,y) = (x,y)\}$ is a subvariety. Further, we have $P_N \subset K$, so it is bounded. But any bounded subvariety of $\mathbf{C}^2$ must be a finite set. □

A periodic point p of period N is a *saddle* if Df_p^N has one eigenvalue with modulus <1 and one eigenvalue with modulus >1. We use the notation *SPer* for the set of saddle (periodic) points and $J^* = \overline{SPer}$ for its closure. By an exercise above, we have:

Proposition 1.6. *If p is a saddle (periodic) point, then $p \in J$. Thus $J^* \subset J$.*

Problem: It is an interesting and basic question to determine whether $J^* = J$ holds for all Hénon maps. However, this is not yet known.

Exercise: Suppose that p is a fixed point, and there is a neighborhood U of p such that $\overline{fU}$ is a compact subset of U. Show that p is a sink, and $\bigcap f^n U = p$.

Theorem 1.7. *If $|a| = 1$, then the volume of $(K^+ \cup K^-) - K$ is zero. If $|a| < 1$, then the volume of K^- is zero.*

Proof. By the previous Corollary, the sets $S_n = K^- - f^n V^+$ are increasing in n. But the volume is $|S_{n+1}| = |a|^2 |S_n|$. If $|a| \le 1$, we have $|S_{n+1}| = |S_n|$, or $|S_{n+1} - S_n| = 0$. Their union is $K^- - V^+ = \bigcup S_n$, so we see that $|K^- - V^+| = 0$. By Theorem, then, we have $|K^- - K| = 0$. If $|a| = 1$, then we apply the argument to f^{-1} to obtain the first statement of the Theorem. If $|a| < 1$, then the volumes are $|K| = |fK| = |a|^2 |K|$, so $|K| = 0$. □

Problem: Suppose that f is a Hénon map with real coefficients, so f is a diffeomorphism of $\mathbf{R}^2$. If $\delta = \pm 1$, then f preserves area. Is it possible for K to have positive volume in $\mathbb{C}^2$? Or if $K \subset \mathbb{R}^2$ can it have positive area?

For a point p, we define the *ω-limit set* $\omega(p)$ to be the set of all limit points $f^{n_j} p \to q$ for subsequences $n_j \to +\infty$. A property of $\omega(p)$ is that it is invariant under both f and f^{-1}.

For a compact set X, define its *stable set* as

$$W^s(X) = \{y : \lim_{n \to \infty} dist(f^n y, f^n X) = 0\},$$

and the unstable set is defined to be the stable set in backward time:

$$W^u(X) = \{y : \lim_{n \to -\infty} dist(f^{-n} y, f^{-n} X) = 0\}.$$

Proposition 1.8. $W^s(K) = K^+$.

Proof. Since K is bounded, it is clear from the definition that $W^s(K) \subset K^+$. On the other hand, let $p \in K^+$ be given. Then by the filtration properties, we have $\omega(p) \subset V$. Since $\omega(p)$ is invariant under f^{-1}, is is contained in K^-. We conclude that $\omega(p) \subset K$, so $dist(f^n p, K) \to 0$ as $n \to \infty$. □

1.3 Fatou Components

We define the *Fatou set* to be the points p for which there is a neighborhood U such that the restrictions of the forward iterates $\{f^n|U : n \geq 0\}$ are equicontinuous. If we consider $\mathbf{C}^2$ as imbedded in the projective plane $\mathbf{P}^2$ (this is defined in 2nd part), then the iterates $f^n : \mathbf{C}^2 \to \mathbf{P}^2$ are locally equicontinuous on $\mathbf{C}^2 - K^+$. That is, they converge locally uniformly to the point $[0:0:1] \in \mathbf{P}^2$, there the y-axis intersects the line at infinity. By the previous section on Filtration, then, we have:

Proposition 1.9. *The Fatou set of a complex Hénon map is* $\mathbf{C^2} - J^+$.

A domain $U \subset \mathbf{C}^2$ is said to be a *Runge domain* if for each compact subset $X \subset U$, the polynomial hull

$$\hat{X} := \{q \in \mathbf{C}^2 : |P(q)| \leq \max_{x \in X} |P(x)| \text{ for all polynomials } P\}$$

is also contained in U.

Proposition 1.10. $int(K^+)$ *is Runge.*

Proof. Suppose that X is a compact subset of $int(K^+)$. By the filtration properties of the previous Section, we know that $\sup_{n \geq 0} \sup_{x \in X} ||f^n(x)|| = C < \infty$. Since the components of f^n are polynomials, it follows from the definition of the polynomial hull that $\sup_{x \in X} ||f^n(x)|| \leq C$. Thus $\hat{X}$ is contained in K^+. Further, since X is compact, we know that the translates $X + v$ are contained in $int(K^+)$ if $||v||$ is sufficiently small. Thus we have $\widehat{X+v} = \hat{X} + v \subset K^+$. It follows that $dist(X, \partial K^+) = dist(\hat{X}, \partial K^+)$ and thus $\hat{X}$ is in the interior of K^+. □

If Ω is a Fatou component (i.e., a connected component of the Fatou set), then $f\Omega$ is again a Fatou component. Let δ denote the (constant) Jacobian of f. There are two distinct cases: $|\delta| = 1$ and $|\delta| \neq 1$. We will first consider the case $|\delta| \neq 1$. In this case, after passing to f^{-1} if necessary, we may suppose that $|\delta| < 1$. If $|\delta| < 1$, then the Fatou components consist of $\mathbf{C}^2 - K^+$ and the connected components of $int(K^+)$.

Problem: Is every Fatou component periodic, i.e., is $f^n\Omega = \Omega$ for some $n > 0$? A non-periodic Fatou component (if it existed) would be said to be *wandering*.

A Fatou component Ω is said to be *recurrent* if there is a point $p \in \Omega$ such that $\Omega \cap \omega(p) \neq \emptyset$. Equivalently, Ω is recurrent if there are a compact set $C \subset \Omega$ and a

point $p \in \Omega$ such that $f^{n_j} p \in C$ for infinitely many $n_j \to +\infty$. Note that a recurrent Fatou component is necessarily periodic.
Exercise: Show that a Fatou component is recurrent if it is a periodic domain and not all points tend to $\bigcup_j f^j \partial\Omega$.

We say that a fixed point p is a *sink* if $p \in Int(W^s(p))$. In this case, we refer to $W^s(p)$ as the *basin* of p. If $\mathscr{O} = \{p, fp, \ldots, f^N p = p\}$ is a periodic orbit, we say that the orbit is a sink if $\mathscr{O} \subset W^s(\mathscr{O})$. In other words, a periodic orbit is a sink orbit for f is p is a sink fixed point for f^N. The following show that every sink is hyperbolic.

Proposition 1.11. *If p is a sink, then the eigenvalues of Df_p all have modulus less than 1.*

Proof. Let B be a ball containing p such that $\bar{B} \subset W^s(p)$. Thus the iterates f^n converge uniformly to p on $\bar{B}$. We may assume that $f^n \bar{B} \subset B$ for $n \geq N$. Let I_ε denote the affine map which fixes p and which dilates $\mathbf{C}^2$ by a factor of $1+\varepsilon$. For $\varepsilon > 0$ small the new map $g := I_\varepsilon \circ f$ maps $\bar{B}$ inside B. Thus the derivatives of g^n are bounded at p. This proves that the derivatives of f^n must tend to 0 as $n \to \infty$. Thus the eigenvalues of Df at p must be less than 1 in modulus. □

Exercise: If p is a periodic point such that $f^n p = p$ and the eigenvalues of $Df^n p$ have modulus < 1, then the basin $W^s(p)$ is a Fatou component. (It is clear that $W^s(p)$ is contained in K^+. The issue is to show that $W^s(p)$ is exactly a connected component of $int(K^+)$.)

Now we give other examples of Fatou components. By a *Siegel disk* we mean a set of the form $\mathscr{D} = \varphi(\Delta)$, where $\varphi : \Delta \to \mathbf{C}^2$ is an injective map of the disk $\Delta \subset \mathbf{C}$ with the property that

$$f(\varphi(\zeta)) = \varphi(\alpha\zeta) \tag{4}$$

holds for $\alpha = e^{\pi i a}$ with fixed irrational a and all $\zeta \in \Delta$. We also call $\mathscr{D}$ a Siegel disk if it is periodic: $f^N \mathscr{D} = \mathscr{D}$ for some $N > 1$, and $\mathscr{D}$ is a Siegel disk for f^N. A *Herman ring* is similar; it is the image of a maximal injective holomorphic mapping of an annulus $\{\zeta \in \mathbf{C} : r_1 < |\zeta| < r_2\}$ such that (4) holds. "Maximal" here means that it is not contained in any larger such annulus.

Theorem 1.12. *Suppose that $|\delta| < 1$ and Ω is a recurrent Fatou component. Then Ω is the basin of either (1) a sink, (2) a Siegel disk, or (3) a Herman ring. In cases (2) and (3), the Siegel disk (or Herman ring) is a subvariety of Ω.*

Proof. Passing to an iterate of f, we may suppose that $f\Omega = \Omega$. Let $p \in \Omega$ be a point with $\omega(p) \cap \Omega \neq \emptyset$. We may choose a sequence $n_j \to \infty$ such that f^{n_j} converges normally to a map $g : \Omega \to \bar{\Omega}$, and we may choose the sequence such that $g(p) \in \Omega$. Since $f \circ f^{n_j} = f^{n_j} \circ f$, it follows that $f \circ g = g \circ f$ on Ω. Thus if $g(\Omega)$ is a single point, then it must be a fixed sink, and Ω is the basin of that sink.

If g is not constant, we set $\Sigma := \Omega \cap g(\Omega) \neq \emptyset$. We may pass to a subsequence so that $n_{j+1} - n_j \to \infty$, and $f^{n_{j+1} - n_j}$ converges normally to a holomorphic map $h : \Omega \to \bar{\Omega}$. Now choose a point $w = g(z) \in \Sigma$. We see that

$$hw = \lim_{j \to \infty} f^{n_{j+1} - n_j}(f^{n_j} z) = \lim_{j \to \infty} f^{n_{j+1}} z = gz = w.$$

Thus, setting $M := \{w \in \Omega : hw = w\}$, we see that $\Sigma \subset M$. Conversely, if $w \in M$, then $gw \in M$ because $h(gw) = g(hw) = g(w)$. Thus $\Sigma = M$.

Again, since $f \circ g = g \circ f$, we have $f\Sigma = \Sigma$. Further, since Σ consists of infinite forward limits of points of K^+, we have $\Sigma \subset K$. Thus Σ is a bounded Riemann surface, so it is covered by the disk. Finally, since the fixed points of f^N are isolated, the restriction $f^N|\Sigma$ is not the identity. Since there is a point of $\omega(p) \cap \Sigma$, we know that the iterates $f^n|\Sigma$ cannot diverge to the boundary of Σ as $n \to \infty$. We conclude that Σ must be uniformized by either the disk or an annulus, and f must be an irrational rotation. Thus Σ must be either a Siegel disk or a Herman ring. □

It is clear that Hénon maps can have sinks. For a Siegel disk, let us consider the diagonal linear map L of $\Delta \times \mathbf{C}$ to itself which is given by $(\zeta, \eta) \mapsto (\alpha\zeta, \delta\eta/\alpha)$, and $\alpha = e^{\pi i a}$ with a irrational. If $\Phi : \Delta \times \mathbf{C} \to \mathbf{C}^2$ is a holomorphic imbedding such that $f \circ \Phi = \Phi \circ L$, then $\mathscr{D} = \Phi(\Delta \times \{0\})$ will be contained in a Siegel disk. Further, if $|\delta| < 1$, we will have that $\Omega := \Phi(\Delta \times \mathbf{C}^2)$ is equal to $W^s(\mathscr{D})$.

Such a map occurs for Hénon maps which have the form

$$f(x,y) = (\alpha x + \cdots, \delta y/\alpha + \cdots)$$

where the $\cdots$ indicate arbitrary terms of higher order. (Or rather we conjugate a Hénon map by an affine map so that the origin is fixed, and the linear part is diagonal.) If $|\delta| < 1$ and if α satisfies a suitable Diophantine condition (see [Z]), then there will exist a linearizing map Φ as in the previous paragraph.

Problem: Is it possible for a Fatou component to be the basin of a Herman ring?

Another sort of Fatou component can arise as follows. Suppose that f is a Hénon map fixing $(0,0)$ and taking the form

$$f(x,y) = (x + x^2 + \cdots, by + \cdots),$$

where $0 < |b| < 1$, and the terms $\cdots$ involve both x and y, but they are "smaller." This is like a parabolic fixed point in the x-direction and an attracting fixed point in the y-direction. We may choose r small so that $D := \{|x+r| < r\} \times \{|y| < r\}$ is mapped inside itself. In fact, for each $p \in D$ there is a neighborhood U such that $f^n \to (0,0)$ uniformly on U as $n \to +\infty$. The forward basin $\mathscr{B} = \{(x,y) \in \mathbf{C}^2 : f^n \to (0,0)$ uniformly in a neighborhood of $(x,y)\}$ is a non-recurrent Fatou component.

Problem: Is such a semi-attracting basin the ony possible sort of non-recurrent Fatou component? More precisely, suppose that $|\delta| \neq 1$ and that Ω is a periodic Fatou component which is not recurrent. Is there necessarily a point $p \in \partial\Omega$ with $f^n p = p$ and such that the multipliers of $Df^n p$ are α and δ^n/α, where α is a root of unity?

The volume preserving case: $|\delta| = 1$.

Except for $\mathbf{C}^2 - K^+$, all Fatou components are contained in the bounded set $int(K^+) = int(K^-) = int(K)$. If Ω is such a bounded component, then there are only finitely many other components with the same volume, so Ω is necessarily periodic.

If U is a Fatou component, we may pass to an iterate of f and assume that $fU = U$. Since $\{f^n|U : n \geq 0\}$ is a normal family, we let $\mathscr{G} = \mathscr{G}(U)$ denote the set of normal limits $g = \lim_{j\to\infty} f^{n_j}$ on U. When a sequence of analytic functions converges, the derivatives converge, too. So it follows that each $g \in \mathscr{G}$ preserves volume, and $g(U)$ is an open set. Thus $g : U \to U$, and not just $g : U \to \bar{U}$. Since the iterates of f commute with each other, so do the elements of $\mathscr{G}$. Finally, we may pass to a subsequence of the $\{n_j\}$ so that $n_{j+1} - 2n_j \to +\infty$. Passing to a further subsequence, we find that $f^{n_{j+1}-2n_j} \to g^{-1}$. Thus $\mathscr{G}$ is a group. By a theorem of H. Cartan, the full automorphism group $Aut(U)$ of a bounded domain is a Lie group (see [N]). Thus $\mathscr{G}$ is a compact abelian subgroup of a Lie group. Since $\mathscr{G}$ is compact, it can have only finitely many connected components. Let $\mathscr{G}_0$ denote the connected component of the identity in $\mathscr{G}$. It follows that $\mathscr{G}_0$ is a torus $\mathbf{T}^d$. Since $f \in \mathscr{G}$, it follows that $\mathscr{G}$ is infinite, so $d \geq 1$. Further, we may choose $m \neq 0$ so that $f^m \in \mathscr{G}_0$. Since $\mathscr{G}_0$ is compact, we have the following:

Proposition 1.13. *For volume-preserving Hénon maps, all Fatou components are periodic and recurrent.*

Since $U \subset \mathbf{C}^2$, it follows that $1 \leq d \leq 2$. Examples of both cases $d = 1$ and $d = 2$ can be obtained by starting with maps of the form $f(x,y) = (\alpha x + \cdots, \beta y + \cdots)$ with α and β jointly Diophantine, so that the map can be linearized in a neighborhood of the fixed points. If $d = 2$, then the action of $\mathscr{G}_0$ on U is equivalent to the standard $\mathbf{T}^2$-action on a Reinhardt domain in $\mathbf{C}^2$.

Theorem 1.14. *Suppose that U is a Fatou component, and suppose that f has a fixed point $a \in U$ such that $A = Df_a$ is unitary. Then there is a holomorphic semi-conjugacy $\Phi : U \to \mathbf{C}^2$, $\Phi(a) = 0$ and taking (f,U) to $(A, \Phi(U))$.*

Proof. Conjugating by a translation, we may assume that $a = 0$. For each compact subset $S \subset U$, the sets $f^j(S)$ are bounded independently of j. Thus $\Phi_j := N^{-1}\sum_{j=0}^{N-1} A^{-j} f^j$ is a bounded sequence of analytic functions. Now let Φ be any limit point of a subsequence Φ_{j_k}, and observe that Φ has the desired property. □

Problem: What $\mathbf{T}^1$ actions can appear as Fatou components? What Reinhardt domains can appear as Fatou components? Is a Fatou component necessarily simply connected?

Notes: Recurrent Fatou components are discussed in [BS2, FS1, BSn]. For a deeper discussion of semi-attracting basins, see [U1,2].

1.4 Hyperbolicity

A compact set $X \subset \mathbf{C}^2$ is said to be *hyperbolic* if there is a continuous splitting $X \ni x \mapsto E^s_x \oplus E^u_x = T_x\mathbf{C}^2$ of the complex tangent space such that $Df_x E^{s/u}_x = E^{s/u}_{fx}$, and if there are constants $c < \infty$ and $\lambda < 1$ such that

$$||Df^n_x|_{E^s_x}|| \leq c\lambda^n, \quad ||Df^{-n}_x|_{E^u_x}|| \leq c\lambda^n.$$

An alternative definition which is sometimes useful is that there are invariant cone fields which are expanding and contracting. For an expanding cone field, there is a neighborhood $U_0 \supset X$, and for each $p \in U_0$ there is an open cone $\mathscr{C}_p \subset T_p\mathbf{C}^2$. This cone field has the property that there is an n such that for each $p \in U_0$ and nonzero $\xi \in \mathscr{C}_p$, we have $Df^n\mathscr{C}_p \subset int(\mathscr{C}_{f^n p})$, and $||Df^n\xi|| \geq 2||\xi||$. A contracting cone field is defined as an expanding cone field for f^{-1}.

Exercise: A finite set is hyperbolic if and only if it is the union of orbits of periodic points, each of which is of saddle, attracting, or repelling type.

Exercise: We say that a hyperbolic set X has *index* i if $\dim E^u_x = i$ for all $x \in X$. Show that if J is a hyperbolic set for f, then J has index 1. (Hint: First subdivide $J = \bigcup_{0\leq i\leq 2} \Lambda_i$, where Λ_i has index i. Then show that Λ_0 is contained in the interior of K^+.)

