# Preface

The theory of holomorphic dynamical systems is a subject of increasing interest in mathematics, both for its challenging problems and for its connections with other branches of pure and applied mathematics.

A holomorphic dynamical system is the datum of a complex variety and a holomorphic object (such as a self-map or a vector field) acting on it. The study of a holomorphic dynamical system consists in describing the asymptotic behavior of the system, associating it with some invariant objects (easy to compute) which describe the dynamics and classify the possible holomorphic dynamical systems supported by a given manifold. The behavior of a holomorphic dynamical system is pretty much related to the geometry of the ambient manifold (for instance, hyperbolic manifolds do no admit chaotic behavior, while projective manifolds have a variety of different chaotic pictures). The techniques used to tackle such problems are of various kinds: complex analysis, methods of real analysis, pluripotential theory, algebraic geometry, differential geometry, topology.

To cover all the possible points of view of the subject in a unique occasion has become almost impossible, and the CIME session in Cetraro on Holomorphic Dynamical Systems was not an exception. On the other hand the selection of the topics and of the speakers made it possible to focus on a number of important topics in the discrete and in the continuous setting, both for the local and for the global aspects, providing a fascinating introduction to many key problems of the current research. The CIME Course aimed to give an ample description of the phenomena occurring in central themes of holomorphic dynamics such as automorphisms and meromorphic self-maps of projective spaces, of entire maps on complex spaces and holomorphic foliations in surfaces and higher dimensional manifolds, enlightening the different techniques used and bringing the audience to the borderline of current research topics. This program, with its interdisciplinary characterization, drew the attention and the participation of young researchers and experienced mathematicians coming from different backgrounds: complex analysis and geometry, topology, ordinary differential equations and number theory. We are sure that the present volume will serve the same purpose. We briefly describe here the papers that stemmed from the courses and constitute the Chapters of this volume.

In his lectures, Marco Abate outlines the local theory of iteration in one and several variables. He studies the structure of the stable set $K_f$ of a selfmap $f$ of a neighborhood $U$ of a fixed point, describing both the topological structure of $K_f$ and the dynamical nature of the (global) dynamical system $(K_f, f_{|K_f})$. One important way to study a local holomorphic dynamical system consists in replacing it by an equivalent but simpler system. Following a traditional approach, Abate considers three equivalence relations - topological, holomorphic and formal conjugacy - and discusses normal forms and invariants in all these cases. He starts surveying the one-dimensional theory, which is fairly complete, even though there are still some open problems, and then he presents what is known in the multidimensional case, that is an exciting mixture of deep results and still unanswered very natural questions.

The lectures of Eric Bedford provide an introduction to the dynamics of the automorphisms of rational surfaces. The first part is devoted to polynomial automorphisms of $\mathbf{C}^2$ and in particular to the complex Hénon maps, the most heavily studied family of invertible holomorphic maps. The investigations of the Hénon maps can be guided by the study of the dynamics of polynomial maps of one variable, a very rich and classical topic. Although the Hénon family is only partially understood, its methods and results provide motivation and guidance for the understanding of other types of automorphisms. In the second part of the notes, Bedford considers the geometry of compact rational surfaces with the illustration of some examples of their automorphisms. In contrast with the case of the polynomial automorphisms of $\mathbf{C}^2$, not much is known about neither the set of all rational surface automorphisms, nor about a dynamical classification of them.

The theory of foliations by Riemann surfaces is central in the study of continuous aspects of holomorphic dynamics. In his lecture notes, Marco Brunella describes the state of the art of the topic and reports on his results on the uniformisation theory of foliations by curves on compact Kähler manifolds. Each leaf is uniformized either by the unit disk or by $\mathbf{C}$ or by the projective line. Brunella explains how the universal covers may be patched together to form a complex manifold with good properties and he studies the analytic properties of this manifold, in particular regarding holomorphic convexity. In turn this leads to results on the distribution of parabolic leaves inside the foliation and to positivity statements concerning the canonical bundle of the foliation, generalizing results of Arakelov on fibrations by algebraic curves.

Sibony's course in the CIME session was based on the lecture notes by Tien-Cuong Dinh and Nessim Sibony that are included in this volume. This contribution, which could be a stand alone monograph for depth and extension, gives a broad presentation to the most recent developments of pluripotential methods, and to the theory of positive closed currents, in dynamics in Several Complex Variables. The notes concentrate on the dynamics of endomorphisms of projective spaces and the polynomial-like mappings. Green currents and equilibrium measure are constructed to study quantitative properties and speed of convergence for endomorphisms of projective spaces; equidistribution problems and ergodic properties are also treated. For polynomial-like mappings, the equilibrium measure of maximal entropy is constructed and equidistribution properties of points are proved, under suitable dynamical degree assumptions. The tools introduced

here are of independent interest and can be applied in other dynamical problems. The presentation includes all the necessary prerequisites about plurisubharmonic functions and currents, making the text self-contained and quite accessible.

In his lectures, Schleicher studies iteration theory of entire holomorphic functions in one complex variable, a field of research that has been quite active in recent years. A review of dynamics of entire maps, which includes the classical and well developed theory of polynomial dynamics, serves to introduce the main topic: the consideration of transcendental maps. The notes study key dynamical properties of large classes of transcendental functions and of special prototypical families of entire maps such as the exponential family $z \mapsto \lambda e^z$ or the cosine family $z \mapsto ae^z + be^{-z}$. It turns out that some aspects of the dynamics of transcendental entire maps are inspired by the polynomial theory, others are very different and exploit all the power of deep results from complex analysis. Transcendental dynamics turns out to be a largely yet unexplored and fascinating area of research where surprising mathematical results - that sometimes had been constructed artificially in other branches of mathematics - arise in a natural way.

It is a real pleasure to thank the speakers for their great lectures and all the authors for the beautiful contributions to this volume. We also like to thank all the participants for their interest and enthusiasm that created a very warm and stimulating scientific atmosphere. We like to express our gratitude to CIME for sponsoring the summer school, and to the LAMI (Laboratorio di Applicazioni della Matematica all'Ingegneria, Università della Calabria) for its financial contribution. Our particular gratitude goes to Pietro Zecca and Elvira Mascolo for their continuous support, and to their collaborators Carla Dionisi and Maria Giulia Bartaloni for their invaluable help.

Graziano Gentili, Jacques Guenot and Giorgio Patrizio

http://www.springer.com/978-3-642-13170-7

Holomorphic Dynamical Systems
Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy, July 7-12, 2008
Sibony, N.; Schleicher, D.; Cuong, D.T.; Brunella, M.; Bedford, E.; Abate, M. - Gentili, G.; Patrizio, G.; Guenot, J. (Eds.)
2010, XII, 348 p., Softcover
ISBN: 978-3-642-13170-7