Stable Manifold Theorem: Let f be a diffeomorphism of a smooth manifold, and let $X \subset M$ be a hyperbolic set for f. For each $x \in X$, the stable set $W^s(x)$ is a submanifold of M, with $T_xW^s(x) = E^s_x$. Further $W^s(X) = \bigcup_{x\in X} W^s(x)$.
From the proof of the Stable Manifold Theorem, it is clear that if f is also holomorphic, then each $W^s(x)$ is a complex submanifold. However, $W^s(x)$ is not necessarily a closed submanifold.

Proposition 1.15. *If f is a complex Hénon map, and if J is a hyperbolic set, then $W^s(J) \subset J^+$ and $W^u(J) \subset J^-$.*

Proof. We will prove that $W^s(J) \subset J^+$; the statement that $W^u(J) \subset J^-$ follows by considering f^{-1}. It is clear that $W^s(J) \subset K^+$, so it will be sufficient to show that $W^s(J)$ is disjoint from the interior of $int(K^+)$. Now the iterates $\{f^n : n \geq 0\}$ are a normal family on the interior of K^+, so if $p \in int(K^+)$, $||Df^n p||$ is bounded for $n \geq 0$. On the other hand, but the hyperbolicity of f, there is an expanding cone field $\mathscr{C}_q$ which is defined for $q \in U_0$. If $p \in W^s(J)$, then $f^m p \in U_0$ for m sufficiently large. Thus there is a vector $\xi \in \mathscr{C}_{f^m p}$ which is uniformly expanded by Df^n. This means that $||Df^{m+kn}\xi||$ is unbounded for k large. The contradiction shows that p, and thus $W^s(J)$, is disjoint from the interior of K^+. □

We say that X has *local product structure* for a map f if $W^s(p_1) \cap W^u(p_2) \subset X$ whenever $p_1, p_2 \in X$.

Proposition 1.16. *If f is hyperbolic on J, then J has local product structure.*

Proof. By the previous Proposition, $W^s(p_1) \subset J^+$, and $W^u(p_2) \subset J^-$. Thus the intersection is contained in $J = J^+ \cap J^-$. □

A consequence of the local product structure and hyperbolicity (see [S, Prop. 8.22]) is the following:

Corollary 1.17. *The set J is locally maximal, which means that there is a neighborhood U of J so that every invariant contained in U is also contained in J.*

Proposition 1.18. *If f is a complex Hénon map, and if J is a hyperbolic set, then each stable manifold $W^s(p)$, $p \in J$ is conformally (biholomorphically) equivalent to* $\mathbf{C}$.

Proof. By the stable manifold theorem, $W^s(p)$ is homeomorphic to a disk, so as a Riemann surface it is conformally equivalent either to the disk or to $\mathbf{C}$. In order to show it is equivalent to $\mathbf{C}$, it suffices to show that $W^s(p)$ contains an infinite increasing family of disjoint annuli $A(k)$ so that $A(1)$ surrounds p, $A(k+1)$ surrounds $A(k)$, and the moduli of the $A(k)$ are bounded below.

Let $B_p(r) = \{q : ||p-q|| < r\}$ be the ball with radius r and center p. Then for r_2 small, the connected component of $B_p(r_2) \cap W^s(p)$ containing p will be a disk, and for $0 < r_1 < r_2, A_p := (B_p(r_2) - B_p(r_1)) \cap W^s(p)$ will be an annulus in $W^s(p)$ which goes around p. By compactness, there is a lower bound on the moduli of the annuli A_p for $p \in J$. By the uniform contraction of hyperbolicity, there is an n such that for every p, $A_p(1) := f^{-n}A_{f^n p}$ contains A_p in its "hole". Thus $A(k) := f^{-nk}A_{f^{nk}p}$ is the desired increasing sequence of annuli in $W^s(p)$. □

Theorem 1.19. *If f is a complex Hénon map with $|\delta| \neq 1$, and if J is a hyperbolic set, then the interior of K^+ consists of the basins of finitely may hyperbolic sink orbits.*

Proof. We may assume that $|\delta| \leq 1$, for if $|\delta| > 1$, then $int(K^+)$ is empty, and there is nothing to prove. The theorem will be a consequence of the following assertions:

(1) Each component is periodic and recurent.
(2) Each component is the basin of a sink.
(3) There are finitely many sink orbits.

Proof of (1). So assume that $|\delta| < 1$. Let C be a wandering component of $int(K^+)$, and let p be a point of C. For n sufficiently large, $f^n p$ will be a point of V. Since V is compact, then, the ω-limit set $\omega(p)$ is a nonempty compact subset of V. Since $\omega(p)$ is invariant under f^{-1} and bounded, it follows that $\omega(p) \subset K \subset K^-$. Since $|\delta| < 1$, we have $J^- = K^-$. Thus $\omega(p) \subset J^-$.

Now we claim that $\omega(p)$ intersects $int(K^+)$. For otherwise, $\omega(p) \subset J^+ = \partial K^+$, which means that $\omega(p) \subset J$. Thus $p \in W^s(J)$, and by the Proposition above, we conclude that $p \in J^+$, which is not the case.

Now choose $q_0 \in \omega(p) \cap int(K^+)$, and let C_0 denote the component of $int(K^+)$ containing q_0. There are n_1 and n_2 so that $f^{n_1} p$ and $f^{n_2} p$ are arbitrarily close to q_0, and so they both belong to C_0. We conclude, then, that $f^{n_1} C = f^{n_2} C$, from which we conclude that C must be periodic, and since it contains a point of $\omega(p)$ it is recurrent.

Proof of (2) Assume $|\delta| < 1$. We Know By Theorem 1.12 that each recurrent component is the basin of either a sink or a Siegel disk or Herman ring. Suppose that is the basin of Σ, which is a Siegel disk or a Herman ring. By Theorem 1.12 again, Σ is a subvariety of C. Since Σ is bounded and invariant under f^{-1}, we have $\Sigma \subset K^-$. Finally, since $\partial\Sigma \subset \partial C \subset \partial K^+$, we have $\partial\Sigma \subset J$. By the rotation property of f on Σ, the closure of the orbit of a point of Σ is a compact curve. If we take $q \in \Sigma$ close to J, then the closure of the orbit will be contained in a small neighborhood of J. However, this contradicts the maximality property of J. We conclude that the Siegel disk Herman ring cases cannot occur, so C must be the basin of a sink.

Proof of 3. The sink orbits are all contained in V, which is compact. Define the set L to be the limit points of sequences of (distinct) sinks. If there are infinitely many sink orbits $\{p_j\}$, then $L \neq \emptyset$. Since $int(K^+)$ is the union of basins of sinks, we see that $L \cap int(K^+) = \emptyset$. Thus we must have $L \subset J$. On the other hand, this means that there are sink orbits which are arbitrarily close to J, which contradicts the maximality of J. □

Corollary 1.20. *If f is hyperbolic on J, and if $|\delta| = 1$, then $int(K^+) = int(K^-) = int(K) = \emptyset$.*

Proof. In the volume-preserving case, we have $int(K^+) = int(K^-)$. We saw in the previous section that all components of $int(K^+)$ are periodic, and if C is such a component, we may pass to f^N and suppose that it is fixed. The iterates of f^N then induce a torus action on C. There will be (compact) orbits if this torus action inside any neighborhood of J, which will contradict the maximality of J. But we saw that in he hyperbolic case, J is always maximal. □

In the following, we assume that the Jacobian δ satisfies $|\delta| \leq 1$, i.e., f decreases volume. Since we may replace f by f^{-1} if necessary, we may always make this assumption.

Theorem 1.21. *Let f be a complex Hénon with $|\delta| \leq 1$, and let J be a hyperbolic set. Then $W^s(J) = J^+$, and if $s_1, \ldots, s_k$ are the sinks of f, then $W^u(J) = J^- - \{s_1, \ldots, s_k\}$.*

Proof. By Proposition 1.15 we have $W^s(J) \subset J^+$. We also know that $J^+ \subset K^+ = W^s(K)$. By Theorem 1.7 and previous corollary we have $int(K^-) = \emptyset$, that is $K^- = J^-$. Since J^+ is invariant $J^+ \subset W^s(K \cap J^+) = W^s(K^- \cap J^+) = W^s(J)$. Thus, we have $W^s(J) = J^+$.

For the other statement, let us note that $J^- = K^-$. We have seen in §2 that this holds for all Hénon maps when $|\delta| < 1$. And the case $|\delta| = 1$ follows from the previous Corollary.

For the second statement, we have $K^- = W^u(K)$, so $J^- \subset W^u(K)$. Suppose first that $p \in J^- - int(K^+)$. Then $p \in W^u(K - int(K^+))$. Since $K^- = J^-$, we have $K - int(K^+) = J$, so $J^- - int(K^+) \subset W^u(J)$.

Now if $p \in J^- \cap int(K^+)$, then by Theorem 1.19, $p \in W^s(s_j)$ is in the basin of some sink s_j. If p is not the sink itself, then the backward iterates converge to the boundary, so $f^{-n}p \to \partial K^+ = J^+$ as $n \to +\infty$. We conclude that if p is not a sink, then $p \in W^u(J^-)$. Since $p \in W^u(J^+)$, we have $p \in W^u(J)$. □

A point p is *wandering* if there is a neighborhood U containing p such that $U \cap f^n U = \emptyset$ for all $n \neq 0$. And p is *nonwandering* if it is not wandering. We let Ω_f denote the set of nonwandering points.

A point p is *chain recurrent* if for each $\varepsilon > 0$ there is a sequence of points $p_0 = p$, $p_1, \ldots, p_{N-1}$ and $p_N = p$ such that $dist(fp_j, p_{j+1}) < \varepsilon$ for $0 \leq j \leq N-1$. The number N and the sequence of points depend on ε. The set of all chain recurrent points is denoted $R(f)$.

Exercise: Show that $\Omega_f \subset R(f)$.

We will use the following transitivity property which will be proved using the convergence theorems in 1.9:

Theorem 1.22. *Let $p_+ \in J^+$ and $p_- \in J^-$ be given. Then for any neighborhoods $U_- \ni p_-$ and $U_+ \ni p_+$, there is an n such that $U_+ \cap f^n U_- \neq \emptyset$.*

Let us note that an immediate consequence of this Theorem is that the restriction $f|_J$ is *topologically mixing*, which means that for any open sets U_1 and U_2 with $U_i \cap J \neq \emptyset$ for $i = 1,2$, there is an $n \neq 0$ such that $U_i \cap f^n U_i \neq \emptyset$. In particular, it follows that $J \subset \Omega_f$.

We will call a Hénon map *hyperbolic* if any of the following sets is a hyperbolic set:

Theorem 1.23. *The following are equivalent:*

(i) f has a hyperbolic splitting over the chain recurrent set.
(ii) f has a hyperbolic splitting over the nonwandering set.
(iii) f has a hyperbolic splitting over J.

Proof. The implication $(i) \Rightarrow (ii)$ follows because $\Omega_f \subset R(f)$. The implication $(ii) \Rightarrow (iii)$ follows because $J \subset \Omega_f$. Now we prove $(iii) \Rightarrow (i)$. Without loss of generality, we may assume that $|\delta| \leq 1$. It will be sufficient to show the claim that $R(f)$ is contained in the union of J with finitely many sink orbits. It is clear that $R(f)$ is contained in K^+ and K^-. Since $|\delta| \leq 1$, we have $J^- = K^-$, so $R(f) \subset J^- \cap K^+$. Now by Theorem 1.19, the interior of K^+ consists of the basins of sink orbits, so the only points of $R(f) \cap int(K^+)$ are the sink orbits themselves. This proves the claim. □

1.5 Rate of Escape

In the second part of these notes, we will look at the Hénon family as maps in projective space. Seen from this point of view, there is an invariant line (the line at infinity), and a superattracting fixed point (the intersection of the line at infinity with the y-axis). Orbits in the basin approach this superattracting fixed point at a super-exponential rate. This rate of escape function is very useful for understanding K^+ (the set of non-escaping points, or the complement of the basin). For the moment, we are able to look at this situation very adequately from within $\mathbf{C}^2$. Let us define

$$q(x,y) = p(y) - y^d = a_{d-2}y^{d-2} + \cdots + a_0 - \delta x$$

Thus we have

$$\frac{y_{n+1}}{y_n^d} = \frac{y_n^d + q(x_n,y_n)}{y_n^d} = 1 + \frac{q(x_n,y_n)}{y_n^d}$$

For $(x,y) \in V^+$, then, we have

$$\left| \frac{y_{n+1}}{y_n^d} - 1 \right| \leq \frac{\kappa}{|y_n|^2}$$

for some $\kappa > 0$. Let us set

$$G_n^+(x,y) = \frac{1}{d^n} \log^+ |y_n|$$

Definition 1.24. A function ψ is *upper-semicontinuous* if $\psi(z_0) \geq \limsup_{z\to z_0} \psi(z)$ at all points z_0. We say that a function ψ is *pluri-subharmonic*, or simply *psh* on a domain $\Omega \subset \mathbf{C}^2$ if ψ is upper-semicontinuous, and if for all $\alpha, \beta \in \mathbf{C}^2$, the function $\zeta \mapsto \psi(\alpha\zeta + \beta)$ is subharmonic in ζ, wherever it is defined. ψ is said to be *pluri-harmonic* if both ψ and $-\psi$ are psh. If ψ is pluri-harmonic, then it is locally the real part of a holomorphic function.

Recall that $\log^+|t| := \max\{\log|t|, 0\} = \log(\max\{|t|, 1\})$ is continuous on all of $\mathbf{C}$.

Proposition 1.25. *$G^+(x,y) := \lim_{n\to\infty} G_n^+(x,y)$ exists uniformly on V^+. Further $G^+ > 0$ on V^+ and is pluri-harmonic there.*

Proof. We have $y_n \neq 0$ on V^+, so $\log^+|y_n| = \log|y_n|$ is pluriharmonic there. It suffices to show uniform convergence of the limit. For this we rewrite it as a telescoping sum

$$\begin{aligned} G_N^+(x,y) &= G_0(x,y) + \sum_{n=1}^{N-1} (G_{n+1}(x,y) - G_n(x,y)) \\ &= \log|y| + \sum_{n=1}^{N-1} \frac{1}{d^n} \left(\frac{1}{d} \log|y_{n+1}| - \log|y_n| \right) \\ &= \log|y| + \sum \frac{1}{d^n} \log \left| \frac{y_{n+1}}{y_n^d} \right|. \end{aligned}$$

Now on V^+ we have
$$\left| \frac{y_{n+1}}{y_n^d} - 1 \right| < \frac{\kappa}{|y_n|^2} \leq \frac{\kappa}{R^2}$$
so the series converges uniformly. □

The formula $G^+ \circ f^n = d^n G^+$ holds on V^+, and this formula may be used to extend G^+ to $U^+ = \bigcup_{n\geq 0} f^{-n} V^+ = \mathbf{C}^2 - K^+$. Recall that $H_1(V^+;\mathbf{Z}) \cong \mathbf{Z}$. Since $f(x,y) = (y, y^d) + \cdots$, we see that the action of f_* on $H_1(V^+;\mathbf{Z})$ to itself is multiplication by d. We may use G^+ to describe the homology of U^+.

Theorem 1.26. *The map defined by $\gamma \mapsto \omega(\gamma) = \frac{1}{\pi i} \int_\gamma \partial G^+$ yields an isomorphism $\omega : H_1(U^+;\mathbf{Z}) \to \mathbf{Z}[\frac{1}{d}]$.*

Proof. Let τ denote the circle inside V^+ which is given by $t \mapsto (0, \rho e^{2\pi i t})$. First we show that τ is nonzero in $H_1(U^+;\mathbf{Z})$. For this we recall that on V^+ we have $G^+(x,y) = \log|y| + O(y^{-1})$. Thus we have

$$\int_\tau \partial G^+ = \int_\tau \partial \log|y| + O(y^{-2}) = \int \frac{dy}{2y} + O(y^{-2}) = i\pi + O(\rho^{-1}).$$

Letting $\rho \to \infty$, we see that the integral is $i\pi \neq 0$, so τ is nonzero in $H_1(U^+)$. Further, τ generates $H_1(V^+;\mathbf{Z})$.

Now if $\gamma \in H_1(U^+;\mathbf{Z})$ is arbitrary, there exists $m \geq 0$ such that $f_*^m \gamma$ is supported in V^+. Thus $f_m^* \gamma \sim k\tau$ for some $k \in \mathbf{Z}$. Thus $\gamma \sim k d^{-m} \tau$. □

Two global convergence theorems will be useful to us. The first concerns locally uniform convergence on $\mathbf{C}^2$.

Theorem 1.27. *The limit* $G^+(x,y) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |y_n|$ *exists uniformly on compact subsets of* $\mathbf{C}^2$*, and the following hold:*

(i) G^+ *is continuous and psh on* $\mathbf{C}^2$*;*
(ii) $\{G^+ = 0\} = K^+$*;*
(iii) G^+ *is pluri-harmonic on* U^+*;*
(iv) $G^+ \circ f = (d) \cdot G^+$.

Proof. The two things that we must prove are the continuity of G^+ and the uniform convergence. Everything else should be clear. We first show that G^+ is continuous on $\mathbf{C}^2$. By Proposition 1.25 we know that G^+ is continuous (and even pluri-harmonic) on U^+, and G^+ is equal to 0 on K^+. Thus we need to show that $\lim_{\zeta\to z} G^+(\zeta) = 0$ for all $z \in \partial K^+$. Let $M = \max_{V^+ \cap fV} G^+$. Without loss of generality, we may assume that $z \in V$. Let $\zeta_j \to z$ be any sequence. If $\zeta_j \notin K^+$, then there is a smallest number n_j such that $f^{n_j}\zeta_j \notin V$. As $\zeta_j \to z \in K^+$, we must have $n_j \to \infty$. Since $f^{n_j}\zeta_j \in V +^{\cap} fV$, we have

$$G^+(\zeta_j) = \frac{1}{d^{n_j}} G^+(f^{n_j}\zeta_{n_j}) \le \frac{M}{d^{n_j}},$$

from which we conclude that $G^+(\zeta_j) \to 0$ as $j \to \infty$.

Now we show uniform convergence on a set of the form $D = \{|x| < \rho', |y| < \rho''\}$. Choosing ρ' large, we may assume that $K^+ \cap \{|x| = \rho', |y| \le \rho''\} = \emptyset$. Let $\varepsilon > 0$ be given, and choose $U \subset D$ such that for any $|y_0| < \rho''$, U contains a neighborhood of $K^+ \cap \{y = y_0\}$ inside $\{y = y_0\}$. We may choose U sufficiently large that $G^+ < \varepsilon$ on $\{|y| < \rho''\} \cap \partial U$. By the Proposition, we may choose n such that $|G_n^+ - G^+| < \varepsilon$ on $D - U$. Thus $G_n^+ \le 2\varepsilon$ on $\partial U \cap D$, so by the maximum principle, G^+ and G_n^+ are both bounded above by 2ε on U. We conclude that $|G^+ - G_n^+|$ is uniformly small on D. □

Theorem 1.28. $d^{-n} \log |y_n|$ *converges to* G^+ *in* L^1_{loc} *as* $n \to \infty$.

Proof. Since $\log^+ |y| = \log |y|$ for $|y| > 1$, it follows from the previous Theorem that $d^{-n} \log |y_n|$ converges locally uniformly to G^+ on $\mathbf{C}^2 - K^+ = \{G^+ > 0\}$. By Hartogs' Theorem, we know that for any sequence $d^{-n_j} \log |y_{n_j}|$ there is a further subsequence which will converge in L^1_{loc} to a psh limit v. If we can show that $v = 0$ on the interior of K^+, then by the upper semicontinuity of G^+, it will follow that $v = G^+$. Since the limit is independent of the subsequence, we will conclude that $d^{-n} \log |y_n|$ converges in L^1_{loc} to G^+.

Now if v is not identically zero on $int(K^+)$, there will be a $\delta > 0$ such that $\{v < -2\delta\}$ is a nonempty open set W. By Hartogs' Theorem again, we may choose a relatively compact open subset $W_0 \subset W$ such that $d^{n_j} \log |y_{n_j}| < -\delta$ holds on W_0 for large j. This means that $f^{n_j} W_0$ is contained in the set $\{\log |y| < -\delta d^{n_j}\}$. Further, since $W_0 \subset int(K^+)$ we may assume that $f^{n_j} W_0 \subset V$. Now the standard comparison between volume and capacity gives us that:

$$Vol(V \cap \{|y| < e^{-\delta d^{n_j}}\}) < Ce^{-\delta d^{n_j}}$$

for some constant C. On the other hand, the Jacobian δ is constant, so we have

$$Vol(f^{n_j}W_0) = |\delta|^{2n_j}Vol(W_0).$$

Since $f^{n_j}W_0 \subset V \cap \{|y| < e^{\delta d^{n_j}}\}$ the first estimate needs to dominate the second estimate. This is not possible when n_j is large, so we conclude that we must have $v = 0$ on $int(K^+)$. □

1.6 Böttcher Coordinate

We would like to define a function $\varphi^+ = \lim_{n\to\infty} y_n^{1/d^n}$, because if we have such a function, then we will have $\varphi^+ \circ f = (\varphi^+)^d$. In order to make this work, we consider the product

$$y_n^{1/d^n} = y_0 \left(\frac{y_1}{y_0^d}\right)^{1/d} \left(\frac{y_2}{y_1^d}\right)^{1/d^2} \cdots \left(\frac{y_n}{y_{n-1}^d}\right)^{1/d^n}.$$

By the estimate (2), $y_n/y_{n-1}^d = 1 + \frac{q(x_j,y_j)}{y_j^d}$ is close to 1 for $(x,y) \in V^+$, so we have the following:

Theorem 1.29. *The limit*

$$\varphi^+(x,y) := y \lim_{n\to\infty} \prod_{j=0}^{n-1} \left(1 + \frac{q(x_j,y_j)}{y_j^d}\right)^{\frac{1}{d^j}}$$

exists uniformly on V^+ and defines a holomorphic function there. Further, we have $\varphi^+ \circ f = (\varphi^+)^d$.

Theorem 1.30.
(1) If γ is a path in U^+ which starts at a point of V^+, then φ^+ has an analytic continuation along γ.
(2) There exists no function ψ which is holomorphic on $f^{-1}V^+$ and which is equal to φ^+ on V^+.

Proof. It is clear from the definitions that $G^+ = \log|\varphi^+|$ on V^+. Now let G^* be a pluriharmonic conjugate for G^+ in a neighborhood of the starting point of γ, such that $\varphi^+ = exp(G^+ + iG^*)$. Now we may continue G^* pluri-harmonically along γ, which gives us the desired analytic continuation of φ^+.

The function $\psi \circ f^{-1}$ is defined on V^+, and it is equal to $\varphi^+ \circ f^{-1}$ on fV^+. Thus $(\psi \circ f^{-1})^d = (\varphi \circ f^{-1})^d = \varphi^+$ on fV^+. Thus $\varphi \circ f$ is a dth root of φ^+ on V^+. On the other hand, by the formula, we see that $\varphi^+ \sim y$ for $(x,y) \in V^+$ for $|y|$ large. Since y does not have a dth root on $\{|y| > M\}$, this is a contradiction. □

Exercises:

1. Re-do the previous discussions in the case where p is not monic. How does the factor $a_d \neq 1$ change things? In particular, if we write
$$G^+(x,y) = \log|y| + c_0 + \frac{c_1}{y} + O\left(\frac{1}{y^2}\right)$$
then what are c_0 and c_1?
2. Re-do the previous discussion for backward time. In particular, construct $G^- := \lim_{n\to\infty} \frac{1}{d^n} \log^+ |x_{-n}|$, as well as a function φ^- on V^- which has the property that $\varphi^- \circ f^{-1} = (\varphi^-)^d$.
3. Let $||\cdot||$ be any norm on $\mathbf{C}^2$. Show that $G^+(x,y) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ ||(x_n,y_n)||$.
4. Show that for all $(x,y) \in V^+$, we have $|y_n| \geq c\, e^{d^n G^+(x,y)}$. Give a value for c.

In dimension 1, we consider a polynomial map $p(z) = z^d + \cdots : \mathbf{C} \to \mathbf{C}$. In this case, the Böttcher coordinate has been fundamental in the description of the dynamics. The dynamical systems that serves as a 1-D model domain is $(\sigma, \mathbf{C} - \bar{D})$, where $\sigma(w) = w^d$, and $\mathbf{C} - \bar{D}$ is the complement of the closed unit disk. The dynamics in the model system is seen in terms of polar coordinates (r,θ): orbits are escaping to infinity by $r \mapsto r^d$, and the recurrent part of the dynamics is given by $\theta \mapsto d\cdot\theta$ (modulo 2π). The Böttcher coordinate φ^+ may be analytically continued to $\mathbf{C} - K$ if and only if K is connected. In this case, it gives a conjugacy $\varphi^+ : (p, \mathbf{C} - K) \to (\sigma, \mathbf{C} - \bar{D})$. The radial lines $\{\theta = c\}$ in the image correspond to the gradient lines of G^+ in $\mathbf{C} - K$. When $d = 2$, the most natural picture represents the level sets $\{G^+ = c\}$ for $c = 2^{-n}$ and gradient lines for G^+, i.e., $(\varphi^+)^{-1}\{\theta = c\}$ for $\theta = j2^{-n}$.

This is shown for the map $p(z) = z^2 - 1$ in Figure 2; the set K is black, and its complement is colored in white/gray. A starting point z is iterated until some iterate $z_n = p^n(z)$ has modulus bigger than 100. The color "white" means that $Im(z_n) > 0$. This is equivalent to the statement on the argument for the bottcher coordinate $\varphi(z)$: we have $2\pi j 2^{-n} < Arg(\varphi(z)) < 2\pi(j+1)2^{-n}$ (mod 2π), with j being even. If n is odd, then the color is "gray". The binary (white/gray) color scheme corresponds to the fact that we may identify the circle with binary expansions $\theta = .b_1b_2b_3\ldots$ in base 2. That is, we may represent the circle as infinite binary sequences $\{0,1\}^{\mathbf{N}}/\sim$, where the equivalence relation $\sim$ identifies two binary expansions $.*1\overline{0} = .*0\overline{1}$ for the same real number.

Looking at Figure 2, the eye of the observer can fill in the curves of the level sets $\{G = 2^{-n}\}$ of the Green function, as well as arcs of the form $\{Arg(\varphi) = j2^{-n} = \theta\}$. Looking more carefully at one of these arcs, we may observe the alternating white/gray sequences as the arc lands, and this lets us deduce part of the binary expansion of θ. If we make a zoom of this picture near the landing point, we would see more binary digits of θ.

The two fixed points $\{fix_\alpha, fix_\beta\}$ of p are repelling; fix_α is the right hand tip of of J, and fix_β is the point of $J \cap [-1,0]$, which is the intersection of the immediate basin of -1 and the basin of 0.

In dimension 1, the subject of representing a Julia set as the quotient of a circle is a very well developed subject, so we would hope to develop something analogous

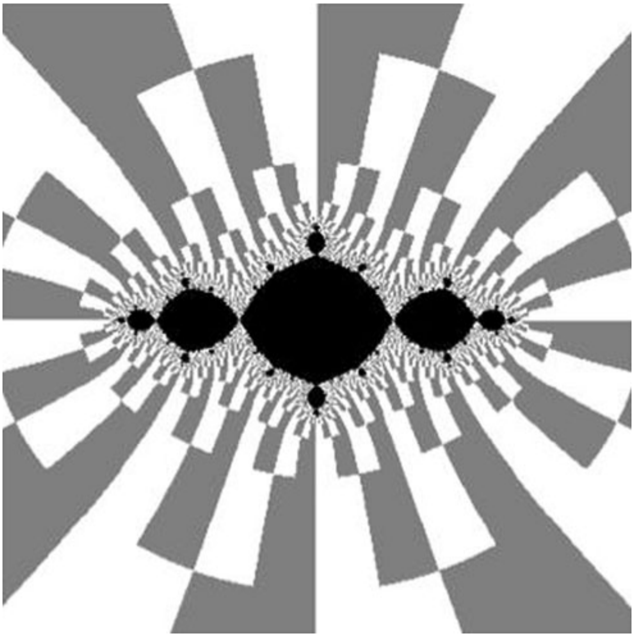

Fig. 2 Julia set for $p(z) = z^2 - 1$

in 2D. It is natural to take the corresponding 2-D model to be the complex solenoid. We set

$$\Sigma_* = \{\zeta = (\zeta_n)_{n\in\mathbf{Z}} : \zeta_n \in \mathbf{C}_*, \zeta_{n+1} = \zeta_n^d\}$$

and we give Σ_* the infinite product topology induced by $\Sigma_* \subset \mathbf{C}_*^\infty$. Also, Σ_* is a group under the operation $\zeta \cdot \eta = (\zeta_n \eta_n)_{n\in\mathbf{Z}}$. The map $\sigma(\zeta) = \zeta^d$ is the same as the (bilateral) shift map, and σ defines a homeomorphism of Σ_*.

Let us write $\pi : \Sigma_* \to \mathbf{C}_*$ for the projection $\pi(\zeta) = \zeta_0$ and set $\Sigma_+ = \pi^{-1}(\mathbf{C} - \bar{D})$. Our model dynamical system is now (σ, Σ_+); in fact this is the projective limit of the dynamical system $(\sigma, \mathbf{C} - \bar{D})$. The 2-D analogue of $\mathbf{C} - \bar{D}$ is $J_+^- := J^- - K$. The following should be clear from the definitions.

Proposition 1.31. *If φ^+ extends holomorphically to a neighborhood of J_+^-, then it yields a semiconjugacy $\Phi^+ : J_+^- \to \Sigma_+$ defined at $p \in J_+^-$ by $\Phi^+(p) = (\varphi^+(f^n p))_{n\in\mathbf{Z}}$.*

Problem: Suppose that f is hyperbolic and that φ^+ extends to J_+^-. Is the map Φ^+ injective, i.e., does it give a conjugacy between (f, J_+^-) and the model (σ, Σ_*)?

The 2-D version of the circle variable θ is the real solenoid $\Sigma_0 := \{\zeta \in \Sigma_* : |\zeta_0| = 1\}$. The path components of Σ_0 are homeomorphic to $\mathbf{R}$. The "natural" picture to draw for a complex Hénon map is to start with a saddle point p. The stable and unstable manifolds $W^{s/u}(p)$ are conformally equivalent to $\mathbf{C}$. Let $\psi : \mathbf{C} \to W^u(p)$ denote a uniformization such that $\psi(0) = p$. Any other such uniformization $\hat{\psi}$ is given by $\hat{\psi}(\zeta) = \psi(\alpha\zeta)$ for some $\alpha \in \mathbf{C}_*$. A useful picture, then, is to draw the

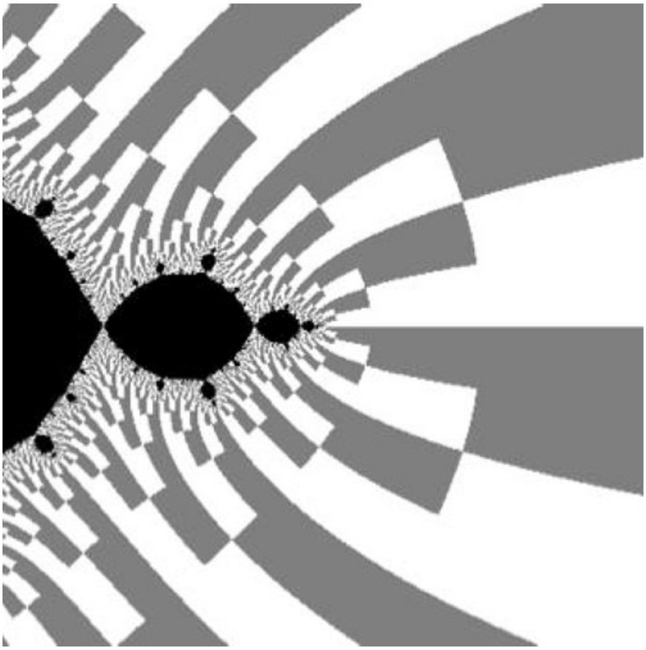

Fig. 3 The map $f(x,y) = (y, y^2 - 1 - .01x)$, slice $W^u(q_\alpha) \cap K^+$

level sets of $G^+ \circ \psi$ and the gradient lines, which are parametrized by a solenoidal variable $s \in \Sigma_0$. If λ is the unstable multiplier for Df at p, then we will have the relationship $d \cdot G^+(\psi(\zeta)) = G^+(\psi(\lambda\zeta))$, which means that the picture will be self-similar about the origin with a factor of λ.

Figures 3 and 4 show this phenomenon for the map $f(x,y) = (y, y^2 - 1 - .01x)$. There are saddle points $\{q_\alpha, q_\beta\}$ which are "close" to fix_α and fix_β. Figure 3 shows the slice through $W^s(q_\alpha)$, and the appearance is consistent with the idea that Figure 3 is obtained by expanding Figure 2, centered at fix_α, by the unstable multiplier infinitely many times until it becomes self-similar. Similarly, Figure 4 looks like Figure 2 made self-similar by expanding about fix_β. The black in Figures 3 and 4 is $W^u(q_\alpha) \cap K^+$; the white/gray parts are the intersection of $W^u(q_\alpha)$ with the basin of infinity, i.e., $W^u(q_\alpha) \cap U^+$. A starting point $(x,y) = \psi(\zeta)$ is iterated until $|y_n| > 1000$. Then the picture is colored white/gray depending on whether $Im(y_n) > 0$ or not. If we want to interpret this in terms of an angular or "argument" variable, we must consider it as an "argument" of Φ^+; this so-called argument, then, would be interpreted as an element of the real solenoid Σ_0. The slices $W^u(q) \cap U^+$ appear to be simply connected in Figures 3 and 4, and the pictures are self-similar with respect to the origin, so they would also be simply connected in the large. Thus we see that it would not work to have an argument (mod 2π); working in the solenoid, the "argument" is obtained as a value in $\mathbf{R}$. The binary nature of the color scheme corresponds to the fact that we may also represent the real solenoid Σ_0 in terms of binary sequences: $\Sigma_0 \cong \{0,1\}^{\mathbf{Z}} / \sim$.

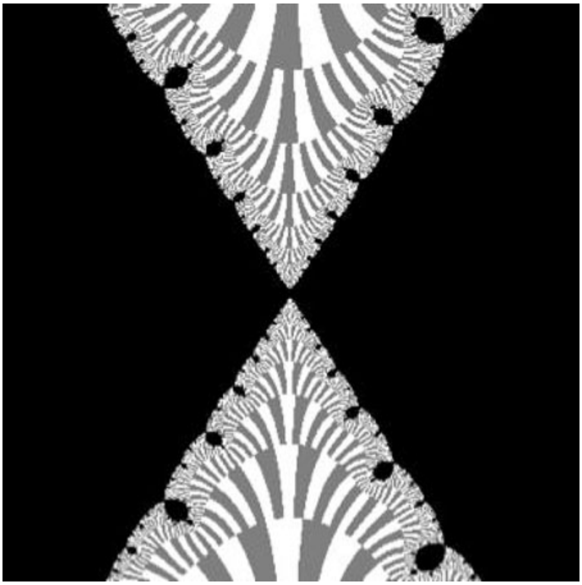

Fig. 4 Slice $W^u(q_\beta) \cap K^+$

Problem Suppose that J is connected, and f is hyperbolic. How do you write J as a quotient of the solenoid? What quotients can appear?

Notes. The geometry of U^+ is discussed further in [HO]. The problem of writing J as a quotient of the solenoid is discussed in [BS7]. The thesis of Oliva [O] gives a number of computer experiments that are helpful in understanding what identifications can appear when writing J as a quotient. The papers [I] and [IS] give a combinatorial/topological approach to this problem.

1.7 Currents on $\mathbf{R}^N$

Let $\mathscr{D}^k$ be the set of smooth k-forms on $\mathbf{R}^N$ with compact support. We define the space of k-currents to be the dual: $\mathscr{D}'_k := (\mathscr{D}^k)'$. A basic example is the *current of integration*. Let $M \subset \mathbf{R}^N$ be a k-dimensional submanifold which is oriented and has locally bounded (k-dimensional) area. Then the current of integration $[M]$ is defined by

$$\mathscr{D}^k \ni \varphi \mapsto \langle [M], \varphi \rangle = \int_M \varphi.$$

If φ is a k-form, and if $\mathbf{t}$ is a k-vector, then $x \mapsto \varphi(x) \cdot \mathbf{t}$ is a scalar-valued function. Now let ν be a Borel measure on $\mathbf{R}^N$, and let $\mathbf{t}(x)$ denote a field of k-vectors on $\mathbf{R}^N$ which is Borel measurable, and which is locally integrable with respect to ν. Then the current $T = \mathbf{t}\,\nu$ is defined by the action

$$\mathscr{D}^k \ni \varphi \mapsto \langle T, \varphi \rangle := \int \varphi(x) \cdot \mathbf{t}(x)\, \nu(x).$$

We see that currents of the form $T = \mathbf{t}\,\nu$ have the property that for each compact set K, there is a constant c_K such that

$$|\langle T, \varphi\rangle| \le c_K \sup_{x\in K} |\varphi(x)|$$

for every $\varphi \in \mathscr{D}^k$ with support in K. Currents with this property are said to be *represented by integration*. Recall that the dual space of the continuous functions may be identified with a space of measures. In a similar way, the currents represented by integration may be identified with polar representations $\mathbf{t}\,\nu$.

Currents of integration may be represented in this form, which is called the *polar representation*. Namely, we let dS_M denote the Euclidean k-dimensional surface measure on M, and at each $x \in M$, we let $\mathbf{t}(x)$ denote the k-vector which has unit Euclidean length, and which gives the orientation of the tangent space of M at x. Thus we have

$$\int_M \varphi = \int_M \mathbf{t}(x) \cdot \varphi(x)\, dS_M(x)$$

Examples. Let $\mathbf{C}^2_{x,y}$, and write $x = t + is$ and $y = u + iv$. Let dS denote the 3-dimensional surface measure on the set $\mathbf{R} \times \mathbf{C} = \{s = 0\}$. We give some examples of currents which correspond to different ways in which we might "laminate" or "stratify" $\mathbf{R} \times \mathbf{C} = \{s = 0\}$. Let us define

$T_0 := dS$. This is a measure, and as a current it has dimension 0.

$T_1 := \partial_x\, dS$. Here ∂_x denotes the 1-vector dual to dx. We may fill the set $\mathbf{R} \times \mathbf{C}$ by the disjoint family of lines $\mathbf{R} \times \{y_0\}$ for all $y_0 \in \mathbf{C}$. The current T_1 may be described as the family of 1-dimensional currents of integration $[\mathbf{R} \times \{y_0\}]$, averaged with respect to area measure on $\mathbf{C}_y$. Thus, as a current, T_1 has dimension 1. We may write $T_1 = \int_{\mathbf{C}} du\, dv\, [\mathbf{R} \times \{y = u + iv\}]$, where we interpret the "current-valued" integral as follows:

$$\langle T_1, \xi\rangle = \int_{\mathbf{C}} du\, dv \left(\int_{\mathbf{R}_x \times \{y=u+iv\}} \xi \right).$$

$T_2 := \partial_u \wedge \partial_v\, dS$. This is a 2-dimensional current which is in some sense dual to T_1. Namely, we consider $\mathbf{R} \times \mathbf{C}$ to be filled by the disjoint family of complex lines $\{t_0\} \times \mathbf{C}$. The current T_2 acts by averaging the currents of integration $[\{t_0\} \times \mathbf{C}]$ with respect to dt. In notation analogous to what we used for T_1, we may write

$$T_2 = \int_{t\in\mathbf{R}} dt\, [\{x = t\} \times \mathbf{C}].$$

$T_3 := \partial_t \wedge \partial_u \wedge \partial_v\, dS = [\mathbf{R} \times \mathbf{C}]$ is the (3-dimensional) current of integration. We may interpret the space of $N-k$-forms, $\mathscr{A}^{N-k}$, as currents. We define $\iota : \mathscr{A}^{N-k} \to \mathscr{D}'_k$ from $N-k$-forms to currents of *degree* $N-k$ or *dimension* k by considering them as densities with respect to the current of integration:

$$\mathscr{A}^{N-k} \ni \eta \mapsto \iota\eta := \eta\, [\mathbf{R}^N] \in \mathscr{D}'_k,$$

which acts on k-forms as

$$\varphi \mapsto \langle \varphi\eta, [\mathbf{R}^N]\rangle = \int_{\mathbf{R}^N} \varphi \wedge \eta.$$

If we say that a current T is *smooth*, we mean that there is a smooth form η such that $T = \iota\eta$. Note that if f is a diffeomorphism, then we may pull back in two different ways: $f^* : \mathscr{D}^k \to \mathscr{D}^k$ or $(f^{-1})^* : \mathscr{D}^k \to \mathscr{D}^k$. The second operation is equivalent to a push-forward f_*. Taking adjoints, we may push forward $f_* : \mathscr{D}'_k \to \mathscr{D}'_k$. And pushing currents forward by f^{-1} is equivalent to pulling them back by f. These operations are natural with respect to taking currents of integration:

$$f^*[M] = [f^{-1}M], \quad f_*[M] = [fM].$$

There is also the exterior derivative $d : \mathscr{D}^k \to \mathscr{D}^{k+1}$, and its adjoint $d : \mathscr{D}'_{k+1} \to \mathscr{D}'_k$. The operator d on currents is a natural generalization of the boundary operator, since if M is a compact manifold-with-boundary, the statement of Stokes theorem becomes simply $d[M] = [\partial M]$.

1.8 Currents on $\mathbf{C}^N$ *and Especially* $\mu^\pm$

The forms $dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}$ are said to have bidegree p,q. We may grade the k-forms according to their bidegrees: $\mathscr{D}^k = \bigoplus_{p+q=k} \mathscr{D}^{p,q}$. We may apply the complex conjugation operator to p,q-forms and obtain forms of bidegree q,p. A form α is *real* if $\bar{\alpha} = \alpha$. If we write $z = x + iy$, then $\frac{i}{2} dz \wedge d\bar{z} = dx \wedge dy$, so this is a real form. On the other hand, if $\beta = dz \wedge d\bar{z}$, then $\bar{\beta} = -\beta$.

A current T is *positive* if it is of type (p,p), and if $\langle T, \bigwedge_{j=1}^{p} \frac{i}{2} \alpha_j \wedge \bar{\alpha}_j \rangle \geq 0$ for all $\alpha_j \in \mathscr{D}^{1,0}$.

Theorem 1.32. *A current of integration $[M]$ over an oriented, 2-dimensional submanifold of $\mathbf{C}^2$ is real if and only if M is complex. In this case $[M]$ is positive if and only if it is real.*

Exercise: Let M be a real, oriented submanifold of $\mathbf{C}^N$. Under what conditions on M is the current $[M]$ real?

Theorem 1.33. *If T is a positive current, then T is represented by integration.*

We may split the operator d into its parts of type 1,0 and 0,1 respectively, which produces the splitting $d = \partial + \bar{\partial}$. Thus we have $\partial = \sum_j \partial_{z_j} dz_j$. We define $d^c = i(\bar{\partial} - \partial)$. In dimension 1, we write $z = x + iy$, and we have

$$dd^c = 2i\partial\bar{\partial} = 2i dz \wedge d\bar{z}\, \partial_z \partial_{\bar{z}} = 4 dx \wedge dy\, \partial_z \partial_{\bar{z}} = dx \wedge dy\, \Delta$$

where $\Delta = \partial_x^2 + \partial_y^2$ is the Laplacian. In dimension $N > 1$, the operator dd^c is not a scalar object, and when it acts on functions, it is essentially equal to the $n \times n$ hermitian matrix of second derivatives $\left(\partial_{z_j} \partial_{\bar{z}_k}\right)$.

The currents we are interested are

$$\mu^+ := \frac{1}{2\pi} dd^c G^+, \;\; \mu^- := \frac{1}{2\pi} dd^c G^-.$$

These are positive, closed currents which satisfy $f^*\mu^+ = d \cdot \mu^+$ and $f^*\mu^- = d^{-1} \cdot \mu^-$. Further, we have

Proposition 1.34. *The support of μ^+ is J^+.*

Proof. We have seen that G^+ is pluri-harmonic on the complement of J^+, so the support of μ^+ is contained in J^+. On the other hand, if $p \in J^+$, then we have $G^+(p) = 0$, but $G^+ > 0$ at points of $\mathbf{C}^2 - K^+$ arbitrarily close to p. Since $G^+ \geq 0$, it follows from the maximum principle that G^+ cannot be pluriharmonic in a neighborhood of p. On the other hand, if p is not in the support of dd^cG^+, then G^+ must be pluriharmonic there. □

Of course, the defining property of currents is that they act on test forms. Another operation that is basic for μ^+ is taking slice measures. That is, suppose that $M \subset \mathbf{C}^2$ is a 1-dimensional complex submanifold. We may define the slice measure

$$\mu^+\Big|_M := \frac{1}{2\pi} dd^c_M G^+\Big|_M$$

That is, we restrict G^+ to the manifold M and we take dd^c intrinsically to M. The continuity of G^+ means that the slice measure depends continuously on M.

If f is a hyperbolic Hénon map, we can present a heuristic picture to show the connection between the geometry of the stable manifolds and the current μ^+. By the Stable Manifold Theorem, the stable manifolds $\mathscr{W}^s = \mathscr{W}^s(J)$ form a lamination in a neighborhood of J. That is there are local charts U for which $\mathscr{W}^s \cap U$ is homeomorphic to $A \times D$, where A is closed, and D is a disk. Figure 5 shows two complex manifold M_1 and M_2 which are transverse to $\mathscr{W}^s$. The set A_j is $M_j \cap \mathscr{W}^s$. For $\alpha \in A$, we define the local stable disk D^s_α to be the component of $W^s(\alpha) \cap U$ containing α. There is also a *holonomy* map χ which takes the intersection points $D^s_\alpha \cap M_1$ to $D^s_\alpha \cap M_2$. A property of the current μ^+ is that the holonomy map transports the slice measure $\mu^+|_{M_1}$ to the slice measure $\mu^+|_{M_2}$. So, within this flow box, all slice measures are equivalent via the holonomy map.

We can use this to generate the local pieces of μ^+. Since D^s_α has locally finite area it defines a current $[D^s_\alpha]$. We define a current by integrating these currents of integration with respect to the slice measure:

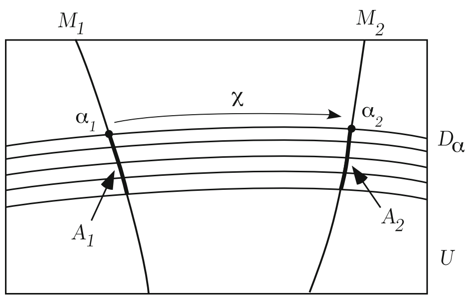

Fig. 5 Holonomy map χ

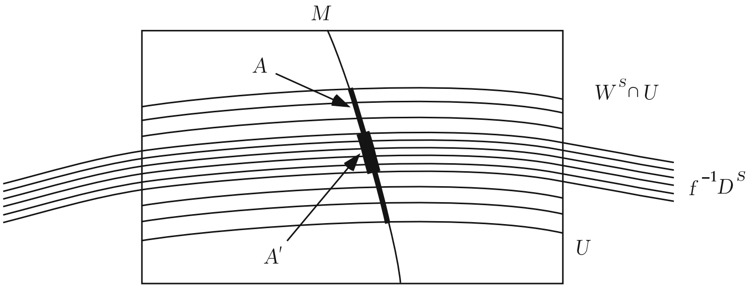

Fig. 6 Expansion in terms of transversal measure

$$\int_{\alpha \in A} [D^s_\alpha]\ \mu^+|_M(\alpha)$$

where we use the notation $\mu^+|_M(\alpha)$ in the integral to mean that we are integrating with respect to the variable α and the measure $\mu^+|_M$. It is shown in [BS1] that the restriction $\mu^+ \llcorner U$ can be written in this manner.

Now let us interpret the relationship between "expansion" of the current μ^+ and a more directly geometrical sort of expansion. First, what is the effect of pulling back the restriction $\mu^+ \llcorner U$ under f^*? If we write $\mu^+ \llcorner U$ in the laminar form above, we may pull the currents of integration to obtain

$$f^*(\mu^+ \llcorner U) = \int_{\alpha \in A} [f^{-1} D^s_\alpha]\ \mu^+|_M(\alpha)$$

Now let us restrict the pullback to U again, and let us suppose that the box U maps across itself as in Figure 6. Then we find that the pullbacks $f^{-1} D^s_\alpha$ will again be disks of $\mathscr{W}^s \cap U$, and these will correspond to a subset of the transversal $A' \subset M$. Thus we may write

$$(f^* \mu^+ \llcorner U) \llcorner U = \int_{\alpha' \in A'} [f^{-1} D^s_{\alpha'}]\ \lambda'(\alpha')$$

where λ' is a measure on A' determined by the pullback. The total mass of λ' on A' is the same as the total mass of $\mu^+|_M$ on A. The fact that $f^*\mu^+ = d \cdot \mu^+$ means the amount of transversal mass of A' will be d^{-1} times the transversal mass of A. Thus the expansion is expressed through the contraction of the "thickness" of bands of stable disks as they are mapped under f^{-1}. Thus the contraction is not precisely geometric, it is expressed by the decrease of measure for transversals.

Now we develop our intuition with some non-dynamical currents. Let us look at some simpler examples of currents obtained by taking dd^c of a psh function. First let us define for $z \in \mathbf{C}$:

$$L(z) = \log|z|, \ \ L_\varepsilon(z) = \log|z|, \text{ if } |z| \geq \varepsilon, \text{ and } \frac{|z|^2}{2\varepsilon^2} + C_\varepsilon, \text{ if } |z| \leq \varepsilon.$$

Thus L_ε is globally C^1 and piecewise C^2. Thus we may compute:

$$dd^c L_\varepsilon = \frac{2}{\varepsilon^2} \cdot \text{area measure}|_{|z|<\varepsilon}.$$

Since L_ε decreases to L, it converges in the topology of $\mathscr{D}_2'$, so $dd^c L_\varepsilon \to dd^c L$ as $\varepsilon \to 0$. Since $dd^c L_\varepsilon$ is a measure with total mass 2π, we see that we obtain

$$dd^c L = 2\pi\delta_0.$$

Now let us apply a similar argument to the function $L_\varepsilon(x)$ on $\mathbf{C}^2$. We find, this time, that

$$dd^c L_\varepsilon = \frac{2}{\varepsilon^2} \int_{|x|<\varepsilon} dA(x)\,[\{x\} \times \mathbf{C}].$$

That is, we obtain an average of the currents of integration over $\{x\} \times \mathbf{C}$. Letting $\varepsilon \to 0$, we obtain the Poincaré-Lelong formula in the case of a line:

$$\frac{1}{2\pi} dd^c \log|x| = [\{x = 0\}].$$

Now we may interpret this for submanifolds which are not necessarily linear. Suppose that $M = \{h = 0\}$, where h is holomorphic, and the gradient of h does not vanish on M. Observe that if $\tilde{h}$ is any other such function, then we have $\tilde{h} = \alpha h$, where α is holomorphic and nonvanishing in a neighborhood of M. It follows that $\log|\tilde{h}| = \log|h| + \log|\alpha|$, and $dd^c \log|\alpha| = 0$. Thus we have $dd^c \log|h| = dd^c \log|\tilde{h}|$.

At each point of $\{x = 0\}$, there is a (locally) biholomorphic map φ of $\mathbf{C}^2$ with $\varphi : \{x = 0\} \to M$. Thus $\varphi^* h = h \circ \varphi$ is a defining function for $\{x = 0\}$, so we have $\varphi^* h = x \cdot \alpha$ for some invertible holomorphic function α. Thus we have $dd^c \log|h| = \varphi_* dd^c \log|\varphi^* h| = \varphi_* dd^c \log|x \cdot \alpha| = \varphi_* dd^c \log|x| = [M]$. We summarize this as:

Theorem 1.35. *(Poincaré-Lelong formula)*

$$\frac{1}{2\pi} dd^c \log|h| = [\{h = 0\}].$$

Exercises: (1) Let $\chi(x,y) = \max\{Re\, x, 0\}$. Show that $\frac{1}{2\pi} dd^c \chi = T_2$. (Hint: You may deal with this like the case with L above. Consider the maps $\chi_\varepsilon(t) = 0$ for $t \le 0$, $\chi_\varepsilon(t) = t^2/(2\varepsilon)$ for $0 < t < \varepsilon$, and $\chi_\varepsilon(t) = t - \varepsilon/2$ for $t \ge \varepsilon$.)
(2) Use an exponential change of variable on (1) to obtain:

$$dd^c \log^+|x| = \int_0^{2\pi} d\theta\,[\{x = e^{i\theta}\} \times \mathbf{C}].$$

Theorem 1.36. *The currents $d^{-n}[f^{-n}\{y = 0\}]$ converge to μ^+ as $n \to +\infty$.*

Proof. By Theorem 1.28, we know that $\lim_{n\to\infty} d^{-n} \log|y_n| = G^+$. If we apply $\frac{1}{2\pi} dd^c$ to both sides of this equation, we obtain the Theorem from the Poincaré-Lelong formula. □

This Theorem gives us a measure-theoretic sense of the location of $f^{-n}\{y=0\}$. Namely, if B is a ball that is disjoint from J^+, we can let M_n denote the number of connected components of $B \cap f^{-n}\{y=0\}$. From the Theorem, we conclude that $d^{-n}M_n \to 0$ as $n \to \infty$.

Notes: Many of the important properties of μ^+ come from its laminar structure, which must be interpreted in a measure-theoretic sense. A good starting place for entering the literature is to look at [Du, §2].

1.9 Convergence to μ^+

Much of the utility of the invariant current μ^+ comes from various convergence theorems. The proofs of these results use some potential theory; the relevant tools have been well developed in [S]. So we will state two results below and refer the reader to [S] for the proofs.

First we consider a pair of complex disks $D \subset \tilde{D}$ such that $\tilde{D}$ contains the closure of D. As we saw in the previous section, we may slice the invariant current μ^- along $\tilde{D}$ to obtain a slice measure $\mu^-|_{\tilde{D}}$. The main convergence theorem is:

Theorem 1.37. *Let D and $\tilde{D}$ be complex disks such that the closure of D is contained in $\tilde{D}$. Suppose that the slice measure $\mu^-|_{\tilde{D}}$ puts no mass on ∂D, and let c be the measure $\mu^-|_{\tilde{D}}(D)$. Then the normalized pullbacks $d^{-n}f^{*n}[D]$ converge in the weak sense of currents to $c\mu^+$.*

For an alternative formulation: if we replace $[D]$ by $\chi[D]$, where χ is a test function with support in D, then we may omit the hypothesis that ∂D has no mass for the slice measure, and we use the value $c = \int_D \chi\, \mu^-|_{\tilde{D}}$.

We get several consequences from this theorem.

Corollary 1.38. *Let p be a saddle point, and let $W^s(p)$ be its stable manifold. Then J^+ is the closure of $W^s(p)$. In particular, J^+ is connected.*

Proof. We have seen already that $W^s(p) \subset J^+$. Let D^s be a disk inside $W^s(p)$. We may assume that $p \in D^s$. It follows that $\mu^-|_{D^s}$ is not the zero measure, so we may choose χ so that $\int_{D^s} \chi\, \mu^-|_{\tilde{D}} > 0$. Each pullback $d^{-n}f^{n*}(\chi[D^s])$ has support in $W^s(p)$. Since the sequence of currents converges to μ^+, and J^+ is the support of μ^+, the Theorem follows. □

Proof of Theorem 1.22. Let D denote a linear complex disk inside U_-. We may choose D so that it intersects both J^- and the complement of K^-. Thus the slice $\mu^-|_D$ is not the zero measure, and we may choose a test function χ such that $c > 0$. Now since the pullbacks converge to μ^+, their supports must be dense in J^+, so we see that for some large n, we will have $f^{-n}D \cap U_+ \neq \emptyset$. This proves the theorem. □

We may wedge the currents μ^+ and μ^- together to obtain a measure μ. In order to define a measure, it suffices to show how to integrate a function χ with respect to μ. If χ is smooth, then we can define

$$\int \chi\, \mu := \int (dd^c\chi) \wedge G^+\mu^- \tag{5}$$

which would be exactly what you would obtain G^+ and G^- were smooth and you integrate by parts: $\int \chi dd^c G^+ \wedge dd^c G^- = \int \chi dd^c(G^+ \wedge \mu^-) = \int (dd^c \chi) \wedge (G^+ \wedge \mu^-)$. This defines μ as a distribution. But since $\mu^\pm$ are a positive currents, we see that (*) defines μ as a positive distribution, and thus a measure.

A variant of the convergence theorem above deals with the normalized pullbacks $d^{-n} f^{n*}(\chi\mu^+) = \chi(f^n)\mu^+$:

Theorem 1.39. *Let χ be a test function, and let $c = \int \chi\, \mu$. Then*

$$\lim_{n\to\infty} d^{-n} f^{*n}(\chi\mu^+) = \lim_{n\to\infty} \chi(f^n)\mu^+ = c\mu^+.$$

A measure ν is said to be *mixing* with respect to a transformation f if $\lim_{n\to\infty} \nu(A \cap f^{-n}B) = \nu(A)\nu(B)$ for all Borel sets A and B. For this, it suffices to show that for every pair of smooth functions φ and ψ, we have

$$\lim_{n\to\infty} \int \varphi(f^n)\psi\,\mu = \int \varphi\,\mu \int \psi\,\mu$$

Corollary 1.40. *The measure μ is mixing (and thus ergodic).*

Proof. We have $\int \varphi(f^n)\,\psi\,\mu = \int (\varphi(f^n)\mu^+) \wedge (\psi\mu^-)$, so as $n \to \infty$, the right hand integral converges to $\int (c\mu^+) \wedge \psi\mu^- = c\int \psi\mu^+ \wedge \mu^- = c\int \psi\,\mu$, and the constant is $c = \int \varphi\,\mu$. □

2 Rational Surfaces

2.1 Blowing Up

In the second section of these notes, we will consider automorphisms of a compact, rational surface M. Given an automorphism $f \in Aut(M)$, we can look at its pullback on cohomology $f^* \in GL(H^2(M))$. The dynamical degree is then defined as

$$\lambda(f) := \lim_{n\to\infty} ||f^{n*}||^{1/n}.$$

A surface M' that is birationally equivalent to M may be topologically different: for instance, $H^2(M')$ and $H^2(M)$ can have different dimensions, and so the induced maps on cohomology will not be conjugate. However, in [DF] it is shown that $\lambda(f)$ is the same for all birationally equivalent maps.

Here we will focus on automorphisms for which $\lambda(f) > 1$. Although we will not discuss entropy here, we note that in this case the entropy is $\log(\lambda(f)) > 0$. Much of the theory for Hénon maps can be carried over to the case of automorphisms of compact surfaces. However, the Hénon family of diffeomorphisms themselves will not be part of this section: for a Hénon map H, there is no compact, complex manifold M which compactifies $\mathbf{C}^2$ in such a way that H becomes a homeomorphism of M.

(Exercise: The most obvious first attempt, the one point compactification $\hat{\mathbf{C}}^2$ of $\mathbf{C}^2$, is not a complex manifold.) We note, too, that we do not discuss are the complex 2-tori or the $K3$ surfaces, which are the other possibilities for projective surfaces with automorphisms for which $\lambda(f) > 1$ (see [C1]). In fact, rational surface automorphisms are more abundant than these other two cases, so we will be drawing from a rich family of dynamical systems.

In §1 we used the dynamical classification of $PolyAut(\mathbf{C}^2)$ which shows that the Hénon diffeomorphisms represent the conjugacy classes with positive entropy. On the other hand, it is not easy to determine the set of all rational surfaces that admit nontrivial automorphisms; and given such a surface, it is not easy to determine its automorphisms. In other words, an analogous dynamical classification of rational surface automorphisms is not yet known.

Here we focus on the surfaces that are obtained from $\mathbf{P}^2$ by blow-ups; this focus is justified by a Theorem of Nagata which is stated in §2.7. However, we note that a surface constructed from $\mathbf{P}^2$ by making "generic" blowups will not have any automorphisms except the identity (see [H, K]). Our starting place will be complex projective space $\mathbf{P}^n$, which is $\mathbf{C}^{n+1} - 0$, modulo the equivalence $(x_0,\ldots,x_n) \sim (\lambda x_0,\ldots,\lambda x_n)$ for any $\lambda \in \mathbf{C}$ with $\lambda \neq 0$. We write $[x_0 : \cdots : x_n]$ (square brackets to denote homogeneous coordinates) for the equivalence class. It is classical that $Aut(\mathbf{P}^n) = PGL(n+1,\mathbf{C})$, and $Aut(\mathbf{P}^1 \times \mathbf{P}^1)$ is the group generated by $PGL(2,\mathbf{C}) \times PGL(2,\mathbf{C})$, together with the map $\tau(x,y) = (y,x)$.

The blow-up of a manifold X at a point $p \in X$ is a manifold $\tilde{X}$, together with a projection $\pi : \tilde{X} \to X$ such that the *exceptional fiber* $E := \pi^{-1}p$ is equivalent to $\mathbf{P}^1$, and $\pi : \tilde{X} - E \to X - p$ is biholomorphic. A concrete presentation of this is the space

$$\Gamma = \{(x,y);[\xi : \eta] \in \mathbf{C}^2 \times \mathbf{P}^1 : x\eta = y\xi\}$$

together with the projection π to the first coordinate, so we define the pair (Γ,π) to be the blow-up of $\mathbf{C}^2$ at 0. This construction is essentially local at the center of blowup. Thus we may use this construction of Γ to define the blowup of X at the point p.

Observe that $\pi^{-1} : \mathbf{C}^2 - 0 \to \Gamma$ is the tautological map $\pi^{-1}(x,y) = (x,y);[x : y]$. We may also write $\pi^{-1}(x,y) = (x,y);[1 : y/x] = (x,y);[x/y : 1]$, and we may use these two representations to define local coordinate charts $U' = \mathbf{C}^2_{s,\eta}$ and $U'' = \mathbf{C}^2_{\xi,t}$, where the coordinates are defined via the form that the projection π takes:

$$\pi' : (s,\eta) \mapsto (s,s\eta) = (x,y), \quad \pi'' : (\xi,t) \mapsto (\xi t,t) = (x,y).$$

It follows that $U' \cup U'' = \Gamma$, and $E' := E \cap U' = \{s = 0\}$ is equivalent to $\mathbf{C}$.

We may pull the 2-form $dx \wedge dy$ back to Γ; in the U' coordinate chart, for instance, we have $(\pi')^* dx \wedge dy = ds \wedge d(s\eta) = s\, ds \wedge d\eta$.

We will use the following notational convention: if C is a curve in X, then C will also denote the *strict transform* which is the curve in $\tilde{X}$ which is obtained by taking the closure of $\pi^{-1}(C - p)$ inside $\tilde{X}$. In the blowup, the strict transform of the x-axis $\{y = 0\}$ inside Γ is given by $\{\eta = 0\}$. Thus we use the (s,η) coordinate system if we want to work in a neighborhood of the x-axis, and we use the (ξ,t) coordinate system if we want to work in a neighborhood of the y-axis.

Exercises:

1. Show that Γ is smooth.
2. Let $f : \mathbf{C}^2, 0 \to \mathbf{C}^2, 0$ be a local biholomorphism. Show that f lifts to a locally invertible holomorphic map f_Γ of Γ, and f_Γ maps E to itself E. Show that if f is not locally invertible at 0, then f_Γ is not everywhere holomorphic on E.
3. Let X_1 denote the space obtained by blowing up $\mathbf{P}^2$ at two points, and let X_2 denote the space obtained by blowing up $\mathbf{P}^1 \times \mathbf{P}^1$ at one point. Show that X_1 and X_2 are biholomorphic, and thus $\mathbf{P}^2$ is birationally equivalent to $\mathbf{P}^1 \times \mathbf{P}^1$. (The easiest way to do this is to let the two points of blowup of $\mathbf{P}^2$ be the points where the x- and y- axes cross the line at infinity; and blowup $\mathbf{P}^1 \times \mathbf{P}^1$ at (∞, ∞).)

We define a divisor to be a linear combination $D = \sum c_j D_j$, where D_j is a hypersurface in X. We say that divisors D' and D'' are linearly equivalent if $D' - D''$ is the divisor of a rational function r. That is, $D' - D''$ is equal to the zero set (with multiplicity) of r minus its pole set. We define the Picard group $Pic(X)$ to be the set of divisors modulo linear equivalence.

Proposition 2.1. $Pic(\mathbf{P}^2) \cong \mathbf{Z}$.

Proof. If D is a hypersurface, then $D = \{p([x_0 : x_1 : x_2]) = 0\}$ is defined by a homogeneous polynomial of some degree d. Thus $r = p/x_0^d$ is a well defined rational function on $\mathbf{P}^2$, so we see that as an element of $Pic(\mathbf{P}^2)$, D is equivalent to d times the line $\{x_0 = 0\}$. Similarly, we see that all lines define the same element, and this generates $Pic(\mathbf{P}^2)$. □

Remark: In the proof of the Proposition, we saw that if $H \in Pic(\mathbf{P}^2)$ is the class of a line, then $C \sim d \cdot H \in Pic(\mathbf{P}^2)$, where d is the degree of C.

Theorem 2.2. *If f is an automorphism of $\mathbf{P}^2$, then f must have degree 1.*

Proof. The action f^* on $Pic(\mathbf{P}^2)$ is multiplication by d, where d is the degree of f. If f is invertible, then the action of f^{-1^*} must be multiplication by d_1, where d_1 is the degree of f^{-1}. On the other hand, we must have $1 = id^* = (f \circ f^{-1})^* = dd_1$, so $d = d_1 = 1$, since d and d_1 are both positive integers. □

Now let $\tilde{\mathbf{P}}^2$ denote $\mathbf{P}^2$ blown up at a point p, and let P denote the exceptional fiber. The rational functions on $\tilde{\mathbf{P}}^2$ are defined to be the pullbacks under π of rational functions on $\mathbf{P}^2$, i.e., $Rat(\tilde{\mathbf{P}}^2) := \pi^* Rat(\mathbf{P}^2)$. It follows that $Pic(\tilde{\mathbf{P}}^2)$ is generated by $\tilde{L}$ and P, where $L = \{\ell = \sum a_j x_j = 0\}$ is a line, and $\tilde{L} := \pi^* L = \{\ell \circ \pi = 0\}$ is the pullback. More generally, we consider N distinct points $p_1, \ldots, p_N$ and let X denote the complex manifold obtained by blowing up $\mathbf{P}^2$ at the points p_j, $1 \le j \le N$. We will let $H_X = \pi^* H$ denote the class of a line; if $H \subset \mathbf{P}^2$ which does not contain any p_j, $1 \le j \le N$, then H_X is represented by the strict transform $\tilde{H}$. We let $P_j = \pi^{-1} p_j$ denote the class in $Pic(X)$ of the exceptional fiber. In general, if $H \subset \mathbf{P}^2$ is a line, then $H_X = \tilde{H} + \sum' P_j$, where the sum is taken over the indices j for which $p_j \in H$. By the discussion above, we see that:

Theorem 2.3. *Suppose that $\pi : X \to \mathbf{P}^2$ is obtained by blowing up N distinct points $p_1,\dots,p_N \in \mathbf{P}^2$. If $P_j = \pi^{-1}p_j$ denotes the exceptional fiber, then $Pic(X)$ is generated by H_X and P_j, $1 \le j \le N$.*

Theorem 2.4. *Let X denote $\mathbf{P}^2$ blown up at distinct points $p_1,\dots,p_N$, and consider two elements $T' = D' + \sum a'_j P_j$ and $T'' = D'' + \sum a''_j P_j$ in $Pic(X)$, where D' and D'' denote strict transforms of divisors in $\mathbf{P}^2$. It follows that $T' \sim T'' \in Pic(X)$ if and only if D' and D'' have the same degrees, and $a'_j = a''_j$ for all j.*

Proof. We may replace D' and D'' by $d' \cdot H$ and $d'' \cdot H$, and we may suppose that H is disjoint from the centers of blowup. If $T' \sim T''$, then there is a rational function $r_X = \pi^* r$ whose divisor is $T' - T''$. It follows that the divisor of r is $(d' - d'')H$. Thus we must have $d' - d'' = 0$ and $a'_j - a''_j = 0$. $\square$

2.2 Cohomology

A basic result is that $H^2(\mathbf{P}^2;\mathbf{Z}) \cong \mathbf{Z}$. By the DeRham theorem, we may represent cohomology classes by closed 2-forms. An example of a global closed 2-form may be written loosely as $\omega := dd^c \log(|x_0|^2 + |x_1|^2 + |x_2|^2)$, which we interpret as follows. On the coordinate chart $\mathbf{C}^2_{x,y} \ni (x,y) \mapsto [1 : x : y] \in \mathbf{P}^2$, we write $\omega' = dd^c \log(1+|x|^2+|y|^2)$. On the coordinate chart $\mathbf{C}^2_{t,v} \ni (t,v) \mapsto [t : 1 : v] \in \mathbf{P}^2$, we have $\omega'' = \log(|t|^2 + 1 + |v|^2)$. These coordinates are related by $[1 : x : y] = [t : 1 : v]$, so $t = 1/x$ and $v = y/x$. Thus $\omega' - \omega'' = dd^c \log|x|^2 = dd^c \log|t|^{-2} = 0$ on the overlap $\{x_0 x_1 \neq 0\}$ of these two coordinate charts, so this definition is well-defined globally. *Exercise:* $\log(1+|x|^2+|y|^2)$ is strictly psh on $\mathbf{C}^2$, and thus $\omega > 0$. Find $c > 0$ such that $\int_L c\omega = 1$ for some (or equivalently, every) line $L \subset \mathbf{P}^2$.

In complex dimension 2, there is a duality on $H^2(X;\mathbf{C})$ which is given by $(\alpha,\beta) = \int \alpha \wedge \bar{\beta}$. We will sometimes write this as $\alpha \cdot \beta$ and call it the *intersection product*. The Poincaré duality theorem says that H^2 is self-dual under this pairing. In the case of Kähler manifolds, we have the following Signature Theorem (see Theorem IV.2.14 of [BHPV]).

Theorem 2.5. *Let X be a compact Kähler surface, and let $h_{1,1}$ denote the dimension of $H^{1,1}(X;\mathbf{R}) \subset H^2(X;\mathbf{R})$. Then the signature of the restriction of the intersection product to $H^{1,1}$ is $(1, h_{1,1} - 1)$. In particular, there is no 2-dimensional linear subspace $L \subset H^{1,1}(X;\mathbf{R})$ with the property that $\omega \cdot \omega = 0$ for all $\omega \in L$.*

For a complex curve C, the current of integration $[C]$ defines an element in the dual of H^2, and thus we may consider C as an element of H^2 as well. With c as in the preceding exercise, we see that $c\omega$ represents the class $\langle L \rangle$, where L is any complex line in $\mathbf{P}^2$. Let $C \subset \mathbf{P}^2$ be a curve of degree μ. Then its class in $H^2(\mathbf{P}^2)$ is $C = \mu H$. We see that if $\tilde{\mathbf{P}}^2$ is the blowup of $\mathbf{P}^2$ at a point p, then we may identify $H^2(\tilde{\mathbf{P}}^2)$ and $Pic(\tilde{\mathbf{P}}^2)$. In the sequel, we will find it convenient to use H^2 and Pic interchangeably. But note that this is a special property of blowups of $\mathbf{P}^2$; for other complex manifolds, it is possible for H^2 and Pic to be very different.

Let C_1 and C_2 be complex curves which intersect transversally. A basic result of intersection theory is that the number of intersection points $C_1 \cap C_2$ is equal to the intersection product (C_1, C_2) of their cohomology classes. We note that for general 2-manifolds, the intersection multiplicity is determined by the orientation. That is, if E_1 and E_2 are smooth 2-manifolds in X which intersect transversally, then $E_1 \cdot E_2$ is equal to the total number of intersection points, counted with multiplicity, which is ± 1, depending on the orientation.

Let us consider the case of the exceptional blowup fiber E in Γ, defined in the previous section. We would like to determine $E \cdot E$, but of course E does not intersect itself transversally. We will perturb E to obtain a new surface $\tilde{E}$ which intersects E transversally. We define $\tilde{E}$ in the U' coordinate system as

$$\tilde{E} \cap U' = \{s = \varepsilon\bar{\eta} : |\eta| \le 1\} \cup \{s = 1/\eta : |\eta| \ge 1\}$$
$$\tilde{E} \cap U'' = \{t = \varepsilon : |\xi| \le 1\} \cup \{t = \varepsilon/|\xi|^2 : |\xi| \ge 1\}$$

By the relation $(x,y) = (s, \eta s)$, for instance, we see that

$$\pi\tilde{E} = \{y = \varepsilon|x|^2 : |x| \le 1\} \cup \{(x,\varepsilon) : |x| \le 1\}.$$

We see that this surface is the union of two smooth surfaces, and it is not hard to smoothen $\tilde{E}$ along the circle where the two surfaces intersect. We see that the intersection point of $E \cap \tilde{E}$ occurs at $(0,0) \in U'$. The canonical 2-form which orients E is $i ds \wedge d\bar{s}$. The canonical 2-form which orients $\tilde{E}$ at $\{s - \bar{\eta} = 0\}$ is $i d(s-\bar{\eta}) \wedge \overline{d(s-\bar{\eta})}$. We wedge these together and find $i ds \wedge d\bar{s} \wedge i d\bar{\eta} \wedge d\eta$, which is a negative multiple of the canonical orientation 4-form on X. Thus we have the intersection number $E \cdot \tilde{E} = -1$.

The cohomology class of $E \in H^2$ is represented by the current of integration $[E]$, which is a positive, closed current. However, if ω is a smooth 2-form which represents the class of $E \in H^2$, then $\int \omega^2 = -1$, so we cannot have $\omega \ge 0$ everywhere. In other words, $[E]$ cannot be approximated by a positive, smooth form.

Theorem 2.6. *Suppose that $\pi : \tilde{X} \to X$ is the blowup of X at the center p. Suppose that C is a smooth curve in X containing p, and suppose that $\tilde{C}$ represents the strict transformation of C inside $\tilde{X}$. Then $\tilde{C}^2 = C^2 - 1$.*

Proof. Since C is smooth, the curves $\tilde{C}$ and P are smooth and intersect transversally, with $\tilde{C} \cdot P = 1$. Thus

$$C^2 = (\pi^* C)^2 = (\tilde{C} + P)^2 = \tilde{C}^2 + 2\tilde{C} \cdot P + P^2 = \tilde{C}^2 + 2 - 1,$$

which gives us what we wanted. □

Now let us suppose that C is a curve of degree μ, that $C \cap \{p_1, \ldots, p_N\} = p_1$ and that C is smooth at p_1. Now $\pi^* C = C + P_1 \in H^2(X)$. We conclude, then, that $C = \mu H - P_1$. Conversely, if C is any curve whose cohomology class is $\mu H - P_1$, then C is a curve of degree μ, which contains p_1 with multiplicity 1. "Containing p_1

with multiplicity 1" means, in particular, that C is regular at p_1. In a similar manner, we see that if C is a curve with the cohomology class $H - P_1 - P_2$, then C must be the line $p_1 p_2$.

If X is a surface obtained by repeated blowups, the total cohomology is given by $H^*(X;\mathbf{C}) = H^0(X;\mathbf{C}) \oplus H^{1,1}(X;\mathbf{C}) \oplus H^4(X;\mathbf{C})$. If f is a holomorphic map, then the total map f^* acts on each of these factors. We have $H^0(X;\mathbf{C}) \cong \mathbf{C}$, and $f^*|_{H^0} = 1$. Similarly, the dimension of H^4 is the number of connected components (which is equal to 1), and $f^*|_{H^4}$ is multiplication the mapping degree of f, which is 1 in the case of an automorphism. Thus the Lefschetz Fixed Point Formula takes the form:

Theorem 2.7. *If X is obtained from $\mathbf{P}^2$ by iterated blowups and if each f^n has isolated fixed points, then*

$$Per_n = 2 + trace(f^{*n})$$

where f^* denotes the restriction of f^* to $H^{1,1}$, and Per_n denotes the number of solutions of $\{p \in X : f^n(p) = p\}$, counted with multiplicity.

2.3 Invariant Currents and Measures

Let f be an automorphism of a compact Kähler surface X. Since $f^* \in GL(H^{1,1};\mathbf{Z})$, the determinant of f^* must be ± 1. The pullback and push-forward preserve the intersection product: $(\omega, \eta) = (f^*\omega, f^*\eta) = (f_*\omega, f_*\eta)$. In fact, the push-forward and pullback are adjoint: $f^*\alpha \cdot \beta = \alpha \cdot f_*\beta$. Further, $(f^{-1})^* = (f^*)^{-1}$. And since $\lim_{n\to\infty} ||f_*^n||^{1/n} = \lim_{n\to\infty} ||f^{*n}||^{1/n}$, we have $\lambda(f) = \lambda(f^{-1})$.

Theorem 2.8. *Let $f \in Aut(X)$ be an automorphism of a Kähler manifold with $\lambda(f) > 1$. Then λ is an eigenvalue of f^* with multiplicity 1, and it is the unique eigenvalue with modulus > 1.*

Proof. Let $\omega_1, \ldots, \omega_k$ denote the eigenvectors for f^* for which the associated eigenvalues μ_j has modulus > 1. For $1 \le j \le k$ we have

$$(\omega_j, \omega_k) = (f^*\omega_j, f^*\omega_k) = \mu_j \bar{\mu}_k (\omega_j, \omega_k),$$

so $(\omega_j, \omega_k) = 0$. Letting L denote the linear span of $\omega_1, \ldots, \omega_k$, we see that each element $\omega = \sum c_j \omega_j \in L$ satisfies $(\omega, \omega) = 0$. By the Signature Theorem, it follows that the dimension of L is ≤ 1. On the other hand, since $\lambda(f) > 1$, L is spanned by a unique nontrivial eigenvector. If ω has eigenvalue μ, then $\bar{\omega}$ has eigenvalue $\bar{\mu}$, so we must have $\mu = \bar{\mu} = \lambda$.

Now we claim that λ has multiplicity one. Otherwise there exists θ such that $f^*\theta = \lambda\theta + c\omega$. In this case,

$$(\theta, \omega) = (f^*\theta, f^*\omega) = (\lambda\theta + c\omega, \lambda\omega) = \lambda^2(\theta, \omega),$$

so $(\theta, \omega) = 0$. Similarly, we have $(\theta, \theta) = 0$, so by the Signature Theorem again, the space spanned by θ and ω must have dimension 1, so λ is a simple eigenvalue. □

Corollary 2.9. *If η is an eigenvalue of f^*, then either $\eta = \lambda, \lambda^{-1}$, or $|\eta| = 1$.*

Proof. We have seen that $\lambda(f)$ is the only eigenvalue of modulus > 1. Now we know that $(f^*)^{-1} = (f^{-1})^*$, so if η is an eigenvalue of f^*, then η^{-1} is an eigenvalue of $(f^{-1})^*$. Applying the Theorem to f^{-1}, we conclude that λ is the only eigenvalue for $(f^{-1})^*$ which is > 1. □

Let χ_f denote the characteristic polynomial of f^*. It follows that χ_f is monic, and the constant term (the determinant of f^*, an invertible matrix) is ± 1. Let ψ_f denote the minimal polynomial of λ. By the Theorem, we see that except for λ and $1/\lambda$, all zeros of χ_f (and thus all zeros of ψ_f) lie on the unit circle. Such a polynomial ψ_f is called a *Salem polynomial*, and λ is a *Salem number*. We may factor $\chi_f = C \cdot \psi_f$, where C is a polynomial whose coefficients belong to $\mathbf{Z}$, and the roots of C lie in the unit circle. It follows by elementary number theory that the zeros of C are roots of unity.

The following is a heuristic argument for the existence of a positive, closed invariant 1,1-current. Suppose that X is a Kähler surface and that $f \in Aut(X)$ has $\lambda(f) > 1$. Let ω^+ be a positive, smooth cohomology class which is an eigenvector of λ. By this we mean that there is a smooth form ω^+ such that the cohomology class is expanded

$$\{f^*\omega^+\} = f^*\{\omega^+\} = \lambda\{\omega^+\}.$$

Thus $f^*\omega^+ - \lambda\omega^+$ is cohomologous to zero, and thus there is a smooth function γ^+ such that

$$\frac{1}{\lambda} f^*\omega^+ - \omega^+ = dd^c\gamma^+.$$

If we can take ω^+ to be poisitive, then $f^*\omega^+$ will be positive, and we see that γ^+ is essentially pluri-subharmonic, i.e., $dd^c\gamma^+ + \omega^+ \geq 0$. Applying $\lambda^{-1}f^*$ repeatedly, we find

$$\begin{aligned}
\frac{1}{\lambda^2} f^{*2}\omega^+ - \frac{1}{\lambda} f^*\omega^+ &= \frac{1}{\lambda} dd^c f^*\gamma^+ = \frac{1}{\lambda} dd^c\gamma^+ \circ f \\
\frac{1}{\lambda^n} f^{*n}\omega^+ - \frac{1}{\lambda^{n-1}} f^{*(n-1)}\omega^+ &= \frac{1}{\lambda^{n-1}} dd^c\gamma^+ \circ f^{n-1}.
\end{aligned}$$

If we define

$$g^+ = \sum_{n=0}^{\infty} \frac{\gamma^+ \circ f^n}{\lambda^n}, \quad T^+ = \omega^+ + dd^c g^+,$$

then we see that g^+ is continuous, since the defining series converges uniformly. Further, T^+ is a positive, closed current with the invariance property $f^*T^+ = \lambda T^+$. We may apply the same argument with f^{-1} and the eigenvalue λ^{-1} obtain a positive, closed current T^- with the property that $(f^{-1})^*T^- = \lambda T^-$. We may obtain an invariant measure $\mu := T^+ \wedge T^-$.

Notes The currents $T^{\pm}$ and measure μ have been shown to have many of the same properties that were found for the Hénon family (see [C2, Du]). Much of this theory may be carried over to more general meromorphic surface mappings; one recent work in this direction is [DDG1–3].

2.4 Three Involutions

Let us apply the preceding discussion to three birational maps that are involutions. These are quadratic maps, presented in the order of increasing degeneracy. All three can be turned into regular involutions after blowing up. The degree of degeneracy will correspond to the depth of blowup that is necessary to remove the exceptional curves and indeterminate points. The linear fractional recurrences, discussed in the following section, are of the form $A \circ J$, where A is linear, and J is the Cremona inversion, our first involution. We note, too, that the general quadratic Hénon map is also given by composing the third involution L with an affine map:

$$(y, y^2 + c - \delta x) = L \circ A, \quad A(x,y) = (y, -\delta x + -c).$$

These involutions play a basic role in the classification of the Cremona group of birational maps of the plane. Namely, if f is a quadratic, birational map of the plane, then f is linearly conjugate to a map of the form $A \circ \phi$, where ϕ is one of the involutions studied in this section (see [CD]). The analogous classification of the cubic Cremona transformations, which is more complicated, is also given in [CD].

The first of these involutions is the Cremona inversion J of $\mathbf{P}^2$ which is defined by

$$J[x_0 : x_1 : x_2] = [J_0 : J_1 : J_2] = [x_0^{-1} : x_1^{-1} : x_2^{-1}] = [x_1x_2 : x_0x_2 : x_0x_1].$$

It is evident that $J = J^{-1}$. We set $p_0 = [1:0:0]$, $p_1 = [0:1:0]$, and $p_2 = [0:0:1]$, and let L_j, $0 \le j \le 2$, denote the side of the triangle $p_0p_1p_2$ which is opposite p_j. We let X denote the manifold obtained by blowing up $\mathbf{P}^2$ at the points p_j, $0 \le j \le 2$. Figure 7 shows that we may visualize this as an inversion in a triangle sending $p_j \leftrightarrow \Sigma_j$ for $j = 0,1,2$.

Exercise: Let $J_X : X \to X$. Show that J_X is holomorphic and maps $L_j \leftrightarrow P_j$ for $0 \le j \le 2$.

In $H^2(X)$, we have $L_0 = H - P_1 - P_2$, etc. Specifically, if we pull back a line $\sum a_j x_j = 0$, then we have $\sum a_j J_j = 0$, which is a quadric which contains all three points p_j, $0 \le j \le 2$. Thus we see that $J^*H = 2H$ on $H^2(\mathbf{P}^2)$, and

$$J_X^* H_X = 2H_X - P_0 - P_1 - P_2 \in H^2(X).$$

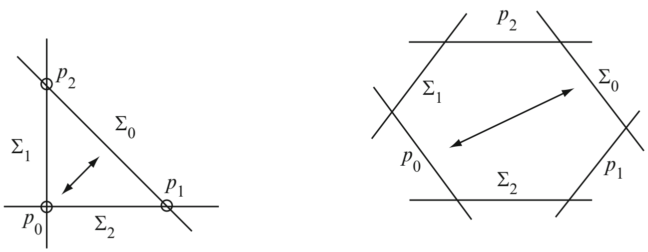

Fig. 7 Inversion in a triangle

By the exercise above, we have $J_X^* P_j = L_j = H_X - \sum_{i \neq j} P_i$. So with respect to the ordered basis $\langle H_X, P_0, P_1, P_2 \rangle$, we find that J_X^* is represented by the matrix

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix} \tag{6}$$

The second involution. The next involution is $K(x,y) = (y^2/x, y)$, which in homogeneous coordinates is given by

$$K[x_0 : x_1 : x_2] = [x_0x_1 : x_2^2 : x_1x_2].$$

We observe that K is quadratic, and its jacobian determinant is $-2x_1x_2^2$. Thus the exceptional locus consists of the y-axis $A_Y = \{x_2 = 0\}$ and the x-axis $A_X = \{x_1 = 0\}$, which has multiplicity 2. They map according to $K: \ A_Y \to p_0 := [1:0:0]$, and $A_X \to p_1 := [0:1:0]$. Now let us construct the space $\pi_1 : \mathscr{X}_1 \to \mathbf{P}^2$ which is the blow up the points p_1 and p_2. The induced map is then $K_{\mathscr{X}_1} = \pi_1^{-1} \circ K \circ \pi_1$.

Exercise: Show that neither A_Y nor P_1 is exceptional for $K_{\mathscr{X}_1}$. In fact they map: $A_Y \leftrightarrow P_1$. One choice for local coordinates at P_1 is $\pi_1(t,\eta) = [t:1:t\eta] = [x_0 : 1 : x_2]$, so $\pi_1^{-1}(x_0, x_2) = (t = x_0, \eta = x_2/x_0)$. Let us look at the behavior of $K_{\mathscr{X}_1}$ on P_2 and A_X. We use the coordinate chart (ξ, s), with projection $\pi_1(\xi, s) = [1 : \xi s : s] \in \mathbf{P}^2$. Thus $\pi_1^{-1}[x_0 : x_1 : x_2] = (\xi = x_1/x_2, s = x_2/x_0)$. In this chart, the blowup fiber is $P_2 = \{s = 0\}$, and $A_X = \{\xi = 0\}$. In this chart we have $K_{\mathscr{X}_1} : (\xi, s) \to [1 : \xi s : s] \to [\xi s : s^2 : \xi s^2] = [1 : s/\xi : s] = [x_0 : x_1 : x_2]$. Thus $\{s = 0\}$ maps to $p_2 = [1:0:0]$. If we follow this map by π_1^{-1}, then we have $K_{\mathscr{X}_1} : (\xi, s) \to (\xi^{-1}, s)$. In other words the mapping $K_{\mathscr{X}_1}|_{P_2}$ of P_2 to itself is given by $\xi \mapsto \xi^{-1}$.

Now we look at the behavior of $K_{\mathscr{X}_1}$ on the set A_X. Since A_X maps to p_2, we look at the map $\pi_1^{-1} \circ K$. We see that this is given by

$$[1 : x : y] \to [x_0 = x : x_1 = y^2 : x_2 = xy] \to (\xi = x_1/x_2 = y/x, s = x_2/x_0 = y)$$

Thus $A_X \to p_3 := (\xi = 0) \in P_2$. Now we create the space $\pi_2 : \mathscr{X}_2 \to \mathscr{X}_1$ by blowing up the point p_3. It is an exercise like the one above to show that A_X is not exceptional for the induced map $K_{\mathscr{X}_2} = \pi_2^{-1} \circ \pi_1^{-1} K \circ \pi_1 \circ \pi_2$. Further, $K_{\mathscr{X}_2}$ has no exceptional curves and no points of indeterminacy. Thus $K_{\mathscr{X}_2} \in Aut(\mathscr{X}_2)$.

Now we find what is happening with the action on the Picard group. We will use $H = H_{\mathscr{X}_2}, P_1, P_2, P_3$ as an ordered basis for $Pic(\mathscr{X}_2)$. Let us write the elements $A_X, A_Y \in Pic(\mathscr{X}_2)$ in terms of this basis. In $\mathbf{P}^2$, we have $A_Y = H$. Pulling this back by π_1^* to $\mathscr{X}_1$, we have $A_Y + P_2 = H \in Pic(\mathscr{X}_1)$. Now we pull this back by π_2^* to $\mathscr{X}_2$ to have $A_Y + P_2 + 2P_3 = H \in Pic(\mathscr{X}_2)$. (Note that p_3 belongs to P_2 and A_Y both, so we have 2 occurrences of P_3.) Reasoning in a similar way, we have $A_X = H \in Pic(\mathbf{P}^2)$, $A_X + P_1 + P_2 = H \in Pic(\mathscr{X}_1)$, and $A_X + P_1 + P_2 + P_3 = H \in Pic(\mathscr{X}_2)$.

From Figure 8, we see that

$$K^* : P_1 \to A_Y = H - P_2 - 2P_3, \quad P_2 \to P_2, \quad P_3 \to A_X = H - P_1 - P_2 - P_3.$$

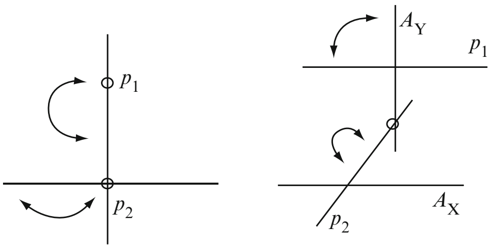

Fig. 8 Involution K and its lifts to $\mathscr{X}_1$ and $\mathscr{X}_2$

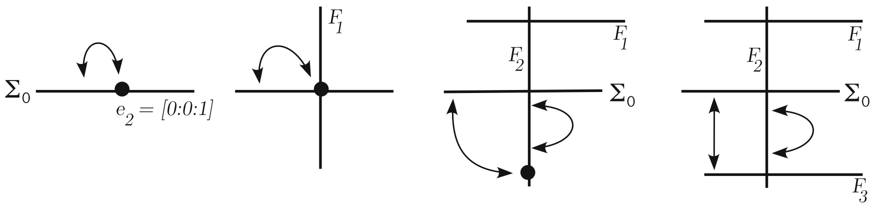

Fig. 9 Involution L and its lifts to $\mathscr{X}_1$, $\mathscr{X}_2$ and $\mathscr{X}_3$

Finally, if H is a line in $\mathbf{P}^2$, then H will intersect both A_X and A_Y. The pullback of H will intersect both points of indeterminacy p_1 and p_2. Looking at Figure 8, we see that in $\mathscr{X}_1$, p_3 blows up to A_X, so the pullback of H will pass through p_3. It follows that we will have $K^*H = 2H - P_1 - P_2 - 2P_3$. Thus we have

$$K^* = \begin{pmatrix} 2 & 1 & 0 & 1 \\ -1 & 0 & 0 & -1 \\ -1 & -1 & 1 & -1 \\ -2 & -2 & 0 & -1 \end{pmatrix} \tag{7}$$

on $Pic(\mathscr{X}_2)$.

Exercise: Define a new ordered basis $\langle H, E_1, E_2, E_3 \rangle$ for $Pic(\mathscr{X}_2)$ by setting $E_1 = P_1$, $E_2 = P_2 + P_3$, $E_3 = P_3$. Show that with respect to this basis, K^* is represented by the matrix (6).

Exercise: Give an analysis of the involutions $K_j(x,y) = (y^j/x, y)$ for $j = 1,2,3,\ldots$ along similar lines. That is, can you construct suitable spaces $\mathscr{X}_j$ such that K_j becomes an automorphism? How does K_j^* act on $Pic(\mathscr{X}_j)$?

The third involution. Finally, we consider $L(x,y) = (x, -y + x^2)$. In homogeneous coordinates, this is written as $L : [x_0 : x_1 : x_2] \to [x_0^2 : x_0x_1 : -x_0x_2 + x_1^2]$. Thus $\Sigma_0 = \{x_0 = 0\}$ is the unique exceptional curve, and $e_2 = [0:0:1]$ is the unique point of indeterminacy.

We will regularize this map L by performing the sequence of three blowups which is shown in Figure 9. Let $\pi_1 : \mathscr{X}_1 \to \mathbf{P}^2$ be the blowup of e_2, and denote the blowup fiber by F_1. We will use the local coordinate chart $\pi_1(\xi, s) = [\xi s : s : 1] = [x_0 : x_1 : x_2]$, so the fiber is $F_1 = \{s = 0\}$, and $\{\xi = 0\} = \Sigma_0$. The inverse is $\pi_1^{-1}[x_0 : x_1 : x_2] = (\xi = x_0/x_1, s = x_1/x_2)$. In this coordinate chart, we have $L_{\mathscr{X}_1} = \pi_1^{-1} \circ L \circ \pi_1(\xi, s) = (\xi, \xi s/(s-\xi))$. The restriction to $F_1 = \{s = 0\}$ is given by $(\xi, 0) \mapsto (\xi, 0)$. Thus each point of F_1 is fixed under $L_{\mathscr{X}_1}$, and F_1 is not exceptional. For the behavior near Σ_0, we map

$$[x_0 : x_1 : x_2] \to \pi_1^{-1}[x_0^2 : x_0x_1 : -x_0x_2 + x_1^2] = (\xi' = x_0/x_1, s' = x_0x_1/(-x_0x_2 + x_1^2)).$$

We see that $\Sigma_0 \to \{\xi = 0\} = F_1 \cap \Sigma_0 \in F_1$.

The next step is to construct $\pi_2 : \mathscr{X}_2 \to \mathscr{X}_1$ by blowing up the point $F_1 \cap \Sigma_0$. We let F_2 denote the exceptional blowup fiber, and we use local coordinates (u, η) with coordinate projection $\pi_2(u, \eta) = (\xi = u, s = u\eta)$. The inverse is $\pi_2^{-1}(\xi, s) = (u = \xi, \eta = s/\xi)$. Thus we use $L_{\mathscr{X}_2} = \pi_2^{-1} \circ \pi_1^{-1} \circ L \circ \pi_1 \circ \pi_2$. We find that $L_{\mathscr{X}_2}(u, \eta) = (u, \eta/(\eta - 1))$, which means that $F_2 = \{u = 0\}$ is mapped to itself in an invertible way. For the image of Σ_0, we see that

$$\pi_2^{-1} \circ \pi_1^{-1} \circ L[x_0 : x_1 : x_2] = (u = x_0/x_1, \eta = x_1^2/(x_1^2 - x_0x_2))$$

Setting $x_0 = 0$ we find that $L_{\mathscr{X}_2} : \Sigma_0 \to \eta = 1 \in F^2$.

Finally, we construct the blowup $\pi_3 : \mathscr{X}_3 \to \mathscr{X}_2$ centered at $\eta = 1 \in F^2$. For this we use local coordinates (t, μ) with coordinate projection $\pi_3(t, \mu) = (t = u, t\mu + 1 = \eta)$. Thus the exceptional fiber F_3 is $\{t = 0\}$. We find that

$$L \circ \pi_3 \circ \pi_2 \circ \pi_1(t, \mu) = [x_0 = t(t\mu + 1) : x_1 = t\mu + 1 : x_2 = \mu].$$

Thus, setting $t = 0$, we see that $L_{\mathscr{X}_3}$ maps $F_3 \ni \mu \to [0 : 1 : \mu]$. We conclude that Σ_0 is no longer exceptional. Since L is an involution, it follows that $F_3 \to \Sigma_0$ is also not exceptional.

Now we discuss the pullback L^* on $Pic(\mathscr{X}_3)$. We use the ordered basis $\langle H, F_1, F_2, F_3 \rangle$. We start by observing that $\Sigma_0 = H$ in $Pic(\mathbf{P}^2)$. Pulling back to $\mathscr{X}_1$, we have $\Sigma_0 + F_1 = H$ in $Pic(\mathscr{X}_1)$. The next center of blowup is $\Sigma_0 \cap F_1 \in \mathscr{X}_1$, so when we pull back, we get 2 copies of F_2, which gives $\Sigma_0 + F_1 + 2F_2 = H$ in $Pic(\mathscr{X}_2)$. Finally, we pull back the point $\eta = 1 \in F_2 - (\Sigma_0 \cup F_1)$, so $\Sigma_0 + F_1 + 2F_2 + 2F_3 = H$ in $Pic(\mathscr{X}_3)$. Now we pull back $H = \{\sum a_j x_j = 0\}$ and have $\{a_0x_0^2 + a_1x_0x_1 + a_2(-x_0x_2 + x_1^2) = 0\}$, which is a quadric, so $L^*H = 2H + \sum m_j P_j$. We need to determine the a_j. For m_1, we pull back $\ell = \sum a_j x_j$ by $L \circ \pi_1$ and find $a_0\xi^2 s^2 + a_1 \xi s^2 + a_2(-\xi s + s^2)$. This vanishes to order 1 on $P_1 = \{s = 0\}$, so $m_1 = 1$. For m_2, we look at $\ell \circ L \circ \pi_1 \circ \pi_2$ to obtain the function $u^2\eta(a_0u^2\eta + a_1u\eta + a_2(-1 + \eta)$, which vanishes to order 2 on $P_2 = \{u = 0\}$. Thus $m_2 = 2$. For m_3, we pull back $\sum a_j x_j$ by $L \circ \pi_1 \circ \pi_2 \circ \pi_3$ to obtain $t^3(1 + t\mu)(a_0t(1 + t\mu) + a_1(1 + t\mu) + a_2\mu)$ which vanishes to order 3 on $F_3 = \{t = 0\}$, so $m_3 = 3$.

In conclusion, we see that L^* is given by

$$H \to 2H - F_1 - 2F_2 - 3F_3, \;\; P_1 \to P_1, \;\; P_2 \to P_2, \;\; P_3 \to \Sigma_0 = H - F_1 - 2F_2 - 2F_3$$

which corresponds to the matrix

$$\begin{pmatrix} 2 & 0 & 0 & 1 \\ -1 & 1 & 0 & -1 \\ -2 & 0 & 1 & -2 \\ -3 & 0 & 0 & -2 \end{pmatrix} \tag{8}$$

As expected, the square of this matrix is the identity.

Exercise: Define a new ordered basis $\langle H, E_1, E_2, E_3 \rangle$ for $Pic(\mathscr{X}_3)$ by setting $E_1 = F_1 + F_2 + F_3$, $E_2 = F_2 + F_3$, $E_3 = F_3$. Show that with respect to this basis, L^* is represented by the matrix (6).

Exercise: Perform an analysis on the maps $L_j(x,y) = (x, -y + x^j)$ for $j = 1, 2, 3, \ldots$

2.5 Linear Fractional Recurrences

Here we consider the possibility of producing a space X and an automorphism $f \in Aut(X)$ by the following procedure. That is, we consider a birational map f of projective space $\mathbf{P}^2$, and we would like to perform some blowups $\pi : X \to \mathbf{P}^2$ such that the induced map f_X will be an automorphism. Now if the original map f fails to be an automorphism, then there will be an exceptional curve C which is blown down to a point p_1. We can try to fix this by blowing up the point p_1 to produce a space $\pi_1 : X_1 \to \mathbf{P}^2$. If we are lucky, then the induced map f_{X_1} will not map C to a point but will map it to the whole curve P_1. However, this is only a temporary success if p_1 is not a point of indeterminacy, because f_{X_1} will map the fiber P_1 to the point $p_2 := f p_1$. Thus we see that the blowing-up procedure cannot work unless we have the property that any exceptional curve C in $\mathbf{P}^2$ is eventually mapped to a point of indeterminacy.

We spend this section looking at the example of linear fractional recurrences:

$$f(x,y) = \left(y, \frac{y+a}{x+b} \right). \tag{9}$$

See [BK3] for further discussion of this family. In projective coordinates this map takes the form

$$f[x_0 : x_1 : x_2] = [x_0(bx_0 + x_1) : x_2(bx_0 + x_1) : x_0(ax_0 + x_2)].$$

We explain that to obtain the homogeneous representation, we write

$$[x_0 : x_1 : x_2] = [1 : x_1/x_0 : x_2/x_0] = [1 : x : y]$$

and so

$$f = \left[1 : y : \frac{y+a}{x+b} \right] = \left[1 : \frac{x_2}{x_0} : \frac{\frac{x_2}{x_0} + a}{\frac{x_1}{x_0} + b} \right] = \left[1 : \frac{x_2}{x_0} : \frac{x_2 + ax_0}{x_1 + bx_0} \right].$$

Inspection shows that the exceptional curves are given by

$$\begin{aligned}
\Sigma_0 &:= \{x_0 = 0\} \mapsto p_1 := [0:1:0] \\
\Sigma_\beta &:= \{x+b=0\} = \{x_1 + bx_0 = 0\} \mapsto p_2 := [0:0:1] \\
\Sigma_\gamma &:= \{y+a=0\} = \{ax_0 + x_2 = 0\} \mapsto q := [1:-a:0].
\end{aligned}$$

This means that Σ_0 minus the indeterminacy locus maps to p_1, etc. In order to find the exceptional curves in a more systematic way, we may take the jacobian of the homogeneous form of f, which gives $2x_0(bx_0+x_1)(ax_0+x_2)$ and which shows us the exceptional curves directly.

If we set $p_* = (-b,-a) = [1:-b:-a]$, then the points of indeterminacy are given by

$$\begin{aligned}
p_2 &= \Sigma_0 \cap \Sigma_\beta \mapsto \overline{p_1 p_2} = \Sigma_0 \\
p_1 &= \Sigma_0 \cap \Sigma_\gamma \mapsto \overline{p_1 q} =: \Sigma_B \\
p_* &= \Sigma_\beta \cap \Sigma_\gamma \mapsto \overline{p_2 q} =: \Sigma_C
\end{aligned}$$

We note that the map f is not actually defined at the points of indeterminacy, and we interpret these formulas to mean $f^{-1} : \Sigma_0 \mapsto p_2$, etc.

Let $\pi : Y \to \mathbf{P}^2$ be the space obtained by blowing up p_1 and p_2. Let us show that Σ_0 is not exceptional for the induced map f_Y. We may use local coordinates $(t,x) \mapsto [t:x:1]$ at $\Sigma_0 = \{t=0\}$. Local coordinate chart U' at $p_1 = [0:1:0]$ is given by $(s_1,\eta_1) \mapsto [s_1 : 1 : s_1\eta_1]$. In these coordinates the map becomes:

$$\begin{aligned}
(t,x) \mapsto [t:x:1] \mapsto f[t:x:1] &= [f_0 : f_1 : f_2] \\
&= [f_0[t:x:1]/f_1[t:x:1] : 1 : f_2[t:x:1]/f_1[t:x:1]] \\
&= [t/x : 1 : (1+at)/(x(x+bt))] = [s_1 : 1 : s_1\eta_1]
\end{aligned}$$

So we have

$$(t,x) \mapsto (s_1,\eta_1) = (t/x, (1+at)(bt+x)),$$

which means that we have $\Sigma_0 = \{t=0\} \ni [0:x:1] \mapsto (0,1/x) \in P_1$, so Σ_0 is not exceptional. Similarly, we have

$$\Sigma_\beta \to P_2 \to \Sigma_0 \to P_1 \to \Sigma_B = \{y=0\}. \tag{10}$$

Exercise: Carry out the details of the remark concerning (10). In particular, show that the only exceptional curve for the rational map $f_Y : Y \dashrightarrow Y$ is Σ_γ, and the only point of indeterminacy is p_*.

Exercise: Compare Figures 7 and 10, and show that a map f corresponding to Figure 10 must be linearly conjugate to a map of the form $L \circ J$, where L is linear, and J is the involution from the previous section.

Now we can look at $H^2(Y) = \langle H, P_1, P_2 \rangle$. We observe that Σ_B is a line which contains exactly one center of blowup, namely p_1. Thus we have $\Sigma_B = H - P_1 \in H^2(Y)$. Similarly, since Σ_0 contains both p_1 and p_2, we have $\Sigma_0 = H - P_1 - P_2$.

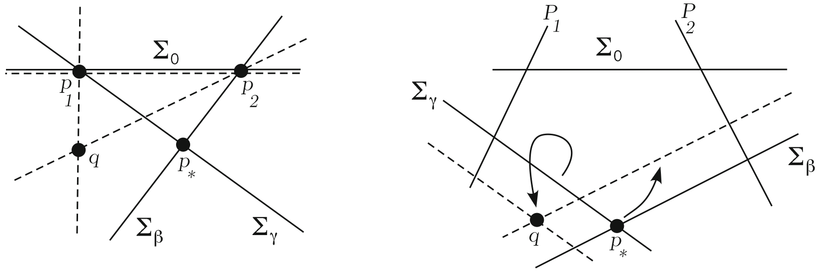

Fig. 10 Linear fractional recurrence on $\mathbf{P}^2$ and on $\mathscr{Y}$

Now, if we want to know what f_{Y*} does on $H^2(Y)$, we must determine the action on H. A generic line intersects Σ_0, Σ_β and Σ_γ. Thus fH will be a quadric passing through the images of these lines, namely p_1, p_2 and q. Thus $f_{Y*}H = 2H - P_1 - P_2$. Using (10), we have that f_{Y*} is given with respect to the ordered basis H, P_1, P_2 by the matrix

$$\begin{pmatrix} 2 & 1 & 1 \\ -1 & -1 & -1 \\ -1 & 0 & -1 \end{pmatrix}$$

The characteristic polynomial of this matrix is $t^3 - t - 1$.

The following is how we may use f_{Y*} to get information about f_Y. Let L be a line, and let $\{L\}$ denote its class in $Pic(Y)$. If L does not intersect P_1 or P_2, then $\{L\} = H$. We have just seen that $f_{Y*}H = 2H - P_1 - P_1$. The $2H$ means that $f_Y L$ has degree 2, and the $-P_1 - P_2$ means that it intersects P_1 and P_2, each with multiplicity 1. If L contains p_*, then $f_Y L$ is the union of two lines, one of which is Σ_C. In addition, if L is disjoint from P_1 and p_*, it must intersect Σ_γ, which means that $f_Y L$ must contain q.

Now suppose that $p_* \notin L \cup f_Y L$. Then $\{f_Y^2 L\} = (2, 0, -1) = 2H - P_2$ is obtained by multiplying the square of the matrix above by $H = (1, 0, 0)$. Thus $f_Y^2 L$ is a quadric intersecting P_2 but not P_1. Similarly, if $p_* \notin L \cup f_Y L \cup f_Y^2 L$, then we multiply $H = (1, 0, 0)$ by the cube of this matrix to find that $\{f_Y^3 L\} = 3H - P_1 - P_2$, so $f_Y^3 L$ is a cubic intersecting P_1 and P_2 with multiplicity 1. If $p_* \notin L \cup \cdots \cup f_Y^{n-1} L$, then the iterates of f_Y are holomorphic in a neighborhood of L, so we have $(f_Y^*)^n H = \{f_Y^n L\}$. We will say that the parameters a and b are *generic* if $p_* \notin \bigcup_{j=0}^{\infty} f_Y^j L$.

Theorem 2.10. *For generic a and b, we have $(f_Y^*)^n = (f_Y^n)^*$, and $\lambda(f) \sim 1.324\ldots$ is the largest root of the polynomial $t^3 - t - 1$.*

2.6 Linear Fractional Automorphisms

Let $C^2_{a,b}$ denote the parameter space of $a, b \in C$, and for $(a,b) \in C^2_{a,b}$ let $f_{a,b}$ be as in (9). We construct the space Y as in the previous section, and let $f_Y : Y \dashrightarrow Y$

be the associated birational map. The point $p_* = (-b,-a)$ is the unique point of indeterminacy for f_Y, and Σ_γ is the exceptional locus. Setting $q = (-a,0)$, let us define the following subset of parameter space:

$$\{\mathscr{V}_n = \{(a,b) \in C^2_{a,b} : f_Y^j q \neq p_*, 0 \leq j < n,\ f_Y^n q = p_*\}.$$

Theorem 2.11. *f_Y is a rational surface automorphism if and only if $(a,b) \in \mathscr{V}_n$ for some n.*

Proof. If $(a,b) \notin \mathscr{V}_n$ for any n, then by Theorem 2.10 we have that $\lambda(f_Y)$ is the largest root of $t^3 - t - 1$. This is not a Salem number, so $f_{a,b}$ is not an automorphism. Conversely, let us suppose that $(a,b) \in \mathscr{V}_n$ for some n. Let Z denote the manifold obtained by blowing up the $n+1$ points in the orbit q, $f_Y q, \ldots, f_Y^n q = p_*$. It follows that the induced map f_Z is an automorphism. □

The action of f_{Z*} on cohomology is given by:

$$P_2 \to \Sigma_0 = H - P_1 - P_2 \to P_1 \to \Sigma_B = H - P_1 - Q \tag{11}$$

which is like what we have seen already from the action of f_Y, except that now the point $q \in \Sigma_B$ has been blown up, so we must subtract Q to obtain the representation of $\Sigma_B = \{y = 0\}$ as an element of $Pic(Z)$. The behavior of the new blowup fibers

$$Q \to fQ \to \cdots \to f^n Q = P_* \to \Sigma_C = \overline{p_2 q} = H - P_2 - Q \tag{12}$$

Finally, since a generic line L intersects all three lines Σ_0, Σ_β, and Σ_γ with multiplicity one, the image fL will be a quadric passing through e_2, e_1, and q. Thus we have

$$H \to 2H - P_1 - P_2 - Q. \tag{13}$$

Theorem 2.12. *If $(a,b) \in \mathscr{V}_n$, then the characteristic polynomial of f_{Z*} is*

$$\chi_n = x^{n+1}(x^3 - x - 1) + x^3 + x^2 - 1.$$

Thus $\delta(f) = \lambda_n$, which is the largest root of χ_n, and $\lambda_n > 1$ if $n \geq 7$.

We note that λ_n increases to the number $\lambda \sim 1.324\ldots$ from the previous section. An interesting consideration is to ask whether $f_{a,b}$ has an invariant curve. The maps which posses invariant curves have a number of interesting properties; we describe one of them below.

There are rational functions $\varphi_j : \mathbf{C} \to \mathbf{C}^2_{a,b}$ such that if $(a,b) = \varphi_j(t)$ for some $t \in \mathbf{C}$, then $f_{a,b}$ has an invariant curve S with j irreducible components. The curve S is a singular cubic. For instance, the first of these functions is

$$\varphi_1(t) = \left(\frac{t - t^3 - t^4}{1 + 2t + t^2}, \frac{1 - t^5}{t^2 + t^3} \right).$$

Theorem 2.13. *Suppose that $(a,b) = \varphi_j(t)$, and j divides n. Then $(a,b) \in \mathscr{V}_n$ if and only if $\chi_n(t) = 0$.*

In every case, the map $f_{a,b}$ has two fixed points. If $(a,b) = \varphi_1(t)$, then one of the fixed points is (x_s, y_s) with $x_s = y_s = t^3/(1+t)$ and is the singular point of S. (There are similar formulas for $j = 2$ and 3.) Thus multipliers of $df_{a,b}$ at (x_s, y_s) are t^2 and t^3.

Now we consider the case where t is a root of χ_n. We may factor $\chi_n = \psi_n \cdot C$ where ψ_n is the minimal polynomial for λ_n, and C is cyclotomic. As we saw earlier, λ_n and λ_n^{-1} are roots of ψ_n, and all the other roots of ψ_n have modulus one and are not roots of unity. In fact, the degree of the cyclotomic may be shown to be bounded by 26. Thus almost all of the roots of χ_n will have modulus 1 and not be roots of unity.

Theorem 2.14. *Suppose that $n \geq 7$, t is a root of χ_n with $|t| = 1$, and t is not a root of unity. If $(a,b) = \varphi_1(t)$, then $f_{a,b}$ has a rank 1 rotation domain about the fixed point (x_s, y_x).*

Notes The linear fractional automorphisms are discussed in [BK1, 2] and [M]. Other automorphisms in the same general spirit have been found by J. Diller [Di].

2.7 Cremona Representation

Let us discuss the representation $\rho : Aut(X) \to GL(H^2(X;\mathbf{Z}))$ which is given by $\rho(f) = f_*$. It is evident that the image of ρ consists of isometries with respect to the intersection product, as well as some other relevant properties. In the most familiar case $X = \mathbf{P}^2$, we see that $\rho(Aut(\mathbf{P}^2))$ consists only of the identity element acting on $H^2(\mathbf{P}^2;\mathbf{Z}) \cong \mathbf{Z}$. Similarly, $\rho(Aut(\mathbf{P}^1 \times \mathbf{P}^1))$ consists of two elements: the identity, and the interchange of coordinates on $\mathbf{Z} \times \mathbf{Z}$.

Exercise: Let $\tilde{\mathbf{P}}^2_j$ denote $\mathbf{P}^2$ blown up at j points in general position, for $1 \leq j \leq 4$. Determine $Aut(\tilde{\mathbf{P}}^2_j)$ and its image under ρ. What happens if the points are not in general position?

The previous exercise shows that if we blow up a small number of points p_j, $1 \leq j \leq 4$ in $\mathbf{P}^2$, then the possible automorphisms are relatively limited. What happens if we also blow up a point of the exceptional fiber P_j over p_j?

Exercise: Let Z_1 be $\mathbf{P}^2$ blown up at a point $p_1 \in \mathbf{P}^2$, and let $p_2, p_3 \in P_1$ denote two points of the blowup fiber P_1 over p_1. Let Z denote the space Z_1 blown up at the points p_2 and p_3. What is the automorphism group of Z? (One way of visualizing Z_1 is as follows. Let $[x:y:z]$ be coordinates on $\mathbf{P}^2$, and let $p_1 = [1:0:0]$ be the point where the x-axis intersects $\{z=0\}$, which we take to be the line at infinity. The points of the blowup fiber P_1 are then identified with the points where the horizontal lines $L_c = \{[x:y:1] : y = c\}$ intersect $\{z=0\}$.)

The automorphism group is limited in the exercises above is because we have created a finite number of rational curves with self intersection -1, and these curves must be mapped around among themselves. That is, for each pair p_1, p_2 of centers of blowup, the strict transform of the line $\overline{p_1 p_2}$ will have self intersection ≤ -1 in X (or more precisely, the self intersection is $1-k$, where k is the number of indices j such that $p_j \in \overline{p_1 p_2}$). If Q is a quadric containing exactly 5 centers of blowup, then the strict transform $\tilde{Q}$ will have self-intersection -1. And as we have seen in the previous section, there can be infinitely many curves with self-intersection -1 if we blow up ≥ 9 "correctly chosen" points. Let us recall a result of Nagata.

Theorem 2.15. *(Nagata) Let X be a rational surface, and let $f \in Aut(X)$ be an automorphism for which f_* has infinite order. Then there is a sequence of holomorphic maps $\pi_{j+1} : X_{j+1} \to X_j$ such that $X_1 = \mathbf{P}^2$, $X_{N+1} = X$, and π_{j+1} is the blowup of a point $p_j \in X_j$.*

This gives a very useful starting point if we are looking for rational surface automorphisms. It says that every one must be given by a "model" birational map of $\mathbf{P}^2$. That is, if $F \in Aut(X)$, then the (birational) projection $\pi = \pi_1 \circ \cdots \circ \pi_{N+1}$ conjugates f to a birational map f of $\mathbf{P}^2$. Or conversely, we can start with birational maps of $\mathbf{P}^2$ which are promising candidates and see whether there might be a blowup X which turns them into automorphisms.

There are further limitations on the image $\rho(Aut(X))$ in $GL(H^2(X;\mathbf{Z}))$. For this, we need to start with a good basis for H^2. Specifically, let us consider a basis $\{e_0, \ldots, e_N\}$ for $H^2(X;\mathbf{Z})$ such that the intersection form with respect to this basis is given by the diagonal matrix with eigenvalues $1, -1, \ldots, -1$. Such a basis is called a *geometric basis*, and has the properties: $e_0 \cdot e_0 = 1$, $e_j \cdot e_j = -1$ if $1 \le j \le N$, and $e_i \cdot e_j = 0$ if $i \ne j$. If X is obtained by blowing up N distinct points of $\mathbf{P}^2$ as in section §B1, then the basis $\langle H_X, P_1, \ldots, P_N \rangle$ as in the Theorem B.1.1 is geometric. Now let us consider a space $X = X_{N+1} \to X_N \to \cdots \to X_1 = \mathbf{P}^2$ as in Nagata's theorem. Let $P_j = \pi_{j+1}^{-1} p_j \subset X_{j+1}$ denote the exceptional fiber, and let $e_j := \pi_{N+1}^* \cdots \pi_{j+1}^* P_j$.

Exercise: Show that if X is as above, and if we set $e_0 = H_X$, then $e_0, \ldots, e_N$ is a geometric basis. Try this first on the space Z constructed in the exercise above. Show that in Z we have $e_1 = P_1 + P_2 + 2P_3$, $e_2 = P_2 + P_3$, $e_3 = P_3$ and that $P_1 \cdot P_1 = -3$, $P_2 \cdot P_2 = -2$. From this, conclude that $\{e_j, 0 \le j \le 3\}$ is a geometric basis.

Exercise: Let X be as above. Show that $\langle H_X, P_1, \ldots, P_N \rangle$ and $\langle e_0, e_1, \ldots, e_N \rangle$ are dual bases. That is, $H_X \cdot e_0 = 1$, $P_j \cdot e_k = 0$ if $j \ne k$, and $P_j \cdot e_j = -1$ for $1 \le j \le N$.

If α is an element of H^2 such that $\alpha \cdot \alpha = -2$, then $R_\alpha(x) = x - (x \cdot \alpha)\alpha$ is a reflection in the direction α: this means that R_α sends $\alpha \to -\alpha$, and R_α fixes all elements of $\alpha^\perp$.

Let us define the vectors

$$\alpha_0 = e_0 - e_1 - e_2 - e_3, \quad \alpha_j = e_{j+1} - e_j, \ 1 \le j \le N-1.$$

It follows that $\alpha_j \cdot \alpha_j = -2$ for $0 \le j \le N-1$. For $1 \le j \le N-1$, the reflection R_{α_j} interchanges $e_j \leftrightarrow e_{j+1}$. Thus the subgroup generated by R_{α_j}, $1 \le j \le N-1$, is exactly the set of permutations on the elements $\{e_1, \ldots, e_N\}$. The reflection R_{α_0}

corresponds to the Cremona inversion. That is, if we write the action of R_{e_0} on the subspace with ordered basis $\langle e_0, e_1, e_2, e_3\rangle$, then the restriction of R_{e_0} is represented by the matrix:

$$\begin{pmatrix} 2 & 1 & 1 & 1 \\ -1 & 0 & -1 & -1 \\ -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & 0 \end{pmatrix}$$

In §2.3, we saw that after the involutions J, K, and L have been regularized to become automorphisms, the actions of J^*, K^* and L^* can all be written in the form of this matrix.

Let W_N denote the group generated by the reflections R_{α_j}, $0 \le j \le N-1$. Thus W_N is generated by the permutations of the e_j's, $1 \le j \le N$, together with the Cremona inversion. The following result is classical (see [Do, Theorem 5.2]):

Theorem 2.16. *Let X be a rational surface, and let $e_0, \ldots, e_N$ be a geometric basis for $H^2(X)$. If $f \in Aut(X)$, then $f_* \in W_N$.*

Problem: It remains unknown which elements of W_N can arise from rational surface automorphisms.

Let us return to the linear fractional automorphisms and see how f^* is related to W_N. Since the space $\mathscr{Z}$ was constructed by simple blowups, we have a geometric basis for $Pic(Z)$ by letting e_0 be the class of a general line and then setting $e_1 = P$, $e_2 = E_2$, $e_3 = E_1$, $e_4 = Q$, $\ldots$, $e_j = f^{j-4}Q$, $4 \le j \le N-4$. We see that cyclic permutation $\sigma = (123\ldots N)$ is equal to the composition $R_{\alpha_1} \circ R_{\alpha_2} \circ \cdots \circ R_{\alpha_{N-1}}$, and that f_* itself (cf. equation (11-13)) is equal to $R_{\alpha_0} \circ \sigma = R_{\alpha_0} \circ \cdots R_{\alpha_{N-1}}$ is a product of the reflections that generate W_N.

2.8 More Automorphisms

We give some more examples of rational surface automorphisms of positive entropy. Our goal here is to give families of maps which illustrate that such automorphisms can occur in continuous families of arbitrarily high dimension. Namely, for each even k we give a family $\{f_a : a \in \mathbf{C}^{\frac{k}{2}-1}\}$ of birational maps of $\mathbf{P}^2$, and for each map f_a there is a rational surface $\pi : \mathscr{X}_a \to \mathbf{P}^2$, and f_a lifts to an automorphism of $\mathscr{X}_a$. We note that the complex structures of the surfaces $\mathscr{X}_a$ are allowed to vary with a, but the smooth structures are locally constant. The smooth dynamical systems $(f_a, \mathscr{X}_a)$, however, may be shown to vary nontrivially with a. The maps we discuss are

$$f(x,y) = \left(y, -x + cy + \sum_{\substack{\ell=2 \\ \ell \text{ even}}}^{k-2} \frac{a_\ell}{y^\ell} + \frac{1}{y^k} \right) \tag{14}$$

where the sum is taken only over even values of ℓ.

We will describe a number of properties of the maps f_a; the details are given in [BK4], and we do not repeat them here. Each f_a maps the line $\{y=0\}$ to the point $[0:1:0]\in\mathbf{P}^2$, and f_a^{-1} maps $\{x=0\}$ to $[1:0:0]$. The line at infinity $\{z=0\}$ is invariant and is mapped according to $f[1:w:0]=[1:c-1/w:0]$. For $1\le s$ we set $f^s[0:1:0]=[1:w_s:0]$. We note that if $(j,n)=1$ and if we set $c=\pm 2\cos(j\pi/n)$, then $f^{n-1}[0:1:0]=[1:0:0]$, and the restriction of f to $\{z=0\}$ has period n. We define $C_n=\{\pm 2\cos(j\pi/n):(j,n)=1\}$, where we choose "+", "−", or both, according to the condition that $w_1\cdots w_{n-2}=1$. With $c\in C_n$, we obtain the surface $\mathscr{X}_a$ by performing $2k+1$ iterated blowups over each point $[1:w_s:0]$, $0\le s\le n-1$. The fibers over $[1:w_s:0]$ are denoted $\mathscr{F}_s^j$, $1\le j\le 2k+1$. From this construction, it follows that the fibers map as follows:

$$\mathscr{F}_0^1\to\cdots\to\mathscr{F}_s^1\to\mathscr{F}_{s+1}^1\to\cdots\to\mathscr{F}_{n-1}^1\to\mathscr{F}_0^1$$

$$\mathscr{F}_0^j\to\cdots\to\mathscr{F}_s^j\to\mathscr{F}_{s+1}^j\to\cdots\to\mathscr{F}_{n-1}^j\to\mathscr{F}_0^{2k+2-j}\to\cdots\to\mathscr{F}_{n-1}^{2k+2-j}\to\mathscr{F}_0^{2k+2-j}$$

$$\{y=0\}\to\mathscr{F}_0^{2k+1}\to\cdots\to\mathscr{F}_{n-1}^{2k+1}\to\{x=0\}$$

A further observation is that these maps give rational surface automorphisms:

Theorem 2.17. *Let $1\le j<n$ satisfy $(j,n)=1$. There is a nonempty set $C_n\subset\mathbf{R}$ such that for even $k\ge 2$ and for all choices of $c\in C_n$ and $a_\ell\in\mathbf{C}$, the map f in (14) is an automorphism.*

Now we let S denote the subgroup of $Pic(\mathscr{X}_a)$ spanned by $\{z=0\}$ and the fibers $\mathscr{F}_s^j$, $0\le s\le n-1$, $1\le j\le 2k+1$. From [BK4] we also have:

Proposition 2.18. *The intersection form of $\mathscr{X}_a$, when restricted to S, is negative definite.*

We let $T:=S^\perp$ denote the vectors of $Pic(\mathscr{X}_a)$ which are orthogonal to S. By the Proposition, we see that $S\cap T=0$, so we have $Pic(\mathscr{X}_a)=S\oplus T$. Since S is invariant, T is also invariant. For each $0\le s\le n-1$, we let γ_s denote the projection of the class $\{\mathscr{F}_s^{2k+1}\}\in Pic(\mathscr{X})$ to T. Thus the γ_s, $0\le s\le n-1$ give a basis of T. Following $\{x=0\}$ through the various blowups in the construction of $\mathscr{X}$, we may show that $\{x=0\}=-\gamma_0+k\gamma_1+\cdots k\gamma_{n-1}$. And from the mapping of the fibers, we see that f_* maps according to

$$\gamma_0\to\gamma_1\to\cdots\to\gamma_{n-1}\to\{x=0\}=-\gamma_0+k\gamma_1+\cdots+k\gamma_{n-1}. \tag{15}$$

Computing the characteristic polynomial for the transformation (14), we obtain:

Theorem 2.19. *The dynamical degree $\delta(f_a)$ is the largest root of the polynomial*

$$\chi_{n,k}(x)=1-k\sum_{\ell=1}^{n-1}x^\ell+x^n. \tag{16}$$

We have seen that Σ_0 is invariant under f_a, and from the mapping of the fibers we see that certain unions of the fibers are invariant. One consequence of the Proposition is the following: If C is an irreducible curve in $\mathscr{X}$, and if the class of its divisor $\{C\}$ belongs to S, then C must be one of the curves Σ_0 or $\mathscr{F}_s^j$, $0 \le s \le n-1$, $1 \le j \le 2k$. A consequence of this is:

Proposition 2.20. *If C is a curve which is invariant under f_a, then C is a union of components from the collection Σ_0 and $\mathscr{F}_s^j$, $0 \le s \le n-1$, $1 \le j \le 2k$.*

Proof. Let t denote the projection of the class of C to T. Since C is invariant, we have $f_*C = C$, and thus $f_*t = t$. But since $\chi_{n,k}(1) = 2 - k(n-1) < 0$, 1 is not an eigenvalue of f_*. Thus $t = 0$, which means that $C \in S$. And we saw above that each element of S is represented uniquely as a union of the generating curves. □

At this point, it may be evident that there is a certain amount of arbitrariness in our choice of blowups. Specifically, if $f \in Aut(\mathscr{X})$, and $a, fa, \ldots, f^k a = a$ is an orbit of period k, then we may construct a new space $\pi : \mathscr{Y} \to \mathscr{X}$ by blowing up this orbit. The induced map $f_{\mathscr{Y}}$ is an automorphism of $\mathscr{Y}$. If $\mathscr{Z}$ is a smooth surface, and $f \in Aut(\mathscr{Z})$, we say that the dynamical system $(f, \mathscr{Z})$ is *minimal* if whenever $\mathscr{W}$ is a smooth surface and $g \in Aut(\mathscr{W})$, and whenever $\pi : \mathscr{Z} \to \mathscr{W}$ is a regular, birational map such that $g \circ \pi = \pi \circ f$, then π is a biregular conjugacy. It is evident that if we blow up a periodic orbit, then the resulting dynamical system is not minimal. On the other hand, we do not obtain uniqueness simply by requiring that our model be minimal. For instance, let $J(x,y) = (1/x, 1/y)$ be the Cremona involution. We have seen that the space $\mathscr{X}$ obtained by blowing up three points gives $J_{\mathscr{X}} \in Aut(\mathscr{X})$. In addition, it is clear that if $\mathscr{Y} = \mathbf{P}^1 \times \mathbf{P}^1$, then $J_{\mathscr{Y}} \in Aut(\mathscr{Y})$.

Here is another example. Let $\mathscr{X}$ denote the space obtained by blowing up $\mathbf{P}^2$ at $[1:1:1]$, and let L denote the involution $[x:y:z] \to [-x:y:z]$, acting on $\mathscr{X}$. Thus the fiber over $[1:1:1]$ is exceptional and is blown down to $[-1:1:1]$, which is indeterminate. We can turn L into an automorphism by blowing up $\mathscr{X}$ at the point $[-1:1:1]$. The result, of course, is not minimal, since we can now blow both exceptional fibers back down, and L will be a linear automorphism of $\mathbf{P}^2$.

Theorem 2.21. *$(f, \mathscr{X}_f)$ is minimal if $n > 2$. If $n = 2$, then it becomes minimal after we blow down Σ_0.*

Proof. Suppose that $\varphi : \mathscr{X}_f \to \mathscr{Y}$ is a morphism. Consider the curve $\mathscr{C}$ consisting of all the varieties in $\mathscr{X}_f$ which are blown down to points under φ. It follows that $\mathscr{C}$ is invariant under f, so by Theorem 3.5, $\mathscr{C}$ must be a union of components coming from Σ_0 and $\mathscr{F}_s^j$. If $n > 2$, then the self-intersection of each of the components Σ_0 and $\mathscr{F}_s^j$ is ≤ -2, so it is not possible to blow any of them down. On the other hand, if $n = 2$, then the self-intersection of Σ_0 is -1, so we can blow it down. This leaves the self-intersection of all the other fibers unchanged, except for $\mathscr{F}_s^1$, which increases to $-k$. This is strictly less than -1, so nothing further can be blown down. □

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Holomorphic Dynamical Systems
Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy, July 7-12, 2008
Sibony, N.; Schleicher, D.; Cuong, D.T.; Brunella, M.; Bedford, E.; Abate, M. - Gentili, G.; Patrizio, G.; Guenot, J. (Eds.)
2010, XII, 348 p., Softcover
ISBN: 978-3-642-13170-7